Title	Double versus triple competitive processes : non-deterministic model (Mathematical Economics)
Author(s)	Kira, Akifumi; Iwamoto, Seiichi; Fujita, Toshiharu
Citation	数理解析研究所講究録 (2009), 1654: 19-34
Issue Date	2009-06
URL	http://hdl.handle.net/2433/140846
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
第 1654 巻 2009 年 19-34

Double versus triple competitive processes : non-deterministic model

Akifumi Kira and Seiichi Iwamoto
Graduate School of Economics
Kyushu University
6-19-1 Hakozaki, Higashi-ku
Fukuoka 812-8581, Japan
E-mail address: ec207004@s.kyushu-u.ac.jp
and
Toshiharu Fujita
Faculty of Engineering
Kyushu Institute of Technology
1-1 Sensui-cho, Tobata-ku
Kitakyusyu 804-8550, Japan
E-mail address: fujita@comp.kyutech.ac.jp

Abstract

In a continuous model of nondeterministic dynamic programming, we arrive at both the optimal value function and its optimal policy by solving a corresponding controlled integral equation. However it is generally difficult to solve the equation. In this paper, we start from a simple linear case:

$$v(x)=\min_{a\in\{2,3\}}\left[ax+\frac{1}{a}\int_0^1 v(y)\,\mathrm{d}y\right],\quad x\in\mathbf{R}.$$

We consider some models on continuous state space in which 'double process' and 'triple process' compete with each other, as can be seen in the above equation. By solving controlled integral equations, we show the structure of their optimization processes.

Keywords: controlled integral equation, nondeterministic dynamic programming
JEL classification: C6

1 Introduction

Dynamic programming (DP) has been extensively studied and applied to a lot of different kinds of fields in the past. Today, stochastic DP (Markov decision process) plays a vital role in the field of dynamic macroeconomic theory [4, 8, 9]. On the other hand, in the field of mathmatical programming and control, the theory of nondeterministic DP has recently been proposed (The original idea is stated in Bellman[1, Chap.IV] in order to apply the method of DP to many more problems. The formulation of nondeterministic DP is quite broad, as it deal with stochastic DP as special case. We can obtain both the optimal value function and its optimal policy by solving a corresponding Bellman equation. Generally the Bellman equation is given by the following controlled integral equation

$$v(x) = \inf_{a \in \mathcal{A}(x)} \left[r(x,a) + \int_{T(x,a)} \beta(x,a,y) v(y)\, dy \right], \quad x \in S. \tag{1}$$

If $\#\mathcal{A}(x) \equiv 1$, then the above equation is reduced to

$$v(x) = r(x) + \int_{T(x)} \beta(x,y) v(y)\, dy, \quad x \in S.$$

which is an ordinary linear integral equation. Needress to say, it is generally difficult to solve the equations. In this paper, we shall start from a simple equation which can be solved. After doing so, by solving some related equations, we show the structure of their optimization processes, and view the variations of their optimal value functions and the optimal policies. As a paticularly simple case, we consider the following equation.

$$v(x) = mx + \frac{1}{m} \int_0^1 v(y)\, dy, \quad x \in \mathbf{R}, \tag{2}$$

where $m > 1$ is a given constant real number. Eq.(2) has the unique solution

$$v(x) = mx + \frac{m}{2(m-1)}, \quad x \in \mathbf{R}.$$

Then, let us consider a decision process on $\mathcal{A} \equiv \mathcal{A}(x) \equiv \{2,3\}$.

Example 1.1. $\mathcal{D} = (S, (\mathcal{A}, \mathcal{A}(\cdot)), r, T, \beta)$, *where*

$$S = \mathbf{R}, \quad \mathcal{A} \equiv \mathcal{A}(x) \equiv \{2,3\}, \quad r(x,a) = ax, \quad T(x,a) \equiv [0,1], \quad \beta(x,a,y) = \frac{1}{a}.$$

(minimization) The Bellman equation

$$v(x) = \min_{a \in \{2,3\}} \left[ax + \frac{1}{a} \int_0^1 v(y)\, dy \right] \quad x \in \mathbf{R} \tag{3}$$

has the solution

$$v(x) = \begin{cases} 3x + 2(\sqrt{11} - 3) & \text{on} \quad (-\infty, \sqrt{11} - 3] \\ 2x + 3(\sqrt{11} - 3) & \text{on} \quad [\sqrt{11} - 3, \infty) \end{cases}$$

$$\fallingdotseq \begin{cases} 3x + 0.6332 & \text{on} \quad (-\infty, 0.3166] \\ 2x + 0.9499 & \text{on} \quad [0.3166, \infty). \end{cases}$$

Thus $\mathcal{D}$ has the minimum value function $v(x)$ at the stationary Markov policy $\hat{\pi} = \hat{f}^{(\infty)}$, where

$$\hat{f}(x) = \begin{cases} 3 & \text{on} \quad (-\infty, \sqrt{11} - 3] \\ 2 & \text{on} \quad [\sqrt{11} - 3, \infty) \end{cases} \fallingdotseq \begin{cases} 3 & \text{on} \quad (-\infty, 0.3166] \\ 2 & \text{on} \quad [0.3166, \infty). \end{cases}$$

(Maximization) The Bellman equation

$$V(x) = \operatorname*{Max}_{a \in \{2,3\}} \left[ax + \frac{1}{a} \int_0^1 V(y)\, dy \right] \quad x \in \mathbb{R} \tag{4}$$

has the solution

$$V(x) = \begin{cases} 2x + 3(4 - \sqrt{13}) & \text{on} \quad (-\infty, 4 - \sqrt{13}] \\ 3x + 2(4 - \sqrt{13}) & \text{on} \quad [4 - \sqrt{13}, \infty) \end{cases}$$

$$\fallingdotseq \begin{cases} 2x + 1.1833 & \text{on} \quad (-\infty, 0.3944] \\ 3x + 0.7889 & \text{on} \quad [0.3944, \infty)\,. \end{cases}$$

Thus $\mathcal{D}$ has the maximum value function $V(x)$ at the stationary Markov policy $\pi^* = (f^*)^{(\infty)}$, where

$$f^*(x) = \begin{cases} 2 & \text{on} \quad (-\infty, 4 - \sqrt{13}] \\ 3 & \text{on} \quad [4 - \sqrt{13}, \infty) \end{cases} \fallingdotseq \begin{cases} 2 & \text{on} \quad (-\infty, 0.3944] \\ 3 & \text{on} \quad [0.3944, \infty)\,. \end{cases}$$

