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Glaeser's type estimates

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Abstract. In this short Note, which partially reproduces the content of the lecture given in the workshop, we present some gradient estimates for nonnegative semiconcave functions and for nonnegative viscosity solutions of fully nonlinear second order elliptic equations of the form $F(D^2u) + H(Du) = f$ with bounded and continuous right-hand side.
The results, mostly taken from the joint paper with A. Vitolo [6], generalize to a nonlinear setting those of Li and Nirenberg about the so-called Glaeser type estimates.

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1 Introduction

A classical inequality giving information on the intermediate derivatives in terms of the higher derivatives and the function itself states that, for a bounded C^2 function $u : \mathbb{R}^n \to \mathbb{R}$ with bounded Hessian D^2u,

$$||Du||_{L^\infty} \leq \sqrt{2\,||u||_{L^\infty}\,||D^2u||_{L^\infty}}$$

In the 1-dimensional case, the result goes back to Landau [17] and Kolmogorov [16], see also [19], [20] and the bibliographies therein for several refinements of the above

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inequality.
If one assumes instead the less restrictive condition

$$D^2u(x)\, h \cdot h \le M\,|h|^2 \quad \text{for all } x, h \in \mathbb{R}^n \tag{1.1}$$

for some constant $M \ge 0$ and the additional requirement that u is *nonnegative*, the pointwise inequality holds

$$|Du(x)| \le \sqrt{2Mu(x)} \quad \text{for all } x \in \mathbb{R}^n \tag{1.2}$$

If $M = 0$, then (1.2) amounts to the well-known fact that concave nonnegative functions on $\mathbb{R}^n$ are costants. The elementary proof of the validity of (1.2) in the case $M > 0$ is as follows: the Taylor's expansion around a point x gives

$$0 \le u(x+h) \le u(x) + Du(x) \cdot h + \frac{M}{2}|h|^2 \tag{1.3}$$

For any fixed x, the convex quadratic polynomial $q(h) = u(x) + Du(x) \cdot h + \frac{M}{2}|h|^2$ attains its minimum value at $h^* = -\frac{1}{M}\, Du(x)$. Thanks to (1.3), one deduces that

$$q(h^*) = u(x) - \frac{1}{2M}\,|Du(x)|^2 \ge 0$$

yielding immediately inequality (1.2). The above inequality in dimension $n = 1$ is reported in a paper by Glaeser [9], and attributed there to Malgrange, in the form

$$|\,(\sqrt{u})'\,(x)| \le \sqrt{\frac{M}{2}}$$

for strictly positive u. Note that the constant $\sqrt{2}$ is optimal in (1.2) as shown by the function $u(x) = \frac{1}{2}|x|^2$.

A sort of localized version of (1.2) in balls with appropriate radius depending on the value of u at the center x_0 and on M can be easily derived. Take, for simplicity, $x_0 = 0$ and any $\gamma > 0$. Using (1.2) with $x = 0$ and the Taylor's expansion we obtain

$$u(x) \le u(0) + \sqrt{2Mu(0)}\,|x| + \frac{M}{2}|x|^2$$

For $x \in B_{\sqrt{\frac{\gamma u(0)}{M}}}$, then $u(x) \le (1 + \sqrt{2\gamma} + \frac{\gamma}{2})\, u(0)$. Insert this in (1.2) to conclude that

$$|Du(x)| \le \sqrt{\left(2 + 2\sqrt{2\gamma} + \gamma\right)\, M\, u(0)} \quad \text{for all } x \in B_{\sqrt{\frac{\gamma u(0)}{M}}} \tag{1.4}$$

In this Note we present some generalizations of inequalities (1.2) and (1.4) to *semiconcave* functions with applications to viscosity solutions of Hamilton-Jacobi equations and to viscosity solutions of a class of second-order fully nonlinear elliptic equations.

