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A Solution to a Problem posed by S. Manni and the Related Topics

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1 Introduction

Throughout this talk we consider only positive definite even unimodular lattces.
S. Manni [12] proved

Theorem 1.1. *In 56 (resp. 72) dimensional even unimodular extremal lattices, the theta series associated to such lattices we can say that in degree 3 their difference is, up to a multiplicative, possibly 0, constant, and equal to* χ_{28} *(resp.* χ_{36}*).*

In 40 dimensional lattices, if two extremal theta series are equal in degree 2, then in degree 3 their difference is up to a multiplicative, possibly 0, constant, and equal to χ_{20}.

He then wrote
Find two even unimodular extremal lattices L_1 and L_2 of rank 40 whose theta series coincide in degree 2 and differ in degree 3. Besides this he posed the problems in ranks 32, 48 and 56.

In the present report we show that there are 40 dimensional two even unimodular extremal lattices coming from two doubly even self-dual extremal codes, whose theta series of degree 2 coincide and theta series of degree 3 differ definitely. We also show an instance of two another even unimodular extremal lattices coming from another two doubly even self-dual extremal codes, whose theta series of degree 2 and degree 3 coincide. These are shown by computing some beginning Fourier coefficients of theta series of the lattices in question combined with some facts on the dimensions of the linear spaces of Siegel modular forms already proved by other people.
S. Manni [12] also proved

Theorem 1.2. *In 32 (resp. 48) dimensional even unimodular extremal lattices, about the theta series associated to such lattices we can say that*

(i) it is unique in degree 3,
(ii) in degree 4 their difference is, up to a multiplicative (possibly 0) constant, equal to a power of Schottky's polynomial J.

He then wrote
Find two even unimodular extremal lattices L_3 and L_4 of rank 32 or 48 whose theta series differ in degree 4.
We dicuss some related trials to this problem.

2 A brief account

2.1 32 dimensional case

Erokhin [6] proved

Theorem 2.1. *If two 32 dimensional even unimodular lattices have identical theta series of degree 1, then they have identical theta series of degrees up to 3.*

Venkov [28], [29] gave a method to compute some Fourier coefficients of Siegel theta series of degree 3 associated with even unimodula extremal 32 dimensional lattices.

2.2 40 or higher dimensional cases

The 40 dimensional case is our present topic. There is not any explicit result for the 48 dimensional and 56 dimensional cases along with Manni's questions. The reasons for this would be the facts that there are few explicit constructions of lattices and that they are constructed through ternary codes. In 32 dimensional case our present method will apply to Manni's problem, but we have not pursued this case since the shapes of minimal vectors in an extremal 32 dimensional lattice are complicated.

3 Some Basics

3.1 Lattice

A lattice L of rank n (or dimension n) is a $\mathbb{Z}$-module generated by the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ in $\mathbb{R}^n$ that are linearly independent over $\mathbb{R}$. The vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are called the basis of L.

L is integral if the inner product $(\mathbf{x}, \mathbf{y})$ belongs to $\mathbb{Z}$ for all pairs $\mathbf{x}$ and $\mathbf{y}$ in L.

The dual lattice $L^\#$ of L is defined to be

$$L^\# = \{\mathbf{y} \in \mathbb{R}^n | (\mathbf{x}, \mathbf{y}) \in \mathbb{Z}, \ ^\forall \mathbf{x} \in L\}.$$

A lattice L is unimodular if it holds that $L = L^\#$.

A lattice L is even if any element $\mathbf{x}$ of L has even norm $(\mathbf{x}, \mathbf{x})$.

Even unimodular lattices exist only when $n \equiv 0 \pmod 8$.

$Min(L) = \min_{0 \neq \mathbf{x} \in L} (\mathbf{x}, \mathbf{x})$
When L is even unimodular of rank n it holds that

$$\mathrm{Min}(L) \leq 2 \left[\frac{n}{24}\right] + 2.$$

A lattice which attains the above maximum is called an extremal lattice.
Let L be an even unimodular lattice of rank n.
$\Lambda_{2m}(L)$: The set of $\mathbf{x}$ in L with $(\mathbf{x}, \mathbf{x}) = 2m$ $(m \geq 1)$.

3.2 Siegel modular forms

The symplectic group $Sp_g(\mathbb{R})$ of degree g over $\mathbb{R}$is defined to be

$$Sp_g(\mathbb{R}) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2g}(\mathbb{R}) \mid {}^tMJM = J, J = \begin{pmatrix} O & -I_g \\ -I_g & O \end{pmatrix} \right\}.$$

Siegel modular group $Sp_g(\mathbb{Z})$ of degree g is a subgroup of $Sp_g(\mathbb{R})$ consisting of elements in $Sp_g(\mathbb{R})$ whose entries are in $\mathbb{Z}$. Let $\mathbb{H}_g$ be the Siegel upper half-space of degree g:

$$\mathbb{H}_g = \{\tau \mid \tau = X + Yi \in M_g(\mathbb{C}), \mid {}^t\tau = \tau, Y \text{is positive definite}\}.$$

A Siegel modular form of degree g $(g \geq 2)$ and weight k is a holomorphic complex valued function $f(\tau)$ defined on $\mathbb{H}_g$ satisfying the condition :

$$f\left((A\tau + B)(C\tau + D)^{-1}\right) = (\det(C\tau + D))^k f(\tau) \quad ^\forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_g(\mathbb{Z}).$$

Note that when $g = 1$ an additional condition of the holomorphicity of f at the cusp is neccessary.

