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Classical and quantum behavior of the integrated density of states for a randomly perturbed lattice

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Abstract. The asymptotic behavior of the integrated density of states for a randomly perturbed lattice at the infimum of the spectrum is investigated. The leading term is determined when the decay of the single site potential is slow. The leading term depends only on the classical effect from the scalar potential. To the contrary, the quantum effect appears when the decay of the single site potential is fast. The corresponding leading term is estimated and the leading order is determined. In the multidimensional cases, the leading order varies in different ways from the known results in the Poisson case. The same problem is considered for the negative potential. These estimates are applied to investigate the long time asymptotics of Wiener integrals associated with the random potentials.

1. Introduction

In this paper, we are concerned with the self-adjoint operator in the form of

$$H_\xi = -h\Delta + \sum_{q\in\mathbb{Z}^d} u(\,\cdot - q - \xi_q) \tag{1.1}$$

defined on the L^2-space on $\mathbb{R}^d \setminus \bigcup_{q\in\mathbb{Z}^d}(q+\xi_q+K)$ with the Dirichlet boundary condition, where h is a positive constant and K is a compact set in $\mathbb{R}^d$ allowed to be empty. Our assumptions on the potential term are the following: (i) $\xi = (\xi_q)_{q\in\mathbb{Z}^d}$ is a collection of independent and identically distributed $\mathbb{R}^d$-valued random variables with

$$\mathbb{P}_\theta(\xi_q \in dx) = \exp(-|x|^\theta)dx/Z(d,\theta) \tag{1.2}$$

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for some $\theta > 0$ and the normalizing constant $Z(d, \theta)$; (ii) u is a nonnegative function belonging to the Kato class K_d (cf. [3] p-53) and satisfying

$$u(x) = C_0 |x|^{-\alpha}(1 + o(1)) \tag{1.3}$$

as $|x| \to \infty$ for some $\alpha > d$ and $C_0 > 0$.

Although we assume the equality in (1.2), it will be easily seen from the proofs that only the asymptotic relation

$$\mathbb{P}_\theta(\xi_q \in x + [0,1]^d) \asymp \exp(-|x|^\theta)$$

is essential for our theory, where $f(x) \asymp g(x)$ means $0 < \underline{\lim}_{|x|\to\infty} f(x)/g(x) \le \overline{\lim}_{|x|\to\infty} f(x)/g(x) < \infty$. In particular, we may replace $|x|^\theta$ by $(1 + |x|)^\theta$ in (1.2). Then the point process $\{q + \xi_q\}_{q\in\mathbb{Z}^d}$ converges weakly to the complete lattice $\mathbb{Z}^d$ as $\theta \to \infty$. Moreover, it is shown in Appendix A of [6] that this point process converges weakly to the Poisson point process with the intensity 1 as $\theta \downarrow 0$. Since the Poisson point process is usually regarded as a completely disordered configuration, our model gives an interpolation between complete lattice and completely disordered media.

We will consider the integrated density of states $N(\lambda)$ ($\lambda \in \mathbb{R}$) of H_ξ defined by the thermodynamic limit

$$\frac{1}{|\Lambda_R|} N_{\xi, \Lambda_R}(\lambda) \longrightarrow N(\lambda) \qquad \text{as } R \to \infty. \tag{1.4}$$

In (1.4) we denote by Λ_R a box $(-R/2, R/2)^d$ and by $N_{\xi, \Lambda_R}(\lambda)$ the number of eigenvalues not exceeding λ of the self-adjoint operator $H^D_{\xi,R}$ defined by restricting H_ξ to $\Lambda_R \setminus \bigcup_{q\in\mathbb{Z}^d}(q + \xi_q + K)$ with the Dirichlet boundary condition. We here note that the potential term in (1.1) belongs to the local Kato class $K_{d,loc}$ (cf. [3] p-53) as we will show in Section 7 below. It is then well known that the above limit exists for almost every ξ and defines a deterministic increasing function $N(\lambda)$ (cf. [3], [11]).

The following are first two main results in this paper.

Theorem 1.1. *If $d < \alpha \le d + 2$ and*

$$\operatorname{ess\,inf}_{|x|\le R} u(x) \textit{ is positive for any } R \ge 1, \tag{1.5}$$

then we have

$$\log N(\lambda) \asymp -\lambda^{-\kappa}, \tag{1.6}$$

where $\kappa = (d + \theta)/(\alpha - d)$, and $f(\lambda) \asymp g(\lambda)$ means $0 < \underline{\lim}_{\lambda\downarrow 0} f(\lambda)/g(\lambda) \le \overline{\lim}_{\lambda\downarrow 0} f(\lambda)/g(\lambda) < \infty$. Moreover if $\alpha < d + 2$, then we have

$$\lim_{\lambda\downarrow 0} \lambda^\kappa \log N(\lambda) = \frac{-\kappa^\kappa}{(\kappa+1)^{\kappa+1}} \left\{ \int_{\mathbb{R}^d} dq \inf_{y\in\mathbb{R}^d} \left(\frac{C_0}{|q + y|^\alpha} + |y|^\theta \right) \right\}^{\kappa+1}, \tag{1.7}$$

where the right hand side is finite by the assumption $\alpha > d$.

Theorem 1.2. *If $d = 1$ and $\alpha > 3$, then we have*

$$\lim_{\lambda \downarrow 0} \lambda^{(1+\theta)/2} \log N(\lambda) = -\frac{\pi^{1+\theta} h^{(1+\theta)/2}}{(1+\theta)2^\theta}. \tag{1.8}$$

If $d = 2$ and $\alpha > 4$, then we have

$$\log N(\lambda) \asymp -\lambda^{-1-\theta/2} \left(\log \frac{1}{\lambda} \right)^{-\theta/2}. \tag{1.9}$$

If $d \geq 3$ and $\alpha > d + 2$, then we have

$$\log N(\lambda) \asymp -\lambda^{-(d+\mu\theta)/2}, \tag{1.10}$$

where $\mu = 2(\alpha - 2)/(d(\alpha - d))$.

These results are generalizations of Corollary 3.1 in [6] to the case that $\mathrm{supp}(u)$ is not compact (cf. Theorem 3.11 below). The results in Theorem 1.1 are independent of the constant h. In fact these asymptotics coincide with those of the corresponding classical integrated density of states defined by

$$N_c(\lambda) = \mathbb{E}_\theta[|\{(x,p) \in \Lambda_R \times \mathbb{R}^d : H_{\xi,c}(x,p) \leq \lambda\}|](2\pi\sqrt{h}R)^{-d}$$

for any $R \in \mathbb{N}$, where $|\cdot|$ is the $2d$-dimensional Lebesgue measure and

$$H_{\xi,c}(x,p) = \sum_{j=1}^d p_j^2 + V_\xi(x)$$

is the classical Hamiltonian (cf. [16]). Therefore we may say that only the classical effect from the scalar potential determines the leading term for $\alpha < d+2$ and the leading order for $\alpha \leq d + 2$. To the contrary, the right hand side of (1.8) depends on h and the right hand sides of (1.9) and (1.10) are strictly less than that of (1.6). Therefore we may say that the quantum effect appears in Theorem 1.2. We here note that the right hand side of (1.6) gives an upper bound and the asymptotics of the classical counterpart not only for $\alpha \leq d + 2$ but also for $\alpha > d + 2$ (see Proposition 2.1 below). For the critical case $\alpha = d+2$, the quantum effect appears at least in some cases. We shall elaborate on this in Section 4 below.

In our model, the single site potentials are randomly displaced from the lattice. As is mentioned in [6], such a model describes the Frenkel disorder in solid state physics and is called the random displacement model in the theory of random Schrödinger operator. Despite of the appropriateness of this model in physics, there are only a few mathematical studies and in particular the displacements have been assumed to be bounded in almost all works. For that case, Kirsch and Martinelli [12] discussed the existence of band gaps and Klopp [14] proved spectral localization in a semi-classical limit. More recently, Baker, Loss and Stolz [1], [2] studied which configuration minimizes the spectrum of (1.1) and also showed that the corresponding integrated density of states increases rapidly at the minimum in a one-dimensional example. On the other hand, our displacements are unbounded. Then the infimum of the spectrum is easily shown to be 0 opposed to the bounded cases. This is an essential condition for our method, by which we investigate the behavior of $N(\lambda)$ at $\lambda = 0$. All our results show that $N(\lambda)$ increases slowly.

In a slightly broader class of models where the potentials are randomly located, the most studied model is the Poisson model, where the random points $(q+\xi_q)_{q\in\mathbb{Z}^d}$ are replaced by the sample points of the Poisson random measure (cf. [3], [20]). In the limit of $\theta \downarrow 0$, the above results coincide with the corresponding results for the Poisson model obtained by Pastur [21], Lifshitz [17], Donsker and Varadhan [4], Nakao [18], and Ôkura [19]. As in the Poisson model, the critical value is always $\alpha = d + 2$ and, in the one-dimensional case, the leading order increases continuously as α increases to $d + 2$ and does not depend on $\alpha \geq d + 2$. However in contrast to the Poisson case, the leading order jumps at $\alpha = d + 2$ for $d = 2$, and it depends on $\alpha \geq d+2$ for $d \geq 3$. These phenomena are due to the fact that the effect from states which have many tiny holes including $\{q + \xi_q\}_q$ in their supports appears in the leading term of the asymptotics, as observed in [6]. This is a characteristic difference with the Poisson case. On the other hand, the decay rates of $N(\lambda)$ explode in the limit $\theta \to \infty$. This reflects the fact that the infimum of the spectrum is positive in the case of a finitely perturbed lattice including the case of the unperturbed lattice.

On the subjects of this paper, we have more results for the alloy type model

$$H_\omega = -h\Delta + \sum_{q\in\mathbb{Z}^d} \omega_q u(x-q)$$

and the same critical value $\alpha = d+2$ is obtained, where $\omega = (\omega_q)_{q\in\mathbb{Z}^d}$ is a collection of independent and identically distributed nonnegative real valued random variables. As for the results, further developments and the relation with other models, refer to a recent survey by Kirsch and Metzger [13].

Our proof of Theorem 1.1 is an extension of that of the corresponding result for the Poisson case (cf. [21], [20]). For the proof of the multidimensional results in Theorem 1.2, we use a method based on a functional analytic approach (cf. [3], [11]). This is different from the method in [6], where a coarse graining method following Sznitman [24] is applied. The method employed here can also be used to give a simpler proof of the results in the compact case in [6]. We will present it in Section 3 below. For the 1-dimensional result, we use a simple effective estimate of the first eigenvalue in [24].

As an application, we study the survival probability of the Brownian motion in a random environment. This was the main motivation in [6]. We recall the connection between this and the integrated density of states, and extend the theory to the present settings. For the results, see Theorem 6.3 below. In the proof, we take the *hard obstacles* K appropriately so that the local singularity of the potential u does not bring difficulty. This is our only motivation to introduce the hard obstacles, and the hard obstacles do not affect the results.

We also consider the operator

$$H_\xi^- = -h\Delta - \sum_{q\in\mathbb{Z}^d} u(\,\cdot - q - \xi_q) \tag{1.11}$$

obtained by replacing the potential u in H_ξ by $-u$. For this operator, we assume $K = \emptyset$ since we are interested only in the effect of the negative potential. The spectrum of this operator extends to $-\infty$. For the asymptotic distribution, we show the following:

Theorem 1.3. *Suppose $K = \emptyset$, $\sup u = u(0) < \infty$ and $u(x)$ is lower semicontinuous at $x = 0$. Then the integrated density of states $N^-(\lambda)$ of H_ξ^- satisfies*

$$\lim_{\lambda\downarrow-\infty} \frac{\log N^-(\lambda)}{(-\lambda)^{1+\theta/d}} = \frac{-C_1}{u(0)^{1+\theta/d}}, \tag{1.12}$$

where $C_1 = d^{1+\theta/d}/\{(d+\theta)|S^{d-1}|^{\theta/d}\}$ and $|S^{d-1}|$ is the volume of the $(d-1)$-dimensional surface S^{d-1}.

