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On Telgárski's formula

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The following formula due to R. Telgársky ([3]) is indispensable in dealing with products of scattered spaces.

Theorem 1. *Let* X, Y *be scattered spaces with* $\mathrm{leng}(X)=\alpha$ *and* $\mathrm{leng}(Y)=\beta$. *Then*
(1) $(X\times Y)^{(\sigma)}=\cup_{\tau\oplus\upsilon=\sigma}X^{(\tau)}\times Y^{(\upsilon)}$ *for every* σ .
(2) $(X\times Y)_{(\sigma)}=\cup_{\tau\oplus\upsilon=\sigma}X_{(\tau)}\times Y_{(\upsilon)}$ *for every* σ .
(3) $\mathrm{leng}(X\times Y)=\sup\{\tau\oplus\upsilon+1\,|\,\tau<\alpha$ and $\upsilon<\beta\}$.

The symbol $\oplus$ means Hessenberg's sum defined as follows :

Definition 1. Let $\alpha>0,\ \beta>0$ be ordinals. Using Cantor's normal form, represent $\alpha,\ \beta$ uniquely as

$$\alpha=\omega^{\gamma_1}n_1+\omega^{\gamma_2}n_2+\cdots+\omega^{\gamma_k}n_k\ ,\quad \beta=\omega^{\gamma_1}m_1+\omega^{\gamma_2}m_2+\cdots+\omega^{\gamma_k}m_k\ ,$$

$\gamma_1>\gamma_2>\cdots>\gamma_k$, $0\leq n_i<\omega$, $0\leq m_i<\omega$, so that $n_i=0=m_i$ does not occur. Define

$$\alpha\oplus\beta=\Sigma_{i=1}^{k}\,\omega^{\gamma_i}(n_i+m_i)\quad .$$

Also define $\alpha\oplus 0=0\oplus\alpha=0$ for every α .

Hessenberg's sum is certainly convinient for describing the derivatives $(X\times Y)^{(\sigma)}$ but not for describing $\mathrm{leng}(X\times Y)$. With an emphasis on the length of product spaces, we define a binary operation π as follows :

Definition 2. Let $\alpha>0,\ \beta>0$ be ordinals. Represent $\alpha,\ \beta$ as in Definition 1. Put

$$l=\min\{\max\{i\,|\,n_i\neq 0\}\ ,\ \max\{j\,|\,m_j\neq 0\}\}$$

and define

$$\pi(\alpha,\beta)=\begin{cases}\Sigma_{i=1}^{l}\,\omega^{\gamma_i}(n_i+m_i) & \text{if } l<k\\ (\Sigma_{i=1}^{k-1}\,\omega^{\gamma_i}(n_i+m_i))+\omega^{\gamma_k}(n_k+m_k-1) & \text{if } l=k\end{cases}\ ,$$

where $l=k$ is, of course, equivalent to $n_k\neq 0\neq m_k$.

For convenience, define $\pi(\alpha,0)=\pi(0,\alpha)=0$ for every ordinal α .

It is to be noted that , unlike Hessenberg's sum, the operation π is a countinuous operation with respect to the order topology.

Now we can restate Telgársky's formula as follows :

Theorem 2. *Let* X, Y *be scattered spaces with* $\text{leng}(X) = \alpha$, $\text{leng}(Y) = \beta$. *Then*
(1) $\text{leng}(X \times Y) = \pi(\alpha, \beta)$.
(2) $(X \times Y)_{(\sigma)} = \cup\{X_{(\tau)} \times Y_{(\upsilon)} \mid \pi(\tau + 1, \upsilon + 1) = \sigma + 1\}$ for every ordinal σ .
(3) $(X \times Y)^{(\sigma)} = \cup\{X^{(\tau)} \times Y^{(\upsilon)} \mid \pi(\tau + 1, \upsilon + 1) = \sigma + 1\}$ for every ordinal σ .

We write simply $\pi(\alpha, \beta) = \alpha * \beta$.

Proposition 1. $\alpha * \beta = \beta * \alpha$. $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$.

Definition 3. A factorization $\alpha = \beta * \gamma$ of an ordinal α is called *trivial* if one of β, γ is 1 (and the other is α) . An ordinal α is called a *prime* ordinal if $\alpha > 1$ and it does not admit a non-trivial factorization.

As far as I know, the following notion was first defined and used by K. Borsuk ([1] also see [2]).

Definition 4. A factorization $X \approx Y \times Z$ of a space X is called *trivial* if one of Y, Z is a one point space (and the other is homeomorphic to X) . A space X is called a *prime* space (or simply a *prime*) if $|X| > 1$ and it does not admit a non-trivial factorization.

