Title	THE INVARIANT SUBSPACES AND SPECTRAL PROPERTIES OF LINEAR OPERATORS (Application of Geometry to Operator Theory)
Author(s)	DJORDJEVIC, SLAVISA V.
Citation	数理解析研究所講究録 (2009), 1632: 81-87
Issue Date	2009-02
URL	http://hdl.handle.net/2433/140419
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
第 1632 巻 2009 年 81-87

THE INVARIANT SUBSPACES AND SPECTRAL PROPERTIES OF LINEAR OPERATORS

SLAVIŠA V. DJORDJEVIĆ

ABSTRACT. In this note we give describe the spectra and essential spectra of a bounded linear operator T from the Banach space X into self using the same spectra of its restrictions to invariant subspaces and mappings induced by T over quotient subspaces.

1. INTRODUCTION

Given normed space X, let $\mathcal{B}(X)$ denote the space of all bounded linear transformations (equivalently, operators) from X into self. For $T\in\mathcal{B}(X)$, let $N(T)$ and $R(T)$ denote, respectively, the null space and the range of the mapping T. Let $n(T)$ and $d(T)$ denote, respectively, the dimension of $N(T)$ and the the codimension of $R(T)$. If the range $R(T)$ of $T\in\mathcal{B}(X)$ is closed and $n(T)<\infty$ (resp. $d(T)<\infty$), then T is said to be an *upper semi-Fredholm* (resp. a *lower semi-Fredholm*) operator. If $T\in\mathcal{B}(X)$ is either upper or lower semi-Fredholm, then T is called a *semi-Fredholm* operator, and then the *index* of T is defined by $ind(T)=n(T)-d(T)$. If both $n(T)$ and $d(T)$ are finite, then T is a *Fredholm* operator. The essential (Fredholm) spectrum $\sigma_e(T)$ is defined by

$$\sigma_e(T)=\{\lambda\in\mathbb{C}:T-\lambda \text{ is not Fredholm}\}.$$

In this paper, we start by considering the invertibility of a linear operator T by considering the restriction $T_{|E}$ of T to an invariant subspace E and the mapping $T_{|X/E}$ determined by T on the quotient space X/E of this invariant subspace.

The motivation for such approach to spectral problems for linear operators we deduced from a study of the spectrum, and distinguished parts thereof, for a upper triangular matrix representation for a linear operators (see [4], [6], [7], [8], [12], [14]). Also, specially for many special classes of a Hilbert space operators, like as, for example, hyponormal, quasi-hyponormal or p-hyponormal, there exists a certain invariant subspaces that determine spectral property of the operator. The next examples will illustrate this:

1991 *Mathematics Subject Classification.* Primary 47A10, 47A15 ; secondary 47A05, 15A29.
Key words and phrases. Invariant subspace, Spectrum of an operator, essential (Fredholm) spectrum.

Example 1.1. Let H is a Hilbert space H, and $T \in \mathcal{B}(H)$. If E is (closed) subspace of H and $T(E) \subseteq E$, then E is complemented in H and T has an upper triangular representation

$$T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} E \\ E^{\perp} \end{pmatrix} \rightarrow \begin{pmatrix} E \\ E^{\perp} \end{pmatrix},$$

where with A we denote the restriction of T on E. If A and B are invertible then T is invertible and

$$T^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{pmatrix}.$$

Hence, $\sigma(T) \subset \sigma(A) \cup \sigma(B)$. Additionally, if $E^{\perp}$ is invariant for T, i.e.

$$T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

then $\sigma(T) = \sigma(A) \cup \sigma(B)$.

For a Banach space X and its complemented closed subspace M that is invariant for an operator $T \in B(X)$, then (once again) T has an upper triangular representation and the discussion is same.

Example 1.2. Let H be a Hilbert space. Then for a hyponormal operator $T \in B(H)$ $(T^*T \leq TT^*)$, and its invariant subspace

$$E = \{x \in H : \|T^k x\| = \|T^{*k} x\|, \text{ for all } k = 1, 2, \ldots\}$$

we have that $T_{|E}$ is normal.