For the Poisson model, Pastur [21] showed that the corresponding integrated density of states $N^-_{\mathrm{Poi}}(\lambda)$ satisfies

$$\lim_{\lambda\downarrow-\infty} \frac{\log N^-_{\mathrm{Poi}}(\lambda)}{(-\lambda)\log(-\lambda)} = \frac{-1}{u(0)}.$$

The power of λ in (1.12) tends to that of the Poisson model as $\theta \downarrow 0$. However, the logarithmic term is not recovered. Therefore, we cannot *interchange the limits* $\lambda \downarrow -\infty$ and $\theta \downarrow 0$ in this case. Both for the Poisson and our cases, only the classical effect from the scalar potential determines the leading terms. The lower semicontinuity of u at 0 is a sufficient condition for the classical behavior: by this condition, the tunneling effect is suppressed. For this subject, refer to Klopp and Pastur [15].

Let us briefly explain the organization of this paper. We prove Theorems 1.1, 1.2, and 1.3 in Sections 2, 3, and 5, respectively. In Section 3 we also give a simple proof of the corresponding results for the case that $\mathrm{supp}(u)$ is compact. In Section 4, we discuss the critical case $\alpha = d + 2$. In Section 6 we study the asymptotic behaviors of certain Wiener integrals.

2. Proof of Theorem 1.1

2.1. Upper estimate

To derive the asymptotics of the integrated density of states, one of the standard ways is to estimate its Laplace transform and use the Tauberian theorem (cf. [5, 18]). We here say the Tauberian theorem by the theorem deducing the asymptotics from that of the Laplace-Stieltjes transform. Let $\widetilde{N}(t)$ be the Laplace-Stieltjes transform of the integrated density of states $N(\lambda)$:

$$\widetilde{N}(t) = \int_0^\infty e^{-t\lambda} dN(\lambda).$$

Then, in view of the exponential Tauberian theorem due to Kasahara [10], the proof of the upper bound is reduced to the following:

Proposition 2.1. *If $K = \emptyset$ and (1.5) is satisfied, then we have*

$$\varlimsup_{t\uparrow\infty} \frac{\log \widetilde{N}(t)}{t^{(d+\theta)/(\alpha+\theta)}} \le -\int_{\mathbb{R}^d} dq \inf_{y\in\mathbb{R}^d} \Big(\frac{C_0}{|q+y|^\alpha} + |y|^\theta \Big) \tag{2.1}$$

for any $\alpha > d$.

Proof. We use the bound

$$\widetilde{N}(t) \le \widetilde{N}_1(t)(4\pi t h)^{-d/2}, \tag{2.2}$$

where

$$\widetilde{N}_1(t) = \int_{\Lambda_1} dx \mathbb{E}_\theta \Big[\exp\Big(-t \sum_{q\in\mathbb{Z}^d} u(\, x - q - \xi_q) \Big) \Big].$$

This is a simple modification of the bound in Theorem (9.6) in [20] for $\mathbb{Z}^d$-stationary random fields. By replacing the summation by integration, we have

$$\log \widetilde{N}_1(t) \le \int_{\mathbb{R}^d} dq \log \mathbb{E}_\theta \Big[\exp\Big(-t \inf_{x\in\Lambda_2} u(x - q - \xi_0) \Big) \Big].$$

We pick an arbitrary $L > 0$ and restrict the integration to $|q| \le L t^\eta$. The assumption (1.3) tells us that for any $\varepsilon_1 > 0$, there exists R_1 such that $u(x) \ge C_0(1-\varepsilon_1)|x|^{-\alpha}$ whenever $|x|_\infty \ge R_1$, where $|x|_\infty = \max_{1\le i\le d} |x_i|$. Thus the right hand side is dominated by

$$\int_{|q|\le Lt^\eta} dq \log \Big\{ \int_{|q+y|_\infty \ge R_1+1} \frac{dy}{Z(d,\theta)} \exp\Big(-t \inf_{x\in\Lambda_2} \frac{C_0(1-\varepsilon_1)}{|x-q-y|^\alpha} - |y|^\theta \Big) + \exp\Big(-t \inf_{\Lambda_{2R_1+4}} u \Big) \Big\}.$$

Thanks to the assumption (1.5), the second term makes only negligible contribution to the asymptotics. By changing the variables (q, y) to $(t^{-\eta} q, t^{-\eta} y)$ with $\eta = 1/(\alpha+\theta)$, we see that this equals

$$t^{d\eta} \int_{|q|\le L} dq \log \Big\{ \widetilde{N}_2(t,q) + \exp\Big(-t \inf_{\Lambda_{2R_1+4}} u \Big) \Big\},$$

where

$$\widetilde{N}_2(t,q) = t^{d\eta} \int_{|q+y|_\infty \ge (R_1+1)t^{-\eta}} \frac{dy}{Z(d,\theta)} \exp\Big(-t^{\theta\eta} \inf_{x\in\Lambda_{2t^{-\eta}}} \frac{C_0(1-\varepsilon_1)}{|x-q-y|^\alpha} - t^{\theta\eta}|y|^\theta \Big).$$

We take L as an arbitrary constant independent of t. Then, taking $\varepsilon_2, \varepsilon_3 > 0$ sufficiently small, we can dominate $\widetilde{N}_2(t,q)$ by $\exp(-t^{\theta\eta}\widetilde{N}_3(q))\varepsilon_2^{-d/\theta}$ for large enough

t, where

$$\widetilde{N}_3(q) = \inf\Big\{\frac{C_0(1-\varepsilon_1)}{|x-q-y|^\alpha} + (1-\varepsilon_2)|y|^\theta : x \in \Lambda_{\varepsilon_3}, y \in \mathbb{R}^d\Big\}.$$

Therefore we obtain

$$\overline{\lim_{t\uparrow\infty}} \frac{\log \widetilde{N}(t)}{t^{(d+\theta)\eta}} \le -\int_{|q|\le L} \widetilde{N}_3(q)dq.$$

Since $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and L are arbitrary, this completes the proof. □

2.2. Lower estimate

To prove the lower estimate, we have only to show the following:

Proposition 2.2. *If $\alpha < d+2$, then we have*

$$\underline{\lim_{t\uparrow\infty}} \frac{\log \widetilde{N}(t)}{t^{(d+\theta)/(\alpha+\theta)}} \ge -\int_{\mathbb{R}^d} dq \inf_{y\in\mathbb{R}^d}\Big(\frac{C_0}{|q+y|^\alpha} + |y|^\theta\Big). \tag{2.3}$$

Moreover, this bound remains valid for $\alpha = d+2$ with a smaller constant in the right hand side.

The case $\alpha = d+2$ will be discussed in more detail in Section 4 below.
Proof of Proposition 2.2. We use the bound

$$\widetilde{N}(t) \ge R^{-d}\exp(-th\|\nabla\psi_R\|_2^2)\widetilde{N}_1(t) \tag{2.4}$$

which holds for any $R \in \mathbb{N}$ and $\psi_R \in C_0^\infty(\Lambda_R)$ such that $\|\psi_R\|_2 = 1$, where $\|\cdot\|_2$ is the L^2-norm, and

$$\widetilde{N}_1(t) = \mathbb{E}_\theta\Bigg[\exp\Big(-t\sum_{q\in\mathbb{Z}^d}\int dx\psi_R(x)^2 u(x-q-\xi_q)\Big) : \bigcup_{q\in\mathbb{Z}^d}(q+\xi_q+K)\cap\Lambda_R = \emptyset\Bigg].$$

This can be proven by the same method as for the corresponding bound in Theorem (9.6) in [20] for $\mathbb{R}^d$-stationary random fields. By replacing the summation by integration, we have

$$\log \widetilde{N}_1(t) \ge \int_{\mathbb{R}^d} \widetilde{N}_2(t,q)dq,$$

where

$$\widetilde{N}_2(t,q) = \log \mathbb{E}_\theta\Big[\exp\Big(-t\int dx\psi_R(x)^2 \sup_{z\in\Lambda_1} u(x-q-z-\xi_0)\Big) \\ : (q+\xi_0+K)\cap\Lambda_R = \emptyset\Big].$$

For any $\varepsilon_1 > 0$, there exists R_1 such that $K \subset B(R_1)$ and $u(x) \le C_0(1+\varepsilon_1)|x|^{-\alpha}$ for any $|x| \ge R_1$ by the assumption (1.3). To use this bound in the above right hand side, we need $\inf\{|x-q-z-\xi_0| : x \in \Lambda_R, z \in \Lambda_1\} \ge R_1$. However we shall deal with a simpler sufficient condition $|\xi_0| \le |q|/2$ and $|q| \ge 2(R_1 + \sqrt{d}R)$ instead. Now fix $\beta > 0$ and take t large enough so that $t^\beta > 2(R_1 + \sqrt{d}R)$. Then we obtain

$$\int_{|q|\ge t^\beta} \widetilde{N}_2(t,q)dq \ge \int_{|q|\ge t^\beta} dq\Big(-\frac{tC_0(1+\varepsilon_1)2^\alpha}{(|q|-2\sqrt{d}R)^\alpha} + \log \mathbb{P}_\theta(|\xi_0| \le |q|/2)\Big). \tag{2.5}$$

By a simple estimate using $\log(1-X) \ge -2X$ for $0 \le X \le 1/2$, we can bound the right hand side from below by $-c_1 t^{1-\beta(\alpha-d)} - c_2 \exp(-c_3 t^{\beta\theta})$. The other part is estimated as

$$\begin{aligned}
&\int_{|q|\le t^\beta} \widetilde{N}_2(t,q)dq \\
&\ge \int_{|q|\le t^\beta} dq \log \int_{|q+y|\ge R_1+\sqrt{d}R} \frac{dy}{Z(d,\theta)} \\
&\quad \times \exp\Big(-\frac{tC_0(1+\varepsilon_1)}{\inf\{|x-q-z-y|^\alpha : x \in \Lambda_R, z \in \Lambda_1\}} - |y|^\theta\Big).
\end{aligned} \tag{2.6}$$

By changing the variables, we find that the right hand side equals

$$t^{d\eta} \int_{|q|\le t^{\beta-\eta}} dq \log \int_{|q+y|\ge (R_1+\sqrt{d}R)t^{-\eta}} \frac{dy t^{d\eta}}{Z(d,\theta)} \exp(-t^{\theta\eta}\widetilde{N}_3(y,q)),$$

where $\eta = 1/(\alpha+\theta)$ and

$$\widetilde{N}_3(y,q) = \frac{C_0(1+\varepsilon_1)}{\inf\{|x-q-z-y|^\alpha : x \in \Lambda_{Rt^{-\eta}}, z \in \Lambda_{t^{-\eta}}\}} + |y|^\theta. \tag{2.7}$$

Let us take $\gamma > 0$ and restrict the integration with respect to y to the ball $B(y_0, t^{-\gamma})$ with center y_0 and radius $t^{-\gamma}$. Then we can bound the integrand with respect to q from below by

$$\log \frac{|B(0,1)|t^{d(\eta-\gamma)}}{Z(d,\theta)} - t^{\theta\eta}\widetilde{N}_4(q,t), \tag{2.8}$$

where

$$\begin{aligned}
\widetilde{N}_4(q,t) = \inf\Big\{ &\sup_{y\in B(y_0,t^{-\gamma})} \widetilde{N}_3(y,q) \\
&: y_0 \in \mathbb{R}^d, d(B(y_0,t^{-\gamma}), -q) \ge (R_1+\sqrt{d}R)t^{-\eta}\Big\}.
\end{aligned} \tag{2.9}$$

