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3個以上の作用素の幾何平均
(Geometric means of more than two operators)

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1. Introduction

The definition of the geometric mean of more than two positive invertible operators on a Hilbert space (or positive definite matrices) has been presented by several researchers ([1], [15], [3], [8], etc.). We here try to give a definition of such a geometric mean related to the Riccati equation for two operators. Let Ω be the set of all positive invertible operators on H (or positive definite $n \times n$ matrices for some n). For $A, B \in \Omega$ the Riccati equation $XA^{-1}X = B$ has a unique solution $X = X_{A,B} \in \Omega$:

$$X = A\sharp B := A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}},$$

which is defined as the geometric mean of A and B. As an extension, a weighted geometric mean $A\sharp_\alpha B$ for $0 \le \alpha \le 1$ is defined by

$$A\sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}}.$$

For $A, B, C \in \Omega$ we can consider a cubic equation

$$X(A\sharp B)^{-1}X(A\sharp B)^{-1}X = C,$$

as an extension of the Riccati equation. Then it has a unique solution $X = X_{A,B,C} \in \Omega$:

$$X = (A\sharp B)\sharp_{\frac{1}{3}}C (= C\sharp_{\frac{2}{3}}(A\sharp B)). \tag{1.1}$$

If A, B, C commute with each other, then $X = (ABC)^{\frac{1}{3}}$, so that X seems a candidate of a geometric mean. However, it lacks permutation invariance, (one of the ten properties required for a reasonable geometric mean in [3]). To supply the property we borrow the symmetrization technique due to [3]: We define sequences $\{A_n\}, \{B_n\}, \{C_n\}$ by $A_1 = A,\ B_1 = B,\ C_1 = C$ and for $n \ge 1$

$$\begin{cases} A_{n+1} = A_n\sharp_\lambda(B_n\sharp C_n), \\ B_{n+1} = B_n\sharp_\lambda(C_n\sharp A_n), \\ C_{n+1} = C_n\sharp_\lambda(A_n\sharp B_n), \end{cases}$$

taking a real $\lambda \in (0, 1]$ (more generally than 2/3 in (1.1) above).
Then they are convergent and have a common limit with respect to Thompson metric defined below. *We define the limit as the geometric mean of* A, B, C and denote by G_λ or $G_\lambda(A, B, C)$. Thompson metric $d(\cdot,\cdot)$ on Ω is defined ([22], [4], [6]) as follows (and Ω is complete with the metric):

$$d(A, B) = \max\{\log M(A/B),\ \log M(B/A)\} \quad (A, B \in \Omega),$$

where

$$M(A/B) = \inf\{\mu > 0 : A \le \mu B\}\ (= \| B^{-1/2}AB^{-1/2} \|).$$

If $\lambda = 1$, then $G_\lambda (= G_1)$ is the geometric mean given by [3], and if $\lambda = 2/3$, $G_\lambda (= G_{\frac{2}{3}})$ is one given in [21]. As mentioned before, in [3], ten properties were posturated for a geometric mean of n operators (or matrices) to be reasonable. Our geometric mean G_λ satisfies all the properties. Starting from the geometric mean of two operators, we can define those of n operators inductively for all integers $n \geq 2$, which satisfy all of the ten properties. In [3], Ando-Li-Mathias stated the following ten postulates for a geometric mean $G(A_1, \ldots, A_k)$ of k (or a k-tuple of) operators $A_1, \ldots, A_k$ to be a reasonable one, (the usual geometric mean $G(A_1, A_2) = A_1 \# A_2$ is reasonable):

P1 Consistency with scalars. If $A_1, A_2, \ldots, A_k$ commute then

$$G(A_1, A_2, \ldots, A_k) = (A_1 A_2 \cdots A_k)^{\frac{1}{k}}.$$

P1' This implies $G(\overbrace{A, \ldots, A}^{k}) = A$.

P2 Joint homogeneity. $G(a_1 A_1, a_2 A_2, \ldots, a_k A_k) = (a_1 a_2 \cdots a_k)^{\frac{1}{k}} G(A_1, A_2, \ldots, A_k)$ for $a_i \geq 0$ with $i = 1, \ldots, k$.

P2' This implies $G(aA_1, aA_2, \ldots, aA_k) = aG(A_1, A_2, \ldots, A_k)$ $(a \geq 0)$.

P3 Permutation invariance. For any permutation $\pi(A_1, A_2, \ldots, A_k)$ of $(A_1, A_2, \ldots, A_k)$, $G(A_1, A_2, \ldots, A_k) = G(\pi(A_1, A_2, \ldots, A_k))$.

P4 Monotonicity. The map $(A_1, A_2, \ldots, A_k) \mapsto G(A_1, A_2, \ldots, A_k)$ is monotone, i.e., if $A_i \geq B_i$ for $i = 1, \ldots, k$, then $G(A_1, A_2, \ldots, A_k) \geq G(B_1, B_2, \ldots, B_k)$.

P5 Continuity from above. If $\{A_1^{(n)}\}, \{A_2^{(n)}\}, \ldots, \{A_k^{(n)}\}$ are monotone decreasing sequences converging to $A_1, A_2, \ldots, A_k$, respectively, then $\{G(A_1^{(n)}, A_2^{(n)}, \ldots, A_k^{(n)})\}$ converges to $G(A_1, A_2, \ldots, A_k)$.

P6 Congruence invariance. For any invertible S,

$$G(S^* A_1 S, S^* A_2 S, \ldots, S^* A_k S) = S^* G(A_1, A_2, \ldots, A_k) S.$$

P7 Joint concavity. The map $(A_1, A_2, \ldots, A_k) \mapsto G(A_1, A_2, \ldots, A_k)$ is jointly concave:

$$G(\lambda A_1 + (1-\lambda)A_1', \lambda A_2 + (1-\lambda)A_2', \ldots, \lambda A_k + (1-\lambda)A_k')$$
$$\geq \lambda G(A_1, A_2, \ldots, A_k) + (1-\lambda) G(A_1', A_2', \ldots, A_k') \quad (0 < \lambda < 1).$$

P8 Self-duality. $G(A_1, A_2, \ldots, A_k)^* = G(A_1, A_2, \ldots, A_k)$. The dual $G(A_1, A_2, \ldots, A_k)^*$ is defined by

$$G(A_1, A_2, \ldots, A_k)^* = G(A_1^{-1}, A_2^{-1}, \ldots, A_k^{-1})^{-1}.$$

P9 (*In case $A_1, A_2, \ldots, A_k$ are matrices.*) Determinant identity.

$$\det G(A_1, A_2, \ldots, A_k) = (\det A_1 \cdot \det A_2 \cdot \cdots \cdot \det A_k)^{\frac{1}{k}}.$$

P10 The arithmetic-geometric-harmonic mean inequaility.

$$\frac{A_1 + A_2 + \cdots + A_k}{k} \geq G(A_1, A_2, \ldots, A_k) \geq \left(\frac{A_1^{-1} + A_2^{-1} + \cdots + A_k^{-1}}{k}\right)^{-1}.$$

In this report, we define a geometric mean of $(k+1)$ operators with a parameter λ which still satisfies the above properties P1-P10 from a given geometric mean of k operators satisfying all properties by induction. For more than two positive operators, in particular, we define the weighted geometric mean as an extension of that of two operators.