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2 Glaeser type estimates for semiconcave functions

A continuous function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ is *semiconcave* if there exists $M \geq 0$ such that $x \longrightarrow u(x) - \frac{M}{2}|x|^2$ is concave on $\mathbb{R}^n$. For semiconcave functions, the superdifferential of u at x, namely

$$D^+u(x) = \left\{p \in \mathbb{R}^n \,:\, \limsup_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \leq 0\right\}$$

is a non empty, closed convex set. We refer to [5] for a general study of semiconcave functions. From the point of view of regularity, let us only recall here that semiconcave functions are locally Lipschitz continuous and twice differentiable almost everywhere as sums of a C^2 function and a concave one.
Observe also that if $u \in C^2(\mathbb{R}^n)$ and (1.1) holds, then u is semiconcave with semiconcavity constant M. The next statement is a simple generalization of estimates (1.2) and (1.4) in the Introduction:

Proposition 2.1 *Assume that $u \in C(\mathbb{R}^n)$ is semiconcave and nonnegative. Then,*

$$|p| \leq \sqrt{2Mu(x)} \quad \textit{for all } p \in D^+u(x),\ x \in \mathbb{R}^n \tag{2.1}$$

$$|p| \leq \sqrt{\left(2 + 2\sqrt{2\gamma} + \gamma\right)\, M\, u(0)} \quad \textit{for all } x \in B_R\,,\ R = \sqrt{\frac{\gamma\, u(0)}{M}} \tag{2.2}$$

It is a well-known fact for semiconcave functions that $p \in D^+u(x)$ if and only if

$$u(y) \leq u(x) + p \cdot (y - x) + \frac{M}{2}|y - x|^2$$

for any $y \in \mathbb{R}^n$. Starting from this the proof of the Proposition proceeds as the one indicated in the Introduction for C^2 functions with bounded above Hessian.

As an application of the estimate (2.1), consider the Hamilton-Jacobi equation

$$u + H(Du) = f \text{ in } \mathbb{R}^n \tag{2.3}$$

with H convex and coercive, f semiconcave. If $H(0) = 0$ and $f \geq 0$, then the unique bounded viscosity solution of (2.3) is Lipschitz continuous, nonnegative and semiconcave for some semiconcavity constant M depending on H and f, see [12], [1].
Therefore, by Proposition 2.1 and the Rademacher's theorem,

$$|Du(x)| \leq \sqrt{2Mu(x)} \quad \text{almost everywhere in } \mathbb{R}^n$$

3 Glaeser estimates for fully nonlinear equations

3.1 A quick review of known results

The condition $\Delta u \leq M$ is of course weaker then (1.1) and, as pointed out in [18], is not sufficient to guarantee the validity of estimates (1.2) and (1.4). On the other hand, various versions of Glaeser's type inequalities for functions satisfying bilateral partial differential constraints have been recently established by Li and Nirenberg.
A model result in [18] is obtained under the bilateral bound

$$-M \leq \Delta u \leq M \tag{3.1}$$

for some $M > 0$. They proved indeed that if u is a nonnegative C^2 function in the ball $B_R = \{x \in \mathbb{R}^n : |x| \leq R\}$ and (3.1) is fulfilled in B_R, then the estimates

$$|Du(x)| \leq C\sqrt{u(0)M} \quad \text{if} \quad 2|x| \leq \sqrt{\frac{u(0)}{M}} \leq R \tag{3.2}$$

$$|Du(x)| \leq C\left(\frac{u(0)}{R} + MR\right) \quad \text{if} \quad 2|x| \leq R \leq \sqrt{\frac{u(0)}{M}} \tag{3.3}$$

hold for some constant C depending only on n, λ, Λ but not on u.
Note that using inequality (3.2) one can easily produce an unusual proof of the Liouville theorem:

$$u \in C^2(\mathbb{R}^n)\,,\ \Delta u = 0\,, u \geq 0 \ \text{ imply } \ u \equiv \text{constant} \tag{3.4}$$

Indeed, if $u(0) = 0$ then, by the Maximum Principle, $u \equiv 0$. The other possible case is $u(0) > 0$: since u is harmonic, then $-\varepsilon \leq \Delta u \leq \varepsilon$ for any arbitrarily small $\varepsilon > 0$ so that (3.2) applies to give

$$\sup_{B_{R_\varepsilon}} |Du(x)| \leq C\sqrt{\varepsilon\, u(0)}\ ,\ R_\varepsilon = \frac{1}{2}\sqrt{\frac{u(0)}{\varepsilon}} > 0$$

for some constant C depending only on n, λ, Λ. Since $R_\varepsilon \to +\infty$ as $\varepsilon \to 0^+$, one can pass to the limit by monotonicity in the above to conclude $\sup_{R^n} |Du(x)| = 0$.