We now specify R as the integer part of $\varepsilon_2 t^\eta$, where ε_2 is an arbitrarily fixed positive number. We take ψ_R as the nonnegative and normalized ground state of the Dirichlet Laplacian on the cube Λ_R and take β between η and $\eta(1+\theta/d)$. Then, for $\alpha < d+2$, we obtain

$$\varliminf_{t\uparrow\infty} \frac{\log \widetilde{N}(t)}{t^{(d+\theta)\eta}} \ge -\varlimsup_{t\uparrow\infty} \int_{|q|\le t^{\beta-\eta}} dq \widetilde{N}_4(q,t), \tag{2.10}$$

since $th\|\nabla\psi_R\|_2 \asymp tR^{-2}$ and (2.5) is negligible compared with $t^{(d+\theta)\eta}$. When $|q| \le t^{\beta-\eta}$, we can dominate $1/t$ by a power of q. Thus, for large $|q|$, by taking

y_0 as 0, we can dominate $\widetilde{N}_4(q,t)$ by $|q|^{-\alpha} + |q|^{-\gamma\theta/(\beta-\eta)}$. This is integrable if we take γ large enough so that $\gamma\theta/(\beta-\eta) > d$. Thus, by the Lebesgue convergence theorem, we have

$$\begin{aligned}&\lim_{t\uparrow\infty}\int_{|q|\le t^{\beta-\eta}} dq\widetilde{N}_4(q,t)\\ &= \int_{\mathbb{R}^d} dq \inf\Big\{\frac{C_0(1+\varepsilon_1)}{\inf\limits_{x\in\Lambda_{\varepsilon_2}}|x-q-y|^\alpha} + |y|^\theta : y\in\mathbb{R}^d, d(y,q)\ge \varepsilon_2\sqrt{d}\Big\}.\end{aligned}$$

Since ε_1 and ε_2 are arbitrary, this completes the proof of the former part of Proposition 2.2. For the case $\alpha = d+2$, we take $\varepsilon_2 = 1$. Then we have $th\|\nabla\psi_R\|_2 \asymp t^{(d+\theta)\eta}$ and the latter part of Proposition 2.2 follows from the same argument as above. □

3. Proof of Theorem 1.2 and the compact case

In this section, we use some additional notations to simplify the presentation. For any self-adjoint operator A, let $\lambda_1(A)$ be the infimum of its spectrum and, for any locally integrable function V and $R>0$, let $(-h\Delta+V)_R^D$ and $(-h\Delta+V)_R^N$ be the self-adjoint operators $-h\Delta+V$ on the L^2-space on the cube Λ_R with the Dirichlet and the Neumann boundary conditions, respectively.

3.1. Proof of Theorem 1.2 (I): One-dimensional case

To obtain the upper estimate, we have only to show the following:

Proposition 3.1. *If $d=1$, $K=\emptyset$,* supp(u) *is compact,*

$$\liminf_{x\downarrow 0}\int_0^x u(y)dy/x > 0, \text{ and } \liminf_{x\downarrow 0}\int_{-x}^0 u(y)dy/x > 0, \tag{3.1}$$

then we have

$$\overline{\lim_{t\uparrow\infty}}\frac{\log\widetilde{N}(t)}{t^{(1+\theta)/(3+\theta)}} \le -\frac{3+\theta}{1+\theta}\Big(\frac{h\pi^2}{4}\Big)^{(1+\theta)/(3+\theta)}. \tag{3.2}$$

Proof. We assume $h=1$ for simplicity. In the well known expression

$$\widetilde{N}(t) = \int_{\Lambda_1}\mathbb{E}_\theta[\exp(-tH_\xi)(x,x)]dx,$$

we apply the Feynman-Kac formula and an estimate for the exit time of the Brownian motion (cf. [9]) to obtain

$$\widetilde{N}(t) \le \int_{\Lambda_1}\mathbb{E}_\theta[\exp(-tH_{\xi,t}^D)(x,x)]dx + c_1e^{-c_2t},$$

where $\exp(-tH_\xi)(x,y)$ and $\exp(-tH_{\xi,t}^D)(x,y)$, $t>0$, $x,y\in\mathbb{R}$, are the integral kernels of the heat semigroups generated by H_ξ and $H_{\xi,t}^D$, respectively. By the eigenfunction expansion of the integral kernel, we have

$$\widetilde{N}(t) \le c_3t\widetilde{N}_1(t) + c_4e^{-c_5t},$$

where $\widetilde{N}_1(t) = \mathbb{E}_\theta[\exp(-t\lambda_1(H^D_{\xi,t}))]$. Thus we have only to prove (3.2) with $\widetilde{N}(t)$ replaced by $\widetilde{N}_1(t)$. Now we use Theorem 3.1 in the page 123 in [24], which states

$$\lambda_1(H^D_{\xi,t}) \geq \pi^2/(\sup_k |I_k| + c_6)^2$$

for large enough t under the assumption (3.1), where $\{I_k\}_k$ are the random open intervals such that $\sum_k I_k = \Lambda_t - \{q + \xi_q : q \in \mathbb{Z}\}$ and $|I_k|$ is the length of I_k. If $\sup_k |I_k| \geq s$ for some $0 \leq s \leq t$, then there exists $p \in \mathbb{Z} \cap \Lambda_t$ such that $\{q + \xi_q : q \in \mathbb{Z}\} \cap [p, p + s - 2] = \emptyset$. The probability of this event is estimated as

$$\begin{aligned}
\mathbb{P}_\theta(\sup_k |I_k| \geq s) &\leq \sum_{p\in\mathbb{Z}\cap\Lambda_t} \prod_{q\in\mathbb{Z}\cap[p,p+s-2]} \mathbb{P}_\theta(q + \xi_q \notin [p, p + s - 2]) \\
&\leq t \prod_{q\in\mathbb{Z}\cap[p,p+s-2]} \exp(-(1-\varepsilon)d(q, [p, p + s - 2]^c)^\theta)/\varepsilon^{1/\theta} \\
&\leq t \exp\Big(-(1-\varepsilon)\int_0^{s-3} d(q, [0, s - 3]^c)^\theta dq + \frac{s}{\theta} \log \frac{1}{\varepsilon}\Big) \\
&\leq t \exp\Big(-\frac{2(1-\varepsilon)}{\theta+1}\Big(\frac{s-3}{2}\Big)^{\theta+1} + \frac{s}{\theta} \log \frac{1}{\varepsilon}\Big)
\end{aligned}$$

if $s \geq 3$, where $0 < \varepsilon < 1$ is arbitrary. Therefore we have

$$\widetilde{N}_1(t) \leq c_7 t^2 \exp\Big(-\inf_{R>3}\Big(t\frac{\pi^2}{(R + c_6)^2} + \frac{(1-\varepsilon)}{2^\theta(\theta+1)}(R-3)^{\theta+1} - \frac{R}{\theta} \log \frac{1}{\varepsilon}\Big)\Big) + c_8 e^{-c_9 t}$$

for large t. Now it is easy to see that the infimum in the right hand side is attained by $R \sim 2(\pi^2 t/4)^{1/(3+\theta)}$ and we obtain (3.2). □

Remark 3.2. We put the additional assumption (3.1) only to use Theorem 3.1 in the page 123 in [24]. These assumptions are not restrictive at all since we can always find a $z \in \mathbb{R}$ such that $u(\,\cdot\, + z)$ satisfies them by the fundamental theorem of calculus and such a finite translation of u does not affect the above argument.

Proposition 3.3. *If $d = 1$ and $\alpha > 3$, then we have*

$$\varliminf_{t\uparrow\infty} \frac{\log \widetilde{N}(t)}{t^{(1+\theta)/(3+\theta)}} \geq -\frac{3+\theta}{1+\theta}\Big(\frac{h\pi^2}{4}\Big)^{(1+\theta)/(3+\theta)}. \tag{3.3}$$

Proof. This is proven by modifying our proof of Proposition 2.2. We take ψ_R as the nonnegative and normalized ground state of $(-\Delta)^D_R$. In (2.6), we restrict the integral with respect to y to $|q + y| \geq R_1 + (R + 1)/2$. In (2.8), we take $\eta = 1/(3+\theta)$ and R as the integer part of $\mathcal{R}t^\eta$ for a positive number $\mathcal{R} > 0$. Then since $t\|\nabla\psi_R\|_2^2 \sim t^{(1+\theta)\eta}(\pi/\mathcal{R})^2$ is not negligible, (2.10) is modified as

$$\varliminf_{t\uparrow\infty} \frac{\log \widetilde{N}(t)}{t^{(1+\theta)\eta}} \geq -h\Big(\frac{\pi}{\mathcal{R}}\Big)^2 - \varlimsup_{t\uparrow\infty} \int_{|q|\leq t^{\beta-\eta}} dq\widetilde{N}_4(q, t),$$

where $\widetilde{N}_4(q, t)$ is defined by replacing $\widetilde{N}_3(y, q)$ and $R_1 + \sqrt{d}$ by

$$\frac{C_0(1+\varepsilon_1)}{t^{(\alpha-3)\eta} \inf\{|x - q - z - y|^\alpha : x \in \Lambda_{Rt^{-\eta}}, z \in \Lambda_{t^{-\eta}}\}} + |y|^\theta$$

and $R_1 + (R+1)/2$, respectively, in (2.9). Since

$$\varlimsup_{t\uparrow\infty} \widetilde{N}_4(q,t) \le \inf_{y\notin\Lambda_{\mathcal{R}}(-q)} |y|^\theta = d(q,\Lambda_{\mathcal{R}}^c)^\theta,$$

we obtain

$$\varliminf_{t\uparrow\infty} \frac{\log \widetilde{N}(t)}{t^{(1+\theta)\eta}} \ge -h\Big(\frac{\pi}{\mathcal{R}}\Big)^2 - \frac{\mathcal{R}^{\theta+1}}{2^\theta(\theta+1)},$$

by the Lebesgue convergence theorem. By taking the supremum over $\mathcal{R} > 0$, we obtain the result. □

3.2. Proof of Theorem 1.2 (II) : Upper estimate for the multidimensional case

In the two-dimensional case, we can simply use Corollary 3.1 in [6] to get the upper bound. Indeed, the integrated density of states increases if we truncate the tail of u and hence the bound for the compactly supported potentials yields

$$N(\lambda) \le c_1 \exp(-c_2\lambda^{-1-\theta/2}(\log(1/\lambda))^{-\theta/2}), \tag{3.4}$$

for $0 \le \lambda \le c_3$, where c_1, c_2 and c_3 are positive constants depending on h and C_0. We give another proof for Corollary 3.1 in [6] in Subsection 3.4 below.

In the rest of this subsection we assume $d \ge 3$. Then our goal is the following:

Proposition 3.4. *Let $\alpha \ge d+2$ and $K = \emptyset$. There exist finite positive function $k_1(h)$ and $k_2(h)$ of h and a positive constant c such that*

$$N(\lambda) \le k_1(h)\exp(-c((h\wedge h^{(\alpha-d)/(\alpha-2)})/\lambda)^{(d+\mu\theta)/2}) \tag{3.5}$$

for $0 \le \lambda \le k_2(h)$.

We first see that Proposition 3.4 follows from the following:

Proposition 3.5. *For sufficiently small $\varepsilon_1, \varepsilon_2 > 0$, there exist a positive constant c independent of (h,R), and positive constants c' and c'' independent of (c_0,h,R) such that $\#\{q \in \mathbb{Z}^d \cap \Lambda_R : |\xi_q| \ge \varepsilon_1 R^\mu\} \le \varepsilon_2 R^d$, $R^{\mu d} \ge c'h/c_0$ and $R^{\mu(\alpha-2-d)} \ge c''c_0/h$ imply*

$$\lambda_1\Big(\Big(-h\Delta + \sum_{q\in\mathbb{Z}^d\cap\Lambda_R} \frac{c_0 1_{B(q+\xi_q,R_0)^c}(x)}{|x-q-\xi_q|^\alpha}\Big)_R^N\Big) \ge c(h\wedge h^{(\alpha-d)/(\alpha-2)})/R^2, \tag{3.6}$$

where c_0 and R_0 are arbitrarily fixed positive constants and 1_D is the characteristic function of $D \subset \mathbb{R}^d$.