Without occurrence of ambiguity, we shall often abbreviate the letter λ. All operators (or matrices) are assumed to be *positive invertible* (or *positive definite*) if stated otherwise.

2. Definition of geometric means of more than two operators

Let Ω be the set of all (positive invertible) operators on H. Then as mentioned above the Thompson metric on Ω is defined by

$$d(A,B) = \max\{\log M(A/B),\ \log M(B/A)\} \quad \text{for } A,B \in \Omega,$$

where

$$M(A/B) = \inf\{\mu > 0 : A \le \mu B\} \ (= \| B^{-1/2}AB^{-1/2} \|).$$

Between $\| A - B \|$ and $d(A,B)$ the following facts hold:

$$\| A - B \| \le \min\{\| A \|, \| B \|\}(e^{d(A,B)} - 1),$$

$$d(A,B) \le \max\{\| A^{-1} \|, \| B^{-1} \|\} \, \| A - B \| .$$

We remark that Ω is complete with respect to the Thompson metric topology. As a basic inequality with respect to the metric, the following inequality for a weighted geometric mean of two operators holds [4], [6]:

$$d(A_1 \#_\alpha A_2, B_1 \#_\alpha B_2) \le (1-\alpha) d(A_1, B_1) + \alpha d(A_2, B_2) \quad \text{for } A_1, A_2, B_1, B_2 \in \Omega \text{ and } \alpha \in (0,1). \tag{2.1}$$

Now in order to define our geometric mean $G_\lambda(A_1, \dots, A_{k+1})$ of $(k+1)$ operators from a given one of k (≥ 2) operators, we want to assume a useful inequality:

$$d(G(A_1, \dots, A_k), G(B_1, \dots, B_k)) \le \frac{1}{k} \sum_{i=1}^{k} d(A_i, B_i) \tag{2.2}$$

for another k-tuple of operators $B_1, \dots, B_k$.

Theorem 2.1. *The geometric mean $G_\lambda(A_1, \dots, A_{k+1})$ is always defined as the common limit of the following $(k+1)$ sequences $\{A_1^{(r)}\}, \dots, \{A_{k+1}^{(r)}\}$ of $(k+1)$ operators $A_1, \dots, A_{k+1}$:*

$$\begin{aligned} &A_i^{(1)} = A_i \ \text{for} \ i = 1, \dots, k+1, \quad \text{and} \\ &A_i^{(r+1)} = A_i^{(r)} \#_\lambda G((A_j^{(r)})_{j \ne i}) (= A_i^{(r)} \#_\lambda G(A_1^{(r)}, \dots, A_{i-1}^{(r)}, A_{i+1}^{(r)}, \dots, A_{k+1}^{(r)})) \\ &\qquad \text{for } r \ge 1, \ i = 1, \dots, k+1. \end{aligned} \tag{2.3}$$

where $\lambda \in (0,1]$ and $G(A_1, \dots, A_k)$ is a geometric mean of k operators satisfying P1-P10 and the inequality (2.2). The geometric mean $G_\lambda(A_1, \dots, A_{k+1})$ satisfies P1-P10, and furthermore, the following inequality holds:

$$d(G_\lambda(A_1, \dots, A_{k+1}), G_\lambda(B_1, \dots, B_{k+1})) \le \frac{1}{k+1} \sum_{i=1}^{k+1} d(A_i, B_i) \tag{2.4}$$

corresponding to (2.2) for another $(k+1)$-tuple $B_1, \dots, B_{k+1}$ of operators.

Proof. To see that all sequences $\{A_i^{(r)}\}$ are convergent with a common limit we first show that for $i, j = 1, \dots, k+1, i \ne j$

$$d(A_i^{(r+1)}, A_j^{(r+1)}) \le \left(1 - \frac{k-1}{k}\lambda\right)^r d(A_i, A_j). \tag{2.5}$$

By the definition (2.3) of $A_i^{(r)}$ and the inequalities (2.1) and (2.4), we have

$$\begin{aligned} d(A_i^{(r+1)}, A_j^{(r+1)}) &= d(A_i^{(r)} \#_\lambda G((A_\ell^{(r)})_{\ell \neq i}), A_j^{(r)} \#_\lambda G((A_\ell^{(r)})_{\ell \neq j})) \\ &\leq (1-\lambda) d(A_i^{(r)}, A_j^{(r)}) + \lambda d(G((A_\ell^{(r)})_{\ell \neq i}), G((A_\ell^{(r)})_{\ell \neq j})) \\ &\leq (1-\lambda) d(A_i^{(r)}, A_j^{(r)}) + \lambda \cdot \frac{1}{k} d(A_i^{(r)}, A_j^{(r)}) \\ &= \left(1 - \frac{k-1}{k}\lambda\right) d(A_i^{(r)}, A_j^{(r)}). \end{aligned}$$

Hence by iteration with respect to r we can obtain the desired inequality. Next we show

$$d(A_i^{(r+1)}, A_i^{(r)}) \leq \frac{\lambda}{k} \left(1 - \frac{k-1}{k}\lambda\right)^{r-1} K_i, \tag{2.6}$$

where $K_i = \sum_{\ell=1, \ell \neq i}^{k+1} d(A_i, A_\ell)$. Note that

$$A_i^{(r)} = A_i^{(r)} \#_\lambda G(\overbrace{A_i^{(r)}, \ldots, A_i^{(r)}}^{k}).$$

Using (2.2), we have

$$d(A_i^{(r+1)}, A_i^{(r)}) \leq \lambda d(G((A_\ell^{(r)})_{\ell \neq i}), G(\overbrace{A_i^{(r)}, \ldots, A_i^{(r)}}^{k})) \leq \lambda \cdot \frac{1}{k} \sum_{\ell=1, \ell \neq i}^{k+1} d(A_i^{(r)}, A_\ell^{(r)}).$$