Incidentally, this shows that the inequality (3.2) cannot hold true if one assumes only the unilateral bound $\Delta u \leq M$. In fact, its validity would imply the Liouville theorem

$$u \in C^2(\mathbb{R}^n)\,,\ \Delta u \leq 0\,, u \geq 0 \ \text{ imply }\ u \equiv \text{constant}$$

which is known to be true for $n \leq 2$ and false in higher dimension as the following simple example shows

$$u(x) = \begin{cases} \frac{1}{8}(15 - 10|x|^2 + 3|x|^4) & \text{if } |x| < 1 \\ |x|^{-1} & \text{if } |x| \geq 1 \end{cases}$$

see [21].

Further extensions considered in the same paper [18] involve either the conditons

$$0 \leq u \in C^2(B_R)\,,\ \ ||\Delta u||_{L^p(B_R)} \leq M \quad \text{in } B_R \tag{3.5}$$

with $p > n$, or

$$0 \leq u \in C^2(B_R)\,,\ \ -M \leq Lu \leq M \quad \text{in } B_R \tag{3.6}$$

where

$$L = a_{ij}(x)\partial_{ij} + b_i(x)\partial_i + c(x)$$

is a second order uniformly elliptic operator with continuous coefficients and $c \leq 0$.

The proofs are not elementary as that of the results in Section 1 since they rely on classical techniques in ellptic pde's such as the Maximum Principle, gradient and $W^{2,p}$ estimates and the Harnack inequality, see [18]. Since most of these techniques are also available in the elliptic nonlinear setting, see [3], it is reasonable to guess that Glaeser's type estimates continue to be valid also for functions satisfying appropriate nonlinear partial differential inequalities A different kind of remark is that if F is a C^1 real-valued function defined on $\mathcal{S}^n$, the set of symmetric $n \times n$ real matrices, which is uniformly elliptic

$$\lambda\,|\xi|^2 \le \frac{\partial F}{\partial X_{ij}}\,\xi_i \xi_j \le \Lambda\,|\xi|^2\,, \quad 0 < \lambda \le \Lambda$$

and $u \in C^2(B_R)$ is a classical solution of $F(D^2 u) = f$ in B_R, then u solves a linear uniformly elliptic equation with continuous coefficients

$$a_{ij}(x) = \int_0^1 \frac{\partial F}{\partial X_{ij}}(t\, D^2 u(x))\, dt$$

Hence, it is immediate to derive the validity of the inequalities (3.2) and (3.3) from one of the previously cited results in [18]. In this case the costant C will depend on n, λ, Λ and the moduli of continuity of the a_{ij}.

3.2 Glaeser's type estimates for reflection-invariant equations

In this Section we present some extension of the Li-Nirenberg results to *nonnegative continuous viscosity solutions* u of quite general partial differential inequalities, comprising possibly *non smooth nonlinearities* F acting on second-order derivatives, such as those arising in Bellman or Bellman-Isaacs operators.
More precisely, we will consider continuous functions u satisfying in the *viscosity sense*, see [8], the partial differential equation

$$F(D^2 u) + H(Du) = f \ \text{ in } \operatorname{int} B_R \tag{3.7}$$

We will assume that F is *uniformly elliptic*

$$\lambda\,\mathrm{Tr}\,(Y) \le F(X+Y) - F(X) \le \Lambda\,\mathrm{Tr}\,(Y) \tag{3.8}$$

for some constants $0 < \lambda \le \Lambda$ and for all $X, Y \in \mathcal{S}^n$ (the space of $n \times n$ symmetric matrices) with $Y \ge 0$, a *linear growth* condition on the first-order term

$$|H(p)| \le b_0\,|p| \ \text{ for all } p \in \mathbb{R}^n \tag{3.9}$$

and

$$f \in C(B_R) \tag{3.10}$$

and also, for simplicity, that

$$F(O) = 0 \tag{3.11}$$

A mapping $F : \mathcal{S}^n \to \mathbb{R}$ is *reflection-invariant* with respect to the direction $\nu \in \mathbb{R}^n$, $|\nu| = 1$, if

$$F(RXR) = F(X) \quad \text{for all} \quad X \in \mathcal{S}^n \tag{3.12}$$

where R is the reflection matrix with respect to the hyperplane of equation $\nu \cdot x = 0$. If, for instance, $\nu = (0, \ldots, 0, 1)$ then

$$R = \begin{pmatrix} \mathbb{I}_{n-1} & 0 \\ 0 & -1 \end{pmatrix}$$

where $\mathbb{I}_{n-1}$ is the $(n-1)$-dimensional identity matrix. Examples of (non smooth) nonlinear functions which are reflection-invariant with respect to n linearly independent directions $\nu^1, \ldots, \nu^n$ are