3.3 Siegel theta series

Siegel theta series of degree g attached to the lattice L is defined by

$$\vartheta_g(\tau, L) = \sum_{\mathbf{x}_1,\dots,\mathbf{x}_g \in L} exp(\pi i \sigma([\mathbf{x}_1, \dots, \mathbf{x}_g]\tau)),$$

where τ is the variable on Siegel upper-half space of degree g, $[\mathbf{x}_1, \dots, \mathbf{x}_g]$ is a g by g square matrix whose (i, j) entry is $(\mathbf{x}_i, \mathbf{x}_j)$ and σ is the trace of the matrix.

Siegel theta series of degree g can be expanded to

$$\vartheta_g(\tau, L) = \sum_T a(T, L)e^{2\pi i \sigma(T\tau)}.$$

Here T runs over the set of positive semi-definite semi-integral symmetric square matrices of degree g, and $a(T, L) = \#\{\langle \mathbf{x}_1, \dots, \mathbf{x}_g \rangle \in L^g \mid [\mathbf{x}_1, \dots, \mathbf{x}_g] = 2T\}$.
Fact: Siegel theta series of degree g associated with an even integral unimodular lattice L of rank $2k$ ($2k$ is a multiple of 8) is a modular form of degree g and weight k.

3.4 Theta Functions with characteristics

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\tau, Z) = \sum_{N \in \mathbb{Z}^g} \exp\left\{2\pi i \left[\frac{1}{2}{}^t\left(N + \frac{\epsilon}{2}\right)\tau\left(N + \frac{\epsilon}{2}\right) + {}^t\left(N + \frac{\epsilon}{2}\right)\left(Z + \frac{\epsilon'}{2}\right)\right]\right\}$$

Here ϵ, ϵ' are integral vectors of length g with entries 0 or 1, Z is a variable on $\mathbb{C}^g$, and τ is a variable on $\mathbb{H}_g$, the Siegel upper half space of genus g. For $g = 2$ case

$$\begin{aligned}
&\theta \begin{bmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_1' & \epsilon_2' \end{bmatrix} (\tau, Z) \\
&= \sum_{\mathbf{n}=(n_1,n_2) \in \mathbb{Z}^2} \exp\left\{\pi i \left(\sum_{i,j=1}^{2} \tau_{ij}(n_i + \frac{\epsilon_i}{2})(n_j + \frac{\epsilon_j}{2}) + 2\sum_{i=1}^{2}(n_i + \frac{\epsilon_i}{2})(z_i + \frac{\epsilon_i'}{2})\right)\right\}. \\
&= \sum_{\mathbf{n}=(n_1,n_2) \in \mathbb{Z}^2} q_1^{n_1^2+n_1\epsilon_1+\epsilon_1^2/4} q_2^{n_2^2+n_2\epsilon_2+\epsilon_2^2/4} q_3^{2n_1n_2+(n_2\epsilon_1+n_1\epsilon_2)+\epsilon_1\epsilon_2/2} \\
&\quad \times \zeta_1^{2(n_1+\epsilon_1/2)} \zeta_2^{2(n_2+\epsilon_2/2)} e^{\pi i[\epsilon_1'(n_1+\epsilon_1/2)+\epsilon_2'(n_2+\epsilon_2/2)]}
\end{aligned}$$

Here τ_{ij} is the ij entry of τ, $Z = (z_1, z_2) \in \mathbb{C}^2$, $q_1 = e^{\pi i \tau_{11}}$, $q_2 = e^{\pi i \tau_{22}}$, $q_3 = e^{\pi i \tau_{12}}$, $\zeta_1 = e^{\pi i z_1}$, $\zeta_2 = e^{\pi i z_2}$.

Two instances.

$$\theta\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}(\tau, Z) = \sum_{\mathbf{n}=(n_1,n_2)\in\mathbb{Z}^2} q_1^{n_1^2} q_2^{n_2^2} q_3^{2n_1n_2} \zeta_1^{2n_1} \zeta_2^{2n_2}$$

$$\theta\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}(\tau, Z) = \sum_{\mathbf{n}=(n_1,n_2)\in\mathbb{Z}^2} q_1^{(n_1+1/2)^2} q_2^{n_2^2} q_3^{2(n_1+1/2)n_2} \zeta_1^{2(n_1+1/2)} \zeta_2^{2n_2}$$

3.5 Binary linear code

Let $\mathbb{F}_2 = GF(2)$ be the field of 2 elements. Let $V = \mathbb{F}_2^n$ be the vector space of dimension n over $\mathbb{F}_2$. A linear $[n, k]$ code $\mathbf{C}$ is a vector subspace of V of dimension k. An element $\mathbf{x}$ in $\mathbf{C}$ is called a codeword of $\mathbf{C}$. In V , the inner product , which is denoted by $\mathbf{x} \cdot \mathbf{y}$ for $\mathbf{x}, \mathbf{y}$ in V, is defined as usual. Two codes $\mathbf{C}_1$ and $\mathbf{C}_2$ are said to be equivalent if and only if after a suitable change of coordinate positins of $\mathbf{C}_1$ all the codewords in both codes coincide.
The dual code $\mathbf{C}^{\perp}$ of $\mathbf{C}$ is defined by

$$\mathbf{C}^{\perp} = \{\mathbf{u} \in V \mid \mathbf{u} \cdot \mathbf{v} = 0 \quad \forall \mathbf{v} \in \mathbf{C}\}.$$

The code $\mathbf{C}$ is called self-orthogonal if it satisfies $\mathbf{C} \subseteq \mathbf{C}^{\perp}$, and the code $\mathbf{C}$ is called self-dual if it satisfies $\mathbf{C} = \mathbf{C}^{\perp}$.