Hence from (2.5)

$$d(A_i^{(r+1)}, A_i^{(r)}) \leq \frac{\lambda}{k} \cdot \sum_{\ell=1, \ell \neq i}^{k+1} \left(1 - \frac{k-1}{k}\lambda\right)^{r-1} d(A_\ell, A_i) = \frac{\lambda}{k} \left(1 - \frac{k-1}{k}\lambda\right)^{r-1} K_i,$$

which is the desired inequality. Now we see that for any i, the sequence $\{A_i^{(r)}\}$ is convergent, or a Cauchy sequence. In fact, for $r \leq s$

$$\begin{aligned} d(A_i^{(r+1)}, A_i^{(s+1)}) &\leq \sum_{\ell=r+1}^{s} d(A_i^{(\ell)}, A_i^{(\ell+1)}) \leq \frac{\lambda}{k} K_i \sum_{\ell=r+1}^{s} \left(1 - \frac{k-1}{k}\lambda\right)^{\ell-1} \\ &\leq \frac{\lambda}{k} K_i \cdot \left(1 - \frac{k-1}{k}\lambda\right)^{r} / \left(\frac{k-1}{k}\lambda\right) = \frac{K_i}{k-1} \left(1 - \frac{k-1}{k}\lambda\right)^{r}. \end{aligned}$$

Hence $d(A_i^{(r+1)}, A_i^{(s+1)}) \to 0$ as $r (< s) \to \infty$, so that $\{A_i^{(r)}\}$ is convergent. From (2.5), we easily see that all $\{A_i^{(r)}\}$ have the same limit, which guarantees the desired geometric mean to be defined.

It is not difficult to see that the geometric mean $G_\lambda(A_1, \ldots, A_{k+1})$ satisfies all properties P1-P10. For example, to see P3, let $\pi(A_1, A_2, \ldots, A_{k+1}) = (A_{\pi(1)}, \ldots, A_{\pi(k+1)})$ be a permutation of $(A_1, A_2, \ldots, A_{k+1})$, and let

$$B_i^{(1)} = A_{\pi(i)}^{(1)} = A_{\pi(i)}, \quad B_i^{(r+1)} = B_i^{(r)} \#_\lambda G((B_j)_{j \neq i}^{(r)})$$
$$\text{for } i = 1, \ldots, k+1, \ r \geq 1.$$

Then we see that $B_i^{(r)} = A_{\pi(i)}^{(r)}$. In fact, assuming that $B_i^{(r)} = A_{\pi(i)}^{(r)}$ $(i = 1, \ldots, k+1)$, we have

$$B_i^{(r+1)} = A_{\pi(i)}^{(r)} \#_\lambda G((A_{\pi(j)})_{j \neq i}) = A_{\pi(i)}^{(r+1)}.$$

Hence $\{B_i^{(r)}\}$ and $\{A_{\pi(i)}^{(r)}\}$ coincide, so that they converge to the same limit, which is desired.

For the inequality (2.4), let the sequences $\{B_1^{(r)}\}, \ldots, \{B_{k+1}^{(r)}\}$ be defined corresponding to $B_1, \ldots, B_{k+1}$, similarly as (2.3) for $A_1, \ldots, A_{k+1}$. Then for each i, from (2.1) and the assumption (2.2), we have

$$\begin{aligned} d(A_i^{(r+1)}, B_i^{(r+1)}) &= d(A_i^{(r)} \#_\lambda G((A_j^{(r)})_{j\neq i}), B_i^{(r)} \#_\lambda G((B_j^{(r)})_{j \neq i})) \\ &\leq (1-\lambda) d(A_i^{(r)}, B_i^{(r)}) + \lambda d(G((A_j^{(r)})_{j\neq i}), G((B_j^{(r)})_{j\neq i})) \\ &\leq (1-\lambda) d(A_i^{(r)}, B_i^{(r)}) + \lambda \cdot \frac{1}{k} \sum_{j=1, j\neq i}^{k+1} d(A_j^{(r)}, B_j^{(r)}) \\ &= \left(1 - \frac{k+1}{k}\lambda\right) d(A_i^{(r)}, B_i^{(r)}) + \frac{\lambda}{k} \sum_{j=1}^{k+1} d(A_j^{(r)}, B_j^{(r)}). \end{aligned}$$

Summing up all $d(A_i^{(r+1)}, B_i^{(r+1)})$ with respect to i, we have

$$\begin{aligned} \sigma_{r+1} := \sum_{i=1}^{k+1} & d(A_i^{(r+1)}, B_i^{(r+1)}) \\ &\leq \left(1 - \frac{k+1}{k}\lambda\right) \sum_{i=1}^{k+1} d(A_i^{(r)}, B_i^{(r)}) + \frac{k+1}{k}\lambda \sum_{j=1}^{k+1} d(A_j^{(r)}, B_j^{(r)}) \\ &= \sum_{i=1}^{k+1} d(A_i^{(r)}, B_i^{(r)}) \ (= \sigma_r). \end{aligned}$$

Hence $\sigma_{r+1} \leq \sigma_r \leq \cdots \leq \sigma_1$, that is, $\sigma_{r+1} \leq \sum_{i=1}^{k+1} d(A_i, B_i)$. Taking the limit as $r \to \infty$, we have the desired inequality since $\sigma_{r+1} \to (k+1) d(G_\lambda(A_1, \ldots, A_{k+1}), G_\lambda(B_1, \ldots, B_{k+1}))$.

Example 2.2. Let

$$A_1 = \begin{bmatrix} 10 & 1 \\ 1 & 0.2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 4.1 & 4.9 \\ 4.9 & 6.1 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then by numerical computation we have, (discarded less than 10^{-6},)

$$G_{1/3} = \begin{bmatrix} 1.647\,281 & 0.613\,824 \\ 0.613\,824 & 0.835\,789 \end{bmatrix} (= A_1^{(r)} = A_2^{(r)} = A_3^{(r)} \text{ for } r \geq 24),$$

$$G_{1/2} = \begin{bmatrix} 1.649\,909 & 0.615\,737 \\ 0.615\,737 & 0.835\,883 \end{bmatrix} (= A_1^{(r)} = A_2^{(r)} = A_3^{(r)} \text{ for } r \geq 13),$$

$$G_{2/3} = \begin{bmatrix} 1.660\,083 & 0.623\,133 \\ 0.623\,133 & 0.836\,280 \end{bmatrix} (= A_1^{(r)} = A_2^{(r)} = A_3^{(r)} \text{ for } r \geq 4)$$

and

$$G_1 = \begin{bmatrix} 1.697\,095 & 0.649\,781 \\ 0.649\,781 & 0.838\,029 \end{bmatrix} (= A_1^{(r)} = A_2^{(r)} = A_3^{(r)} \text{ for } r \geq 24).$$

Now for more convenient expression, denote by $(G,\lambda) = (G,\lambda)(A_1,\dots,A_{k+1})$ the geometric mean constructed as in Theorem 2.1. Then successively we can define

$$(G,\lambda_1,\dots,\lambda_\ell) = ((G,\lambda_1,\dots,\lambda_{\ell-1}),\lambda_\ell).$$

Let $G = \#(A_1,A_2) = A_1\#A_2$. Then $(\#,\overbrace{1,\dots,1}^{k-2})$ is the geometric mean (of k operators) given by Ando-Li-Mathias in [3], and $(\#;\frac{2}{3},\dots,\frac{k-1}{k})$ is one given in [21].

