Contributions to Discrete Mathematics

A NEIGHBORHOOD CONDITION FOR GRAPHS TO HAVE RESTRICTED FRACTIONAL (g, f)-FACTORS

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ABSTRACT. Let h be a function defined on $E(G)$ with $h(e) \in [0, 1]$ for any $e \in E(G)$. Set $d_G^h(x) = \sum_{e \ni x} h(e)$. If $g(x) \leq d_G^h(x) \leq f(x)$ for every $x \in V(G)$, then we call the graph F_h with vertex set $V(G)$ and edge set E_h a fractional (g, f)-factor of G with indicator function h, where $E_h = \{e : e \in E(G), h(e) > 0\}$. Let M and N be two sets of independent edges of G with $M \cap N = \emptyset$, $|M| = m$ and $|N| = n$. If G admits a fractional (g, f)-factor F_h such that $h(e) = 1$ for any $e \in M$ and $h(e) = 0$ for any $e \in N$, then we say that G has a fractional (g, f)-factor with the property $E(m, n)$. In this paper, we present a neighborhood condition for the existence of a fractional (g, f)-factor with the property $E(1, n)$ in a graph. Furthermore, it is shown that the neighborhood condition is sharp.

1. INTRODUCTION

The graphs considered here will be finite and undirected simple graphs. Let G be a graph. We denote by $V(G)$ the vertex set of G and by $E(G)$ the edge set of G. For a vertex x of G, we use $d_G(x)$ to denote the degree of x in G and use $N_G(x)$ to denote the neighborhood of x in G. For a vertex subset X of G, we denote by $G[X]$ the subgraph of G induced by X, and write $G - X = G[V(G) \setminus X]$ and $N_G(X) = \bigcup_{x \in X} N_G(x)$. If $G[X]$ does not admit edges, then we call X an independent set of G. For $E' \subseteq E(G)$, the graph obtained from G by deleting edges of E' is denoted by $G - E'$ The minimum degree of G is denoted by $\delta(G)$. Let c be a real number. Recall that $\lfloor c \rfloor$ is the greatest integer with $\lfloor c \rfloor \leq c$.

Let g and f be two integer-valued functions defined on $V(G)$ with $0 \leq g(x) \leq f(x)$ for every $x \in V(G)$. A (g, f)-factor of a graph G is defined as a spanning subgraph F of G satisfying $g(x) \leq d_F(x) \leq f(x)$ for any

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$x \in V(G)$. A (g, f)-factor is called an $[a, b]$-factor if $g(x) \equiv a$ and $f(x) \equiv b$. A $[k, k]$-factor is simply called a k-factor.

Let h be a function defined on $E(G)$ with $h(e) \in [0, 1]$ for any $e \in E(G)$. Set $d_G^h(x) = \sum_{e \ni x} h(e)$. If $g(x) \le d_G^h(x) \le f(x)$ for every $x \in V(G)$, then we call the graph F_h with vertex set $V(G)$ and edge set E_h a fractional (g, f)-factor of G with indicator function h, where $E_h = \{e : e \in E(G), h(e) > 0\}$. A fractional (g, f)-factor is called a fractional f-factor if $g(x) = f(x)$ for each $x \in V(G)$.

Let M and N be two sets of independent edges of G with $M \cap N = \emptyset$, $|M| = m$, and $|N| = n$. If G admits a fractional (g, f)-factor F_h such that $h(e) = 1$ for any $e \in M$ and $h(e) = 0$ for any $e \in N$, then we say that G has a fractional (g, f)-factor with the property $E(m, n)$. A fractional (g, f)-factor with the property $E(m, n)$ is called a fractional f-factor with the property $E(m, n)$ if $g(x) \equiv f(x)$. Similarly, we may define a (g, f)-factor with the property $E(m, n)$ of G and an f-factor with the property $E(m, n)$ of G.

