Title	Generalized parallelogram law for operators and its application (Application of Geometry to Operator Theory)
Author(s)	Zuo, Hongliang; Fujii, Masatoshi
Citation	数理解析研究所講究録 (2009), 1632: 48-54
Issue Date	2009-02
URL	http://hdl.handle.net/2433/140424
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
第 1632 巻 2009 年 48-54

Generalized parallelogram law for operators and its application

左 紅 亮 (河南師範大学)
Hongliang Zuo (Henan Normal University)
藤井 正俊 (大阪教育大学)
Masatoshi Fujii (Osaka Kyoiku University)

Abstract. The calssical Bohr inequality says that $|a+b|^2 \leq p|a|^2+q|b|^2$ for all scalars a, b and p, $q>0$ with $1/p+1/q=1$. In this note, we improve the accuracy of the estimate given by the original Bohr inequality. Actually, we present:

If A and B are operators on a Hilbert space and $t \neq 0$, then

$$|A+B|^2+\frac{1}{t}|tA-B|^2=(1+t)|A|^2+(1+\frac{1}{t})|B|^2.$$

We discuss applications and further generalizations of it.

§1 Introduction

Let H be a complex separable Hilbert space and $\mathbb{B}(H)$ the algebra of all bounded operators on H. Denote by $|A|$ the absolute value operator of $A \in \mathbb{B}(H)$: $|A|=(A^*A)^{1/2}$, where A^* is the adjoint operator of A.

We say that A is a positive operator, if $(Ax,x) \geq 0$ for all $x \in H$, denoted by $A \geq 0$, and $A \geq B$ if A and B are self-adjoint and $A-B \geq 0$.

The classical Bohr inequality for scalar asserts that for any $a,b \in \mathbb{C}$ and all positive conjugate exponents $p,q \in \mathbb{R}$,

$$|a+b|^2 \leq p|a|^2+q|b|^2 \tag{0}$$

with equality if and only if $(1-p)a=b$ (See [1]).

In 2003, O. Hirzallah[4] proposed an operator version of Bohr inequality as follows:

If $A,B \in \mathbf{B}(H)$ and p,q are both positive real conjugate exponents with $q \geq p$, then

$$|A-B|^2+|(p-1)A+B|^2 \leq p|A|^2+q|B|^2. \tag{1}$$

Mathematical subject classification(2000): 46B20
Keywords and Phrases: Bohr inequality of operators, 3×3(block) operator matrices.
The research of the work was partially supported by National Natural Science Funds of China(10726073) and the loan of Japan Bank for International Cooperation(JBIC).

It is well known that the absolute value operator plays an important role in the polar decomposition $A = U|A|$. Recently, various generalizations of Bohr inequalities have been obtained in [2] and [6].

In this paper, we improve the accuracy of the estimate given by the original Bohr inequality. As a matter of fact, the parallelogram law for absolute value of operators:

$$|A+B|^2 + |A-B|^2 = 2|A|^2 + 2|B|^2 \tag{2}$$

is our viewpoint. An operator version of the Bohr inequality (0) is obtained by a generalization of (2) as follows:

$$|A+B|^2 + |\sqrt{p-1}A - \sqrt{q-1}B|^2 = p|A|^2 + q|B|^2 \tag{3}$$

for operators A, B, and $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, because of $(p-1)(q-1) = 1$.

Furthermore, we extend the Bohr inequality to a three variable case:

If $\frac{1}{p} + \frac{1}{q} + \frac{1}{w} = 1$ for $p, q, w > 0$, then for operators A, B, C, we have

$$|A+B+C|^2 \leq p|A|^2 + q|B|^2 + w|C|^2.$$

§2 Bohr equality for 2 operators

The operator parallelogram law (2) has also the following generalization, which is different from (3) a bit:

Theorem 2.1 If $A, B \in \mathbb{B}(H)$, then $|A+B|^2 + \frac{1}{t}|tA-B|^2 = (1+t)|A|^2 + (1+\frac{1}{t})|B|^2$, for $t \neq 0$.

