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COLORINGS OF FIXED-POINT FREE HOMEOMORPHISMS ON FINITE CONNECTED GRAPHS

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Abstract. In this paper, we give backgrounds why we study the colorings of the fixed-point free maps, and give an announcement for our recent results which calculates the exact value of the color number of a periodic homeomorphism without fixed-points on a finite connected graph.

1. Introduction

Let $f : X \to X$ be a fixed-point free map. A subset (resp. closed subset) A of X is called a *color* (resp. *closed color*) of (X, f) if $f(A) \cap A = \emptyset$. A *coloring* (resp. *closed coloring*) of (X, f) is a finite cover $\mathcal{U}$ of X consisting of colors (resp. *closed colors*). This notion was introduced in [2] and [10], but the idea of the coloring was appeared in the 1950s. Since finite open covers can be shrunk to closed covers, and finite closed covers can be swelled to open covers, the closedness of the coloring is irrelevant. Finite open covers do equally well. Here, we can easily verify the following facts.

Proposition 1.1. *Let X be a regular space and $f : X \to X$ a fixed-point free map.*

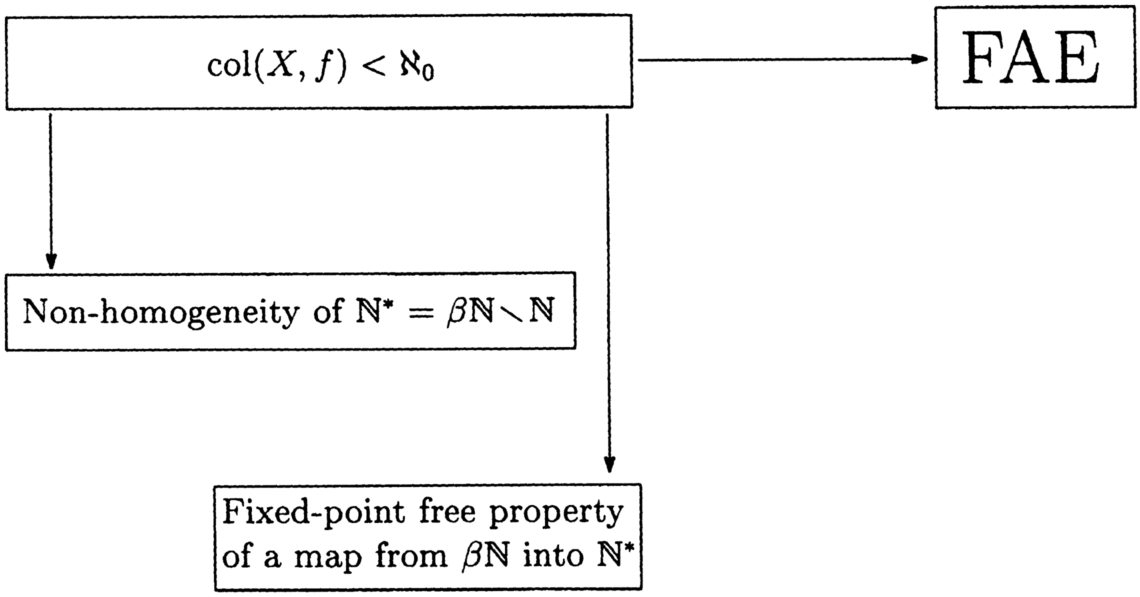
(1) *For every $x \in X$ there exists a closed neighborhood N_x of x such that N_x is a closed color of (X, f).*
(2) *If X is compact, then we can take a closed coloring of (X, f).*

By Proposition 1.1, every fixed-point free map admits a possibly infinite cover consisting closed colors. This explains that we are interested in finite covers only.

The minimal cardinality of a closed coloring is called the *color number* of (X, f), denoted by $\mathrm{col}(X, f)$ (see [1] or [3]), i.e.,

$$\mathrm{col}(X, f) = \min\{|\mathcal{U}| : \mathcal{U} \text{ is a closed coloring of } (X, f)\}.$$

Let $\mathcal{C} = \{C_1, \ldots, C_p\}$ is a coloring (resp. *closed coloring*) of (X, f). If we emphasize the number of colors of $\mathcal{C}$, we say that $\mathcal{C}$ is a *p-coloring* (resp. *p-closed coloring*) of (X, f). Now, we recall old results concerning the coloring. Here, if X is a Tychonoff space for which any autohomeomorphism f with no fixed points has an extension βf without fixed points, we say that X is *fixed-point free autohomeomorphisms extends* or *FAE* (see [15]). At first, we give the following figure which may explain backgrounds why we study the coloring of the fixed-point free self maps, where "$\mathrm{col}(X, f) < \aleph_0 \to \mathcal{P}$" means that for some suitable (X, f) we have $\mathrm{col}(X, f) < \aleph_0$ implies $\mathcal{P}$.