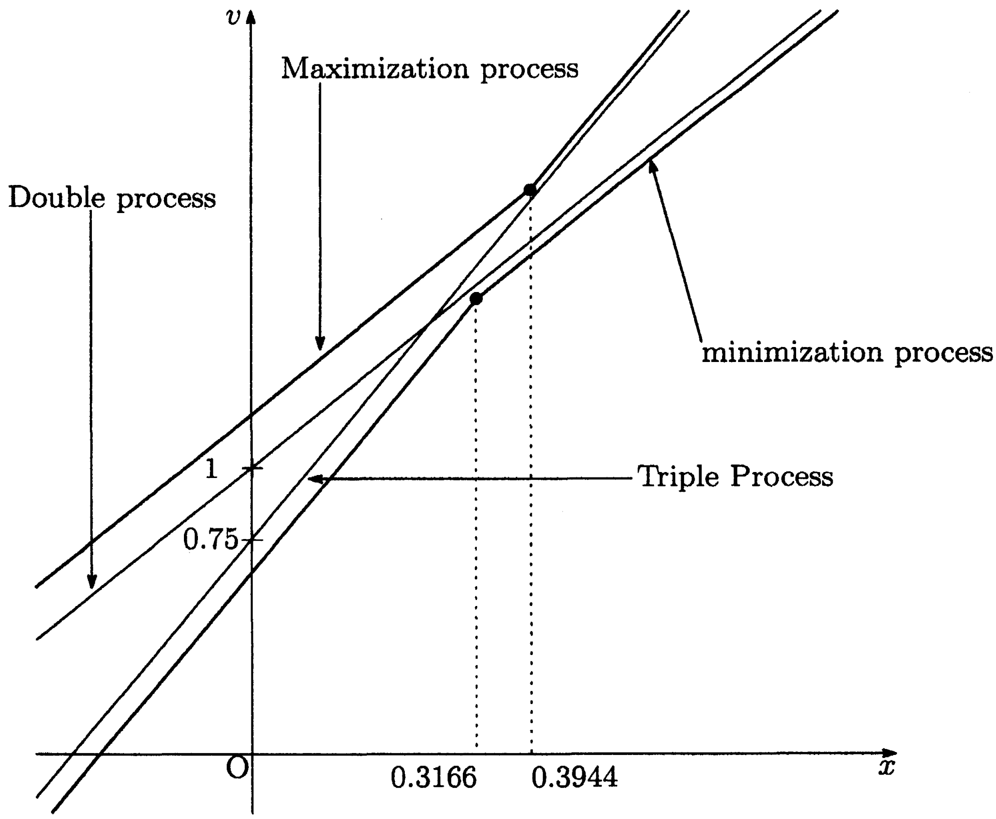

Figure 1: Four value functions in linear case

2 Action Space

The question we have to ask here is how optimal value functions are improved when feasible actions are added. We will now view the question through some examples.

2.1 Action Space with Three Actions

Example 2.1. $\mathcal{D} = (S,\ (\mathcal{A},\ \mathcal{A}(\cdot)),\ r,\ T,\ \beta)$, *where*

$$S = \mathbf{R}, \quad \mathcal{A} \equiv \mathcal{A}(x) \equiv \left\{2,\ \frac{5}{2},\ 3\right\}, \quad r(x,a) = ax, \quad T(x,a) \equiv [0,1], \quad \beta(x,a,y) = \frac{1}{a}.$$

(minimization) The Bellman equation

$$v(x) = \min_{a \in \{2,\, 2/5,\, 3\}} \left[ax + \frac{1}{a} \int_0^1 v(y)\,\mathrm{d}y \right], \quad x \in \mathbf{R} \tag{5}$$

has the solution

$$v(x) = \begin{cases} 3x + \dfrac{5}{13}(\sqrt{277} - 15) & \text{on} \quad \left(-\infty,\ \dfrac{2}{13}(\sqrt{277} - 15)\right] \\ \dfrac{5}{2}x + \dfrac{6}{13}(\sqrt{277} - 15) & \text{on} \quad \left[\dfrac{2}{13}(\sqrt{277} - 15),\ \dfrac{3}{13}(\sqrt{277} - 15)\right] \\ 2x + \dfrac{15}{26}(\sqrt{277} - 15) & \text{on} \quad \left[\dfrac{3}{13}(\sqrt{277} - 15), \infty\right) \end{cases}$$

$$\fallingdotseq \begin{cases} 3x + 0.6320 & \text{on} \quad (-\infty, 0.2528] \\ \dfrac{5}{2}x + 0.7584 & \text{on} \quad [0.2528, 0.3792] \\ 2x + 0.9481 & \text{on} \quad [0.3792, \infty). \end{cases}$$

Thus $\mathcal{D}$ has the minimum value function $v(x)$ at the stationary Markov policy $\hat{\pi} = \hat{f}^{(\infty)}$, where

$$\hat{f}(x) = \begin{cases} 3 & \text{on} \quad \left(-\infty,\ \dfrac{2}{13}(\sqrt{277} - 15)\right] \\ \dfrac{5}{2} & \text{on} \quad \left[\dfrac{2}{13}(\sqrt{277} - 15),\ \dfrac{3}{13}(\sqrt{277} - 15)\right] \\ 2 & \text{on} \quad \left[\dfrac{3}{13}(\sqrt{277} - 15), \infty\right) \end{cases} \fallingdotseq \begin{cases} 3 & \text{on} \quad (-\infty, 0.2528] \\ \dfrac{5}{2} & \text{on} \quad [0.2528, 0.3792] \\ 2 & \text{on} \quad [0.3792, \infty). \end{cases}$$

(maximization) The solution is not improved.