Proof. It follows that

$$\begin{aligned} & |A+B|^2 + \frac{1}{t}|tA-B|^2 \\ = \ & |A|^2 + |B|^2 + A^*B + B^*A + t|A|^2 + \frac{1}{t}|B|^2 - A^*B - B^*A \\ = \ & (1+t)|A|^2 + (1+\frac{1}{t})|B|^2. \end{aligned}$$

It is immediately obtained from the condition of t.

Corollary 2.2 (i) If $0 < t \leq 1$, then $|A+B|^2 + |tA-B|^2 \leq (1+t)|A|^2 + (1+\frac{1}{t})|B|^2$;

(ii) If $t \geq 1$ or $t < 0$, then $|A+B|^2 + |tA-B|^2 \geq (1+t)|A|^2 + (1+\frac{1}{t})|B|^2$.

As an easy consequence, we have Bohr type inequalities obtained in [2] and [4].

Corollary 2.3 [4, Theorem 1] If $A, B \in \mathbb{B}(H)$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p \leq 2$, then

(i) $|A-B|^2 + |(p-1)A+B|^2 \leq p|A|^2 + q|B|^2$.

(ii) $|A-B|^2 + |A+(q-1)B|^2 \geq p|A|^2 + q|B|^2$.

Corollary 2.4 [2, Theorem 2] If $A, B \in \mathbb{B}(H)$, $\frac{1}{p} + \frac{1}{q} = 1$, $p < 1$, then

(iii) $|A-B|^2 + |(p-1)A+B|^2 \geq p|A|^2 + q|B|^2$.

Corollary 2.5 [2, Theorem 3] If $A, B \in \mathbb{B}(H)$, $|\alpha| \geq |\beta|$, then

$$|A-B|^2 + \frac{1}{|\alpha|^2}||\beta|A + |\alpha|B|^2 \leq (1+\frac{|\beta|}{|\alpha|})|A|^2 + (1+\frac{|\alpha|}{|\beta|})|B|^2$$

with equality if and only if $|\alpha| = |\beta|$ or $|\beta|A + |\alpha|B = 0$;

§3 Bohr-type inequalities for 3 operators

Observe that

$$|A+B+C|^2 = \begin{pmatrix} I & I & I \end{pmatrix} \begin{pmatrix} |A|^2 & A^*B & A^*C \\ B^*A & |B|^2 & B^*C \\ C^*A & C^*B & |C|^2 \end{pmatrix} \begin{pmatrix} I \\ I \\ I \end{pmatrix} \geq 0.$$

Then, due to the idea of [6], we may convert a problem of absolute operators to a problem of 3×3 block operator matrices, while the later approach maybe easier to handle.

For the sake of convenience, we cite the following well-known fact:

Lemma 3.1 If $x, y, z \geq 0$, and $a, b, c \in \mathbb{R}$ with

$$\begin{cases} xy \geq a^2, yz \geq c^2, xz \geq b^2; \\ xyz + 2abc \geq xc^2 + yb^2 + za^2. \end{cases}$$

Then

$$\begin{pmatrix} x & a & b \\ a & y & c \\ b & c & z \end{pmatrix} \geq 0.$$

Lemma 3.2 Let $A_i \in \mathbb{B}(H)$, $\alpha_i, \beta_i \in \mathbb{R}$ with $i = 1, 2, 3$. Then positive operator-valued function

$$F(\alpha_1, \alpha_2, \alpha_3) = |\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3|^2$$

is order preserving if the order $\prec$ among $\mathbb{R}$ is defined by

$$(\alpha_1, \alpha_2, \alpha_3) \prec (\beta_1, \beta_2, \beta_3) \Leftrightarrow |\alpha_i| \leq |\beta_i| \quad \text{for all } i \text{ and } \alpha_i\beta_j = \alpha_j\beta_i \quad \text{for} \quad i \neq j.$$

