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Application of the Aubry-Mather theory to a system of Hamilton-Jacobi equations with unilateral implicit obstacles

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1 Introduction

In this note we describe a recent development of the theory of viscosity solutions to a system of Hamilton-Jacobi equations and a recent results obtained by applying the method of Aubry-Mather theory.

In section 2, we give a brief review of the theory of viscosity solutions for systems of Hamilton-Jacobi equations. Section 3 is devoted to the statement of a new result by using Aubry-Mather theory. In section 4 we give a representation formula for the solution of the obstacle problem for Hamilton-Jacobi equation.

2 Brief history

Soon after the notion of viscosity solutions are introduced to the Hamilton-Jacobi equations [5], [6], some people interested in applying this notion to the system of equations.

The main focus is to get the component-wise comparison principle for solutions to such systems.

Since the notion of viscosity solution is based on the maximum principle, the applicable system should have some structural conditions.

In 1984, I. Capuzzo-Dolcetta and L. C. Evans [4] introduced a system

$$\max\{\lambda u^d - g^d \cdot Du^d - f^d,\ u^d - M^d[u]\ \} = 0, \quad d = 1, \ldots, m \tag{1}$$

in the connection with a optimal switching problem for ordinary differential equations. Here, we set

$$M^d[u](x) = \min_{j \neq d}\{u^j(x) + k(j,d)\}$$

for $k(j,d) > 0$ are given constants.

In S. M. Lenhart [15], H. Englar and S. M. Lenhart[7], they treated the system of Hamilton-Jacobi equations which they called weakly coupled system which has the form

$$H_i(x, Du_i) + \sum_{\ell=1}^{m} c_{k\ell}(x)u_\ell(x) = f_k(x) \quad k = 1, \ldots, m.$$

H. Ishii and S. Koike [13], [14] introduced the notion of monotone system or quasi-monotone system for

$$G_k(x, u_k, Du_k, D^2u_k) + \sum_{j=1}^{m} d_{kj}(x)u_j = 0 \qquad (k = 1, \ldots, m).$$

Systems for second order equations of the form:

$$\min\{\max\{G_k(x, u_k, Du_k, D^2u_k),\ u_k - M_k(x,u)\},\ u_k - N_k(x,u)\} = 0$$
$$\max\{G_k(x, u_k, Du_k, D^2u_k),\ u_k - M_k(x,u)\} = 0$$

where

$$M_k(x,u) = \min\{u_j + g_{kj}(x) \mid j = 1, \ldots, m,\ j \neq k\}$$
$$N_k(x,u) = \min\{u_j - h_{kj}(x) \mid j = 1, \ldots, m,\ j \neq k\}$$

are treated by the author [16] and H. Ishii[12].

In these systems the monotonicity assumptions for each system work an essential role to get the comparison principle. We review for these monotonicity assumptions for typical two systems to compare with the main result of this paper.

First consider the simplest weakly coupled system:

$$H_1(Du_1) + d_{11}u_1 + d_{12}u_2 = f_1$$
$$H_2(Du_2) + d_{21}u_1 + d_{22}u_2 = f_2$$

where c_{ij} are some constants satisfying

$$\begin{aligned} d_{11}+d_{12} &\geqq \delta_0 > 0, \qquad & d_{12} \leqq 0 \\ d_{21}+d_{22} &\geqq \delta_0 > 0, \qquad & d_{21} \leqq 0. \end{aligned}$$

This is the monotonicity assumption for this system.

To see how this assumption works in the proof of the uniqueness, we follow the formal argument. Let $u = (u_1, u_2)$, $v = (v_1, v_2)$ be classical solutions. We want to show $u_i \leqq v_i$ $(i = 1, 2)$. By the contrary, we assume

$$(u_1 - v_1)(x_0) = \max_{x,i}(u_i - v_i)(x) > 0.$$

Using $Du_1(x_0) = Dv_1(x_0)$, substitute the equation of v_1 from that of v_1, we get

$$d_{11}(u_1 - v_1)(x_0) + d_{12}(u_2 - v_2)(x_0) = 0.$$

Hence we have

$$\begin{aligned} 0 &= d_{11}(u_1 - v_1)(x_0) + d_{12}(u_2 - v_2)(x_0) \\ &\geqq d_{11}(u_1 - v_1)(x_0) + d_{12}(u_1 - v_1)(x_0) \qquad \text{(by } d_{12} \leqq 0\text{)} \\ &= (d_{11} + d_{12})(u_1 - v_1)(x_0) \\ &\geqq \delta_0(u_1 - v_1)(x_0) > 0. \end{aligned}$$

This is the contradiction.

Next, we would like to describe the monotonicity condition for the system arising from the optimal switching. We restrict ourselves to the simplest case. Consider the following system:

$$\begin{aligned} \max\{H_1(Du_1) + u_1 - f_1,\ u_1 - u_2 - k_1\} = 0 \\ \max\{H_2(Du_2) + u_2 - f_1,\ u_2 - u_1 - k_2\} = 0. \end{aligned}$$

Here k_1, k_2 are positive constants. Arising u_i in each $H_i(Du_i) + u_i$ and the positivity of k_i is the monotonicity condition in this case.