The property that for some Hilbert space operator there exits a invariant subspace that the restriction of operator is one category lest is pretty usual.

An operator $T \in \mathcal{B}(H)$ is said to be p-hyponormal operator, $p \in (0, 1]$, if $(T^*T)^p \leq (TT^*)^p$ and T is a (p, k)-quasihyponormal if $T^{*k}(\|T\|^{2p} - \|T^*\|^{2p})T^k \geq 0$. We have next theorem.

Theorem 1.3. *If $T \in B(H)$ is a (p, k)-quasihyponormal operator and the rank of T^k is not dense, then the restriction of T on space $H_1 = \overline{rank(T^k)}$ is p-hyponormal. Moreover, the transformation $\tilde{T} : H_{|H_1} \rightarrow H_{|H_1}$ define with $\tilde{T}([x]) = [Tx]$ is k-nilpotent.*

Proof. By [13, Lemma 1], T has matrix representation $T = \begin{pmatrix} A & C \\ 0 & B_1 \end{pmatrix}$ where $A = T_{|\overline{R(T^k)}}$ is p-hyponormal and B_1 is k-nilpotent. Now, by the introduction of [1], we have that in this situation $B = T_{|X/\overline{R(T^k)}}$ and B_1 are similar that implies B is k-nilpotent. □

The similar behavior we can find for the same classes of Banach space linear operators, for example for a B-Fredholm operators. For a bounded linear operator

$T \in B(X)$ and for each positive integer n. define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_0 = T$).

If for some positive integer n, the range space $R(T^n)$ is closed and T_n is a Fredholm operator, then T is called a B-Fredholm operator (hence, every Fredholm operator is B-Fredholm). In this case, for any integer m such that $m > n$, T_m is Fredholm operator with $ind(T_m) = ind(T_n)$. For more details see Berkani [2] and [3].

Theorem 1.4. *Let $T \in B(X)$ be a B-Fredholm operator. There exists an invariant subspace $E \subset X$ such that T restricted en E is a Fredholm operator and the transformation $\tilde{T} : X_{|E} \to X_{|E}$ define with $\tilde{T}([x]) = [Tx]$ is nilpotent.*

Proof. By proof of Lemma 4.1. in [2], follows that exist a closed subspaces E and F such that $X = E \oplus F$ and $A = T_{|E}$ is Fredholm operator and $B_1 = T_{|F}$ is nilpotent. Since the operator $B = T_{|X/E}$ is similar to B_1 (see end of the proof of Theorem 1.3), we have that B is nilpotent too. □

2. Spectrum of a linear operator through its invariant subspaces

Let T be a Banach space linear operator, $E \subset X$ closed T-invariant subspace. For the aim of easier notation, with $A \in \mathcal{B}(E)$ we will notate the restriction of T on E, i.e. $A = T_{|E}$ and, similarly, with $B \in \mathcal{B}(X/E)$ we will always notate the mapping determined by T on the quotient space X/E. In this section will be discission relationship between the spectrums of the operators T, A and B.

Theorem 2.1. *If $T \in B(X)$ is a bounded operator and $E \in Inv(T)$, then the following holds.*

(i) $\sigma(T) \subset \sigma(A) \cup \sigma(B)$;

(ii) $\sigma(A) \subset \sigma(T) \cup \sigma(B)$;

(iii) $\sigma(B) \subset \sigma(T) \cup \sigma(A)$.

Moreover,

(vi) *if* $\lambda \in (\sigma(A) \cup \sigma(B)) \setminus \sigma(T)$, *then* $\lambda \in \sigma(A) \cap \sigma(B)$;

(v) *if* $\lambda \in (\sigma(T) \cup \sigma(B)) \setminus \sigma(A)$, *then* $\lambda \in \sigma(T) \cap \sigma(B)$;

(vi) *if* $\lambda \in (\sigma(T) \cup \sigma(A)) \setminus \sigma(B)$, *then* $\lambda \in \sigma(T) \cap \sigma(A)$.