Example 2.3. Let

$$A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} \quad \text{and} \quad A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then by numerical computation, we obtain, (discarded less than 10^{-6},) for $r \geq 4$,

$$(\#;\tfrac{2}{3},\tfrac{3}{4})(A_1,A_2,A_3,A_4) = \begin{bmatrix} 1.412\,693 & 0.706\,627 \\ 0.706\,627 & 1.033\,191 \end{bmatrix}$$
$$(= A_1^{(r)} = A_2^{(r)} = A_3^{(r)} = A_4^{(r)}).$$

3. Weighted geometric means of more than two operators

We introduce two types of weighted geometric means of $k(\geq 3)$ operators as the extensions of weighted geometric means of two operators. Let Ω be the set of all (positive invertible) operators on H. Denote by $\mathbf{G(k)}$ the set of all geometric means of k operators with the properties P1-P10.

3.1 Weighted geometric means of k operators, type I

First for $A_1,\ A_2 \in \Omega$ and for real numbers α_1,α_2 satisfying $\alpha_1 \in [0,1]$ and $\alpha_2 = 1-\alpha_1$, we write the weighted geometric mean by

$$(\tilde{G} =) A_1\#_{\alpha_2}A_2 = G(\alpha_1,\alpha_2;A_1,A_2).$$

Then we see

$$G(\alpha_1,\alpha_2;A_1,A_2) = A_2\#_{\alpha_1}A_1 = G(\alpha_2,\alpha_1;A_2,A_1).$$

This implies that $\tilde{G}$ is a weighted geometric mean with permutation invariance. We want to extend this property for weighted geometric means of more operators.

For three operators A_1,A_2,A_3 on Ω and for real numbers $\alpha_1,\alpha_2,\alpha_3$ satisfying $\alpha_1,\alpha_2,\alpha_3 > 0$ and $\alpha_1+\alpha_2+\alpha_3 = 1$, we define the three sequences $\{B_1^{(r)}\},\{B_2^{(r)}\}$ and $\{B_3^{(r)}\}$, by $B_1^{(1)} = B_1, B_2^{(1)} = B_2, B_3^{(1)} = B_3$, as follows:

$$\begin{cases} B_1 = A_1\#_{1-\alpha_1}G\left(\frac{\alpha_2}{1-\alpha_1},\frac{\alpha_3}{1-\alpha_1};A_2,A_3\right), \\ B_2 = A_2\#_{1-\alpha_2}G\left(\frac{\alpha_3}{1-\alpha_2},\frac{\alpha_1}{1-\alpha_2};A_3,A_1\right), \\ B_3 = A_3\#_{1-\alpha_3}G\left(\frac{\alpha_1}{1-\alpha_3},\frac{\alpha_2}{1-\alpha_3};A_1,A_2\right). \end{cases} \tag{3.1}$$

It is easy to see that if A_1, A_2, A_3 commute with each other then $B_1 = B_2 = B_3 = A_1^{\alpha_1} A_2^{\alpha_2} A_3^{\alpha_3}$.

Now let $\Gamma \in \mathbf{G(3)}$. Then we can obtain a common limit of the sequences $\{B_1^{(r)}\}, \{B_2^{(r)}\}$ and $\{B_3^{(r)}\}$ which we define a weighted geometric mean

$$G_\Gamma(\alpha_1, \alpha_2, \alpha_3 \,; A_1, A_2, A_3) := \Gamma(B_1, B_2, B_3).$$

We want to call it as a weighted geometric mean of A_1, A_2, A_3 with weight $(\alpha_1, \alpha_2, \alpha_3)$.

Here we, parallel to P1-P10, state basic properties for a reasonable weighted geometric mean of k operators: Let $\widetilde{G} = G(\alpha_1, \dots, \alpha_k \,; A_1, \dots, A_k)$ be a weighted geometric mean of $A_1, \dots, A_k \in \Omega$ $(\alpha_1, \dots, \alpha_k \geq 0, \sum_{j=1}^k \alpha_j = 1)$

PW1. $G(\alpha_1, \dots, \alpha_k \,; A, \dots, A) = A$.
PW2. $G(\alpha_1, \alpha_2, \dots, \alpha_k \,; a_1A_1, a_2A_2, \dots, a_kA_k) = a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} \widetilde{G}$.
PW3. $\widetilde{G}$ is permutation invariant with respect to $\mathbf{S(k)}$ (which denote a permutation group of k letters).
PW4. $\widetilde{G}$ is monotone.
PW5. $\widetilde{G}$ is continuous from above.
PW6. $\widetilde{G}$ is congruence invariant.
PW7. $\widetilde{G}$ is jointly concave.
PW8. $\widetilde{G}$ is self-dual.
PW9. (In case of matrices) $\det \widetilde{G} = (\det A_1)^{\alpha_1} \cdots (\det A_k)^{\alpha_k}$.
PW10. The weighted arithmetic-geometric-harmonic mean inequality holds:

$$\alpha_1 A_1 + \cdots + \alpha_k A_k \geq \widetilde{G} \geq \left(\alpha_1 A_1^{-1} + \cdots + \alpha_k A_k^{-1}\right)^{-1}.$$

Now we can see that $G(\alpha_1, \alpha_2, \alpha_3 \,; A_1, A_2, A_3)$ satisfies the above properties PW1-PW10 for $k = 3$, and furthermore if $\Gamma = G_{\#, \frac{2}{3}} \in \mathbf{G(3)}$, then we can obtain

$$G_\Gamma\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \,; A_1, A_2, A_3\right) = G_{\#, \frac{2}{3}}(A_1, A_2, A_3).$$

Generalizing the above result to k (≥ 2) operators, we have

Theorem 3.1.1 *Assume that $G(\lambda_1, \dots, \lambda_k \,; X_1, \dots, X_k)$ $(\lambda_j \geq 0, \sum_{j=1}^k \lambda_j = 1)$ is a weighted geometric mean of k operators with the properties PW1-PW10. Let $A_1, \dots, A_{k+1}$ be $k+1$ operators in Ω. For $\alpha_1, \dots, \alpha_{k+1}$ satisfying $\alpha_1, \dots, \alpha_{k+1} > 0$ and $\sum_{j=1}^{k+1} \alpha_j = 1$, we put*

$$B_i = A_i \#_{1-\alpha_i} G\left(\left(\frac{\alpha_j}{1-\alpha_i}\right)_{j \neq i} \,; (A_j)_{j \neq i}\right).$$

Then for a $\Gamma \in \mathbf{G(k)}$, define

$$(\widetilde{G} =) G_\Gamma(\alpha_1, \dots, \alpha_{k+1} \,; A_1, \dots, A_{k+1}) = \Gamma(B_1, \dots, B_{k+1}).$$

Then we have a "reasonable weighted geometric mean", which satisfies the following:
(i) $\widetilde{G}$ satisfies PW1-PW10 for $(k+1)$ operators.