Self-dual codes exist only if $n \equiv 0 \pmod 2$ and $k = \frac{n}{2}$

Let

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

be a vector in V , then the Hamming weight $wt(\mathbf{x})$ of the vector $\mathbf{x}$ is defined to be the number of i's such that $x_i \neq 0$. The Hamming distance d on V is also defined by $d(\mathbf{x}, \mathbf{y}) = wt(\mathbf{x} - \mathbf{y})$. Let $\mathbf{C}$ be a code ,then the minimum distance d of the code $\mathbf{C}$ is defined by

$$\begin{aligned} d &= \mathrm{Min}_{\mathbf{x},\mathbf{y}\in\mathbf{C},\mathbf{x}\neq\mathbf{y}} d(\mathbf{x}, \mathbf{y}) \\ &= \mathrm{Min}_{\mathbf{x}\in\mathbf{C},\mathbf{x}\neq\mathbf{0}} wt(\mathbf{x}). \end{aligned}$$

Let $\mathbf{C}$ be a self-dual binary $[n, \frac{n}{2}]$ code, then the weight $wt(\mathbf{x})$ of each codeword $\mathbf{x}$ in $\mathbf{C}$ is an even number. Further, if the weight of each codeword $\mathbf{x}$ in $\mathbf{C}$ is divisible by 4, then the code is called a doubly even binary code. It is known that doubly even self-dual binary codes $\mathbf{C}$ exist only when the length n of $\mathbf{C}$ is a multiple of 8.

Let $\mathbf{C}$ be a self-dual doubly even code of length n, which are embedded in $\mathbb{F}_2^n$. Let $\mathbf{u} = (u_1, u_2, \cdots, u_n), \mathbf{v} = (v_1, v_2, \cdots, v_n)$ be any pair of vectors in $\mathbb{F}_2^n$, then the number of common 1's of the corresponding coordinates for $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} * \mathbf{v}$. This is called the intersection number of $\mathbf{u}$ and $\mathbf{v}$, and $\mathbf{u} * \mathbf{u}$ is nothing else $wt(\mathbf{u})$.

3.6 Multiple weight enumerator

Let $\mathbf{C}$ be a doubly even self-dual code of length n, and g be a positive integer and we let α run the set $\mathbb{F}_2^g$ of g-tuple vectors. The 2^g algebraically independent over $\mathbb{C}$variables x_α are parametrized by $\alpha \in\mathbb{F}_2^g$. Let

$\mathbf{u}_1 = (u_1^1, u_1^2, \cdots, u_1^n), \mathbf{u}_2 = (u_2^1, u_2^2, \cdots, u_2^n), \cdots, \mathbf{u}_g = (u_g^1, u_g^2, \cdots, u_g^n)$ be the g-tuple codewords of $\mathbf{C}$. For each $\alpha \in \mathbb{F}^g$ a generalized weight
$wt_\alpha(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_g)$ α is defined to be the number of coordinates j $(1 \leq j \leq n)$ such that the equation $\alpha = (u_1^j, u_2^j, \cdots, u_g^j)$ holds.
The multiple weight enumerator $\mathbf{W}_g(x_\alpha; \mathbf{C})$ of genus g for the code $\mathbf{C}$ is defined by

$$\mathbf{W}_g(x_\alpha; \mathbf{C}) = \sum_{(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_g) \in \mathbf{C}^g} \prod_{\alpha \in \mathbb{F}_2^g} x_\alpha^{wt_\alpha(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_g)}.$$

The multiple weight enumerator of second degree is called a biweight enumerator, and the multiple weight enumerator of third degree is called a triweight enumerator.

3.7 From binary codes to lattices

C : binary self-orthogonal $[n, k]$ code

Construction A_2

$$\begin{array}{ccc} \rho : \mathbb{Z}^n & \to & \mathbb{F}_2^n \\ \Uparrow & & \Uparrow \\ \mathbf{x} & \mapsto & \mathbf{x} \bmod 2 \end{array}$$

$$L(C) = \frac{1}{\sqrt{2}} \rho^{-1}(C).$$

Construction B_2

$$\begin{array}{ccc} \rho : \mathbb{Z}^n & \to & \mathbb{F}_2^n \\ \Uparrow & & \Uparrow \\ \mathbf{x} & \mapsto & \mathbf{x} \bmod 2 \end{array}$$

$$M(C) = \frac{1}{\sqrt{2}} \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \rho^{-1}(C) | \sum_{i=1}^{n} x_i \equiv 0 \pmod 4 \right\}$$

Doubling the density process:

Suppose that $\mathbf{C}$ is a doubly even self-dual binary $[n, n/2]$ code. Put

$$\gamma = \begin{cases} \frac{1}{\sqrt{8}}(1, \ldots, 1, -3) & \text{if } n \equiv 8 \pmod{16}, \\ \frac{1}{\sqrt{8}}(1, \ldots, 1, 1) & \text{if } n \equiv 0 \pmod{16} \end{cases}$$

$$\mathcal{N}(C) = \mathcal{M}(C) \cup (\gamma + \mathcal{M}(C))$$

We pick up peculiar codes. We denote the codes C_1 (respctively C_2, C_3, C_4) the second code in [17],Yorgov's C_5 ,Yorgov's code C_2 and Yorgov's code C_4 [?] respectively. The lattices constructed by the above process are denoted by $M_{11} = \mathcal{N}(C_1), M_{12} = \mathcal{N}(C_2), M_{13} = \mathcal{N}(C_3)$ and $M_{14} = \mathcal{N}(C_4)$ respectively.