Proof. The proof of the theorem we can find partially in [1, Proposition 3 (i)], [9, Theorem 2.1] and [10, Proposition 1.2.4]). □

It is interesting to find conditions when the spectrum of T is equal to union of the spectrums of the operators A and B. The next proposition give some of such conditions.

Proposition 2.2. *Let $T \in B(X)$ and $E \in Inv(T)$. If one of following conditions holds*

(i) E is T-hyperinvariant:

(ii) exists $F \in Inv(T)$ such that $X = E \oplus F$;

(iii) $\sigma(A) \cap \sigma(B) = \emptyset$;

(iv) $\sigma(A) \subset \sigma(T)$ or $\sigma(B) \subset \sigma(T)$.

then $\sigma(T) = \sigma(A) \cup \sigma(B)$.

Proof. (i) [1, Proposition 3(3)].

(ii) Let $T = A \oplus B_1$ on $X = E \oplus F$. Then B_1 and B are similar operator, and $\sigma(T) = \sigma(A) \cup \sigma(B_1) = \sigma(A) \cup \sigma(B)$.

(iii) and (iv) are direct consequences of Theorem 2.1. □

3. Essential spectrum of a linear operator through its invariant subspaces

Given a Banach space X, let $\Phi_+(X)$and $\Phi_-(X)$ denote, respectively, the set of upper and lower semi-Fredholm operators and $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$ denote the set of Fredholm operators. Let $E \in Inv(T)$, and let A and B be defined as in the previous section.

The kernel and rang of an operator take main role in observation of the Fredholmes of an operators. It is clearly that $N(A) \subset N(T)$, and consequently $n(A) \leq n(T)$. Barnes in [1] showed if $n(A) < \infty$ and $n(B) < \infty$, then $n(A) \leq n(T) \leq n(A) + n(B)$, and if $d(A) < \infty$ and $d(B) < \infty$, then $d(B) \leq d(T) \leq d(A) + d(B)$. Also, by [1, Theorem 8], if T is a Fredholm operator, then A is upper semi-Fredholm and B is lower semi-Fredholm. Using the Theorem 8 in [1] we can get next results (see also [9, Theorem 3.1]).

Theorem 3.1. *Let $T \in B(X)$, be a bounded operator and $E \in Inv(T)$. Then the following properties hold.*

(i) $\sigma_e(T) \subset \sigma_e(A) \cup \sigma_e(B)$;

(ii) $\sigma_e(A) \subset \sigma_e(T) \cup \sigma_e(B)$;

(iii) $\sigma_e(B) \subset \sigma_e(T) \cup \sigma_e(A)$.

Moreover,

(vi) if $\lambda \in (\sigma_e(A) \cup \sigma_e(B)) \setminus \sigma_e(T)$, then $\lambda \in \sigma_e(A) \cap \sigma_e(B)$;

(v) if $\lambda \in (\sigma_e(T) \cup \sigma_e(B)) \setminus \sigma_e(A)$, then $\lambda \in \sigma_e(T) \cap \sigma_e(B)$;

(vi) if $\lambda \in (\sigma_e(T) \cup \sigma_e(A)) \setminus \sigma_e(B)$, then $\lambda \in \sigma_e(T) \cap \sigma_e(A)$.

Proof. The proof of (i) is straightforward from [1, Theorem 8].

(ii) Let T and B are Fredholm. Then by [1, Proposition 8] follows A upper semi-Fredholm operator and from [1, Theorem 8] $d(A) < \infty$, i.e. A is Fredholm.

(iii) In the similar way like (ii).