(ii) If $A_1, \ldots, A_{k+1}$ commute each other, then we obtain $\tilde{G} = A_1^{\alpha_1} \cdots A_{k+1}^{\alpha_{k+1}}$.
(iii) If $\Gamma = G_{\#, \frac{2}{3}, \ldots, \frac{k}{k+1}}$, then we obtain

$$G_\Gamma\left(\frac{1}{k+1}, \ldots, \frac{1}{k+1}; A_1, \ldots, A_{k+1}\right) = \Gamma(A_1, \ldots, A_{k+1}).$$

3.2 Weighted geometric means of k operators, type II

We want to construct a weighted geometric mean by another way. For real numbers $\alpha_1, \alpha_2, \alpha_3$ satisfying $\alpha_1, \alpha_2, \alpha_3 > 0$, $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Define $\{A_1^{(r)}\}, \{A_2^{(r)}\}$ and $\{A_3^{(r)}\}$, by $A_1^{(1)} = A_1, A_2(1) = A_2, A_3^{(1)} = A_3$ and

$$\begin{cases} A_1^{(r+1)} = A_1^{(r)} \#_{1-\alpha_1} \left(A_2^{(r)} \#_{\frac{\alpha_3}{1-\alpha_1}} A_3^{(r)}\right), \\ A_2^{(r+1)} = A_2^{(r)} \#_{1-\alpha_2} \left(A_3^{(r)} \#_{\frac{\alpha_1}{1-\alpha_2}} A_1^{(r)}\right), \\ A_3^{(r+1)} = A_3^{(r)} \#_{1-\alpha_3} \left(A_1^{(r)} \#_{\frac{\alpha_2}{1-\alpha_3}} A_2^{(r)}\right). \end{cases} \tag{3.2}$$

We want to show that they converge to the same limit by a method without using the Thompson metric.

Proposition 3.2.1. *Let $\{A_1^{(r)}\}, \{A_2^{(r)}\}$ and $\{A_3^{(r)}\}$ be the sequences given above. Then the sequences converge (with respect to strong operator topology) and have a common limit, which we denoted by*

$$G_s = G_s(\alpha_1, \alpha_2, \alpha_3; A_1, A_2, A_3).$$

Here $S = \{id, (123), (123)^2\}$ is a subset of $\mathbf{S(3)}$. Moreover, the limit G_s is permutation invariant with respect to S, (more precisely, with respect to $\mathbf{S(3)}$.)

Before the proof of the proposition we prepare a useful lemma:

Lemma 3.2.2. *Let $\{A_n^{(r)}\}$ and $\{B_n^{(r)}\}$ be sequences of positive operators such that $0 < mI \leq A_n, B_n \leq MI$ for some scalars m and M, and let h be real number satisfying $0 < h < 1$. If $E_n := (1-h)A_n + hB_n - A_n \#_h B_n \to 0$ then $A_n - B_n \to 0$ (as $n \to \infty$).*

Proof. First note that for any $t \geq 0$,

$$(1-h) + ht - t^h \geq \min\{h, 1-h\}(1 - t^{\frac{1}{2}})^2,$$

From this inequality, replacing t by $A_n^{-\frac{1}{2}} B_n A_n^{-\frac{1}{2}}$ and multiplying both hand sides by $A_n^{\frac{1}{2}}$ from the left and the right, we can obtain

$$(1-h)A_n + hB_n - A_n \#_h B_n \geq \min\{h, 1-h\} A_n^{\frac{1}{2}} \{I - (A_n^{-\frac{1}{2}} B_n A_n^{-\frac{1}{2}})^{\frac{1}{2}}\}^2 A_n^{\frac{1}{2}}.$$

Hence, if $E_n \to 0$ then (putting $C_n = (A_n^{-\frac{1}{2}} B_n A_n^{-\frac{1}{2}})^{\frac{1}{2}}$) we have $A_n^{\frac{1}{2}}(I - C_n)^2 A_n^{\frac{1}{2}} \to 0$, so that also $(I - C_n)A_n^{\frac{1}{2}} \to 0$. Henc we have, using boundedness assumption,

$$A_n - B_n = A_n^{\frac{1}{2}}(I - C_n^2)A_n^{\frac{1}{2}} = A_n^{\frac{1}{2}}(I + C_n)(I - C_n)A_n^{\frac{1}{2}} \to 0.$$

Proof of Proposition 3.2.1. From Young inequality, we have

$$A_1^{(r+1)} \leq \alpha_1 A_1^{(r)} + (1-\alpha_1)\left(A_2^{(r)} \#_{\frac{\alpha_3}{1-\alpha_1}} A_3^{(r)}\right).$$

Put $C_1^{(r)} = A_2^{(r)} \#_{\frac{\alpha_3}{1-\alpha_1}} A_3^{(r)}$, then we obtain

$$A_1^{(r+1)} \leq \alpha_1 A_1^{(r)} + (1-\alpha_1) C_1^{(r)} \leq \alpha_1 A_1^{(r)} + \alpha_2 A_2^{(r)} + \alpha_3 A_3^{(r)} \quad \cdots \textcircled{1}.$$

Similarly we obtain

$$A_2^{(r+1)} \leq \alpha_2 A_2^{(r)} + (1-\alpha_2)\left(A_3^{(r)} \#_{\frac{\alpha_1}{1-\alpha_1}} A_1^{(r)}\right) = \alpha_2 A_2^{(r)} + (1-\alpha_2) C_2^{(r)} \leq \alpha_1 A_1^{(r)} + \alpha_2 A_2^{(r)} + \alpha_3 A_3^{(r)} \quad \cdots \textcircled{2}.$$

$$A_3^{(r+1)} \leq \alpha_3 A_2^{(r)} + (1-\alpha_3)\left(A_1^{(r)} \#_{\frac{\alpha_2}{1-\alpha_1}} A_2^{(r)}\right) = \alpha_3 A_2^{(r)} + (1-\alpha_3) C_3^{(r)} \leq \alpha_1 A_1^{(r)} + \alpha_2 A_2^{(r)} + \alpha A_3^{(r)} \quad \cdots \textcircled{3}.$$

Put $D^{(s)} = \alpha_1 A_1^{(s)} + \alpha_2 A_2^{(s)} + \alpha_3 A_3^{(s)}$. By simple computation of $(\textcircled{1} \times \alpha_1 + \textcircled{2} \times \alpha_2 + \textcircled{3} \times \alpha_3) D^{(r+1)}$, we then obtain the following inequality:

$$\alpha_1 A_1^{(r+1)} + \alpha_2 A_2^{(r+1)} + \alpha_3 A_3^{(r+1)} \leq (*) \leq \alpha_1 A_1^{(r)} + \alpha_2 A_2^{(r)} + \alpha_3 A_3^{(r)} (= D^{(r)}).$$