Proof of Proposition 3.4. It is well known that

$$N(\lambda) \le \frac{c_1}{(R\wedge\sqrt{h})^d}\mathbb{P}_\theta(\lambda_1(H_R^N) \le \lambda)$$

(cf. (10.10) in [20]). We can take c_0 and R_0 so that

$$u(x) \ge c_0 1_{B(R_0)^c}(x)|x|^{-\alpha}.$$

Thus by Proposition 3.5, there exists a constant c_2 such that

$$\begin{aligned} &N(c_2(h \wedge h^{(\alpha-d)/(\alpha-2)})/R^2) \\ &\le \frac{c_1}{(R\wedge\sqrt{h})^d}\mathbb{P}_\theta(\#\{q\in\mathbb{Z}^d\cap\Lambda_R : |\xi_q|\ge\varepsilon_1 R^\mu\}\ge\varepsilon_2 R^d). \end{aligned}$$

We here should take c_0 sufficiently small so that the conditions of Proposition 3.5 are satisfied if $\alpha = d+2$. When the event in the right hand side occurs, we have

$$\sum_{q\in\mathbb{Z}^d\cap\Lambda_R}|\xi_q|^\theta \ge \varepsilon_1^\theta\varepsilon_2 R^{d+\mu\theta}.$$

Thus it is easy to show

$$N(c_2(h\wedge h^{(\alpha-d)/(\alpha-2)})/R^2) \le \frac{c_3}{(R\wedge\sqrt{h})^d}\exp(-c_4R^{d+\mu\theta}),$$

and (3.5) follows immediately. □

We next proceed to the proof of Proposition 3.5. We start with the following:

Lemma 3.6. $\inf\{\lambda_1((-\Delta+1_{B(b,1)})_R^N) : b\in\Lambda_R\}\ge cR^{-d}$.

This lemma follows immediately from the Proposition 2.3 of Taylor [25] using the scaling with the factor R^{-1}. That proposition is stated in terms of the scattering length. We here give an elementary proof following a lemma in the page 378 in Rauch [22] for the reader's convenience.

Proof. We rewrite as $\lambda_1((-\Delta+1_{B(b,1)})_R^N) = \lambda_1((-\Delta+1_{B(1)})_{R,b}^N)$, where, for any locally integrable function V and $R>0$, $(-\Delta+V)_{R,b}^N$ is the self-adjoint operator $-\Delta+V$ on the L^2 space on the cube $\Lambda_R(b) = b+\Lambda_R$ with the the Neumann boundary condition, and $B(1)=B(0,1)$. For any smooth function φ on the closure of $\Lambda_R(b)$, we have

$$\begin{aligned} &\int_{\Lambda_R(b)}\varphi^2(x)dx \\ &= \int_1^{R(b)} drr^{d-1}\int_{\theta\in S^{d-1}:(r,\theta)\in\Lambda_R(b)} dS\Big(\varphi(g(r),\theta)+\int_{g(r)}^r\partial_s\varphi(s,\theta)ds\Big)^2 \\ &\quad+\int_{B(1)\cap\Lambda_R(b)}\varphi^2(x)dx, \end{aligned}$$

where (r,θ) is the polar coordinate, $R(b)=\sup\{|x| : x\in\Lambda_R(b)\}$, dS is the volume element of the $(d-1)$-dimensional surface S^{d-1} and $g(r)=\{(r-1)/(R(b)-1)+1\}/2$. By the Schwarz inequality and a simple estimate, we can show

$$\int_1^{R(b)} drr^{d-1}\int_{\theta\in S^{d-1}:(r,\theta)\in\Lambda_R(b)} dS\Big(\int_{g(r)}^r\partial_s\varphi(s,\theta)ds\Big)^2 \le cR(b)^d\int_{\Lambda_R(b)}|\nabla\varphi|^2(x)dx,$$

where c is a constant depending only on d. By changing the variable, we can also show

$$\int_1^{R(b)} drr^{d-1} \int_{\theta\in S^{d-1}:(r,\theta)\in\Lambda_R(b)} dS\varphi(g(r),\theta)^2 \le c'R(b)^d \int_{B(1)\cap\Lambda_R(b)} \varphi^2(x)dx,$$

where c' is also a constant depending only on d. Since $\sup_{b\in\Lambda_R} R(b) \le \sqrt{d}R$, we can complete the proof. □

Lemma 3.7. *There exist positive constants c, c', and c'' such that*

$$\inf\Big\{\lambda_1\Big(\Big(-h\Delta+\sum_{j=1}^n \frac{c_0 1_{B(b_j,R_0)^c}(x)}{|x-b_j|^\alpha}\Big)_R^N\Big) : b_1,\dots,b_n\in\Lambda_R\Big\}$$
$$\ge c(c_0 n)^{(d-2)/(\alpha-2)} h^{(\alpha-d)/(\alpha-2)}/R^d$$

for $n \ge c'h/c_0$ and $R \ge c''(c_0 n/h)^{1/(\alpha-2)}$.

Proof. Since $\lambda_1(A+B) \ge \lambda_1(A)+\lambda_1(B)$ for any self-adjoint operators A and B, the left hand side is bounded from below by

$$\inf\{\lambda_1((-h\Delta+c_0 n 1_{B(b,R_0)^c}(x)|x-b|^{-\alpha})_R^N) : b\in\Lambda_R\}.$$

A change of the variable shows that this equals

$$hk^{-2}\inf\{\lambda_1((-\Delta+c_0 n k^{2-\alpha}h^{-1}1_{B(b,R_0/k)^c}(x)|x-b|^{-\alpha})_{R/k}^N) : b\in\Lambda_{R/k}\}$$

for any $k>0$. We can bound this from below by

$$hk^{-2}\inf\{\lambda_1((-\Delta+c_0 n k^{2-\alpha}h^{-1}3^{-\alpha}1_{B(b',1)}(x))_{R/k}^N) : b'\in\Lambda_{R/k}\}$$

for $k \ge R_0$ and $R > 4\sqrt{d}k$, and we can use Lemma 3.6 to complete the proof by taking k as $(c_0 n 3^{-\alpha}h^{-1})^{1/(\alpha-2)}$. Indeed, for each $b\in\Lambda_{R/k}$, we set $b' := b-(1+R_0/k)b/|b|$ if b is not the zero vector. If b is the zero vector, we set b' as an arbitrarily chosen vector with the norm $1+R_0/k$. Since $R_0/k \le |x-b| \le 2+R_0/k$ on $B(b',1)$, we have

$$1_{B(b,R_0/k)^c}(x)|x-b|^{-\alpha} \ge (2+R_0/k)^{-\alpha}1_{B(b',1)}(x).$$

We bound this from below by $3^{-\alpha}1_{B(b',1)}(x)$ by assuming $k\ge R_0$. Moreover we claim $b'\in\Lambda_{R/k}$ for all $b\in\Lambda_{R/k}$. A sufficient condition for this is $R\ge 2\sqrt{d}(R_0+k)$, since b' for b with $|b|\ge 1+R_0/k$ is a contraction of b and $\sup\{|b'|_\infty : |b|\le 1+R_0/k\} = \sqrt{d}(1+R_0/k)$. □

Lemma 3.8. *Let V be any locally integrable nonnegative function on $\mathbb{R}^d$. Then any eigenfunction ϕ of $(-h\Delta+V)_R^N$ satisfies*

$$\|\phi\|_\infty \le c(1/R+\sqrt{\lambda/h})^{d/2}\|\phi\|_2,$$

where c is a finite constant depending only on d, λ is the corresponding eigenvalue, and $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are L^∞ and L^2 norms, respectively.

The proof of this lemma is same as that of (3.1.55) in [24]. Now we prove Proposition 3.5:

Proof of Proposition 3.5. We use the following classification:

$$\mathcal{F} = \{a \in \Lambda_R \cap R^\mu \mathbb{Z}^d : \#(\Lambda_{R^\mu}(a) \cap \{q + \xi_q : q \in \mathbb{Z}^d \cap \Lambda_R\}) < R^{\mu d}/2\}$$

and

$$\mathcal{N} = \{a \in \Lambda_R \cap R^\mu \mathbb{Z}^d : \#(\Lambda_{R^\mu}(a) \cap \{q + \xi_q : q \in \mathbb{Z}^d \cap \Lambda_R\}) \geq R^{\mu d}/2\}.$$

By Lemma 3.7,

$$\lambda_1((-h\Delta + \sum_q c_0 1_{B(q+\xi_q, R_0)^c}(x)|x - q - \xi_q|^{-\alpha})^N_{R^\mu, a}) \geq ch^{(\alpha-d)/(\alpha-2)}/R^2$$

for any $a \in \mathcal{N}$.@ Let us write φ for the nonnegative and normalized ground state of the operator $(-h\Delta + \sum_q c_0 1_{B(q+\xi_q,R_0)^c}(x)|x - q - \xi_q|^{-\alpha})^N_R$. Then, applying the Rayleigh–Ritz variational formula, we have

$$\lambda_1\Big(\Big(-h\Delta + \sum_q \frac{c_0 1_{B(q+\xi_q,R_0)^c}(x)}{|x-q-\xi_q|^\alpha}\Big)^N_R\Big) \geq \frac{ch^{(\alpha-d)/(\alpha-2)}}{R^2} \sum_{a\in\mathcal{N}} \int_{\Lambda_{R^\mu}(a)} \varphi^2 dx.$$

If we assume $\lambda_1((-h\Delta + \sum_q c_0 1_{B(q+\xi_q,R_0)^c}(x)|x - q - \xi_q|^{-\alpha})^N_{R^\mu,a}) \leq Mh/R^2$, then Lemma 3.8 implies that the right hand side is bounded from below by

$$cR^{-2}h^{(\alpha-d)/(\alpha-2)}(1 - c' M^{d/2} R^{(\mu-1)d} \#\mathcal{F}). \tag{3.7}$$

Since $\#(\Lambda_{R^\mu}(a) \cap \{q + \xi_q : q \in \mathbb{Z}^d \cap \Lambda_R\}) \geq \#\{q \in \Lambda_{(1-2\varepsilon_1)R^\mu}(a) \cap \mathbb{Z}^d : |\xi_q| \leq \varepsilon_1 R^\mu\}$, we have $\#\{q \in \Lambda_{(1-2\varepsilon_1)R^\mu}(a) \cap \mathbb{Z}^d : |\xi_q| \leq \varepsilon_1 R^\mu\} < R^{\mu d}/2$ and $\#\{q \in \Lambda_{(1-2\varepsilon_1)R^\mu}(a) \cap \mathbb{Z}^d : |\xi_q| \geq \varepsilon_1 R^\mu\} > \{(1-2\varepsilon_1)^d - 1/2\}R^{\mu d}$ for $a \in \mathcal{F}$. Thus, by the assumption of this proposition, we have $\varepsilon_2 R^d \geq (\#\mathcal{F})\{(1-2\varepsilon_1)^d - 1/2\}R^{\mu d}$ and $\#\mathcal{F} \leq R^{d(1-\mu)}\varepsilon_2/\{(1-2\varepsilon_1)^2 - 1/2\}$. By substituting this to (3.7), we complete the proof. □

3.3. Proof of Theorem 1.2 (III) : Lower estimate for the multidimensional case

We shall work with $h = C_0 = 1$ for simplicity.

Proposition 3.9. *Suppose $d = 2$ and $\alpha > 4$ or $d \geq 3$ and $\alpha \geq d + 2$. Then there exist positive constants c_1, c_2, and c_3 such that*

$$N(\lambda) \geq \begin{cases} c_1 \exp\Big(-c_2 \lambda^{-1-\theta/2} (\log(1/\lambda))^{-\theta/2}\Big) & (d = 2), \\ c_1 \exp(-c_2 \lambda^{-(d+\mu\theta)/2}) & (d \geq 3), \end{cases} \tag{3.8}$$

for $0 \leq \lambda \leq c_3$.