- the Pucci extremal operators

$$\mathcal{P}^+_{\lambda,\Lambda}(X) = \Lambda \mathrm{Tr}\,(X^+) - \lambda \mathrm{Tr}\,(X^-)$$

$$\mathcal{P}^-_{\lambda,\Lambda}(X) = \lambda \mathrm{Tr}\,(X^+) - \Lambda \mathrm{Tr}\,(X^-)$$

- any $F = F(X)$ depending only on the eigenvalues of X
- the Bellman operators $\inf_{j \in J} \mathrm{Tr}\,(A_j X)$ for constant symmetric positive definite matrices A_j, provided that A_j commutes with A_i for each $i, j \in J$
- the Bellman-Isaacs operators $\inf_{j \in J} \sup_{k \in K} \mathrm{Tr}\,(A_{jk} X)$ under suitable commutation conditions,

see [6]. Observe that $\mathcal{P}^-_{\lambda,\Lambda}(X) \leq F(X) \leq \mathcal{P}^+_{\lambda,\Lambda}(X)$ for any F satisfying (3.8).

By suitably exploiting reflection-invariance properties of F one can prove the following gradient estimate of Glaeser type:

Theorem 3.1 *Assume that the data F, H, f satisfy conditions (3.8), (3.9), (3.10) and (3.11). Assume also that F is reflection-invariant with respect to n linearly independent directions $\nu^1, \ldots, \nu^n$. Let $u \in C(B_R)$ be a nonnegative viscosity solution of*

$$F(D^2 u) + H(Du) = f \ \textit{in} \ \mathrm{int}\, B_R$$

Then for almost every $x \in B_{R/2}$

$$|Du(x)| \le C\sqrt{u(0)\sup_{B_R}|f|} \quad \textit{if} \quad 2|x| \le \sqrt{\frac{u(0)}{\sup_{B_R}|f|}} \le R \tag{3.13}$$

$$|Du(x)| \le C\left(\frac{u(0)}{R} + R\sup_{B_R}|f|\right) \quad \textit{if} \quad 2|x| \le R \le \sqrt{\frac{u(0)}{\sup_{B_R}|f|}} \tag{3.14}$$

for some constant C depending only on n, λ, Λ but not on u.

It is conceivable that the linear growth assumption (3.9) in Theorem (3.1) could be somewhat relaxed. Observe, however, that if H grows quadratically in $|Du|$ then inequalities (3.13) and (3.14) may continue to hold true but with a constant C depending on u itself. Here is a simple evidence in this direction: suppose that $u \ge 0$ is smooth and such that

$$-M \le \Delta u - k|Du|^2 \le M \ \text{ in } B_R$$

for positive constants k and M. Then, the Hopf-Cole transform $v = 1 - e^{-ku}$ satisfies

$$-kM \le \Delta v \le kM \ \text{ in } B_R$$

By the above mentioned result of [18], $Du = \frac{1}{k}e^{ku}Dv$ can be therefore estimated as

$$|Du(x)| \le C\frac{1}{k}\, e^{k\sup u}\sqrt{(1 - e^{-ku(0)})\, kM} \le Ce^{k\sup u}\sqrt{u(0)\, M}$$

in the case $2|x| \le \sqrt{\frac{u(0)}{M}} \le R$. Hence, (3.13) holds for nonnegative solutions of the viscous Hamilton-Jacobi equation

$$\Delta u - k|Du|^2 = f$$

with bounded, continuous right-hand side with a constant C depending on $\sup u$ (and k). This is due, of course, to the presence of the quadratic term, see [15] for related issues.

A major ingredient in the proof of Theorem 3.1 is the next Lemma which is in fact a viscosity version of a well-known property of smooth solutions of the Poisson equation, see [10]. Its validity in the present context is guaranteed by the reflection-invariance properties of F which replace the symmetry properties of the Laplace operator.