For the simplicity, let $u = (u_1, u_2)$, $v = (v_1, v_2)$ be classical solutions and we want to show $u_i \leqq v_i$ $(i = 1, 2)$. By the contradiction, we assume

$$(u_1 - v_1)(x_0) = \max_{x,i}(u_i - v_i)(x) > 0$$

for some x_0. In the following we argue at x_0.

First note that $H_1(Du_1)+u_1-f_1 \leqq 0$ is always true. We have two cases.

(a) The case $H_1(Dv_1)+v_1-f_1=0$:

In this case, substitute two equations by using $Du_1(x_0) = Dv_1(x_0)$, we have $(u_1-v_1)(x_0) \leqq 0$, which is a contradiction.

(b) The case $v_1-v_2-k_1=0$:

If $v_2-v_1-k_2=0$ in the second equation, we get $-(k_1+k_2)=0$, which is a contradiction. Then it must be $H_2(Dv_2)+v_2-f_2=0$. On the other hand, combining the relations $v_1-v_2-k_1=0$ and $u_1-u_2-k_1 \leqq 0$, we have $u_1-u_2 \leqq v_1-v_2$. This implies $u_1-v_1 \leqq u_2-v_2$, which says

$$(u_2-v_2)(x_0) = \max_{x,i}(u_i-v_i)(x) > 0.$$

Hence we concentrate to the second equation, which satisfies

$$H_2(Du_2)+u_2-f_2 \leqq 0$$
$$H_2(Dv_2)+v_2-f_2 = 0.$$

From this we can get $(u_2-v_2)(x_0) \leqq 0$, which is also a contradiction.

3 Comparison results by applying Aubry-Mather theory

First, note that in both of above examples we use also the fact that the Hamiltonian $H_i(p)+r$ is strictly increasing with respect to r.

There are lot of papers that argue the uniqueness of Hamilton-Jacobi equations without these increasing property. For the system, H. Ishii and S. Koike [13] introduced the notion of "quasi-monotone system" and prove the comparison principle.

The method using Aubry set is the one which is recently introduced by A. Fathi [9]. The Aubry set is first introduced in the connection of dynamic theory, and the relation with PDE is investigated by A. Fathi and A. Siconolfi [10], [11].

We prepare some notations.

Consider the equation $H(x,Du)=0$ and assume that $H(x,p)$ is convex and coercive with respect to p.

Assume that there exist a functions $\psi \in C^1$ and $f(x) \geqq 0$ such that $H(x,D\psi(x)) \leqq -f(x)$. We call the set

$$\mathcal{A} = \{x \mid f(x)=0\}$$

the Aubry set of H.

If $\mathcal{A} = \emptyset$, then there exists strict sub-solution of C^1 class, hence we can get the uniqueness. On the other hand, if $\mathcal{A} \neq \emptyset$, the Aubry set plays a rule as inner boundary in some sense, hence we can get information of the solution from the value on $\mathcal{A}$.

F. Camilli and P. Loreti [2] applied these results to the system of eikonal equations. Their result is as follows: Consider the system

$$H_i(x, Du_i) + \sum_{j=1}^{M} c_{ij}(x)(u_i - u_j) = 0, \quad (i = 1, \ldots, M).$$

Assume that each H_i satisfies the assumption of convexity and coerciveness. Assume also that there exist functions $\psi \in C^1$ and $f_i(x) \geqq 0$ satisfying $H_i(x, D\psi(x)) \leqq -f_i(x)$. Note that we assume that there exists common ψ for all H_i. Let

$$\mathcal{A}_i = \{x \mid f_i(x) = 0\}.$$

Theorem 1 (F. Camilli and P. Loreti [2]) *Assume that one of the following assumption is satisfied, then the uniqueness of the viscosity solutions holds:*

(i) $c_{ij} \geqq 0 \quad (i \neq j), \qquad \mathcal{A}_i = \emptyset \quad (i = 1, \ldots, M)$

(ii) $c_{ij} > 0 \quad (i \neq j), \qquad \bigcap_{i=1}^{M} \mathcal{A}_i = \emptyset$

Note that this assumption includes the case $d_{11} = 0$, $d_{22} = 0$ in the previous example.

Now we are in the position to state our result. Soon after I learned the result of Camilli and Loreti, I asked them how about the associated result to the system of obstacle problem.

We started the joint work and obtained the following result.

Consider the system

$$\max\{H_i(x, Du_i(x)),\ u_i(x) - (M_i u)(x)\} = 0 \quad \text{in } D,\ i = 1, \ldots, M.$$

Here,

$$(M_i u)(x) = \min_{j \neq i}\{u_j + k_{ij}(x)\}.$$

We assume that $H_i(x, p)$ are convex and coercive, and $k_{ij} > 0$ $(i \neq j)$.