Proof. We consider the event

$$\begin{aligned} &\{\text{For any } p \in R_1\mathbb{Z}^d \cap \Lambda_{3R} \text{ and } q \in \mathbb{Z}^d \cap \Lambda_{R_1}(p) \cap \Lambda_{2R}, q + \xi_q \in \Lambda_1(p)\} \\ &\cap \{\text{For any } q \in \mathbb{Z}^d \setminus \Lambda_{2R}, |\xi_q| \leq |q|/4\} \end{aligned} \tag{3.9}$$

where $R_1 = R^\mu$ for $d \geq 3$ and $R_1 = R/\sqrt{\log R}$ for $d = 2$. Then we have

$$N(\lambda) \geq R^{-d}\mathbb{P}_\theta\Big(\|\nabla\Phi_R\|_2^2 + \Big(\Phi_R, \sum_{q\in\mathbb{Z}^d} u(x-q-\xi_q)\Phi_R\Big) \leq \lambda \\ \text{and the event (3.9) occurs}\Big), \tag{3.10}$$

where Φ_R is an element of the domain of the Dirichlet Laplacian on the cube $\Lambda_R \setminus \bigcup_{p\in R_1\mathbb{Z}^d\cap\Lambda_{3R}}(p+K)$ such that $\|\Phi_R\|_2 = 1$ (cf. Theorem (5.25) in [20]). We take Φ_R as $\phi_R\psi_R/\|\phi_R\psi_R\|_2$, where ψ_R is the nonnegative and normalized ground state of the Dirichlet Laplacian on Λ_R and

$$\phi_R(x) = \begin{cases} \Big(2d_\infty\Big(x, \sum_{p\in R^\mu\mathbb{Z}^d\cap\Lambda_R}\Lambda_{R^\nu}(p)\Big)R^{-\nu}\Big)\wedge 1 & (d\geq 3), \\ \dfrac{\Big(\log d_\infty(x, \Lambda_R\cap \dfrac{R\mathbb{Z}^2}{\sqrt{\log R}}) - \dfrac{4}{\alpha}\log R\Big)_+}{\log\dfrac{R}{2\sqrt{\log R}} - \dfrac{4}{\alpha}\log R} & (d=2). \end{cases} \tag{3.11}$$

In (3.11), $d_\infty(\cdot,\cdot)$ is the distance function with respect to the maximal norm, $\nu = 2/(\alpha-d)$, and $(\cdot)_+$ is the positive part. Then it is not difficult to see $\|\nabla\Phi_R\|_2^2 \leq c_4R^{-2}$. On the event (3.9), we have in addition that

$$\sum_{q\in\mathbb{Z}^d} u(x-q-\xi_q) \leq \frac{c_5R_1^d}{d(x, \sum_{p\in R_1\mathbb{Z}^d\cap\Lambda_{2R}}\Lambda_1(p))^\alpha} + c_6R_1^{-(\alpha-d)} \tag{3.12}$$

in Λ_R. Hence we have

$$\Big(\Phi_R, \sum_{q\in\mathbb{Z}^d} u(x-q-\xi_q)\Phi_R\Big) \leq c_7R^{-2}.$$

On the other hand, the probability of the event (3.9) can be estimated as

$$\begin{aligned} &\log\mathbb{P}_\theta(\text{ the event (3.9) occurs }) \\ \geq &-\#(R_1\mathbb{Z}^d\cap\Lambda_{3R})\sum_{q\in\mathbb{Z}^d\cap\Lambda_{R_1}}\log\mathbb{P}_\theta(\xi_0\in\Lambda_1(q)) \\ &+\sum_{q\in\mathbb{Z}^d\setminus\Lambda_{2R}}\log(1-\mathbb{P}_\theta(|\xi_0|\geq|q|/4)) \\ \geq &-c_8R^dR_1^\theta \end{aligned}$$

by using $\log(1-X) \geq -2X$ for $0 \leq X \leq 1/2$ in the last line. Therefore, we have

$$N(c_9R^{-2}) \geq R^{-d}\exp\Big(-c_{10}R^dR_1^\theta\Big)$$

and the proof is finished. □

Remark 3.10. For the manner of taking the function ϕ_R in (3.11) and the event in (3.9), we refer the reader to the notion of the "constant capacity regime" (cf. Section 3.2.B of [24]). The same technique is used in Appendix B of [6].

3.4. Compact case

In this subsection, we adapt the methods in the preceding sections to give a simple proof of the following results in [6]:

Theorem 3.11. *Assume* $\Lambda_{r_1} \subset \mathrm{supp}(u) \cup K \subset \Lambda_{r_2}$ *for some* $0 < r_1 \leq r_2 < \infty$ *instead of* (1.3)*. Then we have*

$$\log N(\lambda) \begin{cases} \sim -(\pi^2 h/\lambda)^{(1+\theta)/2}(1+\theta)^{-1}2^{-\theta} & (d=1), \\ \asymp -\lambda^{-1-\theta/2}(\log(1/\lambda))^{-\theta/2} & (d=2), \\ \asymp -\lambda^{-(d/2+\theta/d)} & (d \geq 3) \end{cases}$$

as $\lambda \downarrow 0$*, where* $f(\lambda) \sim g(\lambda)$ *means* $\lim_{\lambda\downarrow 0} f(\lambda)/g(\lambda) = 1$ *and* $f(\lambda) \asymp g(\lambda)$ *means* $0 < \underline{\lim}_{\lambda\downarrow 0} f(\lambda)/g(\lambda) \leq \overline{\lim}_{\lambda\downarrow 0} f(\lambda)/g(\lambda) < \infty$.

Remark 3.12. The assumption on u in this theorem is only for giving a simple proof in the multidimensional case. If $d = 1$, then the assumption in Proposition 3.1 is sufficient. If $d \geq 3$, then this theorem can be extended to the case that the scattering length of u is positive.

The proof for $d = 1$ is given in Subsection 3.1. The lower estimate for $d = 2$ is given in Subsection 3.3. To prove the lower estimate for $d \geq 3$, we replace R^ν by $2r_2 + 1$ in the proof of Proposition 3.9. Then the rest of the proof is simpler than that of the proposition since

$$\Big(\Phi_R, \sum_{q\in\mathbb{Z}^d} u(x-q-\xi_q)\Phi_R\Big) = 0$$

under the event in (3.9) with $R_1 = R^{2/d}$. To prove the upper estimate for $d \geq 3$, we have only to apply the following instead of Proposition 3.5 in the proof of Proposition 3.4:

Proposition 3.13. *For sufficiently small* $\varepsilon_1, \varepsilon_2 > 0$*, there exists a finite constant* c *such that* $\#\{q \in \mathbb{Z}^d \cap \Lambda_R : |\xi_q| \geq \varepsilon_1 R^{2/d}\} \leq \varepsilon_2 R^d$ *implies*

$$\lambda_1\Big(\Big(-\Delta + c_0 \sum_{q\in\mathbb{Z}^d\cap\Lambda_R} 1_{B(q+\xi_q,r_0)}\Big)_R^N\Big) \geq c/R^2, \tag{3.13}$$

where c_0 *and* r_0 *are arbitrarily fixed positive constants.*

Proof. We use the classification

$$\mathcal{F}_0 = \{a \in \Lambda_R \cap R^{2/d}\mathbb{Z}^d : \Lambda_{R^{2/d}}(a) \cap \{q+\xi_q : q \in \mathbb{Z}^d \cap \Lambda_R\} = \emptyset\}$$

and

$$\mathcal{N}_0 = \{a \in \Lambda_R \cap R^{2/d}\mathbb{Z}^d : \Lambda_{R^{2/d}}(a) \cap \{q+\xi_q : q \in \mathbb{Z}^d \cap \Lambda_R\} \neq \emptyset\},$$

instead of $\mathcal{F}$ and $\mathcal{N}$ in the proof of Proposition 3.5. Then we complete the proof by Lemmas 3.6 and 3.8 without using Lemma 3.7. □

To prove the upper estimate for $d = 2$, we have only to apply the following instead of Proposition 3.5 in the proof of Proposition 3.4:

Proposition 3.14. *For sufficiently small $\varepsilon_1, \varepsilon_2 > 0$, there exists a finite constant c such that $\#\{q \in \mathbb{Z}^2 \cap \Lambda_R : |\xi_q| \geq \varepsilon_1 R/\sqrt{\log R}\} \leq \varepsilon_2 R^2$ implies*

$$\lambda_1\Big(\Big(-\Delta + c_0 \sum_{q \in \mathbb{Z}^2 \cap \Lambda_R} 1_{B(q+\xi_q, r_0)}\Big)_R^N\Big) \geq c/R^2. \tag{3.14}$$

To prove this, we replace $R^{2/d}$ by $R/\sqrt{\log R}$ in the proof of Proposition 3.13 and we further need to extend Lemma 3.6 to the 2-dimensional case. By a simple modification of the proof of Lemma 3.6, we have the following, which is sufficient for our purpose:

Lemma 3.15. *If $d = 2$, then we have $\inf\{\lambda_1((-\Delta + c_0 1_{B(b,r_0)})_R^N) : b \in \Lambda_R\} \geq c/(R^2 \log R)$.*

4. Critical case

In this section we discuss the case of $\alpha = d + 2$. By modifying our proof of Proposition 2.2, we can prove the following:

Proposition 4.1. *If $\alpha = d + 2$, then we have*

$$\varliminf_{t \uparrow \infty} \frac{\log \widetilde{N}(t)}{t^{(d+\theta)/(d+2+\theta)}} \geq -K_0(h, C_0), \tag{4.1}$$

where

$$\begin{aligned} &K_0(h, C_0) \\ &= \inf\Big\{h\|\nabla\psi\|_2^2 + \int_{\mathbb{R}^d} dq \inf_{y \notin \mathrm{supp}(\psi) - q}\Big(\int_{\mathbb{R}^d} \frac{dx C_0 \psi(x)^2}{|x - q - y|^{d+2}} + |y|^\theta\Big) \\ &\quad : \psi \in W_2^1(\mathbb{R}^d), \|\psi\|_2 = 1\Big\} \end{aligned} \tag{4.2}$$

and $W_2^1(\mathbb{R}^d) = \{\psi \in L^2(\mathbb{R}^d) : \nabla\psi \in L^2(\mathbb{R}^d)\}$.

Proof. In (2.4), we replace ψ_R by an arbitrary function $\varphi \in H_0^1(\Lambda_R)$ with $\|\varphi\|_2 = 1$, where $H_0^1(\Lambda_R)$ is the completion of $C_0^\infty(\Lambda_R)$ in $W_2^1(\mathbb{R}^d)$. Then (2.6) is modified as

$$\begin{aligned} &\int_{|q| \leq t^\beta} \widetilde{N}_2(t, q) dq \\ &\geq \int_{|q| \leq t^\beta} dq \log \int_{y \in [\mathrm{supp}(\varphi) : R_1 + \sqrt{d}/2]^c - q} \frac{dy}{Z(d, \theta)} \\ &\quad \times \exp\Big(-\int \frac{dx \varphi(x)^2 t C_0 (1 + \varepsilon_1)}{\inf\{|x - q - z - y|^{d+2} : z \in \Lambda_1\}} - |y|^\theta\Big), \end{aligned}$$

where $[A : r] = \{x \in \mathbb{R}^d : d(x, A) < r\}$ for any $A \subset \mathbb{R}^d$ and $r > 0$. We take η as $1/(d + 2 + \theta)$. Then, by changing the variables, we see that the right hand side equals

$$t^{d\eta}\int_{|q|\le t^{\beta-\eta}} dq\log\int_{y\in[\mathrm{supp}(\varphi_\eta):(R_1+\sqrt{d}/2)/t^\eta]^c-q}\frac{dyt^{d\eta}}{Z(d,\theta)}\exp(-t^{\theta\eta}\widetilde{N}_3(y,q;\varphi_\eta)),$$

where

$$\widetilde{N}_3(y,q;\varphi_\eta)=\int\frac{dx\varphi_\eta(x)^2C_0(1+\varepsilon_1)}{\inf\{|x-q-z-y|^{d+2}:z\in\Lambda_{t^{-\eta}}\}}+|y|^\theta$$

and $\varphi_\eta(x)=t^{d\eta/2}\varphi(t^\eta x)$. We take R as the integer part of $\mathcal{R}t^\eta$ for a positive number $\mathcal{R}$ and take φ so that $\varphi_\eta=\psi$ is a t-independent element of $H_0^1(\Lambda_{\mathcal{R}})$. Since $t\|\nabla\varphi\|_2^2=t^{(d+\theta)\eta}\|\nabla\psi\|_2^2$ is not negligible, (2.10) is modified as

$$\varliminf_{t\uparrow\infty}\frac{\log\widetilde{N}(t)}{t^{(d+\theta)\eta}}\ge -h\|\nabla\psi\|_2^2-\varlimsup_{t\uparrow\infty}\int_{|q|\le t^{\beta-\eta}}dq\widetilde{N}_4(q,t),$$

where

$$\widetilde{N}_4(q,t)=\inf\Big\{\sup_{y\in B(y_0,t^{-\gamma})}\widetilde{N}_3(y,q;\psi):y_0\in\Big[\mathrm{supp}(\psi):\frac{R_1+\sqrt{d}/2}{t^\eta}+\frac{1}{t^\gamma}\Big]^c-q\Big\}.$$