3.7.1 40 dimensional case

We are particularly concerned with the set of minmal vectors $\Lambda_4(N(\mathbf{C}))$ in an extremal even unimodular lattice constructed from binary self-dual extremal $[40, 20, 8]$ code.
When $\mathbf{C}$ is a doubly even self-dual binary $[40, 20, 8]$ code, $\Lambda_4 = \Lambda_4(N(\mathbf{C}))$ consists of two kinds of vectors:

$$\begin{aligned} \Lambda_4^1 &= \left\{\frac{1}{\sqrt{2}}((\pm 2)^2, 0^{38})\right\} \quad \textit{number} = 3120 \\ \Lambda_4^2 &= \left\{\frac{1}{\sqrt{2}}((\pm 1)^8, 0^{32})\right\} \quad \textit{number} = 36480 \end{aligned}$$

The set Λ_4^1 forms a root system of type D_{40} scaled by a factor $\sqrt{2}$, and the vectors in the set Λ_4^2 come from codewords of weight 8 in the code $\mathbf{C}$.
To each $\mathbf{y} \in \Lambda_4$ we associate a binary vector $\mathbf{v} = supp(\mathbf{y}) \in \mathbb{F}_2^{40}$ which corresponds to non zero positions of $\mathbf{y}$.

3.8 Duke-Runge map

We explain the map by using the case $g = 2$.
We put

$$\varphi_\epsilon(\tau) = \theta \begin{bmatrix} \epsilon \\ \mathbf{0} \end{bmatrix} (2\tau, 0).$$

These are theta zero values with the variable τ multiplied by 2. There are 2^g functions $\varphi_\epsilon(\tau)$.

$$\begin{aligned} &\varphi_{00}(\tau) \\ &= \sum_{\mathbf{n}=(n_1,n_2)\in\mathbb{Z}^2} q_1^{2n_1^2} q_2^{2n_2^2} q_3^{4n_1n_2} \\ &= 1 + 2q_1^2 + 2q_2^2 + 2q_1^2q_2^2(q_3^4 + q_3^{-4}) + 2q_1^8 + 2q_2^8 \\ &\quad +2q_1^8q_2^8(q_3^{16} + q_3^{-16}) + 2q_1^8q_2^2(q_3^8 + q_3^{-8}) \\ &\quad +2q_1^2q_2^8(q_3^8 + q_3^{-8}) + \cdots \end{aligned}$$

$$\begin{aligned} &\varphi_{10}(\tau) \\ &= \sum_{\mathbf{n}=(n_1,n_2)\in\mathbb{Z}^2} q_1^{2(n_1+1/2)^2} q_2^{2n_2^2} q_3^{4(n_1+1/2)n_2} \\ &= 2q_1^{\frac{1}{2}} + 2q_1^{\frac{1}{2}}q_2^2(q_3^2 + q_3^{-2}) \\ &\quad +2q_1^{\frac{1}{2}}q_2^8(q_3^4 + q_3^{-4}) \\ &\quad +2q_1^{\frac{9}{2}} + 2q_1^{\frac{9}{2}}q_2^2(q_3^6 + q_3^{-6}) \\ &\quad +2q_1^{\frac{9}{2}}q_2^8(q_3^{12} + q_3^{-12}) + \cdots \end{aligned}$$

Likewise $\varphi_{01}(\tau), \varphi_{11}(\tau)$ can be expanded. Let $\mathbf{W}_g(x_\alpha; \mathbf{C})$ be a multiple weight enumerator of genus g for a doubly even self-dual code $\mathbf{C}$, then $\mathbf{W}_g(\varphi_\epsilon; \mathbf{C})$ is proved to be Siegel theta series of degree g that is associated

with the lattice constructed by using Construction A_2 in Section 3.7.
For instance

$$
\begin{aligned}
&W_2(x_{00}, x_{01}, x_{10}, x_{11}; Ham)\\
&= x_{00}^8 + x_{01}^8 + x_{10}^8 + x_{11}^8 + 14(x_{00}^4 x_{01}^4 + x_{00}^4 x_{10}^4 + x_{00}^4 x_{11}^4 + x_{01}^4 x_{10}^4 + x_{01}^4 x_{11}^4 + x_{10}^4 x_{11}^4) + 168 x_{11}^2 x_{10}^2 x_{01}^2 x_{00}^2
\end{aligned}
$$

is the biweight enumerator of the Hamming $[8,4,4]$ code. And

$$
\begin{aligned}
&W_2(\varphi_{00}(\tau), \varphi_{01}(\tau), \varphi_{10}(\tau), \varphi_{11}(\tau); Ham)\\
&= 1 + 240q_2^2 + 2160q_2^4 + 6720q_2^6 + 17520q_2^8 + 30240q_2^{10}\\
&+q_1^2[240 + 240q_3^4q_2^2 + +240/q_3^4q_2^2 + 13440q_3^2q_2^2 + 13440/q_3^{-2}q_2^2 + 30240q_2^2 + 30240/q_3^8q_2^4 + 30240q_3^8q_2^4 +\\
&13440/q_3^{12}q_2^6 + 181440/q_3^8q_2^6 + 138240q_3^4q_2^4 + 181440q_2^4 + 138240/q_3^4q_2^4 + 13440q_3^{12}q_2^6 +\\
&+362880/q_3^{12}q_2^{10} + 1330560/q_3^4q_2^{10} + 30240/q_3^{16}q_2^{10} + 362880q_3^{12}q_2^{10} + 30240q_3^{16}q_2^{10}\\
&+1814400q_2^{10} + 997920q_3^8q_2^{10} + 997920/q_3^8q_2^{10} + 1330560q_3^4q_2^{10} + 497280/q_3^8q_2^8 + 997920q_2^8\\
&+240/q_3^{16}q_2^8 + 138240q_3^{12}q_2^8 + 240q_3^{16}q_2^8 + 497280q_3^8q_2^8 + 967680/q_3^4q_2^8\\
&+138240/q_3^{12}q_2^8 + 967680q_3^4q_2^8 + 181440q_3^8q_2^6 + 497280q_2^6 + 362880/q_3^4q_2^6 + 362880q_3^4q_2^6+]
\end{aligned}
$$