1.1. **Non-homogeneity of $\mathbb{N}^*$ versus coloring.** During a seminar at the University of Wisconsin in 1955, the following question was raised: If a space X is homogeneous, does it necessarily follow that the growth $X^* = \beta X \setminus X$ is also homogeneous? Under CH, W. Rudin proved that $\mathbb{N}^*$ is not homogeneous (see [13] and [14]), but the question above still had remained at that time. Afterward, in the 1960s, Frolík showed the following theorem (see [16, Theorem 6.25]):

Theorem 1.2 (Frolík [8]). *If f is a one-to-one mapping of an extremally disconnected compact space X into itself, then there exists a 3-clopen coloring of $(X \setminus \mathrm{Fix}(f), f|_{X \setminus \mathrm{Fix}(f)})$ and $\mathrm{Fix}(f) = \{x : f(x) = x\}$ is clopen. In particular, $\mathrm{col}(X \setminus \mathrm{Fix}(f), f|_{X \setminus \mathrm{Fix}(f)}) \leq 3$.*

Applying Theorem 1.2, Frolík showed the following (see [16, Theorem 6.33]):

Corollary 1.3 (Frolík [8]). *No infinite closed subspace of $\mathbb{N}^*$ is homogeneous.*

1.2. **Fixed-point free property versus coloring.** It is well-known that $\mathbb{N}^*$ contains a copy of $\beta\mathbb{N}$. In the 1960s, Katětov was interested in the following question: Let f be a homeomorphism of $\beta\mathbb{N}$ into $\mathbb{N}^*$. Does such an f have fixed-points? In [11], Katětov showed the following:

Theorem 1.4 (de Bruijn and Erdös [4] and Katětov [11]). *Let X be a set and $f : X \to X$ a fixed-point free map (not necessarily continuous). Then there exists a 3-coloring of (X, f), i.e.,* $\mathrm{col}(X, f) \leq 3$.

Applying Theorem 1.4, Katětov showed the following:

Corollary 1.5 (Katětov [11]). *Let f be a homeomorphism of $\beta\mathbb{N}$ into $\mathbb{N}^*$. Then f has no fixed-point.*

1.3. **FAE versus coloring.** By de Bruijn-Erdös-Katětov's theorem, it is naturally to ask whether we can have colors as closed sets whenever X is a topological space? In the 1980s, Błaszczyk and Kim gave the following partial answer:

Theorem 1.6 (Błaszczyk and Kim, [5]). *Let X be a 0-dimensional paracompact space and $f : X \to X$ a fixed-point free homeomorphism. Then there exists a 3-clopen coloring of (X, f), i.e.,* $\mathrm{col}(X, f) \leq 3$.

Afterward, van Douwen (see [6, Theorem 1.1]) showed the following:

Theorem 1.7 (van Douwen [6]). *Let X be an n-dimensional paracompact space and $f : X \to X$ a fixed-point free homeomorphism. Then there exists a $(2n + 3)$-closed coloring of (X, f), i.e.,* $\mathrm{col}(X, f) \leq 2n + 3$.

By Theorem 1.7, van Douwen pointed out that every finite dimensional paracompact space is FAE:

Corollary 1.8 (van Douwen [6]). *Let X be a n-dimensional paracompact space and $f : X \to X$ a fixed-point free homeomorphism. Then βf is fixed-point free. In particular, every finite dimensional paracompact space is FAE.*

Furthermore, the following fact is known:

Theorem 1.9 (Douwen [6], Hartskamp-Mill [9]). *Let X be a normal space and let $f : X \to X$ be a fixed-point free map. Then the following conditions are equivalent:*

(1) *βf is fixed-point free.*
(2) *There exists a closed coloring of (X, f), i.e.,* $\mathrm{col}(X, f) < \aleph_0$.

On the other hand, Theorem 1.9 makes us have the following question: Is there any "nice" space which is not FAE? For the question above, we need the following folklore (see [7]).

Theorem 1.10 (Lusternik-Schnirelmann). *If $\iota : \mathbb{S}^n \to \mathbb{S}^n$ is the antipodal map, then every closed cover $\mathcal{F}$ of $\mathbb{S}^n$ such that $F \cap \iota(F) = \emptyset$ for $F \in \mathcal{F}$ has at least $n + 2$ elements. In particular,* $\mathrm{col}(\mathbb{S}^n, \iota) \geq n + 2$.

By Theorem 1.10, van Douwen found the following example:

Example 1.11 (van Douwen [6]). Let X be $\bigoplus_{n\in\mathbb{N}} \mathbb{S}^n$, the topological sum of the n-sphere $\mathbb{S}^n$, and let f be the topological sum of the antipodal maps, i.e., $f|_{\mathbb{S}^n}$ is the antipodal map of $\mathbb{S}^n$ for each $n \in \mathbb{N}$. Then f is a fixed-point free autohomeomorphism on X such that βf is not fixed-point free because $\mathrm{col}(X, f) = \aleph_0$. In particular, X is not FAE.