2.2 Continuous Action Space

Example 2.2. $\mathcal{D} = (S,\ (\mathcal{A},\ \mathcal{A}(\cdot)),\ r,\ T,\ \beta)$, *where*

$$S = \mathbf{R}, \quad \mathcal{A} \equiv \mathcal{A}(x) \equiv [2,3], \quad r(x,a) = ax, \quad T(x,a) \equiv [0,1], \quad \beta(x,a,y) = \frac{1}{a}.$$

(minimization) The Bellman equation

$$v(x) = \min_{2\le a\le 3}\left[ax + \frac{1}{a}\int_0^1 v(y)\,dy\right], \quad x \in \mathbb{R} \tag{6}$$

has the solution

$$v(x) = \begin{cases} 3x + \dfrac{12}{19} & \text{on} \quad \left(-\infty, \dfrac{4}{19}\right] \\ \dfrac{12}{\sqrt{19}}\sqrt{x} & \text{on} \quad \left[\dfrac{4}{19}, \dfrac{4}{19}\right] \\ 2x + \dfrac{18}{19} & \text{on} \quad \left[\dfrac{4}{19}, \infty\right). \end{cases}$$

Thus $\mathcal{D}$ has the minimum value function $v(x)$ at the stationary Markov policy $\hat{\pi} = \hat{f}^{(\infty)}$, where

$$\hat{f}(x) = \begin{cases} 3 & \text{on} \quad \left(-\infty, \dfrac{4}{19}\right] \\ \dfrac{6}{\sqrt{19x}} & \text{on} \quad \left[\dfrac{4}{19}, \dfrac{4}{19}\right] \\ 2 & \text{on} \quad \left[\dfrac{4}{19}, \infty\right). \end{cases}$$

(maximization) The solution is not improved.

Example 2.3. $\mathcal{D} = (S, (\mathcal{A}, \mathcal{A}(\cdot)), r, T, \beta)$, *where*

$$S = \mathbb{R}_+ \cup \{0\}, \quad \mathcal{A} \equiv \mathcal{A}(x) \equiv \mathbb{R}_+, \quad r(x,a) = ax, \quad T(x,a) \equiv [0,1], \quad \beta(x,a,y) = \frac{1}{a}.$$

(minimization) The Bellman equation

$$v(x) = \min_{0<a<\infty}\left[ax + \frac{1}{a}\int_0^1 v(y)\,dy\right], \quad x > 0, \quad v(0) = 0 \tag{7}$$

has the solution

$$v(x) = \frac{8}{3}\sqrt{x}, \quad x \ge 0.$$

Thus $\mathcal{D}$ has the minimum value function $v(x)$ at the stationary Markov policy $\hat{\pi} = \hat{f}^{(\infty)}$, where

$$\hat{f}(x) = \frac{4}{3\sqrt{x}}, \quad x > 0.$$

(maximization) The solution is not improved.

3 Reward function

3.1 Quadratic Case

We have shown that the simple bi-decision optimization process with linear reward is reduced to solving a quadratic equation. Next we shall consider the simple quadratic case:

$$v(x) = mx^2 + \frac{1}{m}\int_0^1 v(y)\,dy, \quad x \in \mathbb{R}. \tag{8}$$

Eq.(8) has the unique solution

$$v(x) = mx^2 + \frac{m}{3(m-1)}, \quad x \in \mathbf{R}.$$

Let us consider a decision process on $\mathcal{A} \equiv \mathcal{A}(x) \equiv \{2,3\}$.

Example 3.1. $\mathcal{D} = (S, (\mathcal{A}, \mathcal{A}(\cdot)), r, T, \beta)$, *where*

$$S = \mathbf{R}, \quad \mathcal{A} \equiv \mathcal{A}(x) \equiv \{2,3\}, \quad r(x,a) = ax^2, \quad T(x,a) \equiv [0,1], \quad \beta(x,a,y) = \frac{1}{a}.$$

(minimization) The Bellman equation

$$v(x) = \min_{a \in \{2,3\}} \left[ax^2 + \frac{1}{a} \int_0^1 v(y)\, dy \right] \quad x \in \mathbf{R} \tag{9}$$

has the solution

$$v(x) = \begin{cases} 2x^2 + \dfrac{\hat{c}}{2} & \text{on} \quad \left(-\infty, -\sqrt{\dfrac{\hat{c}}{6}}\right] \cup \left[\sqrt{\dfrac{\hat{c}}{6}}, \infty\right) \\ 3x^2 + \dfrac{\hat{c}}{3} & \text{on} \quad \left[-\sqrt{\dfrac{\hat{c}}{6}}, \sqrt{\dfrac{\hat{c}}{6}}\right] \end{cases}$$

$$\fallingdotseq \begin{cases} 2x^2 + 0.6061 & \text{on} \quad (-\infty, -0.4495] \cup [0.4495, \infty) \\ 3x^2 + 0.4041 & \text{on} \quad [-0.4495, 0.4495]\,, \end{cases}$$

where $\hat{\alpha} := \sqrt{\dfrac{\hat{c}}{6}} \fallingdotseq 0.4495$ is a unique solution on $(0,1)$ to

$$2\alpha^3 + 9\alpha^2 - 2 = 0. \tag{10}$$

Then $\hat{c} = 6\hat{\alpha}^2 \fallingdotseq 1.2122$. Thus $\mathcal{D}$ has the minimum value function $v(x)$ at the stationary Markov policy $\hat{\pi} = \hat{f}^{(\infty)}$, where

$$\hat{f}(x) = \begin{cases} 2 & \text{on} \quad \left(-\infty, -\sqrt{\dfrac{\hat{c}}{6}}\right] \cup \left[\sqrt{\dfrac{\hat{c}}{6}}, \infty\right) \\ 3 & \text{on} \quad \left[-\sqrt{\dfrac{\hat{c}}{6}}, \sqrt{\dfrac{\hat{c}}{6}}\right] \end{cases}$$

$$\fallingdotseq \begin{cases} 2 & \text{on} \quad (-\infty, -0.4495] \cup [0.4495, \infty) \\ 3 & \text{on} \quad [-0.4495, 0.4495]\,. \end{cases}$$

(Maximization) The Bellman equation

$$V(x) = \operatorname*{Max}_{a \in \{2,3\}} \left[ax + \frac{1}{a} \int_0^1 V(y)\, dy \right] \quad x \in \mathbf{R} \tag{11}$$

has the solution

$$V(x) = \begin{cases} 3x^2 + \dfrac{c^*}{3} & \text{on} \quad \left(-\infty, -\sqrt{\dfrac{c^*}{6}}\right] \cup \left[\sqrt{\dfrac{c^*}{6}}, \infty\right) \\ 2x^2 + \dfrac{c^*}{2} & \text{on} \quad \left[-\sqrt{\dfrac{c^*}{6}}, \sqrt{\dfrac{c^*}{6}}\right] \end{cases}$$

$$\fallingdotseq \begin{cases} 3x^2 + 0.5478 & \text{on} \quad (-\infty, -0.5233] \cup [0.5233, \infty) \\ 2x^2 + 0.8217 & \text{on} \quad [-0.5233, 0.5233]\,, \end{cases}$$

where $\alpha^* = \sqrt{\dfrac{c^*}{6}} \fallingdotseq 0.5233$ is a unique solution on $(0,1)$ to