Since

$$\varlimsup_{t\uparrow\infty}\widetilde{N}_4(q,t)\le\inf_{y\in(\mathrm{supp}(\psi))^c-q}\Big(\int\frac{dx\psi(x)^2C_0(1+\varepsilon_1)}{|x-q-y|^{d+2}}+|y|^\theta\Big),$$

we obtain

$$\varliminf_{t\uparrow\infty}\frac{\log\widetilde{N}(t)}{t^{(d+\theta)\eta}}\ge -h\|\nabla\psi\|_2^2-\int_{\mathbb{R}^d}dy\inf_{y\in(\mathrm{supp}(\psi))^c-q}\Big(\int\frac{dx\psi(x)^2C_0(1+\varepsilon_1)}{|x-q-y|^{d+2}}+|y|^\theta\Big)$$

by the Lebesgue convergence theorem. By taking the supremum with respect to ε_1, ψ and $\mathcal{R}$, we obtain the result. □

If we apply Donsker and Varadhan's large deviation theory without caring about the topological problems, then the formal upper estimate

$$\varlimsup_{t\uparrow\infty}\frac{\log\widetilde{N}(t)}{t^{(d+\theta)/(d+2+\theta)}}\le -K(h,C_0) \tag{4.3}$$

is expected, where $K(h,C_0)$ is the quantity obtained by removing the restriction $y\notin\mathrm{supp}(\psi)-q$ in the definition (4.2) of $K_0(h,C_0)$. For the corresponding Poisson case, this is rigorously established in Ôkura [19]. In that case, the space $\mathbb{R}^d$ can be replaced by a d-dimensional torus and the Feynman-Kac functional becomes a lower semicontinuous functional, so that Donsker and Varadhan's theory applies. However, verifications of both the replacement of the space and the continuity of the functional seem to be difficult in our case.

From the conjecture (4.3), we expect that the quantum effect appears in the leading term. By Proposition 3.4 in Section 3, we can justify this if $d\ge 3$ and h is large:

Proposition 4.2. *If $d \geq 3$ and $\alpha = d + 2$, then we have*

$$\varlimsup_{h\to\infty} \varlimsup_{\lambda\to 0} \lambda^{(d+\theta)/2} \log N(\lambda) = -\infty. \tag{4.4}$$

In the one-dimensional case we can show the same statement with a more explicit bound

$$\varlimsup_{\lambda\to 0} \lambda^{(1+\theta)/2} \log N(\lambda) \leq -\frac{\pi^{1+\theta} h^{(1+\theta)/2}}{(1+\theta)2^\theta}$$

by Theorem 1.2, since the leading order does not depend on $\alpha \geq 3$. In the two-dimensional case we have no such results.

5. Proof of Theorem 1.3

5.1. Upper estimate

Let $\widetilde{N}^-(t)$ be the Laplace-Stieltjes transform of the integrated density of states $N^-(\lambda)$:

$$\widetilde{N}^-(t) = \int_{-\infty}^{\infty} e^{-t\lambda} dN^-(\lambda).$$

To prove the upper estimate, we have only to show the following:

Proposition 5.1. *Under the condition that $u \geq 0, \sup u = u(0) < \infty$ and $\sup |x|^\alpha u(x) < \infty$ for some $\alpha > d$, we have*

$$\varlimsup_{t\uparrow\infty} \frac{\log \widetilde{N}^-(t)}{t^{1+d/\theta}} \leq u(0)^{1+d/\theta} \int_{|q|\leq 1} dq(1 - |q|^\theta). \tag{5.1}$$

Proof. We use the bound

$$\widetilde{N}^-(t) \leq \widetilde{N}_1^-(t)(4\pi th)^{-d/2}$$

as in (2.2), where

$$\widetilde{N}_1^-(t) = \int_{\Lambda_1} dx \mathbb{E}_\theta \left[\exp\Big(t \sum_{q\in\mathbb{Z}^d} u(x - q - \xi_q)\Big)\right].$$

Here we have used the path integral expression of $\widetilde{N}^-(t)$ in Theorem VI.1.1 of [3]. The assumption required in that theorem will be checked in Lemma 7.2 in Section 7. By replacing the summation by integration, we have

$$\log \widetilde{N}_1^-(t) \leq \int_{\mathbb{R}^d} dq \log \widetilde{N}_2^-(t, q),$$

where

$$\widetilde{N}_2^-(t, q) = \mathbb{E}_\theta \Big[\exp\Big(t \sup_{x\in\Lambda_2} u(x - q - \xi_0)\Big)\Big].$$

Now we fix an arbitrary small number $\varepsilon > 0$ and let $C = \sup |x|^\alpha u(x)$. When $|q| > (1+\varepsilon)(u(0)t)^{1/\theta}$, we estimate as

$$\begin{aligned}\widetilde{N}_2^-(t,q) \le& \exp(t\sup\{u(x-y) : x \in \Lambda_2, |y| \ge \delta|q|\}) \\ &+ \exp(tu(0))\mathbb{P}_\theta(|\xi_0| \ge (1-\delta)|q|),\end{aligned} \tag{5.2}$$

where $\delta > 0$ is chosen to satisfy $(1-\delta)^{\theta+2}(1+\varepsilon)^\theta = 1$. For the first term in the right hand side, we use an obvious bound

$$\sup\{u(x-y) : x \in \Lambda_2, |y| \ge \delta|q|\} \le C(\delta|q| - \sqrt{d})^{-\alpha}.$$

For the second term, it is easy to see

$$\mathbb{P}_\theta(|\xi_q| \ge (1-\delta)|q|) \le M(\delta,\theta)\exp(-(1-\delta)^{\theta+1}|q|^\theta)$$

for some large $M(\delta,\theta) > 0$. Moreover, we have

$$(1-\delta)^{\theta+1}|q|^\theta = (1-\delta)^{\theta+2}|q|^\theta + \delta(1-\delta)^{\theta+1}|q|^\theta \ge u(0)t + \delta(1-\delta)^{\theta+1}|q|^\theta$$

thanks to $|q| > (1+\varepsilon)(u(0)t)^{1/\theta}$ and our choice of δ. Combining above three estimates, we get

$$\widetilde{N}_2^-(t,q) \le \exp(tC(\delta|q| - \sqrt{d})^{-\alpha})(1 + M(\delta,\theta)\exp(-\delta(1-\delta)^{\theta+1}|q|^\theta)) \tag{5.3}$$

and thus

$$\log \widetilde{N}_2^-(t,q) \le tC(\delta|q| - \sqrt{d})^{-\alpha} + M(\delta,\theta)\exp(-\delta(1-\delta)^{\theta+1}|q|^\theta), \tag{5.4}$$

using $\log(1+X) \le X$. Since the integral of the right hand side over $\{|q| > (1+\varepsilon)(u(0)t)^{1/\theta}\}$ is easily seen to be $o(t^{1+d/\theta})$, we can neglect this region.

For q with $|q| \le (1+\varepsilon)(u(0)t)^{1/\theta}$, we estimate as

$$\begin{aligned}\widetilde{N}_2^-(t,q) \le& \exp(t\sup\{u(x-y) : x \in \Lambda_2, |y| \ge L\}) \\ &+ \exp(tu(0))\mathbb{P}_\theta(|q+\xi_0| \le L),\end{aligned} \tag{5.5}$$

where $L = 2\varepsilon(u(0)t)^{1/\theta}$. We use obvious bounds

$$\sup\{u(x-y) : x \in \Lambda_2, |y| \ge L\} \le C(L - \sqrt{d})_+^{-\alpha}$$

for the first term and

$$\mathbb{P}_\theta(|q+\xi_0| \le L) \le \exp(-(|q|-L)_+^\theta)|B(0,L)|/Z(d,\theta)$$

for the second term. Note also that we have

$$tc(L-\sqrt{d})_+^{-\alpha} \le tu(0) - (|q|-L)_+^\theta$$

for large t, from $|q| \le (1+\varepsilon)(u(0)t)^{1/\theta}$ and our choice of L. Using these estimates, we obtain

$$\begin{aligned}&\int_{|q|\le(1+\varepsilon)(u(0)t)^{1/\theta}} dq \log \widetilde{N}_2^-(t,q) \\ \le& \int_{|q|\le(1+\varepsilon)(u(0)t)^{1/\theta}} dq \Big\{ \log\Big(\frac{|B(0,L)|}{Z(d,\theta)} + 1\Big) + tu(0) - (|q|-L)_+^\theta \Big\}.\end{aligned}$$

By changing the variable and taking the limit, we arrive at

$$\overline{\lim_{t\uparrow\infty}} \frac{\log \widetilde{N}(t)}{t^{1+d/\theta}} \le u(0)^{1+d/\theta} \int_{|q|\le 1+\varepsilon} dq\{1-(|q|-2\varepsilon)_+^\theta\}.$$

This completes the proof of Proposition 5.1 since $\varepsilon > 0$ is arbitrary. □

5.2. Lower estimate

To prove the lower estimate, we have only to show the following:

Proposition 5.2. *Suppose* $u \ge 0, \sup u = u(0) < \infty$*, and* $u(x)$ *is lower semicontinuous at* $x = 0$*. Then we have*

$$\underline{\lim_{t\uparrow\infty}} \frac{\log \widetilde{N}^-(t)}{t^{1+d/\theta}} \ge u(0)^{1+d/\theta} \int_{|q|\le 1} dq(1-|q|^\theta). \tag{5.6}$$

Proof. For any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$u(x) \ge u(0) - \varepsilon \text{ for } |x| < R_\varepsilon \tag{5.7}$$

by the lower semicontinuity of u. We use the bound

$$\widetilde{N}^-(t) \ge \exp(-th\|\nabla\psi_\varepsilon\|_2)\widetilde{N}_1^-(t),$$

for any $\psi_\varepsilon \in C_0^\infty(\Lambda_\varepsilon)$ such that the L^2-norm of ψ_ε is 1, where

$$\widetilde{N}_1^-(t) = \mathbb{E}_\theta\left[\exp\left(t\sum_{q\in\mathbb{Z}^d}\inf_{x\in\Lambda_\varepsilon} u(\,x-q-\xi_q)\right)\right]. \tag{5.8}$$

This is proven by the same estimate as used in (2.4). We take ψ_ε as the nonnegative and normalized ground state of the Dirichlet Laplacian on the cube Λ_ε. Since a sufficient condition for $\sup_{x\in\Lambda_\varepsilon}|x-q-\xi_q| \le R_\varepsilon$ is $|q+\xi_q| \le R_\varepsilon - \varepsilon\sqrt{d}/2$, we restrict the expectation to this event and deduce from (5.7) that

$$\log \widetilde{N}_1^-(t) \ge \sum_{q\in\mathbb{Z}^d} \log \int_{|q+y|\le R_\varepsilon-\varepsilon\sqrt{d}/2} \frac{dy}{Z(d,\theta)} \exp(t(u(0)-\varepsilon)-|y|^\theta).$$