Kano [5] showed a neighborhood condition for a graph to admit an $[a, b]$-factor. Zhou [14] improved and generalized Kano's result, and proved a theorem that is generally stronger than Kano's result. Porteous and Aldred [11] first introduced the concept of 1-factors with the property $E(m, n)$, and obtained some results on the existence of 1-factors with the property $E(m, n)$ in graphs. Plummer and Saito [10] presented a binding number condition for the existence of 1-factors with the property $E(m, n)$ in graphs, and put forward a toughness condition for the existence of 1-factors with the property $E(m, n)$ in graphs. Zhou [17] and Zhou, Sun, and Pan [21] obtained two sufficient conditions for a graph to admit a fractional (g, f)-factor with the property $E(1, n)$. More results on factors and fractional factors in graphs can be found in Piummer [9], Zhou and Sun [19, 20], Wang and Zhang [12], Lv [8], Zhou [16, 18, 15, 13], Cai, Wang and Yan [1], Zhou, Yang and Xu [24], Zhou, Zhang and Xu [25], Gao et al. [2, 3, 4], Zhou, Xu and Sun [23], Zhou, Sun and Ye [22], and Liu and Lu [7].

In this paper, we proceed to study fractional (g, f)-factors with the property $E(m, n)$, and show a neighborhood condition that guarantees a graph admitting a fractional (g, f)-factor with the property $E(1, n)$.

Theorem 1.1. *Let $r \ge 0$, $n \ge 0$ and $2 \le a \le b - r$ be four integers, let G be a graph of order p with*

$$p \ge \frac{(a + b - 1)(a + 2b - r - 4) + a + r + 2}{a + r} + \frac{2n}{a + r - 1},$$

and let g, f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) - r \le b - r$ for every $x \in V(G)$. Suppose for any subset $X \subset V(G)$, we have

$$N_G(X) = V(G), \ if \ |X| \ge \left\lfloor \frac{((a + r)(p - 1) - 2n - 2)p}{(a + b - 1)(p - 1)} \right\rfloor;$$

or

$$|N_G(X)| \geq \frac{(a+b-1)(p-1)}{(a+r)(p-1)-2n-2}|X|, \ \ if$$

$$|X| < \left\lfloor \frac{((a+r)(p-1)-2n-2)p}{(a+b-1)(p-1)} \right\rfloor.$$

Then G contains a fractional (g,f)-factor with the property $E(1,n)$.

If $n = 0$ in Theorem 1.1, then we have the following corollary.

Corollary 1.2. *Let $r \geq 0$ and $2 \leq a \leq b-r$ be three integers, let G be a graph of order p with*

$$p \geq \frac{(a+b-1)(a+2b-r-4)+a+r+2}{a+r},$$

and let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) - r \leq b - r$ for every $x \in V(G)$. Suppose for any subset $X \subset V(G)$, we have

$$N_G(X) = V(G), \ \ if \ |X| \geq \left\lfloor \frac{((a+r)(p-1)-2)p}{(a+b-1)(p-1)} \right\rfloor; \ \ or$$

$$|N_G(X)| \geq \frac{(a+b-1)(p-1)}{(a+r)(p-1)-2}|X|, \ \ if \ |X| < \left\lfloor \frac{((a+r)(p-1)-2)p}{(a+b-1)(p-1)} \right\rfloor.$$

Then G contains a fractional (g,f)-factor with the property $E(1,0)$.

If $n = 1$ in Theorem 1.1, then we have the following corollary.

Corollary 1.3. *Let $r \geq 0$ and $2 \leq a \leq b-r$ be three integers, let G be a graph of order p with*

$$p \geq \frac{(a+b-1)(a+2b-r-4)+a+r+2}{a+r} + \frac{2}{a+r-1},$$

and let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) - r \leq b - r$ for every $x \in V(G)$. Suppose for any subset $X \subset V(G)$, we have

$$N_G(X) = V(G), \ \ if \ |X| \geq \left\lfloor \frac{((a+r)(p-1)-4)p}{(a+b-1)(p-1)} \right\rfloor; \ \ or$$

$$|N_G(X)| \geq \frac{(a+b-1)(p-1)}{(a+r)(p-1)-4}|X|, \ \ if \ |X| < \left\lfloor \frac{((a+r)(p-1)-4)p}{(a+b-1)(p-1)} \right\rfloor.$$

Then G contains a fractional (g,f)-factor with the property $E(1,1)$.

If $r = 0$ in Theorem 1.1, then we obtain the following corollary.