Assume that there exist functions $\psi \in C^1$ and $f_i(x) \geqq 0$ such that $H_i(x, D\psi(x)) \leqq -f_i(x)$. We denote the Aubry sets of H_i by

$$\mathcal{A}_i = \{x \mid f_i(x) = 0\}.$$

Theorem 2 *If sub- and super-solutions u and v satisfy*

$$u_i \leqq v_i \quad on\ \mathcal{A}_i \cup \partial D\ (i = 1, \ldots, M),$$

then we have

$$u_i \leqq v_i \quad in\ D\ (i = 1, \ldots, M).$$

The idea of the proof is a combination of the previous results. First consider

$$u_\lambda = (\lambda u_1 + (1-\lambda)\psi, \ldots, \lambda u_M + (1-\lambda)\psi),$$

for $\lambda \in (0, 1)$. If we drive $u_\lambda \leqq v$, then we let $\lambda \to 0$ to get the theorem.

First we note that u_λ is a sub-solution of the following system:

$$\max\{H_i(x, Du_i),\ u_i(x) - (M_i u)(x)\} = -f_{\lambda,i}(x)$$

where

$$f_{\lambda,i}(x) = (1-\lambda)\min\{f_i(x),\ \min_{j \neq k}\{k_{i,j}(x)\}\}.$$

We use the convexity of H_i in this part.

To prove $u_\lambda \leq v$, assume by contradiction that this is not true. Hence there exist $i_0 \in \{1, \ldots, M\}$, $x_0 \in D$ and $\delta > 0$ such that

$$u_{\lambda,i_0}(x_0) - v_{i_0}(x_0) = \max_{x,i}\{u_{\lambda,i}(x) - v_i(x)\} = \delta.$$

Now we divide into two cases.

(i) $v_{i_0}(x_0) - (M_{i_0}v)(x_0) < 0$,

(ii) $v_{i_0}(x_0) - (M_{i_0}v)(x_0) \geq 0$.

In the case (i), we can discuss as same as single equation and get

$$-f_{\lambda,i_0}(x_0) < 0.$$

Then we can argue as usual to get a contradiction.

In the case (ii), we can argue as described in the section about monotonicity conditions.

4 A representation formula

It is known that the Aubry set plays a role in some sense as an inner boundary.

Reflecting this property, it is known a representation formula for the solution of Hamilton-Jacobi equation by using the given value on the Aubry set.

Let us introduce some notations.

Consider the Hamilton-Jacobi equation with Dirichlet condition

$$\begin{aligned} H(x, Du) = 0 \quad \text{in } D \subset \mathbb{R}^n, \\ u(x) = g(x) \quad \text{on } \partial D. \end{aligned} \tag{2}$$

Let

$$\begin{aligned} Z(x) &= \{p \in \mathbb{R}^n \mid H(x,p) \leqq 0\}, \\ \sigma(x,q) &= \sup\{p \cdot q \mid p \in Z(x)\} \end{aligned}$$

for $x \in \overline{D}$, $q \in \mathbb{R}^n$ and we put

$$S(x,y) = \inf \left\{ \int_0^1 \sigma(\xi(s), \dot{\xi}(s)) ds \,\middle|\, \xi \in \mathrm{Lip}(0,1),\ \xi(0) = x,\ \xi(1) = y \right\}.$$

We list some properties by A. Fathi and A. Siconolfi [11]:

(1) $S(x,y) \geqq 0$, $S(x,x) = 0$,

$S(x,y) \leqq S(x,z) + S(z,y)$,

$S(x,y) \leqq {}^{\exists}M|x-y|$.

(2) $S(x,\cdot)$ is a sub-solution on D.

$S(x,\cdot)$ is a super-solution on $D \backslash \{x\}$.

(3) It is equivalent that v is a sub-solution and that $-S(y,x) \leqq v(x) - v(y) \leqq S(y,x)$.

This means that $S(x,y)$ has a similar properties with distance function.

Assume that continuous functions $g : \partial D \cup \mathcal{A} \to \mathbb{R}$ satisfy the compatibility condition $-S(y,x) \leqq g(x) - g(y) \leqq S(y,x)$, then the solution of (2)is represented as

$$u(x) = \min_{y \in \partial D \cup \mathcal{A}} \{g(y) + S(y,x)\}.$$

We can get a similar representation formula for a obstacle problem

$$\max\{H(x, Du),\ u - \phi\} = 0.$$

Theorem 3 *Assume that continuous functions $g : \partial D \cup \mathcal{A} \to \mathbb{R}$ satisfy the compatibility condition $-S(y,x) \leqq g(x) - g(y) \leqq S(y,x)$. Then the solution of*

$$\begin{aligned} &\max\{H(x, Du),\ u - \phi\} = 0 \quad \text{in } D, \\ &u(x) = g(x) \quad \text{on } \partial D \end{aligned}$$

satisfying $u(x) = g(x)$ on $\mathcal{A}$ is unique and is given by

$$u(x) = \min\big\{ \min_{y \in \partial D \cup \mathcal{A}} \{g(y) + S(y,x),\ \min_{y \in \overline{D}}\{\phi(x) + S(y,x)\}\big\}.$$

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