is the Siegel theta series of degree 2 for the root lattice E_8.

The multiple weight enumerators for the class of doubly even self-dual codes are invariant under the action of certain finite group G of linear transformations. Runge discussed the ring $\mathcal{R}$ of invariants under a special subgroup H of G and extended the mapping Φ to $\mathcal{R}$.

4 Preliminary results

Table 1 The dimensions of the linear space of Siegel modular forms of degree g and weight k.

$n\backslash k$	4	6	8	10	12	14	16	18	20
1	1	1	1	1	2	1	2	2	2
2	1	1	1	2	3	2	4	4	5
3	1	1	1	2	4	3	7	8	11
4	1	1	2	3	6	6	14		

Proposition 4.1. *Siegel theta series $\vartheta_g(\mathbf{Z}, L)$ of degree g associated with an even unimodular lattice of rank $2k$ ($k \equiv 0 \pmod 2$) is determined uniquely if the Fourier coefficients $a(T, L)$ are known for T's given in the Table 2-1~ 2-3.*

Table 2-1 $g = 1$ case

$2k$	8	16	24	32	40
T	0	0	0	0	1
			1	1	1

Table 2-2 $g = 2$ case

$2k$	8	16	24	32	40
$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Table 2-3 $g = 3$ case

2k	24	32	40
$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
		$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
		$\begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
		$\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
			$\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}$
			$\begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 1/2 & 1/2 & 2 \end{pmatrix}$
			$\begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$
			$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

even unimodular lattices K_i of rank 32, whose underlining root lattices are denoted below:
$K_1 : 3E_8$, $K_2 : D_{24}$, $K_3 : A_{24}$, $K_4 : A_{17} \oplus E_7$

$m \backslash j$	0	1	2	5
1	1	720	436320	219024000
2	1	1104	1022304	781393536
3	1	600	303600	127512000
4	1	432	158112	48263040

even unimodular lattices L_i of rank 32, whose underlining root lattices are denoted below:
$L_1 : 4E_8$, $L_2 : D_{24} \oplus E_8$, $L_3 : A_{24} \oplus E_8$, $L_4 : E_7 \oplus A_{17} + E_8, L_5 : D_{32}, L_6 : A_1 \oplus A_{31}, L_7 : A_{16} \oplus A_{16}$

Table of the Fourier coefficients of Siegel theta series of degree 3
$\vartheta_3(\mathbf{Z}, L_m)$ $(1 \le m \le 8)$

$m \backslash j$	0	1	2	3	5	6	7
1	1	960	812160	53760	600307200	42577920	1451520
2	1	1344	1582464	110592	1619421696	120729600	4540416
3	1	840	621840	41040	402350400	28406880	970080
4	1	672	395712	27264	203109120	14736384	570240
5	1	1984	3456128	238080	5240378880	386641920	14046720
6	1	994	867008	59520	657636480	8449280	1726080
7	1	544	262208	16320	111041280	7409280	228480

even unimodular lattices M_i of rank 40, whose underlining root lattices are denoted below:
$M_1 : E_8^5$, $M_2 : D_{24} \oplus E_8$, $M_3 : A_{24} \oplus E_8^2$, $M_4 : E_7 \oplus A_{17} \oplus E_8^2$, $M_5 : D_{32} \oplus E_8$,
$M_6 : A_1 \oplus A_{31} \oplus E_8$, $M_7 : A_{16}^2 \oplus E_8$, $M_8 : D_{20}^2$, $M_9 : D_{40}$, $M_{10} : D_{28} \oplus D_{12}$. The lattices M_{11}, M_{12} and M_{13} respectively are the ones ciming from doubly even self-dual [40, 20, 8] codes: Iorgov's C_2 [9], a code in [17], Iorgov's code C_5 [9] respectively.

Table of the Fourier coefficients of Siegel theta series of degree 3
$\vartheta_3(\mathbf{Z}, M_m)$ $(1 \le m \le 13)$

$m \backslash j$	0	1	2	3	4	5	6
1	1	1200	1303200	67200	226106935200	1273968000	69350400
2	1	1584	2257824	124032	383278632864	2882537856	166302720
3	1	1080	1055280	54480	185566037280	928094400	50513760
4	1	912	748512	40704	135668986272	550800000	31279104
5	1	2224	4438688	251520	734060529568	7910593920	471413760
6	1	1234	1374368	72960	238658059648	1373972320	77061120
7	1	784	553568	29760	102348818848	350997120	19605120
8	1	1520	2088480	109440	357220647840	2579975040	142709760
9	1	3120	8779680	474240	1422569435040	22161709440	1263375360
10	1	1776	2815008	167808	474354791328	3964519296	247698432
11	1	0	0	0	994281120	0	0
12	1	0	0	0	994281120	0	0
13	1	0	0	0	1035568800	0	0
14	1	0	0	0	1035568800	0	0