Corollary 1.4. *Let $n \geq 0$ and $2 \leq a \leq b$ be three integers, let G be a graph of order p with*

$$p \geq \frac{(a+b-1)(a+2b-4)+a+2}{a} + \frac{2n}{a-1},$$

and let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for every $x \in V(G)$. Suppose for any subset $X \subset V(G)$, we have

$$N_G(X) = V(G), \ \ if \ |X| \geq \left\lfloor \frac{(a(p-1) - 2n - 2)p}{(a+b-1)(p-1)} \right\rfloor; \ \ or$$

$$|N_G(X)| \geq \frac{(a+b-1)(p-1)}{a(p-1) - 2n - 2}|X|, \ \ if \ |X| < \left\lfloor \frac{(a(p-1) - 2n - 2)p}{(a+b-1)(p-1)} \right\rfloor.$$

Then G contains a fractional (g, f)-factor with the property $E(1, n)$.

2. THE PROOF OF THEOREM 1.1

The proof of Theorem 1.1 depends heavily on the following lemmas.

Lemma 2.1 (Li, Yan, and Zhang [6])**.** *Let G be a graph, and let g, f be two integer-valued functions defined on $V(G)$ such that $0 \leq g(x) \leq f(x)$ for every $x \in V(G)$. Then G has a fractional (g, f)-factor with the property $E(1, 0)$ if and only if*

$$\gamma_G(S, T) = f(S) + d_{G-S}(T) - g(T) \geq \varepsilon(S, T)$$

for any $S \subseteq V(G)$, where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq g(x)\}$ and $\varepsilon(S, T)$ is defined as follows:

$$\varepsilon(S, T) = \begin{cases} 2, & \text{if } S \text{ is not independent,} \\ 1, & \text{if } S \text{ is independent and there is an edge joining } S \\ & \text{and } V(G) \setminus (S \cup T), \text{ or there is an edge } e = uv \\ & \text{joining } S \text{ and } T \text{ such that } d_{G-S}(v) = g(v) \text{ for } v \in T, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.2. *Let G be a graph of order p that satisfies the hypothesis of Theorem 1.1. Then*

$$\delta(G) \geq \frac{(b - r - 1)p + a + r + 2n + 2}{a + b - 1}.$$

Proof. Let $v \in V(G)$ with degree $\delta(G)$. Let $Q = V(G) \setminus N_G(v)$. Obviously, $v \notin N_G(Q)$, that is, $N_G(Q) \neq V(G)$. Thus we obtain

$$\begin{aligned} (a + b - 1)(p - 1)|Q| &\leq & ((a + r)(p - 1) - 2n - 2)|N_G(Q)| \\ &\leq & ((a + r)(p - 1) - 2n - 2)(p - 1), \end{aligned}$$

which implies

$$(a + b - 1)|Q| \leq (a + r)(p - 1) - 2n - 2.$$

Note that $|Q| = p - \delta(G)$. Thus we have

$$(a + b - 1)(p - \delta(G)) \leq (a + r)(p - 1) - 2n - 2,$$

that is,

$$\delta(G) \geq \frac{(b - r - 1)p + a + r + 2n + 2}{a + b - 1}.$$

This finishes the proof of Lemma 2.2. □

Proof of Theorem 1.1. Suppose that a graph G satisfies the hypothesis of Theorem 1.1, but does not have a fractional (g, f)-factor with the property $E(1, n)$. Then there exist a set of independent edges $\{e_1, e_2, \ldots, e_n\}$ and an edge e of G such that G does not contain a fractional (g, f)-factor F_h with $h(e_i) = 0$ for $1 \leq i \leq n$ and $h(e) = 1$. Set $N = \{e_1, e_2, \ldots, e_n\}$ and $H = G - N$. Obviously, H does not have a fractional (g, f)-factor with the property $E(1, 0)$. By Lemma 2.1, there exists a subset $S \subseteq V(H)$ such that

$$\gamma_H(S, T) = f(S) + d_{H-S}(T) - g(T) \leq \varepsilon(S, T) - 1, \tag{1}$$

where $T = \{x : x \in V(H) \setminus S, d_{H-S}(x) \leq g(x)\}$. It is easy to see that $T \neq \emptyset$ by (1). Hence, we may define

$$h = \min\{d_{G-S}(x) : x \in T\}.$$

Claim 2.3. $0 \leq h \leq b - r + 1$.