2. Motivation

In the 1990s, an upper bound of the color number is improved as follows.

Theorem 2.1 (van Hartskamp and Vermeer [10]). *Let X be a paracompact Hausdorff space with* $\dim X \leq n$. *If $f : X \to X$ is a fixed-point free homeomorphism, then* $\mathrm{col}(X, f) \leq n + 3$.

In [12], van Mill gives a simple proof of the theorem above. By Theorem 2.1, we can easily verify the following:

Corollary 2.2. *Let X be a 0-dimensional paracompact space and $f : X \to X$ a fixed-point free homeomorphism. If there exists an $x \in X$ such that $f^3(x) = x$, then* $\mathrm{col}(X, f) = 3$.

Furthermore, for a fixed-point free involution, the upper bound of the color number can be improved.

Theorem 2.3 (Aarts, Fokkink, and Vermeer [2]). *Let X be a paracompact Hausdorff space with* $\dim X \leq n$ *and $f : X \to X$ a fixed-point free homeomorphism. If f is an involution, i.e., $f^2(x) = x$ for all $x \in X$, then* $\mathrm{col}(X, f) \leq n + 2$.

By Theorem 2.3, we have the following:

Corollary 2.4. *Let X be a 0-dimensional paracompact space and $f : X \to X$ a fixed-point free involution. Then* $\mathrm{col}(X, f) = 2$.

Furthermore, Theorem 2.3 indicates that Lusternik-Schnirelmann's theorem can be improved as follows:

Corollary 2.5 (Lusternik-Schnirelmann). *If $\iota : \mathbb{S}^n \to \mathbb{S}^n$ is the antipodal map, then* $\mathrm{col}(\mathbb{S}^n, \iota) = n + 2$.

More generally, the extension of Theorem 2.1 to fixed-point free continuous maps. However, this requires extra conditions on the space.

Theorem 2.6 (van Hartskamp and Vermeer [10]). *Let X be a compact Hausdorff space with* $\dim X \leq n$. *If $f : X \to X$ is a fixed-point free continuous map, then* $\mathrm{col}(X, f) \leq n + 3$.

For example, the color number of the rotation through 120 degree on a circle is 4, and the color number of the rotation through 90 degree on a circle is 3. Moreover, let S_Y^n be the n-dimensional Y-sphere and $\gamma^{n+1} : S_Y^n \to S_Y^n$ the period 3 homeomorphism defined in [3, p.258]. Then $\operatorname{col}(S_Y^n, \gamma^{n+1}) = n+3$ ([3, Theorem 4]). Here, S_Y^1 is the bipartite cubic graph on six nodes $K(3,3)$.

By Theorem 2.1, it is naturally to ask the question whether $\operatorname{col}(X, f) = n+3$ or not. Then, we have concentrated the following question.

Question 2.7. *Let X be a finite connected graph and $f : X \to X$ a fixed-point free homeomorphism on X. Which is true,* $\operatorname{col}(X, f) = 3$ *or* $\operatorname{col}(X, f) = 4$*?*

In the rest of paper, we are going to give an announcement of our recent results which give exact values of color numbers of periodic homeomorphisms.

3. Fixed-point free homeomorphisms with a period three point

Let X be a connected space and $f : X \to X$ a fixed-point free homeomorphism. Clearly, $\operatorname{col}(X, f) \geq 3$. Moreover, if $f^3(x) = x$ for each $x \in X$, then $\operatorname{col}(X, f) \geq 4$ (cf. [2, Example 7(1)]). In fact, suppose that there is a coloring $\{U_1, U_2, U_3\}$ of (X, f). We may assume that $U_1 \cap U_2 \neq \emptyset$, and let $a \in U_1 \cap U_2$. Then we have $f(a) \in U_3$, so $f^2(a) \in U_1 \cup U_2$. However, $f^3(a) = a \in U_1 \cap U_2$, we have a contradiction.

The next proposition below asserts that if a fixed-point free homeomorphism on an arcwise-connected space has a point of period 3, then its color number is at least 4.

Proposition 3.1. *Let X be an arcwise-connected space and $f : X \to X$ a fixed-point free homeomorphism with $f^n = \mathrm{id}_X$ for some $n \in \mathbb{N}$. If f has a period* 3 *point in X, then* $\operatorname{col}(X, f) \geq 4$.

By Theorem 2.1 and Proposition 3.1, we have the following.