$$-2\alpha^3 + 12\alpha^2 - 3 = 0\,. \tag{12}$$

Then $c^* = 6(\alpha^*)^2 \fallingdotseq 1.2122$. Thus $\mathcal{D}$ has the maximum value function $V(x)$ at the stationary Markov policy $\pi^* = (f^*)^{(\infty)}$, where

$$f^*(x) = \begin{cases} 3 & \text{on} \quad \left(-\infty, -\sqrt{\dfrac{c^*}{6}}\right] \cup \left[\sqrt{\dfrac{c^*}{6}}, \infty\right) \\ 2 & \text{on} \quad \left[-\sqrt{\dfrac{c^*}{6}}, \sqrt{\dfrac{c^*}{6}}\right] \end{cases}$$

$$\fallingdotseq \begin{cases} 3 & \text{on} \quad (-\infty, -0.5233] \cup [0.5233, \infty) \\ 2 & \text{on} \quad [-0.5233, 0.5233]\,. \end{cases}$$

3.2 Logarithmic Case

The example of a controlled integral equation with logarithmic reward is also considered:

$$v(x) = m \log(x+1) + \frac{1}{m} \int_0^1 v(y)\, dy\,, \quad x > -1\,. \tag{13}$$

Eq.(13) has the unique solution

$$v(x) = m \log(x+1) + \frac{m}{m-1}(2\log 2 - 1)\,.$$

Let us consider the bi-decision process on $\mathcal{A} \equiv \mathcal{A}(x) \equiv \{2,3\}$.

Example 3.2. $\mathcal{D} = (\,S,\ (\mathcal{A},\ \mathcal{A}(\cdot\,)\,),\ r,\ T,\ \beta\,)$, *where*

$$S = (-1, \infty), \quad \mathcal{A} \equiv \mathcal{A}(x) \equiv \{2,3\}, \quad r(x,a) = a \log(x+1),$$
$$T(x,a) \equiv [0,1], \quad \beta(x,a,y) = \frac{1}{a}\,.$$

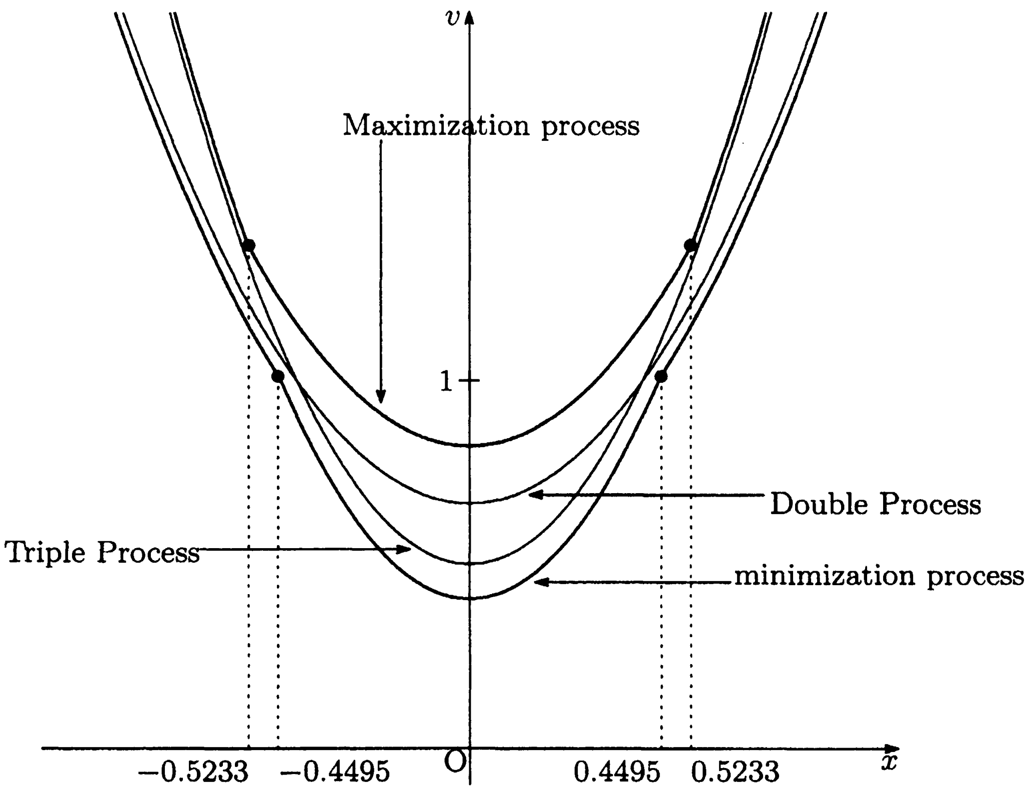

Figure 2: Four value functions in quadratic case

(minimization) The Bellman equation

$$v(x) = \min_{a \in \{2,3\}} \left[a \log(x+1) + \frac{1}{a} \int_0^1 v(y)\, dy \right] \quad -1 < x < \infty \tag{14}$$

has the solution

$$v(x) = \begin{cases} 3\log(x+1) + \dfrac{\hat{c}}{3} & \text{on} \quad (-1, e^{\frac{\hat{c}}{6}} - 1] \\ 2\log(x+1) + \dfrac{\hat{c}}{2} & \text{on} \quad [e^{\frac{\hat{c}}{6}} - 1, \infty) \end{cases}$$

$$\risingdotseq \begin{cases} 3\log(x+1) + 0.4930 & \text{on} \quad (-1,\ 0.2796] \\ 2\log(x+1) + 0.7395 & \text{on} \quad [0.2796,\ \infty)\,, \end{cases}$$

where $\hat{c} \risingdotseq 1.4791$ is a unique solution to

$$e^{\frac{x}{6}} + \frac{1}{3}x + 1 - 4\log 2 = 0\,. \tag{15}$$

Thus $\mathcal{D}$ has the minimum value function $v(x)$ at the stationary Markov policy $\hat{\pi} = \hat{f}^{(\infty)}$, where

$$\hat{f}(x) = \begin{cases} 3 & \text{on} \quad (-1, e^{\frac{\hat{c}}{6}} - 1] \\ 2 & \text{on} \quad [e^{\frac{\hat{c}}{6}} - 1, \infty) \end{cases} \risingdotseq \begin{cases} 3 & \text{on} \quad (-1,\ 0.2796] \\ 2 & \text{on} \quad [0.2796,\ \infty)\,. \end{cases}$$