Proof. Since $|A_1 + A_2 + A_3|^2 = \begin{pmatrix} I & I & I \end{pmatrix} \begin{pmatrix} A_1^* \\ A_2^* \\ A_3^* \end{pmatrix} \begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix} \begin{pmatrix} I \\ I \\ I \end{pmatrix}$,

it suffice to show that $(\alpha_1, \alpha_2, \alpha_3) \prec (\beta_1, \beta_2, \beta_3)$ implies that

$$\begin{pmatrix} \alpha_1 A_1^* \\ \alpha_2 A_2^* \\ \alpha_3 A_3^* \end{pmatrix} \begin{pmatrix} \alpha_1 A_1 & \alpha_2 A_2 & \alpha_3 A_3 \end{pmatrix} \leq \begin{pmatrix} \beta_1 A_1^* \\ \beta_2 A_2^* \\ \beta_3 A_3^* \end{pmatrix} \begin{pmatrix} \beta_1 A_1 & \beta_2 A_2 & \beta_3 A_3 \end{pmatrix},$$

that is,

$$\begin{pmatrix} \alpha_1^2|A_1|^2 & \alpha_1\alpha_2 A_1^*A_2 & \alpha_1\alpha_3 A_1^*A_3 \\ \alpha_1\alpha_2 A_2^*A_1 & \alpha_2^2|A_2|^2 & \alpha_2\alpha_3 A_2^*A_3 \\ \alpha_1\alpha_3 A_3^*A_1 & \alpha_2\alpha_3 A_3^*A_2 & \alpha_3^2|A_3|^2 \end{pmatrix} \leq \begin{pmatrix} \beta_1^2|A_1|^2 & \beta_1\beta_2 A_1^*A_2 & \beta_1\beta_3 A_1^*A_3 \\ \beta_1\beta_2 A_2^*A_1 & \beta_2^2|A_2|^2 & \beta_2\beta_3 A_2^*A_3 \\ \beta_1\beta_3 A_3^*A_1 & \beta_2\beta_3 A_3^*A_2 & \beta_3^2|A_3|^2 \end{pmatrix}.$$

By the definition and Lemma 3.1, we have

$$\begin{pmatrix} \beta_1^2-\alpha_1^2 & \beta_1\beta_2-\alpha_1\alpha_2 & \beta_1\beta_3-\alpha_1\alpha_3 \\ \beta_1\beta_2-\alpha_1\alpha_2 & \beta_2^2-\alpha_2^2 & \beta_2\beta_3-\alpha_2\alpha_3 \\ \beta_1\beta_3-\alpha_1\alpha_3 & \beta_2\beta_3-\alpha_2\alpha_3 & \beta_3^2-\alpha_3^2 \end{pmatrix} \geq 0,$$

which implies the conclusion.

Theorem 3.3 Let $A, B, C \in \mathbb{B}(H)$, $\alpha_i \in \mathbb{R}$, $p, q, w > 0$ with $i = 1, 2, 3$. If

$$\begin{cases} p \geq \alpha^2; \\ q \geq \beta^2; \\ w \geq \gamma^2. \end{cases} \quad \begin{cases} (p-\alpha^2)(q-\beta^2) \geq (\alpha\beta)^2; \\ (q-\beta^2)(w-\gamma^2) \geq (\beta\gamma)^2; \\ (w-\gamma^2)(p-\alpha^2) \geq (\gamma\alpha)^2; \\ pqw \geq \alpha^2 qw + \beta^2 pw + \gamma^2 pq. \end{cases}$$

Then

$$|\alpha A + \beta B + \gamma C|^2 \leq p|A|^2 + q|B|^2 + w|C|^2.$$

Proof. As in above, we have to show that

$$\begin{pmatrix} \alpha^2|A|^2 & \alpha\beta A^*B & \alpha\gamma A^*C \\ \alpha\beta B^*A & \beta^2|B|^2 & \beta\gamma B^*C \\ \alpha\gamma C^*A & \beta\gamma C^*B & \gamma^2|C|^2 \end{pmatrix} \leq \begin{pmatrix} p|A|^2 & 0 & 0 \\ 0 & q|B|^2 & 0 \\ 0 & 0 & w|C|^2 \end{pmatrix}.$$

Therefore, what we should do is just to prove that

$$\begin{pmatrix} p-\alpha^2 & \alpha\beta & \alpha\gamma \\ \alpha\beta & q-\beta^2 & \beta\gamma \\ \alpha\gamma & \beta\gamma & w-\gamma^2 \end{pmatrix} \geq 0,$$

which is obtained by the assumption and Lemma 3.1.

