数理解析研究所講究録
第 1626 巻 2009 年 93-98

Harmonic Univalent Functions for Which Analytic Part is Starlike

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Abstract

In a simply connected domain $\mathcal{U} \subset \mathbb{C}$ a complex-valued harmonic function f has the representation $f = h + \bar{g}$, where h and g are analytic function in $\mathcal{U}$, and are called analytic part and co-analytic part of f, respectively.

Let $h(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ and $g(z) = b_0 + b_1 z + b_2 z^2 + \cdots$ be analytic functions in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$, if $(|h'(z)|^2 - |g'(z)|^2) > 0$ or $(|h'(z)|^2 - |g'(z)|^2) < 0$ then f is called a sense-preserving harmonic function in $\mathbb{D}$. The class of all sense-preserving harmonic functions in $\mathbb{D}$ with $a_0 = b_0 = 0$, and $a_1 = 1$ will be denote by $\mathcal{S_H}$. Thus $\mathcal{S_H}$ contains the standard class of analytic univalent functions $\mathcal{S}$.

The aim of this paper is to investigate the subclass of $\mathcal{S_H}$. By choosing only these functions whose analytic parts are starlike functions for such mappings we will find distortion theorems.

1 Introduction

Let $h(z) = z + a_2 z^2 + \cdots$ be an analytic function in the open unit disc $\mathbb{D}$, if $h(z)$ satisfies the condition

$$Re\left(z\frac{h'(z)}{h(z)}\right) > 0, \tag{1.1}$$

then $h(z)$ is called starlike function in $\mathbb{D}$, and the class of starlike functions in $\mathbb{D}$ is denoted by $\mathcal{S}^*$ [3].

2000 *Mathematics Subject Classification:* Primary 30C45.
Key words and phrases: harmonic univalent functions, starlike functions, sense-preserving.

Next Ω be the family of functions $\phi(z)$ which are regular and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for every $z \in \mathbb{D}$, and let $\Omega(a)$ be the class of functions $w(z)$ which are analytic in $\mathbb{D}$ and satisfying the conditions $w(0) = a$, $|w(z)| < 1$ for all $z \in \mathbb{D}$. We note that $\Omega_{\mathcal{U}}(a)$ be the union of all classes $\Omega(a)$ whereas a ranges over $[0, 1)$ [4].

Moreover, a function f is said to be a complex-valued harmonic function in $\mathbb{D}$ if both Ref and Imf are real harmonic in $\mathbb{D}$. Every such f can be uniquely represented by

$$f = h + \bar{g}, \tag{1.2}$$

where h and g are analytic in $\mathbb{D}$ with $g(0) = 0$. A complex valued harmonic function f not identically constant, satisfying (1.2) is said to be sense-preserving in $\mathbb{D}$ if and only if satisfying the equation

$$g'(z) = w(z)h'(z), \tag{1.3}$$

where $w(z)$ is analytic in $\mathbb{D}$ with $|w(z)| < 1$ for every $z \in \mathbb{D}$ [2], and the function $w(z)$ is called the second dilatation of f. It is closely related to the Jacobian of f is defined by

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2. \tag{1.4}$$

Finally, let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=2}^{\infty} b_n z^n$ be analytic functions in $\mathbb{D}$. Choose $g(0) = 0$, i.e, $b_0 = 0$, so the representation (1.2) is unique in $\mathbb{D}$ and is called the canonical representation of f. For univalent and sense-preserving harmonic functions f in $\mathbb{D}$, it is convenient to make further normalization (with no loss generality) $h(0) = 0$, i.e, $a_0 = 0$ and $h'(0) = 1$, i.e, $a_1 = 1$. The family of all such functions f is denoted by $\mathcal{S}_{\mathcal{H}}$ [1], [2], [5].

In this paper we will study on the subclass of $\mathcal{S}_{\mathcal{H}}$ consisting of all univalent harmonic functions for which analytic part is starlike, and this class will be denoted by $\mathcal{S}^*_{\mathcal{H}}$.

2 Main Results

Lemma 2.1. *Let $w(z)$ be element of $\Omega(a)$, then*

$$\frac{|a-r|}{1-ar} \le |w(z)| \le \frac{a+r}{1+ar} \tag{2.1}$$

for all $z \in \mathbb{D}$, and $0 \le a < 1$.

Proof. Let $w(z) \in \Omega(a)$, then $w(z)$ is analytic in $\mathbb{D}$ and satisfies the condition $w(0) = a$. Now we consider the function

$$\phi(z) = \frac{w(z) - w(0)}{1 - \overline{w(0)}w(z)}, \quad z \in \mathbb{D}. \tag{2.2}$$

Therefore $\phi(z)$ satisfies the conditions of Schwarz lemma. Using the estimate the Schwarz lemma $|\phi(z)| \le |z| = r$, which gives

$$|\phi(z)| = \left|\frac{w(z) - w(0)}{1 - \overline{w(0)}w(z)}\right| \le r. \tag{2.3}$$

The inequality (2.3) can be written in the following form

$$\left|\frac{w(z) - a}{1 - aw(z)}\right| \le r \Leftrightarrow |w(z) - a| \le r|aw(z) - 1|. \tag{2.4}$$

The inequality (2.4) is equivalent

$$\left|w(z) - \frac{a(1-r^2)}{1-a^2r^2}\right| \le \frac{r(1-a^2)}{1-a^2r^2}. \tag{2.5}$$

The equality holds in the inequality (2.5) only for the function

$$w(z) = e^{i\beta}\frac{e^{i\theta}z + a}{1 + ae^{i\theta}z}, \quad z \in \mathbb{D}.$$