Proof. Note that $N = \{e_1, e_2, \ldots, e_n\}$ is a set of independent edges of G and $H = G - N$. Combining these with the definition of T, we have

$$0 \leq d_{G-S}(x) \leq d_{H-S}(x) + 1 \leq g(x) + 1 \leq b - r + 1$$

for each $x \in T$. In terms of the definition of h, we get $0 \leq h \leq b - r + 1$. Claim 2.3 is proved. □

Claim 2.4. $d_{H-S}(T) \geq d_{G-S}(T) - \min\{2n, |T|\}$.

Proof. We write $D = V(G) \setminus (S \cup T)$ and $E_G(T) = \{e : e = xy \in E(G), x, y \in T\}$. It is obvious that $2|N \cap E_G(T)| + |N \cap E_G(T, D)| \leq \min\{2n, |T|\}$. Thus, we have

$$
\begin{aligned}
d_{H-S}(T) &= d_{G-N-S}(T) \\
&= d_{G-S}(T) - (2|N \cap E_G(T)| + |N \cap E_G(T, D)|) \\
&\geq d_{G-S}(T) - \min\{2n, |T|\}.
\end{aligned}
$$

Claim 2.4 is verified. □

We shall consider four cases.
Case 1: $h = 0$.

Set $\lambda = |\{x : x \in T, d_{G-S}(x) = 0\}|$. It is obvious that $\lambda \geq 1$ by $h = 0$. We write $X = V(G) \setminus S$. Clearly, $N_G(X) \neq V(G)$ since $\lambda \geq 1$. It follows from the hypothesis of Theorem 1.1 that

$$
\begin{aligned}
p - \lambda &\geq |N_G(X)| \geq \frac{(a + b - 1)(p - 1)}{(a + r)(p - 1) - 2n - 2}|X| \\
&= \frac{(a + b - 1)(p - 1)}{(a + r)(p - 1) - 2n - 2}(p - |S|),
\end{aligned}
$$

which implies

$$|S| \geq p - \frac{(p - \lambda)((a + r)(p - 1) - 2n - 2)}{(a + b - 1)(p - 1)}. \tag{2}$$

It follows from (1), (2),

$$p \geq \frac{(a + b - 1)(a + 2b - r - 4) + a + r + 2}{a + r} + \frac{2n}{a + r - 1},$$

$\lambda \geq 1$, $|S| + |T| \leq p$, and Claim 2.4 that

$$
\begin{aligned}
\varepsilon(S, T) - 1 \quad &\geq \quad \gamma_H(S, T) = f(S) + d_{H-S}(T) - g(T) \\
&\geq \quad f(S) + d_{G-S}(T) - \min\{2n, |T|\} - g(T) \\
&\geq \quad (a + r)|S| + |T| - \lambda - 2n - (b - r)|T| \\
&= \quad (a + r)|S| - (b - r - 1)|T| - \lambda - 2n \\
&\geq \quad (a + r)|S| - (b - r - 1)(p - |S|) - \lambda - 2n \\
&= \quad (a + b - 1)|S| - (b - r - 1)p - \lambda - 2n \\
&\geq \quad (a + b - 1)\left(p - \frac{(p - \lambda)((a + r)(p - 1) - 2n - 2)}{(a + b - 1)(p - 1)}\right) \\
&\quad -(b - r - 1)p - \lambda - 2n \\
&= \quad \frac{p(2n + 2)}{p - 1} + \left(\frac{(a + r)(p - 1) - 2n - 2}{p - 1} - 1\right)\lambda - 2n \\
&\geq \quad \frac{p(2n + 2)}{p - 1} + \frac{(a + r)(p - 1) - 2n - 2}{p - 1} - 1 - 2n \\
&= \quad a + r + 1 > 2 \geq \varepsilon(S, T),
\end{aligned}
$$

which is a contradiction.

Case 2: $h = 1$.

Subcase 2.1: $|T| > \left\lfloor \frac{((a+r)(p-1)-2n-2)p}{(a+b-1)(p-1)} \right\rfloor$.