The following corollary is symmetric to Theorem 3.3.

Corollary 3.4 Let $A, B, C \in \mathbb{B}(H)$, $\alpha_i \in \mathbb{R}$, $p, q, w > 0$ with $i = 1, 2, 3$. If

$$\begin{cases} p \leq \alpha^2; \\ q \leq \beta^2; \\ w \leq \gamma^2. \end{cases} \quad \begin{cases} (p-\alpha^2)(q-\beta^2) \geq (\alpha\beta)^2; \\ (q-\beta^2)(w-\gamma^2) \geq (\beta\gamma)^2; \\ (w-\gamma^2)(p-\alpha^2) \geq (\gamma\alpha)^2; \\ pqw \geq \alpha^2 qw + \beta^2 pw + \gamma^2 pq. \end{cases}$$

Then

$$|\alpha A + \beta B + \gamma C|^2 \geq p|A|^2 + q|B|^2 + w|C|^2.$$

Now we have Bohr inequality for 3 operators.

Corollary 3.5 If $p, q, w > 0$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{w} = 1$, then $|A + B + C|^2 \leq p|A|^2 + q|B|^2 + w|C|^2$.

Proof. Given $p, q, w > 0$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{w} = 1$, the following is

$$\begin{cases} p \geq 1; \\ q \geq 1; \\ w \geq 1. \end{cases} \quad \begin{cases} (p-1)(q-1) \geq 1; \\ (q-1)(w-1) \geq 1; \\ (w-1)(p-1) \geq 1; \\ pqw = qw + pw + pq. \end{cases}$$

Therefore, Theorem 3.3 implies that

$$|A+B+C|^2 \leq p|A|^2 + q|B|^2 + w|C|^2.$$

§4 Bohr equality for multiple operators

For this, we begin with the reformulation of (3). As a matter of fact, it is just the first step in this section:

Lemma 4.1 Let $A_1, A_2 \in \mathbb{B}(H)$. $\frac{1}{r_1} + \frac{1}{r_2} = 1$ with $r_1, r_2 \geq 1$.

$$r_1|A_1|^2 + r_2|A_2|^2 - |A_1 + A_2|^2 = |\sqrt{\frac{r_1}{r_2}}A_1 - \sqrt{\frac{r_2}{r_1}}A_2|^2.$$

Theorem 4.2 Suppose that $A_i \in \mathbb{B}(H)$, and $r_i \geq 1$ with $\sum_{i=1}^{n} \frac{1}{r_i} = 1$ for $i = 1, 2, ..., n, n \in \mathbb{N}$. Then

$$\sum_{i=1}^{n} r_i|A_i|^2 - |\sum_{i=1}^{n} A_i|^2 = \sum_{1\leq i<j\leq n} \left|\sqrt{\frac{r_i}{r_j}}A_i - \sqrt{\frac{r_j}{r_i}}A_j\right|^2. \tag{4}$$

Proof. We show it by the induction on n. Note that it is true for $n = 2$ by Lemma 4.1. Now suppose that it is true for $n = k$, then we take $A_1, \cdots, A_{k+1} \in \mathbb{B}(H)$ and $r_1, \cdots, r_{k+1} > 1$ satisfying $\sum_{i=1}^{k+1} \frac{1}{r_i} = 1$. If we put $r_i' = r_i(1 - \frac{1}{r_{k+1}})$ for $i = 1, \cdots, k$ for convenience, then $r_i' > 1$ and $\sum_{i=1}^{k} \frac{1}{r_i'} = 1$. Hence we have