Here we put

$$(*) = \alpha_1^2 A_1^{(r)} + \alpha_2^2 A_2^{(r)} + \alpha_3^2 A_3^{(r)} + \alpha_1(1-\alpha_1) C_1^{(r)} + \alpha_2(1-\alpha_2) C_2^{(r)} + \alpha_3(1-\alpha_3) C_3^{(r)}.$$

Note that $E^{(r)} := D^{(r)} - (*) \leq D^{(r)} - D^{(r+1)} \to 0$ (as $r \to \infty$) since $\{D^{(r)}\}$ is decreasing and convergent, which is

$$\begin{aligned} E^{(r)} = \alpha_3 & \overbrace{\left\{\alpha_1 A_1^{(r)} + \alpha_2 A_2^{(r)} - (\alpha_1+\alpha_2)\left(A_1^{(r)} \#_{\frac{\alpha_2}{\alpha_1+\alpha_2}} A_2^{(r)}\right)\right\}}^{I_1^{(r)}} \\ & + \alpha_2 \overbrace{\left\{\alpha_3 A_3^{(r)} + \alpha_1 A_1^{(r)} - (\alpha_3+\alpha_1)\left(A_3^{(r)} \#_{\frac{\alpha_1}{\alpha_3+\alpha_1}} A_1^{(r)}\right)\right\}}^{I_2^{(r)}} \\ & + \alpha_1 \overbrace{\left\{\alpha_2 A_2^{(r)} + \alpha_3 A_3^{(r)} - (\alpha_2+\alpha_3)\left(A_2^{(r)} \#_{\frac{\alpha_3}{\alpha_2+\alpha_3}} A_3^{(r)}\right)\right\}}^{I_3^{(r)}} \\ = {} & \alpha_3 I_1^{(r)} + \alpha_2 I_2^{(r)} + \alpha_1 I_3^{(r)}. \end{aligned}$$

We can see the following fact:

$$I_1^{(r)} = (\alpha_1+\alpha_2)\{(1-h)A_1^{(r)} + hA_2^{(r)} - A_1^{(r)} \#_h A_2^{(r)}\} \geq 0,$$

where $h = \frac{\alpha_2}{\alpha_1+\alpha_2}$.

In the same manner, we can obtain $I_2^{(r)}, I_3^{(r)} \geq 0$.

Hence we can see that $I_1^{(r)}, I_2^{(r)}, I_3^{(r)}$ converge to 0 (as $r \to \infty$), respectively. Hence from Lemma 3.2.2 $\{A_1^{(r)}\}, \{A_2^{(r)}\}, \{A_3^{(r)}\}$ converge to a common limit, which is desired.

Remark 3.2.3. We used the inequality:

$$\#(\alpha, \beta, \gamma; A, B, C) (= A \#_{1-\alpha} (B \#_{\frac{\gamma}{1-\alpha}} C)) \leq \alpha A + (1-\alpha)\left(B \#_{\frac{\gamma}{1-\alpha}} C\right).$$

But the following inequality doesn't hold (by computer simulation).

$$G_{\#,\frac{2}{3}}(\alpha,\beta,\gamma;A,B,C) \leq \alpha A + (1-\alpha)\left(B\#_{\frac{\gamma}{1-\alpha}}C\right). \tag{3.3}$$

Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 10 & 1 \\ 1 & 0.2 \end{bmatrix}, C = \begin{bmatrix} 4.1 & 4.9 \\ 4.9 & 6.1 \end{bmatrix}$$

and for real numbers α,β,γ satisfying $\alpha=\beta=\gamma=\frac{1}{3}$. Then

Left side of (3.3) $= G_{\#,\frac{2}{3}}(A,B,C) \left(= G_{\#,\frac{2}{3}}\left(\frac{1}{3},\frac{1}{3},\frac{1}{3};A,B,C\right)\right) = \begin{bmatrix} 1.660\,083 & 0.623\,133 \\ 0.623\,133 & 0.836\,280 \end{bmatrix}$.

Right side of (3.3)$= \frac{1}{3}A + \frac{2}{3}(B\#C) = \begin{bmatrix} 1.612\,274 & 0.535\,159 \\ 0.535\,159 & 0.904\,775 \end{bmatrix} \ngeq$ Left side of (3.3).

For k operators $A_1,\ldots,A_k$ on Ω and real numbers $\alpha_1,\ldots,\alpha_k$ satisfying $\alpha_1,\ldots,\alpha_k>0$ and $\alpha_1+\cdots+\alpha_k=1$, we define

$$\#(\alpha_1,\ldots,\alpha_k\,;A_1,\ldots,A_k) := A_1\#_{x_1}(A_2\#_{x_2}\cdots(A_{k-1}\#_{x_{k-1}}A_k\overbrace{)\cdots)}^{k-2}.$$

Here the above real numbers $x_1,\ldots,x_{k-1}$ are solutions of the following equations:

$$\begin{cases} 1-x_1=\alpha_1, \\ x_1(1-x_2)=\alpha_2, \\ \cdots\cdots\cdots\cdots, \\ x_1\cdots x_{k-2}(1-x_{k-1})=\alpha_{k-1}, \\ x_1\cdots x_{k-1}=\alpha_k. \end{cases} \tag{3.4}$$

(i) If $A_1,\ldots,A_k$ commute with each other, then

$$\#(\alpha_1,\ldots,\alpha_k\,;A_1,\ldots,A_k) = A_1^{\alpha_1}\cdots A_k^{\alpha_k}.$$

(ii) $\#(\alpha_1,\ldots,\alpha_k\,;A_1,\ldots,A_k) := A_1\#_{1-\alpha_1}\left(\#\left(\frac{\alpha_2}{1-\alpha_1},\ldots,\frac{\alpha_k}{1-\alpha_1}\,;A_2,\ldots,A_k\right)\right)$.

Before we show a main result in this section, we state a lemma which extends Lemma 3.2.2. (We can prove it by induction.)

Lemma 3.2.4. *Let $\{A_1^{(n)}\},\ldots,\{A_k^{(n)}\}$ be sequences of positive operators such that $0<mI\leq A_i\leq MI$ $(i=1,\ldots,k)$ and let h_i be real numbers satisfying $0<h_i<1, \sum_{i=1}^k h_i=1$. If*

$$E_n := \sum_{i=1}^k h_iA_i^{(n)} - \#(h_1,\ldots,h_k\,;A_1^{(n)},\ldots,A_k^{(n)}) \to 0,$$

then for all i,j $(i\neq j)$, $A_i^{(n)}-A_j^{(n)}\to 0$ (as $n\to\infty$).