(iv) From an argument of type:

$$\begin{aligned}\lambda \notin (\sigma_e(A) \cup \sigma_e(B)) &\iff A - \lambda \text{ and } B - \lambda \text{ are Fredholm}\\ &\iff T - \lambda \text{ , and } A - \lambda \text{ or } B - \lambda \text{ are Fredholm}\\ &\iff \lambda \notin \sigma_e(T) \cup \{\sigma_e(A) \cap \sigma_e(B)\}.\end{aligned}$$

follows that $\sigma_e(A) \cup \sigma_e(B) = \sigma_e(T) \cup \{\sigma_e(A) \cap \sigma_e(B)\}$, i.e. $(\sigma_e(A) \cup \sigma_e(B)) \setminus \sigma_e(T) \subset \sigma_e(A) \cap \sigma_e(B)$.

(v) and (vi) in the same way like (iv). □

Corollary 3.2. *If two of the operators A, B and T are Fredholm, then the third one is Fredholm too.*

Theorem 3.1 gives us some conditions that the essential spectrum of T is union of the essential spectrums of A and B.

Corollary 3.3. *Let* $T \in B(X)$ *and* $E \in Inv(T)$. *If one of following conditions holds:*

(i) $\sigma_e(A) \cap \sigma_e(B) = \emptyset$;

(ii) $\sigma_e(A) \subset \sigma_e(T)$ *or* $\sigma_e(B) \subset \sigma_e(T)$;

then $\sigma_e(T) = \sigma_e(A) \cup \sigma_e(B)$.

Remark 3.4. In the way of Proposition 2.2 we can to find some new conditions that imply $\sigma_e(T) = \sigma_e(A) \cup \sigma_e(B)$.

If E is T-hyperinvariant, then it is easily seen that $T^{-1}(E) = E$; applying [1, Corollary 9] it follows that $\sigma_e(T) = \sigma_e(A) \cup \sigma_e(B)$.

Also, if E has direct complement F and $T = A \oplus B_1$ on $X = E \oplus F$. Then B_1 and B are similar operators, and $\sigma_e(T) = \sigma_e(A) \cup \sigma_e(B_1) = \sigma_e(A) \cup \sigma_e(B)$.

If $T \in \mathcal{B}(X)$ is a Fredholm operator with index zero, then T is called a *Weyl* operator. The Weyl spectrum, in notation $\sigma_w(T)$, is defined by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}.$$

The relationship between the Weyl spectra of A, B and T is a bit more delicate, and relationship of type of Proposition 2.2, Corollary 3.3 or Remark 3.4 is not possible for the Weyl spectrum. Even in the case when the invariant closed subspace E has closed complement F that is also invariant for T, $\sigma_w(T)$ is not equal to union of the Weyl spectrums of A and B (see [11. Lemma 1]). However, with additional hypotheses one is able to relate the Weyl spectra of A, B and T.

Theorem 3.5. *Let $T \in B(X)$, be a bounded operator and $E \in Inv(T)$. Then if one of the following equivalent conditions holds*

(a) $T^{-1}(E) = N(T) + E$, *or*

(b) $T(E) = R(T) \cap E$,

then

(i) $\sigma_w(T) \subset \sigma_w(A) \cup \sigma_w(B)$;

(ii) $\sigma_w(A) \subset \sigma_w(T) \cup \sigma_w(B)$;

(iii) $\sigma_w(B) \subset \sigma_w(T) \cup \sigma_w(A)$.

Moreover,

(vi) if $\lambda \in (\sigma_e(w) \cup \sigma_w(B)) \setminus \sigma_w(T)$, *then* $\lambda \in \sigma_w(A) \cap \sigma_w(B)$;

(v) if $\lambda \in (\sigma_w(T) \cup \sigma_w(B)) \setminus \sigma_w(A)$, *then* $\lambda \in \sigma_w(T) \cap \sigma_w(B)$;

(vi) if $\lambda \in (\sigma_w(T) \cup \sigma_w(A)) \setminus \sigma_w(B)$, *then* $\lambda \in \sigma_w(T) \cap \sigma_w(A)$.