It is easy to see that

$$|T| \geq \left\lfloor \frac{((a + r)(p - 1) - 2n - 2)p}{(a + b - 1)(p - 1)} \right\rfloor + 1. \tag{3}$$

Note that $h = 1$. Hence, there exists $v \in T$ with $d_{G-S}(v) = h = 1$, and so

$$v \notin N_G(T \setminus N_G(v)),$$

which implies

$$N_G(T \setminus N_G(v)) \neq V(G).$$

Combining this with $d_{G-S}(v) = h = 1$ and the hypothesis of Theorem 1.1, we get

$$|T| - 1 \leq |T \setminus N_G(v)| < \left\lfloor \frac{((a + r)(p - 1) - 2n - 2)p}{(a + b - 1)(p - 1)} \right\rfloor,$$

that is,

$$|T| < \left\lfloor \frac{((a+r)(p-1) - 2n - 2)p}{(a+b-1)(p-1)} \right\rfloor + 1,$$

which contradicts (3).

Subcase 2.2: $|T| \leq \left\lfloor \frac{((a+r)(p-1)-2n-2)p}{(a+b-1)(p-1)} \right\rfloor$.

In terms of $h = 1$ and Lemma 2.2, we obtain

$$
\begin{aligned}
|S| &\geq \delta(G) - 1 \geq \frac{(b-r-1)p + a + r + 2n + 2}{a+b-1} - 1 \\
&= \frac{(b-r-1)(p-1) + 2n + 2}{a+b-1},
\end{aligned}
$$

that is,

$$|S| \geq \frac{(b-r-1)(p-1) + 2n + 2}{a+b-1}. \tag{4}$$

Claim 2.5. $|T| \leq \frac{(a+r)(p-1)-2n-2}{a+b-1}$.

Proof. We assume that

$$|T| > \frac{(a+r)(p-1) - 2n - 2}{a+b-1}.$$

In light of (4), we have

$$|S| + |T| > \frac{(b-r-1)(p-1) + 2n + 2}{a+b-1} + \frac{(a+r)(p-1) - 2n - 2}{a+b-1} = p - 1.$$

On the other hand, $|S| + |T| \leq p$. Thus, we obtain

$$|S| + |T| = p. \tag{5}$$

According to (4), (5),

$$|T| \leq \left\lfloor \frac{((a+r)(p-1) - 2n - 2)p}{(a+b-1)(p-1)} \right\rfloor \leq \frac{((a+r)(p-1) - 2n - 2)p}{(a+b-1)(p-1)},$$

and Claim 2.4, we obtain

$$
\begin{aligned}
\gamma_H(S,T) &= f(S) + d_{H-S}(T) - g(T) \\
&\geq f(S) + d_{G-S}(T) - \min\{2n, |T|\} - g(T) \\
&\geq (a+r)|S| + |T| - 2n - (b-r)|T| \\
&= (a+r)|S| - (b-r-1)|T| - 2n \\
&= (a+r)(p - |T|) - (b-r-1)|T| - 2n \\
&= (a+r)p - (a+b-1)|T| - 2n \\
&\geq (a+r)p - (a+b-1) \cdot \frac{((a+r)(p-1) - 2n - 2)p}{(a+b-1)(p-1)} - 2n \\
&= (a+r)p - \left((a+r)p - \frac{p(2n+2)}{p-1} \right) - 2n
\end{aligned}
$$

$$= \frac{p(2n+2)}{p-1} - 2n$$
$$> 2n + 2 - 2n = 2 \geq \varepsilon(S, T),$$

which contradicts (1). Claim 2.5 is verified. $\qquad \square$

We write $\beta = |\{x : x \in T, \ d_{G-S}(x) = 1\}|$. It is obvious that $\beta \geq 1$ and $|T| \geq \beta$. In light of (4) and Claims 2.4–2.5, we have

$$
\begin{aligned}
\gamma_H(S, T) &= f(S) + d_{H-S}(T) - g(T) \\
&\geq f(S) + d_{G-S}(T) - \min\{2n, |T|\} - g(T) \\
&\geq (a+r)|S| + 2|T| - \beta - 2n - (b-r)|T| \\
&= (a+r)|S| - (b-r-2)|T| - \beta - 2n \\
&\geq (a+r) \cdot \frac{(b-r-1)(p-1) + 2n + 2}{a+b-1} \\
&\quad -(b-r-2) \cdot \frac{(a+r)(p-1) - 2n - 2}{a+b-1} - \beta - 2n \\
&= \frac{(a+r)(p-1) + (a+b-2)(2n+2)}{a+b-1} - \beta - 2n \\
&= \frac{(a+r)(p-1) - 2n - 2}{a+b-1} + 2n + 2 - \beta - 2n \\
&\geq |T| - \beta + 2 \geq 2 \geq \varepsilon(S, T),
\end{aligned}
$$

which contradicts (1).