Corollary 3.2. *Let X be a* 1*-dimensional arcwise-connected space and $f : X \to X$ a fixed-point free homeomorphism with $f^n = \mathrm{id}_X$ for some $n \in \mathbb{N}$. If f has a period* 3 *point in X, then* $\operatorname{col}(X, f) = 4$.

Example 3.3. Let $Z_n = \{x_0, x_1, \ldots, x_{n-1}\}$ be an n-points discrete space, $Z_m * Z_n$ a join of Z_m and Z_n. Define $f_n : Z_n \to Z_n$ by $f_n(x_i) = x_{i+1}$ modulo n for $i = 0, \ldots, n-1$ and $f_m * f_n : Z_m * Z_n \to Z_m * Z_n$ the natural map constructing f_m and f_n. By Corollary 3.2, $\operatorname{col}(Z_3 * Z_n, f_3 * f_n) = 4$ for all $n \in \mathbb{N}$ with $n \geq 2$.

4. Fixed-point free homeomorphisms without period three point

In this section, we calculate the exact value of the color number for a fixed-point free homeomorphism without period 3 points on a finite connected graph.

For any homeomorphism $f : X \to X$ and any periodic point $x \in X$, we write $n_x = \min\{m : f^m(x) = x\}$. Set $\mathrm{P}(f) = \{x : x \text{ is a periodic point of } f\}$ and $\mathrm{Per}(f) = \{n_x : x \in \mathrm{P}(f)\}$. Now, we give the following main lemma without proof.

Lemma 4.1. *Let $\mathcal{T}$ be a triangulation of a finite connected graph X and $f : X \to X$ a fixed-point free homeomorphism with $\mathrm{P}(f) \neq \emptyset$. If there exists an $n \in \mathbb{N} \setminus \{1, 3\}$ such that n_x is a multiple of n for each $x \in \mathrm{P}(f)$, then* $\mathrm{col}(X, f) = 3$.

Let $\{a_1, \dots, a_m\} \subset \mathbb{N}$, and let us denote by $\gcd\{a_1, \dots, a_m\}$ the great common divisor of $a_1, a_2, \dots, a_m$.

Theorem 4.2. *Let $f : X \to X$ be a fixed-point free homeomorphism on a finite connected graph X with $\mathrm{Per}(f) \neq \emptyset$. If $\gcd(\mathrm{Per}(f)) \neq 1, 3$, then $\mathrm{col}(X, f) = 3$.*

Proof. At first, we need the following fact:
Fact. Let $\{a_1, \dots, a_m\}$ be a subset of natural numbers. Then the following conditions are equivalent:

(1) There exists an $n \in \mathbb{N} \setminus \{1, 3\}$ such that a_k is a multiple of n for each $k = 1, \dots, m$.
(2) $\gcd\{a_1, \dots, a_m\} \neq 1, 3$.

By Lemma 4.1 and the fact above, the proof is complete. □

Corollary 4.3. *Let X be a finite connected graph and $f : X \to X$ a fixed-point free homeomorphism. If there exists an $m \in \mathbb{N} \setminus \{1, 3\}$ such that $f^p(x) \neq x$ with $1 \leq p < m$ and $f^m(x) = x$ for each $x \in X$, then $\mathrm{col}(X, f) = 3$.*

Corollary 4.4. *Let X be a finite connected graph and $f : X \to X$ a fixed-point free homeomorphism. Then $\mathrm{col}(X, f) = 3$ if either the following conditions is fulfilled:*

(1) $\mathrm{Per}(f)$ *consists of even numbers.*
(2) $\mathrm{Per}(f)$ *consists of the power of some prime number p except* 3.

Example 4.5. (1) Let $\mathbb{S}^1 = \{(\cos\theta, \sin\theta) \in \mathbb{R}^2 \mid 0 \leq \theta \leq 2\pi\}$, and let $R_n : \mathbb{S}^1 \to \mathbb{S}^1$ be defined by $R_n(\cos\theta, \sin\theta) = (\cos(\theta + 2\pi/n), \sin(\theta + 2\pi/n))$ for $n \in \mathbb{N}$. If $n \neq 1, 3$, by Corollary 4.3, $\mathrm{col}(\mathbb{S}^1, R_n) = 3$. On the other hand, by Theorem 2.1 and Proposition 3.1, $\mathrm{col}(\mathbb{S}^1, R_3) = 4$.
(2) Let $Z_4 * Z_4$ be as in Example 3.3. By Corollary 4.4, $\mathrm{col}(Z_4 * Z_4, f) = 3$ for any fixed-point free homeomorphism $f : Z_4 * Z_4 \to Z_4 * Z_4$. This shows that a condition that $\mathrm{col}(X, f) \leq n + 3$ for any fixed-point free homeomorphism $f : X \to X$ of period k for some $k \in \mathbb{N}$ does not imply $\dim X \leq n$.

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