Proof. The equivalency of the conditions (a) and (b) follows from [1, Proposition 7, (1)] and from same proposition we have that

$$\mathrm{ind}(T-\lambda) = \mathrm{ind}(A-\lambda) + \mathrm{ind}(B-\lambda).$$

(i) Let A and B are Weyl. Then by Corollary 3.2, T is Fredholm, and by first part of theorem

$$\mathrm{ind}(T-\lambda) = \mathrm{ind}(A-\lambda) + \mathrm{ind}(B-\lambda) = 0,$$

i.e. T is Weyl too.

The proofs of (ii) and (iii) are similar to (i).

(iv) Whenever either the left hand side or the right hand side in the equality $\mathrm{ind}(T-\lambda) = \mathrm{ind}(A-\lambda) + \mathrm{ind}(B-\lambda)$ is finite, then $\sigma_w(A) \cup \sigma_w(B) = \sigma_w(T) \cup \{\sigma_w(A) \cap \sigma_w(B)\}$: this follows from the following implications.

$$\begin{aligned} & \lambda \notin \sigma_w(A) \cup \sigma_w(B) \\ \iff & A-\lambda \text{ and } B-\lambda \text{ are Weyl} \\ \iff & T-\lambda \text{ and } A-\lambda \text{ , or, } T-\lambda \text{ and } B-\lambda \text{ are Weyl} \\ \iff & \lambda \notin \sigma_w(T) \cup \{\sigma_w(A) \cap \sigma_w(B)\}. \end{aligned}$$

The proofs of (v) and (vi) are similar to (iv). □

References

[1] B.A. Barnes, *Spectral and spectral theory involving the diagonal of bounded linear operator,* Acta Math. (Szeged), **73** (2007), 237–250.

[2] M. Berkani, *Index of B-Fredholm operators and generalization of the Weyl theorem*, Proc.Amer.Math.Soc **130** (2002), 1717–1723

[3] M. Berkani and M. Sarih, *An Atkinson-type theorem for B-Fredholm operators*, Studia Math. **148** (2001), 251-257.

[4] X.H. Cao, M.Z. Guo and B. Meng, *Semi-Fredholm spectrum and Weyl's theorem for operator matrices*, Acta Math. Sin. (Engl. Ser.) **22** (2006), no. 1, 169–178.

[5] D.S. Djordjević, *Perturbation of spectra of operator matrices*, J. Operator Theory **48** (2002), 467–486.

[6] S. Djordjević and Y.M. Han, *Operator matrices and spectral continuity*, Glasgow Math. J. **43** (2001), 487–490.

[7] S. Djordjević and Y.M. Han, *a-Weyl's theorem for operator matrices*, Proc. Amer. Math. Soc. **130** (2002), 715–722.

[8] S. Djordjević and Y.M. Han, *A note on Weyl's theorem for operator matrices*, Proc. Amer. Math. Soc. **131** (2002), 2543–2547.

[9] S.V. Djordjevic and B.P. Duggal, *Spectral properties of linear operators*, submitted

[10] K.B. Laursen and M.M. Neumann, An Intruduction to Local Spectra Theory, London Mathematical Society Monographs, New Series 20, Clarendon Press, Oxford 2000.

[11] W.Y. Lee, *Weyl spectra of operator matrices*, Integ. Eq. Op.Th. **32** (1998), 319–331.

[12] J.K. Han, H.Y. Lee and W.Y. Lee, *Invertible completions of* 2×2 *upper triangular operator matrices*, Proc. Amer. Math. Soc. **128** (1999), 119–123.

[13] I.H. Kim, *On* (p, k)*-quasihyponormal operators*, Math. Ineq. Appl. **49** (2004), 629–63.

[14] R. Harte and C. Stack, *Separation of spectra for block triangles*, Proc. Amer. Math. Soc. **136** (2008), 3159–3162.

S. V. Djordjević : Facultad de Ciencias Físico-Matemáticas, BUAP, Río Verde y Av San Claudio, San Manuel, Puebla, Pue. 72570, Mexico.

E-mail address: slavdj@fcfm.buap.mx