Since a sufficient condition for $\inf\{u(0)-\varepsilon)-|y|^\theta \le R_\varepsilon : |q+y| \le R_\varepsilon - \varepsilon\sqrt{d}/2\} \ge 0$ is $|q| \le \{t(u(0)-\varepsilon)\}^{1/\theta} - R_\varepsilon + \varepsilon\sqrt{d}/2$, we restrict the range of q and deduce

$$\begin{aligned}
&\log \widetilde{N}_1^-(t)\\
&\ge \int_{|q|\le h(t)} \left\{c'\log\frac{|B(0,R_\varepsilon-\varepsilon\sqrt{d}/2)|}{Z(d,\theta)} + t(u(0)-\varepsilon) - (|q|+R_\varepsilon-c))^\theta\right\}\\
&= h(t)^d \int_{|q|\le 1} \left\{c'\log\frac{|B(0,R_\varepsilon-\varepsilon\sqrt{d}/2)|}{Z(d,\theta)} + t(u(0)-\varepsilon) - (h(t)|q|+R_\varepsilon+c))^\theta\right\}
\end{aligned}$$

for large t and small ε, where $h(t) = \{t(u(0)-\varepsilon)\}^{1/\theta} - R_\varepsilon - c$ and c and c' are positive constants. Then we obtain

$$\underline{\lim_{t\uparrow\infty}} \frac{\log \widetilde{N}^-(t)}{t^{1+d/\theta}} \ge (u(0)-\varepsilon)^{1+d/\theta} \int_{|q|\le 1} dq(1-|q|^\theta).$$

Since ε is arbitrary, this completes the proof of Proposition 5.2. □

6. Asymptotics for associated Wiener integrals

In the previous work [6], the asymptotic behaviors of the integrated density of states were derived from those of certain Wiener integrals. In this section, we recall the connection and derive the asymptotic behaviors of the associated Wiener integrals in our settings. Let $h = 1/2$ for simplicity and E_x denote the expectation with respect to the standard Brownian motion $(B_s)_{0\le s\le\infty}$ starting at x. Then the Laplace-Stieltjes transform of the integrated density of states can be expressed as follows (cf. Chapter VI of [3]):

$$\begin{aligned}\widetilde{N}(t) = (2\pi t)^{-d/2}\int_{\Lambda_1} dx\mathbb{E}_\theta\otimes E_x\bigg[&\exp\bigg\{-\int_0^t \sum_{q\in\mathbb{Z}^d} u(B_s - q - \xi_q)ds\bigg\}\\ &: B_s \notin \bigcup_{q\in\mathbb{Z}^d}(q+\xi_q+K) \text{ for } 0\le s\le t\,\bigg|\, B_t = x\bigg].\end{aligned} \tag{6.1}$$

We can also express $\widetilde{N}^-(t)$ in the same form by changing the sign of u and setting $K=\emptyset$ in the right hand side. In view of (6.1), $\widetilde{N}(t)$ seems, and indeed will be proven below, to be asymptotically comparable to the Wiener integral

$$\begin{aligned}S_{t,x} = \mathbb{E}_\theta\otimes E_x\bigg[&\exp\bigg\{-\int_0^t \sum_{q\in\mathbb{Z}^d} u(B_s - q - \xi_q)ds\bigg\}\\ &: B_s \notin \bigcup_{q\in\mathbb{Z}^d}(q+\xi_q+K) \text{ for } 0\le s\le t\bigg],\end{aligned} \tag{6.2}$$

which was the main object in [6]. This quantity is of interest itself since not only it gives the average of the solution of a heat equation with random sinks but also can be interpreted as the annealed survival probability of the Brownian motion among killing potentials. Similarly, $\widetilde{N}^-(t)$ is asymptotically comparable to the average of the solution

$$S_{t,x}^- = \mathbb{E}_\theta\otimes E_x\bigg[\exp\bigg\{\int_0^t \sum_{q\in\mathbb{Z}^d} u(B_s - q - \xi_q)ds\bigg\}\bigg], \tag{6.3}$$

of a heat equation with random sources which can also be interpreted as the average number of the branching Brownian motions in random media. We refer the readers to [8, 7, 24] about the interpretations of $S_{t,x}$ and $S_{t,x}^-$. The connection between the asymptotics of $\widetilde{N}(t)$ and $S_{t,x}$ can be found in the literature for the case that $\{q+\xi_q\}_q$ is replaced by an $\mathbb{R}^d$-stationary random field (see e.g. [18], [23]). However our case is only $\mathbb{Z}^d$-stationary.

We first prepare a lemma which gives upper bounds on $\log S_{t,x}$ and $\log S_{t,x}^-$ in terms of $\log\widetilde{N}(t)$ and $\log\widetilde{N}^-(t)$, respectively. We shall state the results only for $x\in\Lambda_1$ since they automatically extend to the whole space by the $\mathbb{Z}^d$-stationarity.

Lemma 6.1. *For any $x \in \Lambda_1$ and $\varepsilon > 0$, we have*

$$\log S_{t,\,x} \le \log \widetilde{N}(t-\varepsilon)(1+o(1)) \tag{6.4}$$

and

$$\log S^-_{t,\,x} \le \log \widetilde{N}^-(t-t^{-2d/\theta})(1+o(1)) \tag{6.5}$$

as $t \to \infty$.

Proof. We give the proof of (6.5) first. Let $V_\xi(x)$ denotes the potential $\sum_{q\in\mathbb{Z}^d} u(x-q-\xi_q)$ for simplicity. We divide the expectation as

$$\begin{aligned} S^-_{t,\,x} =& \mathbb{E}_\theta \otimes E_x\left[\exp\left\{\int_0^t V_\xi(B_s)ds\right\} : \sup_{0\le s\le t}|B_s|_\infty < [t^{1+d/\theta}]\right] \\ &+ \sum_{n>[t^{1+d/\theta}]} \mathbb{E}_\theta \otimes E_x\left[\exp\left\{\int_0^t V_\xi(B_s)ds\right\} : n-1 \le \sup_{0\le s\le t}|B_s|_\infty < n\right]. \end{aligned} \tag{6.6}$$

The summands in the second term can be bounded from above by

$$\begin{aligned} &\mathbb{E}_\theta\left[\exp\left\{t\sup_{y\in\Lambda_{2n}} V_\xi(y)\right\}\right] P_x\left(n-1 \le \sup_{0\le s\le t}|B_s|_\infty\right) \\ \le& c_1 n^d \mathbb{E}_\theta\left[\exp\left\{t\sup_{y\in\Lambda_1} V_\xi(y)\right\}\right]\exp\{-c_2n^2/t\} \\ \le& c_1 n^d\exp\{c_3 t^{1+d/\theta} - c_2 n^2/t\}, \end{aligned} \tag{6.7}$$

where we have used a standard Brownian estimate (cf. [9] Section 1.7) and the $\mathbb{Z}^d$-stationarity in the second line, and Lemma 7.2 below in the third line. Then, it is easy to see that the second term in (6.6) is bounded from above by a constant and hence it is negligible compared with $\widetilde{N}^-(t)$.

Now let us turn to the estimate of the first term in (6.6). Note first that we can derive an upper large deviation bound

$$\mathbb{P}_\theta\left(\sup_{y\in\Lambda_{[t^{1+d/\theta}]}} V_\xi(y) \ge v\right) \le [t^{1+d/\theta}]^d\,\mathbb{P}_\theta\left(\sup_{y\in\Lambda_1} V_\xi(y) \ge v\right) \le \exp(-c_4 v^{1+\theta/d}) \tag{6.8}$$

which is valid for all sufficiently large t and $v \geq t$, from the exponential moment estimate in Lemma 7.2 below. Using this estimate, we get

$$\begin{aligned}
&\mathbb{E}_\theta \otimes E_x \bigg[\exp\bigg\{\int_0^t V_\xi(B_s)ds\bigg\} : \sup_{0\leq s\leq t} |B_s|_\infty < [t^{1+d/\theta}], \\
&\qquad\qquad \sup_{y\in\Lambda_{2[t^{1+d/\theta}]}} V_\xi(y) \geq t^{2d/\theta}\bigg] \\
\leq& \mathbb{E}_\theta \bigg[\exp\bigg\{t \sup_{y\in\Lambda_{2[t^{1+d/\theta}]}} V_\xi(y)\bigg\} : \sup_{y\in\Lambda_{2[t^{1+d/\theta}]}} V_\xi(y) \geq t^{2d/\theta}\bigg] \\
\leq& \sum_{n\geq t^{2d/\theta}} \exp\{tn\}\mathbb{P}_\theta\bigg(n-1 \leq \sup_{y\in\Lambda_{2[t^{1+d/\theta}]}} V_\xi(y) < n\bigg) \\
\leq& \sum_{n\geq t^{2d/\theta}} \exp\Big\{tn - c_4(n-1)^{1+\theta/d}\Big\}.
\end{aligned} \tag{6.9}$$

Since the last expression converges to 0 as $t \to \infty$, we can restrict ourselves on the event $\{\sup V_\xi(x) \leq t^{2d/\theta}\}$. Hereafter, we let $T = [t^{1+d/\theta}]$ since its exact form will be irrelevant in the sequel. Then, the Markov property at time $\varepsilon = t^{-2d/\theta}$ yields

$$\begin{aligned}
&\mathbb{E}_\theta \otimes E_x \bigg[\exp\bigg\{\int_0^t V_\xi(B_s)ds\bigg\} : \sup_{0\leq s\leq t} |B_s|_\infty < T, \sup_{y\in\Lambda_{2T}} V_\xi(y) < t^{2d/\theta}\bigg] \\
\leq& e\int_{\Lambda_{2T}} \frac{dy}{(2\pi\varepsilon)^{d/2}} \exp\Big(-\frac{|x-y|^2}{2\varepsilon}\Big) \\
&\times \mathbb{E}_\theta \otimes E_y \bigg[\exp\bigg\{\int_0^{t-\varepsilon} V_\xi(B_s)ds\bigg\} : \sup_{0\leq s\leq t-\varepsilon} |B_s|_\infty < T\bigg] \\
\leq& \frac{e}{(2\pi\varepsilon)^{d/2}} \int_{\Lambda_{2T}} dy \int_{\Lambda_{2T}} dz \mathbb{E}_\theta[\exp(-(t-\varepsilon)H^{-,D}_{\xi,2T})(y,z)],
\end{aligned} \tag{6.10}$$

where $\exp(-tH^{-,D}_{\xi,2T})(x,y)$, $t > 0$, $x, y \in \Lambda_{2T}$, is the integral kernel of the heat semigroup generated by the self-adjoint operator H^-_ξ on the L^2-space on the cube Λ_{2T} with the Dirichlet boundary condition.

Finally, we use the estimate

$$\exp(-tH^D_{\xi,2T})(y,z) \leq \big\{\exp(-tH^D_{\xi,2T})(y,y)\exp(-tH^D_{\xi,2T})(z,z)\big\}^{1/2}$$

for the kernel of self-adjoint semigroup and the Schwarz inequality to dominate the right hand side in (6.10) by $T^{2d}\widetilde{N}^-(t-\varepsilon)$ multiplied by some constant.

Combining all the estimates above, we finish the proof of (6.5). We can also prove (6.4) in the same way as (6.10). However it is much simpler since we do not have to care about $\sup V_\xi(\,\cdot\,)$ and thus we omit the details. □

The next lemma gives the converse relation between $\log S_{t,x}$ and $\log \widetilde{N}(t)$, while the lower estimate of $\log S^-_{t,x}$ will be derived directly. (See the proof of Theorem 6.3.)