$$\begin{aligned}
&\sum_{i=1}^{k+1} r_i|A_i|^2 - |\sum_{i=1}^{k+1} A_i|^2 = \sum_{i=1}^{k} r_i|A_i|^2 + r_{k+1}|A_{k+1}|^2 - |\sum_{i=1}^{k} A_i + A_{k+1}|^2 \\
= \ & (1 - \frac{1}{r_{k+1}})\sum_{i=1}^{k} r_i|A_i|^2 - |\sum_{i=1}^{k} A_i|^2 \\
& +(r_{k+1}-1)|A_{k+1}|^2 + \frac{1}{r_{k+1}}\sum_{i=1}^{k} r_i|A_i|^2 - (\sum_{i=1}^{k} A_i)^* A_{k+1} - A_{k+1}^* \sum_{i=1}^{k} A_i \\
= \ & \left(\sum_{i=1}^{k} r_i'|A_i|^2 - |\sum_{i=1}^{k} A_i|^2\right) + \sum_{i=1}^{k} \frac{r_i}{r_{k+1}}|A_i|^2 - (\sum_{i=1}^{k} A_i)^* A_{k+1} - A_{k+1}^* \sum_{i=1}^{k} A_i + (r_{k+1}-1)|A_{k+1}|^2 \\
= \ & \sum_{1\leq i<j\leq k} \left|\sqrt{\frac{r_i}{r_j}}A_i - \sqrt{\frac{r_j}{r_i}}A_j\right|^2 + \sum_{i=1}^{k} \frac{r_i}{r_{k+1}}|A_i|^2 - (\sum_{i=1}^{k} A_i)^* A_{k+1} - A_{k+1}^* \sum_{i=1}^{k} A_i + \sum_{i=1}^{k} \frac{r_{k+1}}{r_i}|A_{k+1}|^2 \\
= \ & \sum_{1\leq i<j\leq k+1} \left|\sqrt{\frac{r_i}{r_j}}A_i - \sqrt{\frac{r_j}{r_i}}A_j\right|^2.
\end{aligned}$$

Therefore, the equality (4) holds for all $n \in \mathbb{N}$.

Corollary 4.3 [6, Theorem 7] Suppose that $A_i \in \mathbb{B}(H)$, and $r_i \geq 1$ with $\sum_{i=1}^{n} \frac{1}{r_i} = 1$ for $i = 1, 2, ..., n$. Then

$$|\sum_{i=1}^{n} A_i|^2 \leq \sum_{i=1}^{n} r_i|A_i|^2.$$

Equivalently, we can say that $K(z) = |z|^2$ satisfies (operator) Jensen inequality, in the sense that

$$K(\sum_{i=1}^{n} t_i A_i) \leq \sum_{i=1}^{n} t_i K(A_i)$$

for $t_1, \cdots, t_n \geq 0$ with $\sum_{i=1}^{n} t_i = 1$.

Corollary 4.4 Let $A_i \in \mathbb{B}(H)$, $\sum_{i=1}^{n} \frac{1}{r_i} = 1$, and $r_i \neq 0$ with $\sum_{i=1}^{n} \frac{1}{r_i} = 1$ for $i = 1, 2, ..., n$, $n \in \mathbb{N}$. Then

$$\sum_{i=1}^{n} r_i |A_i|^2 - |\sum_{i=1}^{n} A_i|^2 = \sum_{1 \leq i \leq j \leq n} \frac{r_j}{r_i} \left| \frac{r_i}{r_j} A_i - A_j \right|^2.$$