(Maximization) The Bellman equation

$$V(x) = \underset{a\in\{2,3\}}{\mathrm{Max}} \left[a\log(x+1) + \frac{1}{a}\int_0^1 V(y)\,\mathrm{d}y \right] \quad -1 < x < \infty \tag{16}$$

has the solution

$$V(x) = \begin{cases} 2\log(x+1) + \dfrac{c^*}{2} & \text{on} \quad (-1, e^{\frac{c^*}{6}} - 1] \\ 3\log(x+1) + \dfrac{c^*}{3} & \text{on} \quad [e^{\frac{c^*}{6}} - 1, \infty) \end{cases}$$

$$\fallingdotseq \begin{cases} 2\log(x+1) + 0.9072 & \text{on} \quad (-1,\ 0.3531] \\ 3\log(x+1) + 0.6048 & \text{on} \quad [0.3531,\ \infty)\,, \end{cases}$$

where $c^* \fallingdotseq 1.8144$ is a unique solution to

$$e^{\frac{x}{6}} - \frac{5}{6}x + 6\log 2 - 4 = 0\,. \tag{17}$$

Thus $\mathcal{D}$ has the maximum value function $V(x)$ at the stationary Markov policy $\pi^* = (f^*)^{(\infty)}$, where

$$f^*(x) = \begin{cases} 2 & \text{on} \quad (-1, e^{\frac{c^*}{6}} - 1] \\ 3 & \text{on} \quad [e^{\frac{c^*}{6}} - 1, \infty) \end{cases} \fallingdotseq \begin{cases} 2 & \text{on} \quad (-1,\ 0.3531] \\ 3 & \text{on} \quad [0.3531,\ \infty)\,. \end{cases}$$

3.3 s times vs t times competitive processes

In this section, we consider s times versus t times competitive processes. By considering the problem, we can clearly understand the relationship between minimization and maximization in bi-decision process.

Example 3.3. *$\mathcal{D} = (\,S,\ (\mathcal{A},\ \mathcal{A}(\,\cdot\,)\,),\ r,\ T,\ \beta\,)$, where*

$$S = \mathbf{R}, \quad \mathcal{A} \equiv \mathcal{A}(x) \equiv \{s,t\}, \quad r(x,a) = ax, \quad T(x,a) \equiv [0,1], \quad \beta(x,a,y) = \frac{1}{a}\,.$$

and where s and t are given real constant numbers such that

$$2 \le s < t \le 3\,.$$

(minimization) The Bellman equation

$$v(x) = \min_{a\in\{s,t\}} \left[ax + \frac{1}{a}\int_0^1 v(y)\,\mathrm{d}y \right] \quad x \in \mathbf{R}$$

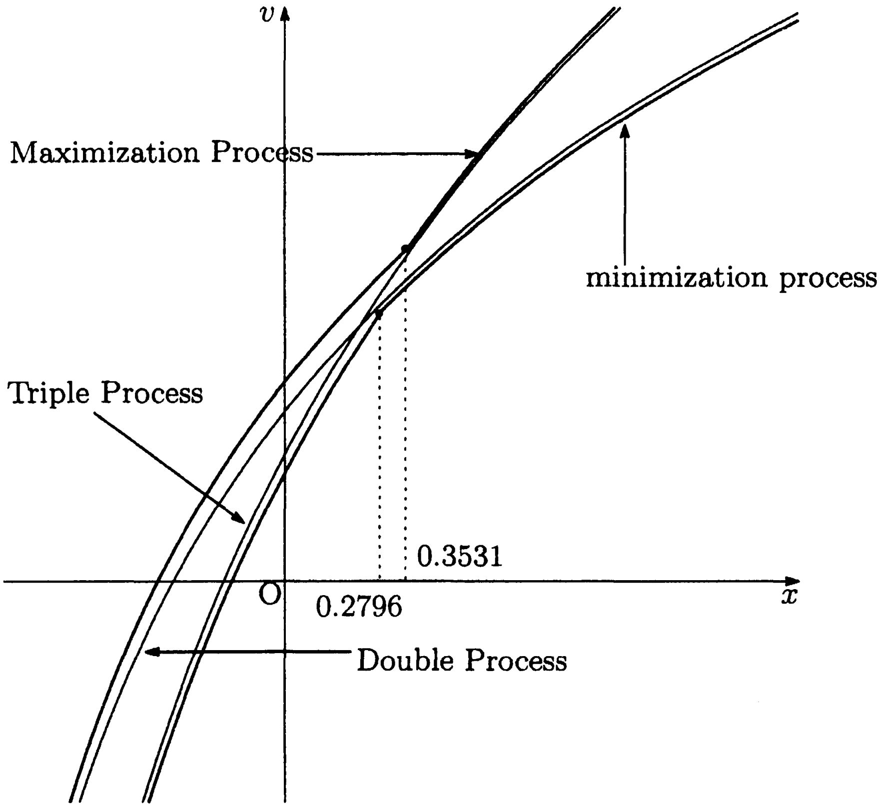

Figure 3: Four value functions in logarithmic case

has the solution

$$v(x) = \begin{cases} tx + \dfrac{\hat{c}}{t} & \text{on } \left(-\infty, \dfrac{\hat{c}}{st}\right] \\ sx + \dfrac{\hat{c}}{s} & \text{on } \left[\dfrac{\hat{c}}{st}, \infty\right), \end{cases} \tag{18}$$

where

$$\hat{c} = \frac{st\left\{\sqrt{(s-1)^2t^2 + s(t-s)} - (s-1)t\right\}}{t-s}. \tag{19}$$

Thus $\mathcal{D}$ has the minimum value function $v(x)$ at the stationary Markov policy $\pi = \hat{f}^{(\infty)}$, where

$$\hat{f}(x) = \begin{cases} t & \text{on } \left(-\infty, \dfrac{\hat{c}}{st}\right] \\ s & \text{on } \left[\dfrac{\hat{c}}{st}, \infty\right). \end{cases} \tag{20}$$

(Maximization) The Bellman equation

$$V(x) = \underset{a \in \{s,t\}}{\text{Max}} \left[ax + \frac{1}{a} \int_0^1 V(y)\, dy \right] \quad x \in \mathbf{R}$$

has the solution

$$V(x) = \begin{cases} sx + \dfrac{c^*}{s} & \text{on} \quad \left(-\infty, \dfrac{c^*}{st}\right] \\ tx + \dfrac{c^*}{t} & \text{on} \quad \left[\dfrac{c^*}{st}, \infty\right), \end{cases} \tag{21}$$

where

$$c^* = \frac{st\left\{\sqrt{(t-1)^2s^2 + t(s-t)} - (t-1)s\right\}}{s-t}. \tag{22}$$

Thus $\mathcal{D}$ has the maximum value function $V(x)$ at the maximization policy $\pi = (f^*)^{(\infty)}$, where

$$f^*(x) = \begin{cases} s & \text{on} \quad \left(-\infty, \dfrac{c^*}{st}\right] \\ t & \text{on} \quad \left[\dfrac{c^*}{st}, \infty\right). \end{cases} \tag{23}$$

As shown above, by reversing the role of s and t, the minimum value function (18), the value of the integration (19), and the minimization policy (20) are transformed into the maximum value function (21), the value of the integration (22), and the maximization policy (23) respectively.