Case 3: $2 \leq h \leq b - r$.

Note that $h = \min\{d_{G-S}(x) : x \in T\}$. Then there exists $x_1 \in T$ such that $d_{G-S}(x_1) = h$. Hence, we obtain

$$\delta(G) \leq d_G(x_1) \leq d_{G-S}(x_1) + |S| = h + |S|.$$

Combining this with Lemma 2.2, we have

$$|S| \geq \delta(G) - h \geq \frac{(b-r-1)p + a + r + 2n + 2}{a+b-1} - h. \qquad (6)$$

In terms of (6), $|S| + |T| \leq p$, and Claim 2.4, we have

$$
\begin{aligned}
\gamma_H(S, T) &= f(S) + d_{H-S}(T) - g(T) \\
&\geq f(S) + d_{G-S}(T) - \min\{2n, |T|\} - g(T) \\
&\geq (a+r)|S| + h|T| - 2n - (b-r)|T| \\
&= (a+r)|S| - (b-r-h)|T| - 2n \\
&\geq (a+r)|S| - (b-r-h)(p - |S|) - 2n \\
&= (a+b-h)|S| - (b-r-h)p - 2n \\
&\geq (a+b-h) \cdot \left(\frac{(b-r-1)p + a + r + 2n + 2}{a+b-1} - h \right) \\
&\quad -(b-r-h)p - 2n.
\end{aligned}
$$

Let

$$\varphi(h) = (a + b - h) \cdot \left(\frac{(b - r - 1)p + a + r + 2n + 2}{a + b - 1} - h \right) - (b - r - h)p - 2n.$$

Then we have

$$\gamma_H(S, T) \geq \varphi(h). \tag{7}$$

By $2 \leq h \leq b - r$ and

$$p \geq \frac{(a + b - 1)(a + 2b - r - 4) + a + r + 2}{a + r} + \frac{2n}{a + r - 1},$$

we get

$$
\begin{aligned}
\varphi'(h) &= -\left(\frac{(b - r - 1)p + a + r + 2n + 2}{a + b - 1} - h \right) - (a + b - h) + p \\
&= \frac{(a + r)p - a - r - 2n - 2}{a + b - 1} + 2h - (a + b) \\
&\geq a + 2b - r - 4 + 4 - (a + b) \\
&= b - r \geq a > 0,
\end{aligned}
$$

and so, $\varphi(h)$ attains its minimum value at $h = 2$ by $2 \leq h \leq b - r$. Combining this with (7) and

$$p \geq \frac{(a + b - 1)(a + 2b - r - 4) + a + r + 2}{a + r} + \frac{2n}{a + r - 1},$$

we obtain

$$
\begin{aligned}
\gamma_H(S, T) \geq \varphi(h) &\geq \varphi(2) \\
&= (a + b - 2) \cdot \left(\frac{(b - r - 1)p + a + r + 2n + 2}{a + b - 1} - 2 \right) \\
&\quad - (b - r - 2)p - 2n \\
&= \frac{(a + r)p}{a + b - 1} + \frac{(a + b - 2)(a + r + 2n + 2)}{a + b - 1} \\
&\quad - 2(a + b - 2) - 2n \\
&\geq \frac{(a + b - 1)(a + 2b - r - 4) + a + r + 2n + 2}{a + b - 1} \\
&\quad + \frac{(a + b - 2)(a + r + 2n + 2)}{a + b - 1} - 2(a + b - 2) - 2n \\
&= 2 \geq \varepsilon(S, T),
\end{aligned}
$$

which contradicts (1).

Case 4: $h = b - r + 1$.