From the inequality (2.5) we have

$$\begin{aligned} |w(z)| &\ge \left|\frac{a(1-r^2)}{1-a^2r^2} - \frac{r(1-a^2)}{1-a^2r^2}\right| = \frac{|a-r|}{1-ar}, \\ |w(z)| &\le \frac{a(1-r^2)}{1-a^2r^2} + \frac{r(1-a^2)}{1-a^2r^2} = \frac{a+r}{1-ar}. \end{aligned} \tag{2.6}$$

□

Corollary 2.2. *Let $f \in S^*_{\mathcal{H}}$, then*

$$\frac{(1-r)|a-r|}{(1+r)^3(1-ar)} \le |g'(z)| \le \frac{(1+r)(a+r)}{(1-r)^3(1+ar)}. \tag{2.7}$$

Proof. Let $f \in S^*_{\mathcal{H}}$, then

$$g'(z) = w(z)h'(z) \Leftrightarrow \frac{g'(z)}{h'(z)} = w(z). \tag{2.8}$$

Using lemma 2.1 we obtain

$$|h'(z)|\frac{|a-r|}{(1-ar)} \leq |g'(z)| \leq |h'(z)|\frac{a+r}{(1+ar)}. \tag{2.9}$$

On the other hand, since $h(z)$ is starlike, then $h(z)$ satisfies the following inequality

$$\frac{1-r}{(1+r)^3} \leq |h'(z)| \leq \frac{1+r}{(1-r)^3}. \tag{2.10}$$

Considering the inequalities (2.9) and (2.10) together we obtain (2.7). □

Theorem 2.3. *Let $f \in S^*_{\mathcal{H}}$, then*

$$|g(z)| \leq \log\left(\frac{1+ar}{1-r}\right)^{\left(\frac{a-1}{a+1}\right)^2} + \frac{a-3}{a+1}\cdot\frac{r}{1-r} + \frac{2r^2}{(1-r)^2}. \tag{2.11}$$

Proof. Applying the estimate (2.7) we have

$$|g(z)| = \left|\int_C g'(\zeta)d\zeta\right| \leq \int_C |g'(\zeta)||d\zeta| \leq \int_0^r \frac{(1+\rho)(a+\rho)}{(1-\rho)^3(1+a\rho)}d\rho.$$

Integrating, we obtain the estimate (2.11), where $C = [0, z]$ is Jordan arc. □

Theorem 2.4. *Let f be element of $S^*_{\mathcal{H}}$, then*

$$\log\left(\frac{1+r}{1+ar}\right)^{\left(\frac{1+a}{1-a}\right)^2} - \frac{2r[(1+a)r+2a]}{(1-a)(1+r)^2} \leq |f(z)|$$
$$\leq \frac{r(1+2r)}{(1-r)^2} + \frac{a-3}{a+1}\cdot\frac{r}{1-r} + \log\left(\frac{1+ar}{1-r}\right)^{\left(\frac{a-1}{a+1}\right)^2}.$$

Proof. Let $z \in \mathbb{D}$. We denote $|z| = r$ and $M(r) = \inf\{|f(z)||\zeta| = r\}$. Then $|f(z)| \geq M(r)$ and $\{w||w| \leq M(r)\} \subset f(\{\zeta||\zeta| < r\}) \subset f(\mathbb{D})$, hence there exist z_r satisfying $|z_r| = r$ such that $M(r) = |f(z_r)|$ and $\gamma(t) = tf(z_r)$,

$t \in [0,1]$. Therefore $f^{-1}(\gamma(t)) = \Gamma(t), t \in [0,1]$ is well defined Jordan arc, and

$$|f(z)| \geq \int_{f^{-1}(\gamma(t))} (|h'(\zeta)| - |g'(z)|)|d\zeta| = \int_{f^{-1}(\gamma(t))} (|h'(\zeta)| - |w(\zeta)||h'(z)|)|d\zeta|$$
$$= \int_{f^{-1}(\gamma(t))} |h'(\zeta)|(1 - |w'(z)|)|d\zeta| \geq \int_0^r \left(\frac{1-\rho}{(1+\rho)^3}\right)\left(1 - \frac{a+\rho}{1+a\rho}\right) d\zeta \Rightarrow$$

$$|f(\zeta)| \geq \log\left(\frac{1+r}{1+ar}\right)^{\left(\frac{1+a}{1-a}\right)^2} - \frac{2r[(1+a)r+2a]}{(1-a)(1+r)^2}.$$

On the other hand, we have

$$|f(z)| = |h(z) + \overline{g(z)}| \leq |h(z)| + |g(z)|,$$

then using the following inequality and theorem 2.3, we have

$$|f(z)| \leq \frac{r(1+2r)}{(1-r)^2} + \frac{a-3}{a+1} \cdot \frac{r}{1-r} + \log\left(\frac{1+ar}{1-r}\right)^{\left(\frac{a-1}{a+1}\right)^2}.$$

□

Theorem 2.5. *Let $f \in S^*_{\mathcal{H}}$, then*

$$\frac{(1-a^2)(1-r)}{(1+ar)^2(1+r)^5} \leq J_f(z) \leq \frac{(1-ar)^2 - |a-r|^2}{(1-ar)^2(1-r)^6}.$$

Proof. Using lemma 2.1 and the following relations,

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2,$$

$$g'(z) = w(z)h'(z),$$

and after the simple calculations we get

$$\frac{(1-a^2)(1-r)}{(1+ar)^2(1+r)^5} \leq |h'(z)|^2(1 - |w(z)|^2) \leq \frac{(1-ar)^2 - |a-r|^2}{(1-ar)^2(1-r)^6}.$$

□

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