Theorem 3.2.5. *Let $A_1,\ldots,A_k$ be k operators in Ω. For real numbers $\alpha_1,\ldots,\alpha_k$ satisfying $\alpha_1,\ldots,\alpha_k>0$, $\alpha_1+\cdots+\alpha_k=1$, and $S=\{\pi_1,\ldots,\pi_k\}\subset \mathbf{S(k)}$, we define the sequences $\{A_1^{(r)}\},\ldots,\{A_k^{(r)}\}$ as follows:*

$$\begin{cases} A_i^{(1)}=A_i \quad (i=1,\ldots,k),\ \textit{for } r\geq 1, \\ A_1^{(r+1)} = \#\pi_1(\alpha_1,\ldots,\alpha_k\,;A_1,\ldots,A_k) = \#(\alpha_{\pi_1(1)},\ldots,\alpha_{\pi_1(k)}\,;A_{\pi_1(1)},\ldots,A_{\pi_1(k)}), \\ \cdots\cdots\cdots\cdots, \\ A_k^{(r+1)} = \#\pi_k(\alpha_1,\ldots,\alpha_k\,;A_1,\ldots,A_k) = \#(\alpha_{\pi_k(1)},\ldots,\alpha_{\pi_k(k)}\,;A_{\pi_k(1)},\ldots,A_{\pi_k(k)}). \end{cases} \tag{3.5}$$

Then the above k sequences converge and have a common limit (denoted by)

$$G_s = G_s(\alpha_1, \dots, \alpha_k \, ; A_1, \dots, A_k).$$

For this mean G_s, the following facts hold.
(i) If $A_1, \dots, A_k$ commute with each other, then $G_s = A_1^{\alpha_1} \cdots A_k^{\alpha_k}$.
(ii) G_s has the properties PW1-PW10 except PW3.
(iii) If the subset S is a subgroup of $\mathbf{S(k)}$ with order k, and if for $\sigma \in S$

$$(\pi_1\sigma, \dots, \pi_k\sigma) = (\pi_{\sigma(1)}, \dots, \pi_{\sigma(k)}),$$

then G_S is permutation invariant with respect to σ (σ-p.i.).

Proof. First by using Young inequality, we can see (by induction) that

$$\begin{aligned} A_1^{(r+1)} &\le \alpha_{\pi_1(1)} A_{\pi_1(1)}^{(r)} + (1-\alpha_{\pi_1(1)})\{\#(\alpha'_{\pi_1(2)}, \dots, \alpha'_{\pi_1(k)} \, ; A_{\pi_1(2)}, \dots, A_{\pi_1(k)})\} \\ &\le \alpha_1 A_1^{(r)} + \cdots + \alpha_k A_k^{(r)}. \\ &\dots \\ A_k^{(r+1)} &\le \alpha_{\pi_k(1)} A_{\pi_k(1)}^{(r)} + (1-\alpha_{\pi_k(1)})\{\#(\alpha'_{\pi_k(2)}, \dots, \alpha'_{\pi_k(k)} \, ; A_{\pi_k(2)}, \dots, A_{\pi_k(k)})\} \\ &\le \alpha_1 A_1^{(r)} + \cdots + \alpha_k A_k^{(r)}. \end{aligned}$$

Here $\alpha'_{\pi_i(j)} = \dfrac{\alpha_{\pi_i(j)}}{1-\alpha_{\pi_i(1)}}$. If we write
$C_i^{(r)} = \#(\alpha'_{\pi_i(1)}, \dots, \alpha'_{\pi_i(k)} \, ; A_{\pi_i(2)}, \dots, A_{\pi_i(k)})$ and $D^{(s)} = \sum_{j=1}^k \alpha_j A_j^{(s)}$,
then from the above inequalities

$$\begin{aligned} D^{(r+1)} &= \alpha_1 A_1^{(r+1)} + \cdots + \alpha_k A_k^{(r+1)} \\ &\le \alpha_1 \left\{\alpha_{\pi_1(1)} A_{\pi_1(1)}^{(r)} + (1-\alpha_{\pi_1(1)}) C_1^{(r)}\right\} + \cdots + \alpha_k \left\{\alpha_{\pi_k(1)} A_{\pi_k(1)}^{(r)} + (1-\alpha_{\pi_k(1)}) C_k^{(r)}\right\} \\ &\le \alpha_1 D^{(r)} + \cdots + \alpha_k D^{(r)} = D^{(r)}. \end{aligned}$$

We then see that $\{D^{(r)}\}$ is a decreasing sequence (with a limit which we shall define as G_s), so that if we put

$$E^{(r)} = \alpha_1 \left\{\alpha_{\pi_1(1)} A_{\pi_1(1)}^{(r)} + (1-\alpha_{\pi_1(1)}) C_1^{(r)}\right\} + \cdots + \alpha_k \left\{\alpha_{\pi_k(1)} A_{\pi_k(1)}^{(r)} + (1-\alpha_{\pi_k(1)}) C_r^{(r)}\right\},$$

then $D^{(r)} - E^{(r)} \to 0$ as $r \to \infty$. Note that

$$D^{(r)} - E^{(r)} = \sum_{j=1}^k \alpha_j I_j^{(r)},$$

where

$$\begin{aligned} I_j^{(r)} &= D^{(r)} - \alpha_{\pi_j(1)} A_{\pi_j(1)}^{(r)} - (1-\alpha_{\pi_j(1)}) C_j^{(r)} \\ &= \sum_{\ell=1, \ell\ne\pi_j(1)}^k \alpha_\ell A_\ell^{(r)} - \left(\sum_{\ell=1,\ell\ne\pi_j(1)}^k \alpha_\ell\right) \cdot \{\#((\alpha_\ell)')_{\ell\ne\pi_j(1)} \, ; (A_\ell^{(r)})_{\ell\ne\pi_j(1)}\} \\ &= \left(\sum_{\ell=1,\ell\ne\pi_j(1)}^k \alpha_\ell\right) \cdot \left\{ \sum_{\ell=1,\ell\ne\pi_j(1)}^k (\alpha_\ell)' A_\ell^{(r)} - \#((\alpha_\ell)')_{\ell\ne\pi_j(1)} \, ; (A_\ell^{(r)})_{\ell\ne\pi_j(1)} \right\}. \end{aligned}$$

Hence since $I_j^{(r)} \geq 0$ for each j by Young inequality, we see that $I_j^{(r)} \to 0$ (from $D^{(r)} - E^{(r)} \to 0$). Hence by Lemma 3.2.4 we have $A_i^{(r)} - A_j^{(r)} \to 0$ for all $i, j, i \neq j$. Now

$$D^{(r)} - A_j^{(r)} = \sum_{\ell=1, \ell \neq j}^{k} \alpha_\ell (A_\ell^{(r)} - A_j^{(r)}) \to 0,$$

which implies that all $A_j^{(r)}$ ($j = 1, \ldots, k+1$) have the same limit as $D^{(r)}$.