By the definition of the operation π we have

Proposition 2. *An ordinal* α *is a prime ordinal if and only if* $\alpha = \omega^\gamma + 1$ *with* γ *an ordinal.*

The compact countable metric space X satisfying $\text{leng}(X) = \alpha$ and $|X^{(\alpha-1)}| = |X_{(\alpha-1)}| = n$ is denoted by $MS(\alpha, n)$. By the uniqueness of $MS(\alpha, n)$ we have $MS(\alpha, n) \approx MS(\alpha, 1) \times MS(1, n)$ and, by Theorem 2, $MS(\alpha, 1) \times MS(\beta, 1) \approx MS(\alpha * \beta, 1)$. These facts are combined with Proposition 2 to give

Proposition 3. *A compact countable metric space* X *is a prime if and only if* $X \approx MS(1, p)$ *with* p *a prime number or* $X \approx MS(\omega^\gamma + 1, 1)$ *with* γ *an ordinal.*

Theorem 3. *Every non-limit ordinal* $\alpha > 1$ *is factorized uniquely as*

$$\alpha = \alpha_1 * \alpha_2 * \cdots * \alpha_j$$

into prime ordinals $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_j$. *In full detail, if the normal form of* α *is*

$$\alpha = \omega^{\zeta_1} n_1 + \omega^{\zeta_2} n_2 + \cdots + \omega^{\zeta_k} n_k$$

with $\zeta_k = 0$ *then*

$$\begin{aligned}\alpha = \ & \overbrace{(\omega^{\zeta_1}+1)*(\omega^{\zeta_1}+1)*\cdots*(\omega^{\zeta_1}+1)}^{n_1}*\\ & \overbrace{(\omega^{\zeta_2}+1)*(\omega^{\zeta_2}+1)*\cdots*(\omega^{\zeta_2}+1)}^{n_2}*\\ & \vdots \\ & \overbrace{(\omega^{\zeta_{k-1}}+1)*(\omega^{\zeta_{k-1}}+1)*\cdots*(\omega^{\zeta_{k-1}}+1)}^{n_{k-1}}*\\ & \overbrace{2*2*\cdots*2}^{n_k-1} \ . \quad \textit{(Do not confuse } n_{k-1} \textit{ with } n_k - 1 \textit{ .)}\end{aligned}$$

A translation of the theorem into product spaces is as follows :

Corollary 1. *Every compact countable metric space* X *with* $|X| > 1$ *is factorized uniquely as*

$$X \approx X_1 \times X_2 \times \cdots \times X_j$$

into primes $X_1, X_2, \ldots, X_j$. *In further detail, if* $X = MS(\alpha, n)$, $\alpha = \omega^{\zeta_1}n_1 + \omega^{\zeta_2}n_2 + \cdots + \omega^{\zeta_k}n_k$ *with* $\zeta_k = 0$, *then*

$$\begin{aligned}X \approx \ & \overbrace{MS(\omega^{\zeta_1}+1,\,1)\times MS(\omega^{\zeta_1}+1,\,1)\times\cdots\times MS(\omega^{\zeta_1}+1,\,1)}^{n_1}\times\\ & \overbrace{MS(\omega^{\zeta_2}+1,\,1)\times MS(\omega^{\zeta_2}+1,\,1)\times\cdots\times MS(\omega^{\zeta_2}+1,\,1)}^{n_2}\times\\ & \vdots \\ & \overbrace{MS(\omega^{\zeta_{k-1}}+1,\,1)\times MS(\omega^{\zeta_{k-1}}+1,\,1)\times\cdots\times MS(\omega^{\zeta_{k-1}}+1,\,1)}^{n_{k-1}}\times\\ & \overbrace{MS(2,\,1)\times MS(2,\,1)\times\cdots\times MS(2,\,1)}^{n_k-1}\times\\ & MS(1,\,p_1)\times MS(1,\,p_2)\times\cdots\times MS(1,\,p_r) \ ,\end{aligned}$$

where $n = p_1 p_2 \cdots p_r$ *is the usual factorization of the natural number* n *into primes.*

Examples. Put, for example, $\alpha = \omega^2 + \omega 2 + 3$. Then

$$\omega^2 + \omega 2 + 3 = (\omega^2+1)*(\omega+1)*(\omega+1)*2*2 \ .$$

Thus

$$\begin{aligned}MS(\omega^2+\omega 2+3,\,1) = \ & MS(\omega^2+1,\,1)\times MS(\omega+1,\,1)\times MS(\omega+1,\,1)\times\\ & MS(2,\,1)\times MS(2,\,1) \ .\end{aligned}$$

$$MS(\omega^2+\omega 2+3,\, 6) = \begin{array}{l} MS(\omega^2+1,\, 1)\times MS(\omega+1,\, 1)\times MS(\omega+1,\, 1)\times \\ MS(2,\, 1)\times MS(2,\, 1)\times MS(1,\, 3)\times MS(1,\, 2)\;. \end{array}$$

References

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[3] R. Telgársky, *Derivatives of Cartesian product and dispersed spaces*, Colloq. Math. **19** (1968), 59-66.