$m \backslash j$	7	8	9	10
1	1814400	5378688000	4410980582400	
2	4903296	11702768640	9419495777280	
3	1332960	3945926400	16132530547200	
4	933120	2366742528	1991438493696	
5	14409600	31503139840	24945629414400	
6	2088960	5677661440	4703505845760	
7	591360	1516810240	1343172216960	
8	3830400	10705566720	8231241262080	
9	35568000	88857400320	67244838036480	
10	8220288	15670729728	12915085943808	
11	0	0	0	15596332778880
12	0	0	0	15596205376896
13	0	0	0	17448486307200
14	0	0	0	17448486307200

5 Main result

Theorem 5.1. *There are a pair of even unimodular 40 dimensional lattices L_1 and L_2 such that their Siegel theta series of degrees 1 and 2 coincide and their theta series of degree 3 differ.*

Theorem 5.2. *There are a pair of even unimodular 40 dimensional non-isomorphic lattices L_3 and L_4 such that their Siegel theta series of degrees 1, 2 and 3 coincide.*

6 A brief sketch of computing the Fourier coefficients of $\vartheta_3(Z, L)$

We compute

$$a(T, L) = \#\{\langle x, y, z\rangle \in L^3 \mid [x, y, z] = 2T\},$$

for the case when L is an even unimodular 40-dimensional extremal lattice constructed from binary code. This quantity is expressed as

$$a(T, L) \quad = \sum_{\langle x,y\rangle\in L^2, [x,y]=\begin{pmatrix} 2t_1 & 2t_{12} \\ 2t_{12} & 2t_2 \end{pmatrix}} \mu(x, y; t_1, t_{13}, t_{23}),$$

where

$$\mu(x, y; t_1, t_{13}, t_{23}) = \#\{z \in \Lambda_{2t_3} \mid (x, z) = 2t_{13}, (y, z) = 2t_{23}.\}$$

We need to compute $a(T, L)$ for particular T's given in the Table 2-3. For

$$T = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

We see that

$$a(T, L) \quad = \sum_{\langle x,y\rangle\in L^2, [x,y]=\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}} \mu(x, y; 2, 0, 0),$$

and

$$\begin{aligned} \mu(x,y;2,0,0) &= \#\{z \in \Lambda_4(L) | (x,z)=0,(y,z)=0\} \\ &= \mu_A(x,y;2,0,0)+\mu_B(x,y;2,0,0), \end{aligned}$$

where

$$\begin{aligned} \mu_A(x,y;2,0,0) &= \#\{z \in A | (x,z)=0,(y,z)=0\}, \\ \mu_B(x,y;2,0,0) &= \#\{z \in B | (x,z)=0,(y,z)=0\}. \end{aligned}$$

Further we get

$$\begin{aligned} a(T,L) &= \sum_{x\in A, y\in A} \{\mu_A(x,y;2,0,0)+\sum_{x\in A, y\in A} \mu_B(x,y;2,0,0)\} \\ &\quad + \sum_{x\in A, y\in B} \{\mu_A(x,y;2,0,0)+\sum_{x\in A, y\in A} \mu_B(x,y;2,0,0)\} \\ &\quad \sum_{x\in B, y\in A} \{\mu_A(x,y;2,0,0)+\sum_{x\in A, y\in A} \mu_B(x,y;2,0,0)\} \\ &\quad \sum_{x\in B, y\in B} \{\mu_A(x,y;2,0,0)+\sum_{x\in A, y\in A} \mu_B(x,y;2,0,0)\} \end{aligned}$$

We can easily prove that

Proposition 6.1. *It holds that*

$$\begin{aligned} &\sum_{x\in A, y\in A} \mu_B(x,y;2,0,0)\} \\ &= \sum_{x\in A, y\in B} \mu_A(x,y;2,0,0) \\ &= \sum_{x\in B, y\in A} \mu_A(x,y;2,0,0), \end{aligned}$$

and

$$\begin{aligned} &\sum_{x\in A, y\in B} \mu_B(x,y;2,0,0)\} \\ &= \sum_{x\in B, y\in A} \mu_B(x,y;2,0,0) \\ &= \sum_{x\in B, y\in B} \mu_A(x,y;2,0,0). \end{aligned}$$

By the above proposition we get an expression:

$$\begin{aligned} &a(T,L) \\ &= \sum_{x\in A, y\in A} \{\mu_A(x,y;2,0,0)+3\sum_{x\in B, y\in A} \{\mu_A(x,y;2,0,0) \\ &\quad +3\sum_{x\in B, y\in B} \{\mu_A(x,y;2,0,0)+\sum_{x\in B, y\in B} \{\mu_B(x,y;2,0,0) \end{aligned}$$

Computation of $\sum_{x\in A,y\in A}\mu_A(x,y;2,0,0)$

We get

$$\sum_{x\in A,y\in A}\mu_A(x,y;2,0,0)=3120(2\cdot 2812+2812\cdot 2524)=22161709440.$$

Computation of $\sum_{x\in B,y\in A}\mu_A(x,y;2,0,0)$

We get

$$\sum_{x\in B,y\in A}\mu_A(x,y;2,0,0)=36480\cdot(56\cdot 2014+1984\cdot 1798)=134246983680.$$