§5 Further generalization of Bohr inequality

Theorem 5.1 Let $A, B, C \in \mathbb{B}(H)$, $\alpha_i \in \mathbb{R}$, $\gamma_i > 0$ with $i = 1, 2, 3$. If

$$\begin{cases} \gamma_1 \geq 1 + \alpha_1^2; \\ \gamma_2 \geq 1 + \alpha_2^2; \\ \gamma_3 \geq 1 + \alpha_3^2. \end{cases} \quad \begin{cases} [\gamma_1 - (1 + \alpha_1^2)][\gamma_2 - (1 + \alpha_2^2)] \geq (1 + \alpha_1\alpha_2)^2; \\ [\gamma_2 - (1 + \alpha_2^2)][\gamma_3 - (1 + \alpha_3^2)] \geq (1 + \alpha_2\alpha_3)^2; \\ [\gamma_1 - (1 + \alpha_1^2)][\gamma_3 - (1 + \alpha_3^2)] \geq (1 + \alpha_1\alpha_3)^2. \end{cases}$$

with $[\gamma_1 - (1 + \alpha_1^2)][\gamma_2 - (1 + \alpha_2^2)][\gamma_3 - (1 + \alpha_3^2)] - 2(1 + \alpha_1\alpha_2)(1 + \alpha_1\alpha_3)(1 + \alpha_2\alpha_3) \geq - [\gamma_3 - (1 + \alpha_3^2)](1 + \alpha_1\alpha_2)^2 - [\gamma_2 - (1 + \alpha_2^2)](1 + \alpha_1\alpha_3)^2 - [\gamma_1 - (1 + \alpha_1^2)](1 + \alpha_2\alpha_3)^2$.
Then

$$|A_1 + A_2 + A_3|^2 + |\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3|^2 \leq \gamma_1 |A_1|^2 + \gamma_2 |A_2|^2 + \gamma_3 |A_3|^2. \tag{5}$$

Proof. Notice that both sides of the inequality (5) correspond to

$$\begin{pmatrix} (1 + \alpha_1^2)|A_1|^2 & (1 + \alpha_1\alpha_2)A_1^* A_2 & (1 + \alpha_1\alpha_3)A_1^* A_3 \\ (1 + \alpha_1\alpha_2)A_2^* A_1 & (1 + \alpha_2^2)|A_2|^2 & (1 + \alpha_2\alpha_3)A_2^* A_3 \\ (1 + \alpha_1\alpha_3)A_3^* A_1 & (1 + \alpha_2\alpha_3)A_3^* A_2 & (1 + \alpha_3^2)|A_3|^2 \end{pmatrix}$$

and

$$\begin{pmatrix} \gamma_1 |A_1|^2 & 0 & 0 \\ 0 & \gamma_2 |A_2|^2 & 0 \\ 0 & 0 & \gamma_3 |A_3|^2 \end{pmatrix}$$

respectively. Hence, it is suffice to show that

$$\begin{pmatrix} \gamma_1 - (1 + \alpha_1^2) & -1 - \alpha_1\alpha_2 & -1 - \alpha_1\alpha_3 \\ -1 - \alpha_1\alpha_2 & \gamma_2 - (1 + \alpha_2^2) & -1 - \alpha_2\alpha_3 \\ -1 - \alpha_1\alpha_3 & -1 - \alpha_2\alpha_3 & \gamma_3 - (1 + \alpha_3^2) \end{pmatrix} \geq 0,$$

which is implied by the assumption and Lemma 3.1.

Corollary 5.2 Let $A, B, C \in \mathbb{B}(H)$, $\alpha_1, \alpha_2, \beta_i \in \mathbb{R}$, $\gamma_i > 0$ with $i = 1, 2, 3$. If

$$\begin{cases} \gamma_1 \geq 1 + \alpha_1^2; \\ \gamma_2 \geq 1 + \alpha_2^2; \\ \gamma_3 \geq 1. \end{cases} \quad \begin{cases} [\gamma_2 - (\alpha_2^2 + 1)](\gamma_3 - 1) \geq 1; \\ [\gamma_1 - (\alpha_1^2 + 1)](\gamma_3 - 1) \geq 1; \\ [\gamma_1 - (\alpha_1^2 + 1)][\gamma_2 - (\alpha_2^2 + 1)] \geq (1 + \alpha_1\alpha_2)^2; \\ 1 + \alpha_1\alpha_2 \leq 0. \end{cases}$$

Then

$$|A_1 + A_2 + A_3|^2 + |\alpha_1 A_1 + \alpha_2 A_2|^2 \leq \gamma_1 |A_1|^2 + \gamma_2 |A_2|^2 + \gamma_3 |A_3|^2.$$