4 Finite Stage Problem

In this section, let us consider a nondeterministic decision process with a finite number of stages.

(minimization) Let $B(S)$ denote the set of all bounded measurable functions on S. then Let us define a integral operator $T : B(S) \to B(S)$ as follows:

$$(Tu)(x) := \min_{a\in\{2,3\}} \left[ax + \frac{1}{a}\int_0^1 u(y)\,dy\right] \quad u(x) \in B(S),$$

and let

$$T^n u = T(T^{n-1}u), \quad T^1 u = Tu.$$

We note that T is a contraction mapping. $(T^n u)(x)$ represents the minimu total weighted reward if we use a minimization policy but we are tarmiated after n periods and gain a terminal reward $u(\cdot)$. For the sake of simplicity, we take $u(x) \equiv 0$, Then we obtain

$$(T^n 0)(x) = \begin{cases} 3x + \dfrac{\hat{c}_n}{3} & \text{on} \quad \left(-\infty, \dfrac{\hat{c}_n}{6}\right] \\ 2x + \dfrac{\hat{c}_n}{2} & \text{on} \quad \left[\dfrac{\hat{c}_n}{6}, \infty\right), \end{cases}$$

where $\{\hat{c}_n\}$ is the sequence on $[0, 1)$ defined by the recurence relation

$$\hat{c}_{n+1} = -\frac{1}{72}\hat{c}_n^2 + \frac{1}{2}\hat{c}_n + 1 \quad \hat{c}_1 = 0.$$

This solution converges to that of infinite stage problem as the limit [3].

(Maxiimization) Let us define a integral operator as follows:

$$(Tu)(x) := \operatorname*{Max}_{u\in\{2,3\}} \left[ax + \frac{1}{a}\int_0^1 u(y)\,\mathrm{d}y \right] \quad u(x) \in B(S)\,.$$

$(T^n u)(x)$ represents the maximum total weighted reward if we use a maximization policy but we are tarmiated after n periods and gain a terminal reward $u(\cdot)$. We obtain

$$(T^n 0)(x) = \begin{cases} 3x + \dfrac{c_n^*}{3} & \text{on} \quad \left(-\infty,\, \dfrac{c_n^*}{6}\right] \\ 2x + \dfrac{c_n^*}{2} & \text{on} \quad \left[\dfrac{c_n^*}{6},\, \infty\right) \end{cases},$$

where $\{c_n^*\}$ is the sequence on $[0, 1)$ defined by the recurence relation

$$c_{n+1}^* = \frac{1}{72}(c_n^*)^2 + \frac{1}{3}c_n^* + \frac{3}{2} \quad c_1^* = 0\,.$$

5 Volterra Type

We can also consider a controlled Volterra equation as follows:

$$v(x) = mx + \frac{1}{m}\int_0^x v(y)\,\mathrm{d}y\,, \quad x \in \mathbf{R}\,. \tag{24}$$

Eq.(24) has the unique solution

$$v(x) = m^2(e^{\frac{x}{m}} - 1), \quad x \in \mathbf{R}\,.$$

Let us consider a decision process on $\mathcal{A} \equiv \mathcal{A}(x) \equiv \{2, 3\}$.

Example 5.1. *$\mathcal{D} = (\,S,\ (\mathcal{A},\ \mathcal{A}(\,\cdot\,)),\ r,\ T,\ \beta\,)$, where*

$$S = \mathbf{R}, \quad \mathcal{A} \equiv \mathcal{A}(x) \equiv \{2,3\}, \quad r(x,a) = ax, \quad T(x,a) = [0,x], \quad \beta(x,a,y) = \frac{1}{a}\,.$$

(minimization) The Bellman equation

$$v(x) = \min_{a\in\{2,3\}} \left[ax + \frac{1}{a}\int_0^x v(y)\,\mathrm{d}y \right], \quad x \in \mathbf{R} \tag{25}$$

has the solution

$$v(x) = \begin{cases} 9(e^{\frac{x}{3}} - 1) & \text{on} \quad (-\infty, 0] \\ 4(e^{\frac{x}{2}} - 1) & \text{on} \quad [0, \hat{\alpha}] \\ (5\hat{\alpha} + 9)e^{\frac{x-\hat{\alpha}}{3}} - 9 & \text{on} \quad [\hat{\alpha}, \infty)\,, \end{cases}$$

where $\hat{\alpha} \risingdotseq 3.2376$ is a unique solution to

$$e^{\frac{x}{2}} - 1 - \frac{5}{4}x = 0\,. \tag{26}$$

Thus $\mathcal{D}$ has the minimum value function $v(x)$ at the stationary Markov policy $\hat{\pi} = \hat{f}^{(\infty)}$, where

$$\hat{f}(x) = \begin{cases} 3 & \text{on} \quad (-\infty,\, 0] \\ 2 & \text{on} \quad [0,\, \hat{\alpha}] \\ 3 & \text{on} \quad [\hat{\alpha},\, \infty)\,. \end{cases}$$

(Maximization) The Bellman equation

$$V(x) = \underset{a \in \{2,3\}}{\mathrm{Max}} \left[ax + \frac{1}{a} \int_0^x V(y)\, dy \right] \quad x \in \mathbf{R} \tag{27}$$

has the solution

$$V(x) = \begin{cases} 4(e^{\frac{x}{2}} - 1) & \text{on} \quad (-\infty,\, 0] \\ 9(e^{\frac{x}{3}} - 1) & \text{on} \quad [0,\, \alpha^*] \\ (5\alpha^* + 4)e^{\frac{x-\alpha^*}{3}} - 4 & \text{on} \quad [\alpha^*,\, \infty) \end{cases}$$

where $\alpha^* \fallingdotseq 2.8422$ is a unique solution on $(0, \infty)$ to

$$e^{\frac{x}{3}} - \frac{5}{9}x - 1 = 0\,. \tag{28}$$

Thus $\mathcal{D}$ has the maximum value function $V(x)$ at the stationary Markov policy $\pi^* = (f^*)^{(\infty)}$, where

$$f^*(x) = \begin{cases} 2 & \text{on} \quad (-\infty,\, 0] \\ 3 & \text{on} \quad [0,\, \alpha^*] \\ 2 & \text{on} \quad [\alpha^*,\, \infty)\,. \end{cases}$$