Computation of $\sum_{x\in B,y\in B}\mu_A(x,y;2,0,0)$

The biweight enumerator of a linear code of length n is defined to be

$$\mathcal{BW}(\mathbf{C},X_{11},X_{10},X_{01},X_{00})=\sum_{\mathbf{u},\mathbf{v}\in\mathbf{C}}X_{11}^{w_{11}(\mathbf{u},\mathbf{v})}X_{10}^{w_{10}(\mathbf{u},\mathbf{v})}X_{01}^{w_{01}(\mathbf{u},\mathbf{v})}X_{00}^{w_{00}(\mathbf{u},\mathbf{v})},$$

where X_{11},X_{10},X_{01} and X_{00} are algebraically independent variables over the field of complex numbers, and $w_{ij}(\mathbf{u},\mathbf{v})$ $(0\le i,j\le 1)$ is the number of the coordinates k $(1\le k\le n)$ such that the kth component of $\mathbf{u}$ takes the value i and the k-th component $\mathbf{v}$ takes the value j. We exibit the biweight enumerators of the codes $\mathbf{C}_i$ $(1\le i\le 4)$:

$$\begin{aligned}
&\mathcal{BW}(\mathbf{C}_1,X_{11},X_{10},X_{01},X_{00})=\\
&\mathcal{BW}(\mathbf{C}_2,X_{11},X_{10},X_{01},X_{00})=\\
=\ &\cdots+285X_{11}^{8}X_{00}^{32}+5040X_{11}^{4}X_{10}^{4}X_{01}^{4}X_{00}^{28}+\\
&+53760X_{11}^{2}X_{10}^{6}X_{01}^{6}X_{00}^{26}+22140X_{10}^{8}X_{01}^{8}X_{00}^{24}+\cdots\\
&\mathcal{BW}(\mathbf{C}_3,X_{11},X_{10},X_{01},X_{00})=\\
&\mathcal{BW}(\mathbf{C}_4,X_{11},X_{10},X_{01},X_{00})=\\
=\ &\cdots+285X_{11}^{8}X_{00}^{32}+11760X_{11}^{4}X_{10}^{4}X_{01}^{4}X_{00}^{28}+\\
&+40320X_{11}^{2}X_{10}^{6}X_{01}^{6}X_{00}^{26}+28860X_{10}^{8}X_{01}^{8}X_{00}^{24}+\cdots
\end{aligned}$$

In the above we display all the terms for both $\mathbf{u}$ and $\mathbf{v}$ are of weight 8.

After all we get

$$\begin{aligned}
&\sum_{x\in B,y\in B}\mu_A(x,y;2,0,0)\\
=\ &2^7\cdot(285\cdot 70\cdot 2008+5040\cdot 48\cdot 1540+53760\cdot 64\cdot 1360+22140\cdot 128\cdot 1216)\\
=\ &1092855490560 \text{ for codes } \mathbf{C}_1,\mathbf{C}_2\\
=\ &2^7\cdot(285\cdot 70\cdot 2008+11760\cdot 48\cdot 1540+40320\cdot 64\cdot 1360+28860\cdot 128\cdot 1216)\\
=\ &1140584048640 \text{ for codes } \mathbf{C}_3,\mathbf{C}_4
\end{aligned}$$

Computation of $\sum_{x\in B,y\in B}\mu_B(x,y;2,0,0)$

If we dare to explain every detail of the computation, it may take too much space, therefore we only describe the inner product relations of the vectors x, y and z in B. The description is well-controled by some terms of the triweight enumerator of a code $\mathbf{C}$:

$$\mathcal{TW}(\mathbf{C}, X_{111}, X_{110}, X_{101}, X_{011}, X_{100}, X_{010}, X_{001}, X_{000}) =$$
$$\sum_{\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{C}} X_{111}^{w_{111}(\mathbf{u},\mathbf{v},\mathbf{w})} X_{110}^{w_{110}(\mathbf{u},\mathbf{v},\mathbf{w})} X_{101}^{w_{101}(\mathbf{u},\mathbf{v},\mathbf{w})} X_{011}^{w_{011}(\mathbf{u},\mathbf{v},\mathbf{w})} X_{100}^{w_{100}(\mathbf{u},\mathbf{v},\mathbf{w})} X_{010}^{w_{010}(\mathbf{u},\mathbf{v},\mathbf{w})} X_{001}^{w_{001}(\mathbf{u},\mathbf{v},\mathbf{w})} X_{000}^{w_{000}(\mathbf{u},\mathbf{v},\mathbf{w})},$$

where $X_{111}, X_{110}, X_{101}, X_{011}, X_{100}, X_{010}, X_{001}$ and X_{000} are algebraically independent variables over the field of complex numbers, and $w_{ijh}(\mathbf{u},\mathbf{v},\mathbf{w})$ $(0 \le i, j, h \le 1)$ is the number of the coordinates k $(1 \le k \le n)$ such that the kth component of $\mathbf{u}$ takes the value i and the k-th component $\mathbf{v}$ takes the value j, and k-th component of $\mathbf{w}$ takes the value h.
For our present computation we only need the terms coming from the codewords $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of weight 8. For instance, in case of $\mathbf{C}_1$ terms such as $11760X_{111}^2X_{110}^2X_{101}^2X_{011}^2X_{100}^2X_{010}^2X_{001}^2X_{000}^{26}$ and $42000X_{111}^4X_{111}^4X_{100}^4X_{010}^4X_{001}^4$. There are 50 types of terms that correspond to triples of codewords of weight8. For a fixed $x \in B$ we want to count the vectors $y, z\iota B$ such that $(x, y) = (x, z) = (y, z) = 0$. However the frequencies of the pairs $< y, z >$vary according to the intersection relation among *suppx*, *suppy*, *suppz*. We omit the details.