Another extension of Bohr inequality is presented:

Corollary 5.3 If $A, B, C \in \mathbb{B}(H)$, and $\frac{1}{p}+\frac{1}{q}+\frac{1}{w}=1, p, q, w \geq 0$, then

$$|A+B+C|^2+|\frac{p}{\sqrt{p+q}}A-\frac{q}{\sqrt{p+q}}B|^2 \leq p|A|^2+q|B|^2+w|C|^2.$$

Theorem 5.4 Let $A_i \in \mathbb{B}(H)$, $\alpha_i, \beta_i \in \mathbb{R}$, $\gamma_i > 0$ with $i=1,2,3$. If

$$\begin{cases} \gamma_1 \geq 1+\alpha_1^2; \\ \gamma_2 \geq 1+\alpha_2^2; \\ \gamma_3 \geq 1+\alpha_3^2. \end{cases} \quad \begin{cases} [\gamma_1-(\alpha_1^2+1)][\gamma_2-(\alpha_2^2+1)] \geq (\alpha_1\alpha_2+1)^2; \\ [\gamma_2-(\alpha_2^2+1)][\gamma_3-(\alpha_3^2+1)] \geq (\alpha_2\alpha_3+1)^2; \\ \alpha_1\alpha_3+1=0. \end{cases}$$

Then

$$|A_1+A_2+A_3|^2+|\alpha_1A_1+\alpha_2A_2+\alpha_3A_3|^2 \leq \gamma_1|A_1|^2+\gamma_2|A_2|^2+\gamma_3|A_3|^2.$$

Corollary 5.5 If $A, B, C \in \mathbb{B}(H)$, and $\frac{1}{p}+\frac{1}{q}+\frac{1}{w}=1, p, q, w \geq 0$, then

$$|A+B+C|^2+|\frac{1}{\sqrt{w-1}}A+\frac{1}{\sqrt{w-1}}B-\sqrt{w-1}C|^2 \leq p|A|^2+q|B|^2+w|C|^2.$$

Related to [3], we have the following inequalities. As a matter of fact, the right-hand sides are regarded as the weighted arithmetic mean of $|A|^2$, $|B|^2$ and $|C|^2$ in [3, Lemma 1].

Corollary 5.6 If $A, B, C \in \mathbb{B}(H)$, and $t \in (0, 1)$, then we have

$$|A+B+C|^2+|\sqrt{t}A+\sqrt{t}B-\frac{1}{\sqrt{t}}C|^2 \leq \frac{2-t}{1-t}\frac{1+t}{t}|A|^2+\frac{2-t}{1-t}|B|^2+\frac{1+t}{t}|C|^2;$$

$$|A+B+C|^2+|\sqrt{1-t}(A+B)-\frac{1}{\sqrt{1-t}}C|^2 \leq \frac{1+t}{t}|A|^2+\frac{2-t}{1-t}\frac{1+t}{t}|B|^2+\frac{2-t}{1-t}|C|^2.$$

Acknowledgement The authors would like to express their thanks to Professor J.I. Fujii for introducing the reference [2].

References

[1] Zur theorie der fastperiodischen funktionmen I, Acta Math., 45(1927), 29-127.

[2] W.-S. Cheung and J.Pecaric, Bohr's inequalities for Hilbert sapce operators, J. Math. Anal. Appl., 323(2006), 403-412.

[3] J. I. Fujii, M. Fujii. M. Nakamura, J. Pecaric and Y. Seo, A reverse inequality for the weighted geometric mean due to Lawson-Lim, Linear Algebra Appl., 427(2007), 272-284.

[4] O. Hirzallah, Non-commutative operator Bohr inequalities, J. Math. Anal. Appl., 282(2003), 578-583.

[5] N.Young, An Introduction to Hilbert Space, Cambridge University Press, 1988.

[6] F. Zhang, On the Bohr inequality of operators, J. Math. Anal. Appl., 333(2007), 1264-1271.