6 System of Equations

Example 6.1. $\mathcal{D} = (\,S,\ (\mathcal{A},\ \mathcal{A}(\cdot\,)),\ r,\ T,\ \beta\,)$, *where*

$$S = \{\, s = (w, x) \mid w \in \{1, 2\},\ x \in \mathbf{R} \,\} = \{1, 2\} \times \mathbf{R},$$

$$\mathcal{A} = \{2, 3, 4, 5\},\ \mathcal{A}(s) = \begin{cases} \{2, 3\} & s \in \{1\} \times \mathbf{R} \\ \{4, 5\} & s \in \{2\} \times \mathbf{R}, \end{cases} \quad r(s, a) = ax, \quad T(s, a) \equiv \{1, 2\} \times [0, 1],$$

$$\beta(s, a, s') = \begin{cases} 1/6 & (s, a, s') \in (\{1\} \times \mathbf{R}) \times \{2\} \times (\{1\} \times \mathbf{R}) \cup (\{1\} \times \mathbf{R}) \times \{3\} \times (\{2\} \times \mathbf{R}) \\ 2/3 & (s, a, s') \in (\{1\} \times \mathbf{R}) \times \{2\} \times (\{2\} \times \mathbf{R}) \cup (\{1\} \times \mathbf{R}) \times \{3\} \times (\{1\} \times \mathbf{R}) \\ 1/3 & (s, a, s') \in (\{2\} \times \mathbf{R}) \times \{4\} \times (\{1\} \times \mathbf{R}) \cup (\{2\} \times \mathbf{R}) \times \{5\} \times (\{2\} \times \mathbf{R}) \\ 1/2 & (s, a, s') \in (\{2\} \times \mathbf{R}) \times \{4\} \times (\{2\} \times \mathbf{R}) \cup (\{2\} \times \mathbf{R}) \times \{5\} \times (\{1\} \times \mathbf{R})\,. \end{cases}$$

(minimization) The Bellman equation

$$\begin{cases} v_1(x) = \min \left[2x + \dfrac{1}{6} \displaystyle\int_0^1 v_1(y)\, dy + \dfrac{2}{3} \displaystyle\int_0^1 v_2(y)\, dy,\ 3x + \dfrac{2}{3} \displaystyle\int_0^1 v_1(y)\, dy + \dfrac{1}{6} \displaystyle\int_0^1 v_2(y)\, dy \right] \\ v_2(x) = \min \left[4x + \dfrac{1}{3} \displaystyle\int_0^1 v_1(y)\, dy + \dfrac{1}{2} \displaystyle\int_0^1 v_2(y)\, dy,\ 5x + \dfrac{1}{2} \displaystyle\int_0^1 v_1(y)\, dy + \dfrac{1}{3} \displaystyle\int_0^1 v_2(y)\, dy \right] \end{cases}$$

has the solution

$$v_1(x) = \begin{cases} 3x + \dfrac{147\sqrt{33} - 349}{64} & \text{on } \left(-\infty, \dfrac{3(7-\sqrt{33})}{8}\right] \\ 2x + \dfrac{123\sqrt{33} - 181}{64} & \text{on } \left[\dfrac{3(7-\sqrt{33})}{8}, \infty\right) \end{cases}$$

$$\fallingdotseq \begin{cases} 3x + 7.7414 & \text{on } (-\infty, 0.4708] \\ 2x + 8.2122 & \text{on } [0.4708, \infty) \end{cases}$$

$$v_2(x) = \begin{cases} 5x + \dfrac{139\sqrt{33} - 293}{64} & \text{on } \left(-\infty, \dfrac{7-\sqrt{33}}{8}\right] \\ 4x + \dfrac{131\sqrt{33} - 237}{64} & \text{on } \left[\dfrac{7-\sqrt{33}}{8}, \infty\right) \end{cases}$$

$$\fallingdotseq \begin{cases} 5x + 7.8983 & \text{on } (-\infty, 0.1569] \\ 4x + 8.0553 & \text{on } [0.1569, \infty) . \end{cases}$$

Thus $\mathcal{D}$ has the minimum value functions $v_1(x)$ and $v_2(x)$ at the stationary Markov policy $\hat{\pi} = (\hat{f}^{(\infty)}, \hat{g}^{(\infty)})$, where

$$\hat{f}(x) = \begin{cases} 3 & \text{on } \left(-\infty, \dfrac{3(7-\sqrt{33})}{8}\right] \\ 2 & \text{on } \left[\dfrac{3(7-\sqrt{33})}{8}, \infty\right) \end{cases} \fallingdotseq \begin{cases} 3 & \text{on } (-\infty, 0.4708] \\ 2 & \text{on } [0.4708, \infty) \end{cases}$$

$$\hat{g}(x) = \begin{cases} 5 & \text{on } \left(-\infty, \dfrac{7-\sqrt{33}}{8}\right] \\ 4 & \text{on } \left[\dfrac{7-\sqrt{33}}{8}, \infty\right) \end{cases} \fallingdotseq \begin{cases} 5 & \text{on } (-\infty, 0.1569] \\ 4 & \text{on } [0.1569, \infty) . \end{cases}$$