For the facts (i)-(iii), (i) is easy and (ii) can be shown by induction without difficulty. So it suffices to show (iii). Let $S = \{\pi_1, \ldots, \pi_k\}$ be a subgroup of $\mathbf{S(k)}$, and let σ be an element in S. Put

$$(\beta_1, \ldots, \beta_k) = \sigma(\alpha_1, \ldots, \alpha_k) = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)}), \text{ i.e., } \beta_i = \alpha_{\sigma(i)},$$

and

$$(B_1, \ldots, B_k) = \sigma(A_1, \ldots, A_k) = (A_{\sigma(1)}, \ldots, A_{\sigma(k)}), \text{ i.e., } B_i = A_{\sigma(i)}.$$

We define sequences $\{B_1^{(r)}\}, \ldots, \{B_k^{(r)}\}$, similarly as, $\{A_1^{(r)}\}, \ldots, \{A_k^{(r)}\}$ by (3.5), that is,

$$B_i^{(1)} = B_i \quad (i = 1, \ldots, k), \text{ and for } r \geq 1,$$
$$B_i^{(r+1)} = \#(\pi_i(\beta_1, \ldots, \beta_k; B_1^{(r)}, \ldots, B_k^{(r)})).$$

We then want to show, by induction on r, that

$$B_i^{(r)} = A_{\sigma(i)}^{(r)} \text{ for } i = 1, \ldots, k, \text{ and for } r \geq 1, \tag{3.6}$$

which implies that all sequences $\{B_i^{(r)}\}$, as a whole, coinside with those of $\{A_i^{(r)}\}$, so that G_S is invariant with respect to σ. Now for (3.6), it is clear for $r = 1$. So assume that (3.6) holds (for r). Then

$$\begin{aligned} B_i^{(r+1)} &= \#\pi_i(\beta_1, \ldots, \beta_k\ ; B_1^{(r)}, \ldots, B_k^{(r)}) \\ &= \#\pi_i(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)}\ ; A_{\sigma(1)}^{(r)}, \ldots, A_{\sigma(k)}^{(r)}) \\ &= \#\pi_i\sigma(\alpha_1, \ldots, \alpha_k\ ; A_1^{(r)}, \ldots, A_k^{(r)}) \\ &= \#\pi_{\sigma(i)}(\alpha_1, \ldots, \alpha_k\ ; A_1^{(r)}, \ldots, A_k^{(r)}) \\ &= A_{\sigma(i)}^{(r+1)}. \end{aligned}$$

Example 3.2.6. Let $S = \{\pi_1, \pi_2, \pi_3, \pi_4\} \subset \mathbf{S(4)}$, with

$$(\pi_1, \pi_2, \pi_3, \pi_4) = (id, (12)(34), (13)(24), (14)(23)).$$

If $\sigma = \pi_2$, then

$$(\pi_1\sigma, \pi_2\sigma, \pi_3\sigma, \pi_4\sigma) = (\pi_2, \pi_1, \pi_4, \pi_3),$$

and

$$(\pi_{\sigma(1)}, \pi_{\sigma(2)}, \pi_{\sigma(3)}, \pi_{\sigma(4)}) = (\pi_2, \pi_1, \pi_4, \pi_3).$$

Hence by Theorem 3.2.5 (iii), G_S is σ-p.i..

Example 3.2.7 Let $p = (12 \cdots k) \in \mathbf{S(k)}$ be a cyclic permutation of k letters, and let $S = \{\pi_1, \ldots, \pi_k\}$ with $\pi_i = p^{i-1}$. If $\sigma = p^j$, then

$$(\pi_1\sigma, \ldots, \pi_k\sigma) = (p^j, \ldots, p^{k+j-1}) = (\pi_{j+1}, \ldots, \pi_{k+j}).$$

For $(p_{\sigma(1)}, \dots . p_{\sigma(k)})$, since

$$\sigma^j = (12\cdots k)^j = \begin{pmatrix} 1 & 2 & \dots & k \\ 1+j & 2+j & \dots & k+j \end{pmatrix} \quad (k+\ell(>k) \text{ is identified with } \ell),$$

we see $\sigma(1) = 1+j, \dots, \sigma(k) = k+j$, so that

$$(\pi_{\sigma(1)}, \dots, \pi_{\sigma(k)}) = (\pi_{j+1}, \dots, \pi_{k+j}).$$

Hence G_S is σ-p.i..

Example 3.2.8. Let

$$A_1 = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then by numerical computation we have, (discarded less than 10^{-6},)

$$G_\Gamma(\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{6}\ ; A_1, A_2, A_3) = \begin{bmatrix} 2.039\ 159 & 0.903\ 343 \\ 0.903\ 343 & 0.890\ 577 \end{bmatrix} (= B_1^{(r)} = B_2^{(r)} = B_3^{(r)} \ \text{ for } r \geq 3).$$

$$\text{for } \Gamma = G_{\#, \frac{2}{3}} \in \mathbf{G(3)}.$$

$$G_S(\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{6}\ ; A_1, A_2, A_3) = \begin{bmatrix} 2.050\ 390 & 0.911\ 941 \\ 0.911\ 941 & 0.893\ 311 \end{bmatrix} (= A_1^{(r)} = A_2^{(r)} = A_3^{(r)} \ \text{ for } r \geq 4)$$

$$\text{for } S = \{id, (123), (123)^2\} \subset \mathbf{S(3)}.$$

Example 3.2.9. Let

$$A_1 = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \ A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then by numerical computation we have, (discarded less than 10^{-6},)

$$\begin{aligned} G_{S_1} &= G_S(\tfrac{1}{12}, \tfrac{1}{6}, \tfrac{1}{4}, \tfrac{1}{2}; A_1, A_2, A_3, A_4) \\ &= \begin{bmatrix} 1.241\ 669 & 0.467\ 074 \\ 0.467\ 074 & 0.981\ 064 \end{bmatrix} (= A_1^{(r)} = A_2^{(r)} = A_3^{(r)} = A_4^{(r)} \ \textit{for } r \geq 4) \end{aligned}$$

$$\textit{for } S_1 = \{id, \ (1234), \ (1234)^2, \ (1234)^3\}.$$

$$\begin{aligned} G_{S_2} &= G_S(\tfrac{1}{12}, \tfrac{1}{6}, \tfrac{1}{4}, \tfrac{1}{2}; A_1, A_2, A_3, A_4) \\ &= \begin{bmatrix} 1.254\ 198 & 0.486\ 200 \\ 0.486\ 200 & 0.985\ 801 \end{bmatrix} (= A_1^{(r)} = A_2^{(r)} = A_3^{(r)} = A_4^{(r)} \ \text{for } r \geq 4) \end{aligned}$$

$$\textit{for } S_2 = \{(23), \ (34), \ (243), \ (123)\}.$$

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