Lemma 6.2. *For any $x \in \Lambda_1$ and $\varepsilon > 0$, we have*

$$\log \widetilde{N}(t) \leq \log S^{v,K'}_{t-\varepsilon,\,x}(1+o(1)) \tag{6.11}$$

as $t \to \infty$, where $S^{v,K'}_{t,\,x}$ is the expectation defined by replacing K and u by $K' = \{x \in K : d(x, K^c) \geq \sqrt{d}\}$ and $v(y) = \inf\{u(y-x+z) : z \in \Lambda_1\}$ respectively in (6.2). *Note that if u is a function satisfying the conditions in Theorem* 1.1 *or* 1.2, *then so is v.*

Proof. Let $\varepsilon > 0$ be an arbitrarily small number. By the Chapman-Kolmogorov identity, we have

$$\widetilde{N}(t) \leq (2\pi\varepsilon)^{-d/2} \int_{\Lambda_1} dz \mathbb{E}_\theta \otimes E_z \Bigg[\exp\Big\{ -\int_0^{t-\varepsilon} \sum_{q\in\mathbb{Z}^d} u(B_s - q - \xi_q) ds \Big\} \\ : B_s \notin \bigcup_{q\in\mathbb{Z}^d} (q + \xi_q + K) \text{ for } 0 \leq s \leq t-\varepsilon \Bigg].$$

The right hand side is dominated by $(2\pi\varepsilon)^{-d/2} S^{v,K'}_{t-\varepsilon,\,x}$ and the proof of (6.11) is completed. □

We now state our results on the asymptotics of $S_{t,\,x}$ and $S^-_{t,\,x}$:

Theorem 6.3. (i) *Assume $d = 1$ and* (1.5) *if $\alpha \leq 3$. Then we have*

$$\log S_{t,\,x} \begin{cases} \sim -t^{(1+\theta)/(\alpha+\theta)} \displaystyle\int_{\mathbb{R}} dq \inf_{y\in\mathbb{R}} \Big(\frac{C_0}{|q+y|^\alpha} + |y|^\theta \Big) & (1 < \alpha < 3), \\ \asymp -t^{(1+\theta)/(3+\theta)} & (\alpha = 3), \\ \sim -t^{(1+\theta)/(3+\theta)} \dfrac{3+\theta}{1+\theta} \Big(\dfrac{\pi^2}{8}\Big)^{(1+\theta)/(3+\theta)} & (\alpha > 3) \end{cases} \tag{6.12}$$

as $t \to \infty$, where $f(t) \sim g(t)$ means $\lim_{t\to\infty} f(t)/g(t) = 1$ and $f(t) \asymp g(t)$ means $0 < \underline{\lim}_{t\to\infty} f(t)/g(t) \leq \overline{\lim}_{t\to\infty} f(t)/g(t) < \infty$.

(ii) *Assume $d = 2$ and* (1.5) *if $\alpha \leq 4$. Then we have*

$$\log S_{t,\,x} \begin{cases} \sim -t^{(2+\theta)/(\alpha+\theta)} \displaystyle\int_{\mathbb{R}^2} dq \inf_{y\in\mathbb{R}^2} \Big(\frac{C_0}{|q+y|^\alpha} + |y|^\theta \Big) & (2 < \alpha < 4), \\ \asymp -t^{(2+\theta)/(4+\theta)} & (\alpha = 4), \\ \asymp -t^{(2+\theta)/(4+\theta)} (\log t)^{-\theta/(4+\theta)} & (\alpha > 4) \end{cases} \tag{6.13}$$

as $t \to \infty$.

(iii) *Assume $d \geq 3$ and* (1.5) *if $\alpha \leq d+2$. Then we have*

$$\log S_{t,\,x} \begin{cases} \sim -t^{(d+\theta)/(\alpha+\theta)} \displaystyle\int_{\mathbb{R}^d} dq \inf_{y\in\mathbb{R}^d} \Big(\frac{C_0}{|q+y|^\alpha} + |y|^\theta \Big) & (d < \alpha < d+2), \\ \asymp -t^{(d+\theta\mu)/(d+2+\theta\mu)} & (\alpha \geq d+2) \end{cases} \tag{6.14}$$

as $t \to \infty$, where $\mu = 2(\alpha-2)/(d(\alpha-d))$ as in Theorem 1.2.

(iv) *Assume* $\sup u = u(0) < \infty$ *and the existence of* $R_\varepsilon > 0$ *for any* $\varepsilon > 0$ *such that* $\operatorname{ess\,inf}_{B(R_\varepsilon)} u \geq u(0) - \varepsilon$. *Then we have*

$$\log S^-_{t,\,x} \sim t^{1+d/\theta} u(0)^{1+d/\theta} \int_{|q|\leq 1} dq(1 - |q|^\theta) \tag{6.15}$$

as $t \to \infty$.

Proof. We first consider the corresponding results for $\widetilde{N}(t)$ and $\widetilde{N}^-(t)$: the estimates (6.12)–(6.15) with $S_{t,\,x}$ and $S^-_{t,\,x}$ replaced by $\widetilde{N}(t)$ and $\widetilde{N}^-(t)$, respectively. These are already proven in earlier sections except for the case of $\alpha > d+2$ and $d \geq 2$. The results for the remaining case follow from Propositions 3.4 and 3.9 and Abelian theorems in [10]. Then by Lemma 6.1, we obtain the upper estimates of $S_{t,\,x}$ and $S^-_{t,\,x}$. For the lower estimates of $S_{t,\,x}$, we set $u^\#(y) = \sup\{u(y+x+z) : z \in \Lambda_1\} 1_{B(R_1)^c}(y) + 1_{B(R_1)}(y)$ with $R_1 \geq 0$. If u satisfies the conditions in Theorems 1.1 and 1.2, and R_1 is sufficiently large, then $u^\#$ also satisfies the same conditions. Therefore we obtain the corresponding lower estimates of $\widetilde{N}(t)$ where K is replaced by $B(R_2)$ with any $R_2 \geq R_1$ and u is replaced by $u^\#$. Then by Lemma 6.2, we obtain the corresponding lower estimates of $S^{v^\#, B(R_2+\sqrt{d})}_{t,\,x}$, where $v^\#(y) = \inf\{u^\#(y-x+z) : z \in \Lambda_1\}$. Since $K \subset B(R_2+\sqrt{d})$ and $v^\# \geq u$ on $B(R_2)^c$ for some $R_2 \geq R_1$, we obtain the corresponding lower estimates of $S_{t,\,x}$. For the lower estimate of $S^-_{t,\,x}$, we restrict the expectation to the event $B_s \in \Lambda_\varepsilon$ for any $s \in [1,t]$ to obtain

$$S^-_{t,\,x} \geq \int_{\Lambda_\varepsilon} dy e^{\Delta/2}(x,y) \int_{\Lambda_\varepsilon} dz e^{(t-1)\Delta^D_\varepsilon/2}(y,z) \widetilde{N}^-_1(t-1) \geq c_1 e^{-c_2 t} \widetilde{N}^-_1(t-1),$$

where $\widetilde{N}^-_1(t)$ is the function defined in (5.8), and $\exp(t\Delta/2)(x,y)$, $(t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and $\exp(t\Delta^D_\varepsilon/2)(x,y)$, $(t,x,y) \in (0,\infty) \times \Lambda_\varepsilon \times \Lambda_\varepsilon$ are the integral kernels of the heat semigroups generated by the Laplacian and the Dirichlet Laplacian on Λ_ε, respectively, multiplied by $-1/2$. Therefore the lower estimate of $S^-_{t,\,x}$ is given by our proof of Proposition 5.2. □

7. Appendix

We here state and prove two lemmas which we used before. The first one is to define the integrated density of states $N(\lambda)$ and to represent it by the Feynman-Kac formula:

Lemma 7.1. *Let* u *be a nonnegative function belonging to the class* K_d *and satisfying* (1.3). *Let* $\xi = (\xi_q)_{q\in\mathbb{Z}^d}$ *be a collection of independently and identically distributed* $\mathbb{R}^d$*-valued random variables satisfying* (1.2). *Then almost all sample functions of the random field defined by* $V_\xi(x) = \sum_{q\in\mathbb{Z}^d} u(x-q-\xi_q)$ *belong to the class* $K_{d,loc}$.

Proof. For any $\varepsilon, \delta > 0$, by the Chebyshev inequality, we have

$$\mathbb{P}_\theta(|\xi_q| \geq |q|^\varepsilon) \leq \mathbb{E}_\theta[(|\xi_q|/|q|^\varepsilon)^\delta] \leq c_1/|q|^{\varepsilon\delta}.$$

For any ε, there exists δ such that

$$\sum_{q\in\mathbb{Z}^d} \mathbb{P}_\theta(|\xi_q| \geq |q|^\varepsilon) < \infty.$$

By the Borel-Cantelli lemma, for almost all ξ, we have $N_\xi \in \mathbb{N}$ such that $|\xi_q| < |q|^\varepsilon < |q|/3$ for any $q \in \mathbb{Z} - B(N_\xi)$. By the condition (1.3) we also have R_ε such that $u(x) \leq (C_0 + \varepsilon)/|x|^\alpha$ for any $x \in B(R_\varepsilon)^c$. We now take $R > 0$ arbitrarily. If $x \in B(R)$ and $q \in \mathbb{Z}^d - B(3(R \vee R_\varepsilon) \vee N_\xi)$, then

$$|x - q - \xi_q| \geq |q| - |\xi_q| - |x| \geq |q|/3 \geq R_\varepsilon$$

and

$$V_\xi(x) \leq \sum_{q\in\mathbb{Z}^d\cap B(3(R\vee R_\varepsilon)\vee N_\xi)} u(x - q - \xi_q) + c_2.$$

Since the right hand side is a finite sum, we have $1_{B(R)}V_\xi \in K_d$. Since R is arbitrary, we complete the proof. □

The second is to define the integrated density of states $N^-(\lambda)$ and represent it by the Feynman-Kac formula. The following is enough to apply Theorem VI.1.1 in [3]. This lemma was also used in (6.8).

Lemma 7.2. *Let u be a bounded nonnegative function satisfying* (1.3). *Then there exist finite constants c_1 and c_2 such that*

$$\mathbb{E}_\theta\Big[\exp\Big(r \sup_{x\in\Lambda_1} V_\xi(x)\Big)\Big] \leq c_1 \exp(c_2 r^{1+d/\theta})$$

for any $r \geq 0$, where ξ and V_ξ are same as in the last lemma.

Proof. We first dominate as

$$\log \mathbb{E}_\theta\Big[\exp\Big(r \sup_{x\in\Lambda_1} V_\xi(x)\Big)\Big] \leq \int_{\mathbb{R}^d} \log I(q)dq,$$

where

$$I(q) = \mathbb{E}_\theta\Big[\exp\Big(r \sup_{x\in\Lambda_2} u(x - q - \xi_0)\Big)\Big].$$

For sufficiently large $R > 0$, we have $u(x) \leq 2C_0|x|^{-\alpha}$ for $|x| \geq R_0$. A sufficient condition for $\inf_{x\in\Lambda_2} |x - q - \xi_0| \geq R$ is $|q + \xi_0| \geq R + \sqrt{d}$. Then, for $q \in B(2(R + \sqrt{d}))^c$, we dominate as

$$\begin{aligned} I(q) \leq& \mathbb{E}_\theta\Big[\exp\Big(\sup_{x\in\Lambda_2} \frac{2rC_0}{|x - q - \xi_0|^\alpha}\Big) : |q + \xi_0| \geq \frac{|q|}{2}\Big] + e^{r \sup u}\mathbb{P}_\theta\Big(|q + \xi_0| < \frac{|q|}{2}\Big) \\ \leq& \exp\Big(\frac{2rC_0}{(|q|/2 - \sqrt{d})^\alpha}\Big)(1 + c_1 \exp(r \sup u - c_2|q|^\theta)) \end{aligned}$$

Since $\log(1+X) \le X$ for any $X \ge 0$, we have

$$\begin{aligned}&\int_{B(2(R+\sqrt{d}))^c} \log I(q)dq \\ &\le \int_{B(2(R+\sqrt{d}))^c} \frac{2rC_0}{(|q|/2-\sqrt{d})^\alpha}dq + \int_{B(2(R+\sqrt{d}))^c} c_1 \exp(r\sup u - c_2|q|^\theta))dq \\ &\le \frac{c_3 r}{R^{\alpha-d}} + c_4 \exp(r \sup u - c_5 R^\theta).\end{aligned}$$

By a simple uniform estimate, we have

$$\int_{B(2(R+\sqrt{d}))} \log I(q)dq \le c_6 r \sup u R^d.$$

Setting $R = (r\sup u/c_5)^{1/\theta}$, we have

$$\int \log I(q)dq \le c_7 r^{1+d/\theta}$$

for sufficiently large $r > 0$. □

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