(Maximization) The Bellman equation

$$\begin{cases} V_1(x) = \text{Max}\left[2x + \dfrac{1}{6}\displaystyle\int_0^1 V_1(y)\,dy + \dfrac{2}{3}\int_0^1 V_2(y)\,dy,\ 3x + \dfrac{2}{3}\int_0^1 V_1(y)\,dy + \dfrac{1}{6}\int_0^1 V_2(y)\,dy\right] \\ V_2(x) = \text{Max}\left[4x + \dfrac{1}{3}\displaystyle\int_0^1 V_1(y)\,dy + \dfrac{1}{2}\int_0^1 V_2(y)\,dy,\ 5x + \dfrac{1}{2}\int_0^1 V_1(y)\,dy + \dfrac{1}{3}\int_0^1 V_2(y)\,dy\right] \end{cases}$$

has the solution

$$V_1(x) = \begin{cases} 2x + \dfrac{3(535 - 51\sqrt{41})}{64} & \text{on} \quad \left(-\infty, \dfrac{3(\sqrt{41}-5)}{8} \right] \\ 3x + \dfrac{3(575 - 59\sqrt{41})}{64} & \text{on} \quad \left[\dfrac{3(\sqrt{41}-5)}{8}, \infty \right) \end{cases}$$

$$\risingdotseq \begin{cases} 2x + 9.7707 & \text{on} \quad (-\infty, 0.5262] \\ 3x + 9.2445 & \text{on} \quad [0.5262, \infty) \end{cases}$$

$$V_2(x) = \begin{cases} 4x + \dfrac{7(235 - 23\sqrt{41})}{64} & \text{on} \quad \left(-\infty, \dfrac{\sqrt{41}-5}{8} \right] \\ 5x + \dfrac{1685 - 169\sqrt{41})}{64} & \text{on} \quad \left[\dfrac{\sqrt{41}-5}{8}, \infty \right) \end{cases}$$

$$\risingdotseq \begin{cases} 4x + 9.5953 & \text{on} \quad (-\infty, 0.1754] \\ 5x + 9.4199 & \text{on} \quad [0.1754, \infty)\,. \end{cases}$$

Thus $\mathcal{D}$ has the maximum value functions $V_1(x)$ and $V_2(x)$ at the stationary Markov policy $\pi^* = (\,(f^*)^{(\infty)}, (g^*)^{(\infty)}\,)$, where

$$f^*(x) = \begin{cases} 2 & \text{on} \quad \left(-\infty, \dfrac{3(\sqrt{41}-5)}{8} \right] \\ 3 & \text{on} \quad \left[\dfrac{3(\sqrt{41}-5)}{8}, \infty \right) \end{cases} \risingdotseq \begin{cases} 2 & \text{on} \quad (-\infty, 0.5262] \\ 3 & \text{on} \quad [0.5262, \infty) \end{cases}$$

$$g^*(x) = \begin{cases} 4 & \text{on} \quad \left(-\infty, \dfrac{\sqrt{41}-5}{8} \right] \\ 5 & \text{on} \quad \left[\dfrac{\sqrt{41}-5}{8}, \infty \right) \end{cases} \risingdotseq \begin{cases} 5 & \text{on} \quad (-\infty, 0.1754] \\ 4 & \text{on} \quad [0.1754, \infty)\,. \end{cases}$$

7 Bynamic Process

(Bynamic process 1)

Finally, the example of a controlled equation equation including both minimization and maximization is considered:

$$\begin{cases} v_1(x) = \min\left[2x + \dfrac{1}{6}\displaystyle\int_0^1 v_1(y)\,dy + \dfrac{2}{3}\int_0^1 v_2(y)\,dy,\ 3x + \dfrac{2}{3}\int_0^1 v_1(y)\,dy + \dfrac{1}{6}\int_0^1 v_2(y)\,dy\right] \\ v_2(x) = \mathrm{Max}\left[4x + \dfrac{1}{3}\displaystyle\int_0^1 v_1(y)\,dy + \dfrac{1}{2}\int_0^1 v_2(y)\,dy,\ 5x + \dfrac{1}{2}\int_0^1 V_1(y)\,dy + \dfrac{1}{3}\int_0^1 V_2(y)\,dy\right]. \end{cases} \tag{29}$$

This is a Bellman equation of **bynamic programming** [7]. This (29) has the following approximate solution.

$$v_1(x) \risingdotseq \begin{cases} 3x + 8.4964 & \text{on} \quad (-\infty, 0.6507] \\ 2x + 9.1471 & \text{on} \quad [0.6507, \infty)\,, \end{cases} \qquad v_2(x) \risingdotseq \begin{cases} 4x + 8.9302 & \text{on} \quad (-\infty, 0.2169] \\ 5x + 8.7133 & \text{on} \quad [0.2169, \infty)\,. \end{cases}$$

Thus $\mathcal{D}$ has the optimum value functions $v_1(x)$ and $v_2(x)$ at the stationary Markov policy $\pi = (\tilde{f}^{(\infty)}, \tilde{g}^{(\infty)})$, where

$$\tilde{f}(x) \fallingdotseq \begin{cases} 3 & \text{on} \quad (-\infty, 0.6507] \\ 2 & \text{on} \quad [0.6507, \infty), \end{cases} \qquad \tilde{g}(x) \fallingdotseq \begin{cases} 4 & \text{on} \quad (-\infty, 0.2169] \\ 5 & \text{on} \quad [0.2169, \infty). \end{cases}$$

(Bynamic process 2)

By reversing the optimization operators, the following problem is also considered:

$$\begin{cases} v_1(x) = \text{Max}\left[2x + \frac{1}{6}\int_0^1 v_1(y)\,dy + \frac{2}{3}\int_0^1 v_2(y)\,dy,\ 3x + \frac{2}{3}\int_0^1 v_1(y)\,dy + \frac{1}{6}\int_0^1 v_2(y)\,dy\right] \\ v_2(x) = \min\left[4x + \frac{1}{3}\int_0^1 v_1(y)\,dy + \frac{1}{2}\int_0^1 v_2(y)\,dy,\ 5x + \frac{1}{2}\int_0^1 V_1(y)\,dy + \frac{1}{3}\int_0^1 V_2(y)\,dy\right] \end{cases}$$

has the following approximate solution.

$$v_1(x) \fallingdotseq \begin{cases} 2x + 8.7609 & \text{on} \quad (-\infty, 0.3297] \\ 3x + 8.4312 & \text{on} \quad [0.3297, \infty), \end{cases} \qquad v_2(x) \fallingdotseq \begin{cases} 5x + 8.5411 & \text{on} \quad (-\infty, 0.1099] \\ 4x + 8.6510 & \text{on} \quad [0.1099, \infty). \end{cases}$$

Thus $\mathcal{D}$ has the optimum value functions $v_1(x)$ and $v_2(x)$ at the stationary Markov policy $\pi = (\check{f}^{(\infty)}, \check{g}^{(\infty)})$, where

$$\check{f}(x) \fallingdotseq \begin{cases} 2 & \text{on} \quad (-\infty, 0.3297] \\ 3 & \text{on} \quad [0.3297, \infty), \end{cases} \qquad \check{g}(x) \fallingdotseq \begin{cases} 5 & \text{on} \quad (-\infty, 0.1099] \\ 4 & \text{on} \quad [0.1099, \infty). \end{cases}$$

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