It follows from (1) and Claim 2.4 that

$$
\begin{aligned}
\varepsilon(S, T) - 1 \geq \gamma_H(S, T) &= f(S) + d_{H-S}(T) - g(T) \\
&\geq f(S) + d_{G-S}(T) - \min\{2n, |T|\} - g(T) \\
&\geq (a+r)|S| + h|T| - |T| - (b-r)|T| \\
&= (a+r)|S| - (b-r+1-h)|T| \\
&= (a+r)|S| \geq |S| \geq \varepsilon(S, T),
\end{aligned}
$$

which is a contradiction. This completes the proof of Theorem 1.1. $\qquad \square$

3. Remark

In this section, we claim that the assumption on the neighborhood in Theorem 1.1 is the best possible, which cannot be replaced by $N_G(X) = V(G)$ or

$$
|N_G(X)| \geq \frac{(a+b-1)(p-1)}{(a+r)(p-1) - 2n - 2}|X|
$$

for any $X \subseteq V(G)$.

Let a, b, r, n be four nonnegative integers such that $2 \leq a = b - r$, b is odd and $(2n+2)/b$ is an integer. We construct a graph

$$
G = K_{(a-1)m + \frac{2n+2}{b}} \vee \left(\frac{bm+1}{2} K_2 \right)
$$

of order p, where m is an enough large positive integer, m is odd, and $\vee$ means "join". It is easy to see that $p = (a-1)m + bm + 1 + ((2n+2)/b)$. We write

$$
S = V\left(K_{(a-1)m + \frac{2n+2}{b}} \right)
$$

and

$$
T = V\left(\frac{bm+1}{2} K_2 \right).
$$

We first show that the assumption $N_G(X) = V(G)$ or

$$
|N_G(X)| \geq \frac{(a+b-1)(p-1)}{(a+r)(p-1) - 2n - 2}|X|
$$

for any $X \subseteq V(G)$ holds. Let any $X \subseteq V(G)$. It is easy to see that if $|X \cap S| \geq 2$, or $|X \cap S| = 1$ and $|X \cap T| \geq 1$, then $N_G(X) = V(G)$. Of course, if $|X| = 1$ and $X \subseteq S$, then we easily obtain

$$
|N_G(X)| = |V(G)| - 1 = p - 1 > \frac{(a+b-1)(p-1)}{(a+r)(p-1) - 2n - 2}|X|.
$$

Therefore, we may assume that $X \subseteq T$. Note that

$$
|N_G(X)| = |S| + |X| = (a-1)m + \frac{2n+2}{b} + |X|.
$$

Hence,

$$
|N_G(X)| \geq \frac{(a+b-1)(p-1)}{(a+r)(p-1) - 2n - 2}|X|
$$

holds if and only if

$$(a-1)m + \frac{2n+2}{b} + |X| \geq \frac{(a+b-1)(p-1)}{(a+r)(p-1)-2n-2}|X|.$$

This inequality is equivalent to $|X| \leq bm$. Thus if $X \neq T$ and $X \subset T$, then we have

$$|N_G(X)| \geq \frac{(a+b-1)(p-1)}{(a+r)(p-1)-2n-2}|X|$$

for any $X \subset V(G)$. If $X = T$, then it is obvious that $N_G(X) = V(G)$. Consequently, $N_G(X) = V(G)$ or

$$|N_G(X)| \geq \frac{(a+b-1)(p-1)}{(a+r)(p-1)-2n-2}|X|$$

for any $X \subseteq V(G)$ holds.

Let g, f be two integer-valued functions defined on $V(G)$ with $g(x) = a$ and $f(x) = b = a + r$ for every $x \in V(G)$. Let $N = \{e_1, e_2, \ldots, e_n\} \subseteq E(\frac{bm+1}{2}K_2)$ be a set of independent edges of G. We write $H = G - N$. Next, we show that H does not have a fractional (g, f)-factor with the property $E(1, 0)$. For above S and T, we have $|S| = (a-1)m + ((2n+2)/b)$, $|T| = bm + 1$, $d_{H-S}(T) = bm + 1 - 2n$ and $\varepsilon(S, T) = 2$. Thus, we have

$$\begin{aligned}
\gamma_H(S, T) &= f(S) + d_{H-S}(T) - g(T) \\
&= b|S| + d_{H-S}(T) - a|T| \\
&= b \cdot \left((a-1)m + \frac{2n+2}{b}\right) + bm + 1 - 2n - a \cdot (bm+1) \\
&= 3 - a \leq 1 < 2 = \varepsilon(S, T).
\end{aligned}$$

In light of Lemma 2.1, H does not admit a fractional (g, f)-factor with the property $E(1, 0)$, and so, G does not admit a fractional (g, f)-factor with the property $E(1, n)$.

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