After all we get

$$a(\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, L(C_1)) = \quad 15596332778880,$$
$$a(\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, L(C_2)) = \quad 15596205376896.$$
$$a(\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, L(C_3)) = a(\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, L(C_4)) = \quad 17448486307200$$

In the same way the values in the last table in Section 4 are determined. These value are the base of our Theorems in Section 5.

7 Further Research

7.1 Some Basic Difficulties

7.1.1 Graded Ring Structure

In genus (degree) 2 case the theory of Siegel modular forms has rich tools.
In genus 3 case thanks to Tsuyumine the graded ring structure of Siegel modular forms is available. However if we fix the weight k we seems not to have the explicit method to determine the linear basis of the space of Siegel modular forms of genus 3 and weight k, although we could know the dimension of the space. We do not have the way to compute the Fourier expansion of those Siegel modular forms.
In genus 4 case the graded ring structure is not determined. Oura, Poor and Yuen [13] initiate to study this case.

7.1.2 Computational Difficulties

Duke-Runge map does not directly produce the Siegel theta series of even unimodular extremal lattice from the multiple weight enumerator of doubly even self-dual extremal binary code.
The weight enumarator of $[24, 12, 8]$ binary Golay code is given by

$$W_{G_{24}}(x, y) = x^{24} + 759x^{16}y^8 + 2576x^{12}y^{12} + 759x^8y^{16} + y^{24}.$$

In $g=1$ case the mapping from weight enumerators to modular forms is known as Broué-Enguehard map (cf. [2])

Example1. In 24 dimension case.

$$\begin{aligned} & W_{G_{24}}(\varphi_0(\tau),\varphi_1(\tau)) \\ = \ & 1+48q_1^2+195408q_1^4+16785216q_1^6+397963344q_1^8+4629612960q_1^{10}+\cdots \end{aligned}$$

This is theta series of degree 1 associated with even unimodular lattice of root type $24\times A_1$. The polynomial

$$\begin{aligned} & \hat{W}_{G_{24}} \\ = \ & x^{24}+759x^{16}y^8+2576x^{12}y^{12}+759x^8y^{16}+y^{24} \\ & -3(x^{20}y^4-4x^{16}y^8+6x^{12}y^{12}-4x^8y^{16}+x^4y^{20}) \end{aligned}$$

leads to theta series of degree 1 associated with the Leech lattice:

$$\begin{aligned} & W_{G_{24}}(\varphi_0(\tau),\varphi_1(\tau)) \\ = \ & 1+196560q_1^4+16773120q_1^6+398034000q_1^8+4629381120q_1^{10}+\cdots \end{aligned}$$

Example 2. In 32 dimension there are five classes of doubly even self-dual binary linear codes, and they have identical weight enumerator:

$$W_{C_{32}}(x,y)=x^{32}+620x^{24}y^8+13888x^{20}y^{12}+36518x^{16}y^{16}+13888x^{12}y^{20}+620x^8y^{24}+y^{32}.$$

The image of this polynomial under Broué-Enguehard map is

$$\begin{aligned} & W_{C_{32}}(\varphi_0(\tau),\varphi_1(\tau)) \\ = \ & 1+64q_1^2+160704q_1^4+64543488q_1^6+4845725632q_1^8+137699222400q_1^{10}+\cdots \end{aligned}$$

This is theta series of degree 1 associated with even unimodular 32 dimensional lattice of root type $32\times A_1$. Another polynomial:

$$\begin{aligned} & \hat{W}_{C_{32}} \\ = \ & x^{32}+620x^{24}y^8+13888x^{20}y^{12}+36518x^{16}y^{16} \\ & +13888x^{12}y^{20}+620x^8y^{24}+y^{32} \\ & -4(-10x^{24}y^8-49x^{20}y^{12}+76x^{16}y^{16} \\ & -49x^{12}y^{20}+10x^8y^{24}+x^{28}y^4+y^{28}x^4) \end{aligned}$$

leads to

$$\begin{aligned} & W^{\sharp}_{C_{32}}(\varphi_0(\tau),\varphi_1(\tau)) \\ = \ & 1+167360q_1^4+65740800q_1^6+4867610560q_1^8+138035363840q_1^{10}+\cdots \end{aligned}$$

which is theta series of degree 1 associated with even unimodular 32 dimensional extremal lattice.

In $g=2$ case. We utilize the polynomials P_8,P_{12},P_{20},P_{24} that are described in [14]. The biweight enumerator of extremal binary self-dula doubly even self-dual $[32,16,8]$ code is

$$\frac{182}{729}P_8P_{24}+\frac{13}{27}P_8^4+\frac{49}{729}P_8P_{12}^2+\frac{49}{243}P_{12}P_{20}$$

The image under the Duke-Runge map is the Siegel theta series of degree 2 for the even unimodular lattice of root lattice type $32 \times A_1$.
The polynomial which corresponds to the Siegel theta series for even unimodular exremal lattice constructed from the above extremal code is

$$\frac{20}{81}P_8P_{24} + \frac{4}{9}P_8^4 + \frac{25}{324}P_8P_{12}^2 + \frac{25}{108}P_{12}P_{20},$$

which is not the biweight enumerator of a code, since it has negative coefficients.
A last remark: the reporter has downsized the total report, since he realizes the strong constraint that the number of pages should be under 16 posed by the organizer. The reader who wants to read this report more precisely may take enlarged copies.

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