Lemma 3.2 *Assume that F and f satisfy, respectively, conditions (3.8), (3.11) and (3.10). Let $u \in C(B_d)$ be a viscosity solution of*

$$F(D^2u) = f \text{ in } \operatorname{int} B_d \tag{3.15}$$

If the directional derivative D_ν of u with respect to ν exists at $x = 0$, then

$$|D_\nu u(0)| \leq \frac{n}{d}\sqrt{\lambda+\Lambda}\sup_{B_d}|u| + \frac{d}{2\sqrt{\lambda(\lambda+\Lambda)}}\sup_{B_d}|f| \tag{3.16}$$

For the proof, let us observe preliminarily that since F is invariant by reflection with respect to the n independent directions ν^i, a simple argument in linear algebra shows that there exists an orthogonal matrix Q such that

$$G(X) = F(Q^tXQ)$$

is reflection-invariant with respect to the standard basis vectors $e^i = (0, \ldots, 1, \ldots 0)$. Also, the uniform ellipticity of F implies the same property for G. Observe finally that if u is a viscosity solution of equation (3.15), then the function $v(x) = u(Q^tx)$ is a viscosity solution of $G(D^2v(x)) = f(Q^tx)$.
We can assume therefore that F is invariant by reflection with respect to the e^i's. Using some viscosity calculus, uniform ellipticity and the assumption of reflection-invariance with respect to e^n, one can check that the function

$$\tilde{u}(x) = \frac{u(x', x_n) - u(x', -x_n)}{2}, \quad x = (x', x_n)$$

satisfies the inequalities

$$\mathcal{P}^+_{\lambda,\Lambda}\left(D^2(\tilde{u} - \Phi)\right) \geq 0 \geq \mathcal{P}^-_{\lambda,\Lambda}\left(D^2(\tilde{u} + \Phi)\right)$$

in the cylinder

$$K^+ = \left\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < \frac{d\sqrt{\Lambda}}{\sqrt{\lambda+\Lambda}}, 0 < x_n < \frac{d\sqrt{\lambda}}{\sqrt{\lambda+\Lambda}}\right\} \subset B_d$$

Here, Φ is the smooth comparison function

$$\Phi(x) = \frac{\sup|u|}{d'^2}\left[\frac{|x'|^2}{\Lambda} + \frac{x_n}{\sqrt{\lambda}}\left(nd' - (n-1)\frac{x_n}{\sqrt{\lambda}}\right)\right] + \frac{M}{2}\frac{x_n}{\sqrt{\lambda}}\left(d' - \frac{x_n}{\sqrt{\lambda}}\right)$$

where we set $d' = \frac{d}{\sqrt{\lambda+\Lambda}}$ and $M = \sup_{B_d}|f|$.
Since $\tilde{u} - \Phi \leq 0 \leq \tilde{u} + \Phi$ on ∂K^+, then from Comparison Principles for viscosity solutions, see [8], it follows that $\tilde{u} - \Phi \leq 0 \leq \tilde{u} + \Phi$ on K^+ and, in particular, at points

$(0, x_n)$. These inequalities yield the conclusion after dividing by $x_n > 0$ and letting $x_n \to 0^+$.

The Lemma does not require $u \geq 0$. For *nonnegative* solutions of equation (3.15) we can derive Theorem 3.1 from Lemma 3.2. We will use at this purpose the Harnack inequality

$$\sup_{B_{\frac{3}{4}r}} u \leq C \left(\inf_{B_{\frac{3}{4}r}} u + r \, \|f\|_{L^n(B_r)} \right) \tag{3.17}$$

which holds for all nonnegative viscosity solutions of equation $F(D^2u) = f$ in B_r with some universal constant C depending only on n, λ, Λ, see [3].

To realize that, take $0 < r < R$ and any $x \in B_{r/2}$ and observe that the inclusion $B_d(x) \subset B_{\frac{3}{4}r}$ holds , for $d = r/4$. By translation invariance we can use (3.16) and then the Harnack inequality to deduce

$$|Du(x)| \leq C \left(\frac{\sup_{B_d} u}{r} + r \sup_{B_R} |f| \right) \leq C \left(\frac{u(0) + Mr^2}{r} + Mr \right) \leq C \left(\frac{u(0)}{r} + r \sup_{B_R} |f| \right)$$

at those $x \in B_R$ where u is differentiable. In the above, C denotes different positive constants depending only on n, λ, Λ.
By the regularity results in [13], u is Lipschitz continuous and therefore is differentiable almost everywhere in int B_R.
At this point, the Glaeser's inequalities (3.13), (3.14) are deduced by optimizing the right-hand side of the above with respect to $r \in [0, R]$.
Once the Theorem is proved in the case $H \equiv 0$, the general case of a non-zero first order term H with linear growth can be treated by more or less standard perturbation arguments.

A final remark is that the result of Theorem 3.1 continue to hold if we adopt the slightly stronger notion of L^n-viscosity solutions which makes use of $W^{2,n}_{loc}$ rather than on C^2 test functions, see [2], [4]. In this setting the assumption of continuity of f can be relaxed to $f \in L^\infty$.

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