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LAGRANGIAN FLOER THEORY ON COMPACT TORIC MANIFOLDS, I

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Abstract

We introduced the notion of weakly unobstructed *Lagrangian submanifolds and constructed their* potential function *($\mathfrak{PO}$) purely in terms of A-model data in [FOOO3]. In this article, we carry out explicit calculations involving $\mathfrak{PO}$ on toric manifolds and study the relationship between this class of Lagrangian submanifolds with the earlier work of Givental [G1], which advocates that the quantum cohomology ring is isomorphic to the Jacobian ring of a certain function, called the* Landau-Ginzburg superpotential. *Combining this study with the results from [FOOO3], we also apply the study to various examples to illustrate its implications to symplectic topology of Lagrangian fibers of toric manifolds. In particular, we relate it to the Hamiltonian displacement property of Lagrangian fibers and to Entov-Polterovich's symplectic quasi-states.*

Contents

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1. Introduction

The Floer theory of Lagrangian submanifolds has played an important role in symplectic geometry since Floer's invention [Fo] of the Floer cohomology and subsequent generalization to the class of *monotone* Lagrangian submanifolds [O1]. After the introduction of A_∞-structure in Floer theory [F1] and Kontsevich's homological mirror symmetry proposal [K], it has also played an essential role in a formulation of mirror symmetry in string theory.

In [FOOO1], we analyzed the anomaly $\partial^2 \neq 0$ and developed an obstruction theory for the definition of Floer cohomology and introduced the class of *unobstructed* Lagrangian submanifolds for which one can deform Floer's original definition of the "boundary" map by a suitable bounding cochain denoted by b. Expanding the discussion in [FOOO1, Section 7] and motivated by the work of Cho and Oh [CO], we also introduced the notion of *weakly unobstructed* Lagrangian submanifolds in [FOOO3, Chapter 3] which turns out to be the right class of Lagrangian submanifolds to look at in relation to the mirror symmetry of Fano toric A-model and Landau-Ginzburg B-model proposed by physicists (see [H], [HV]). In this article, we study the relationship between this class of Lagrangian submanifolds and the earlier work of Givental [G1], which advocates that the quantum cohomology ring is isomorphic to the Jacobian ring of a certain function, which is called the *Landau-Ginzburg superpotential*. Combining this study with the results from [FOOO3], we also apply this study to symplectic topology of Lagrangian fibers of toric manifolds.

While the appearance of bounding cochains is natural from the point of view of deformation theory, explicit computation thereof has not been carried out. One of the main purposes of this article is to perform this calculation in the case of fibers of toric manifolds and to draw its various applications. Especially, we show that each fiber $L(u)$ at $u \in \mathfrak{t}^*$ is weakly unobstructed for *any* toric manifold $\pi : X \to \mathfrak{t}^*$ (see Proposition 4.3), and we then show that the set of the pairs $(L(u), b)$ of a fiber $L(u)$ and a weak bounding cochain b with nontrivial Floer cohomology can be calculated from the quantum cohomology of the ambient toric manifold, at least in the Fano case. Namely, the set of such pairs $(L(u), b)$ is identified with the set of ring homomorphisms from quantum cohomology to the relevant Novikov ring. We also show by a variational analysis that for any compact toric manifold there exists at least one pair of (u, b)'s for which the Floer cohomology of $(L(u), b)$ is nontrivial.

We call a Lagrangian fiber (that is, a T^n-orbit) *balanced*, roughly speaking, if its Floer cohomology is nontrivial (see Definition 4.11 for its precise definition). The main result of this article can be summarized as follows.

(1) When X is a compact Fano toric manifold, we give a method to locate all the balanced fibers.

(2) Even when X is not Fano, we can still apply the same method to obtain a finite set of Lagrangian fibers. We prove that this set coincides with the set of balanced Lagrangian fibers under certain nondegeneracy condition. This condition can be easily checked when a toric manifold is given.

Now, a more precise statement of the main results is in order.

Let X be an n-dimensional smooth compact toric manifold. We fix a T^n-equivariant Kähler form on X, and we let $\pi : X \to \mathfrak{t}^* \cong (\mathbb{R}^n)^*$ be the moment map. The image $P = \pi(X) \subset (\mathbb{R}^n)^*$ is called the *moment polytope*. For $u \in \mathrm{Int}\, P$, we denote $L(u) = \pi^{-1}(u)$. The fiber $L(u)$ is a Lagrangian torus that is an orbit of the T^n-action (see Section 2; we refer readers to, e.g., [Au], [Fu] for the details on toric manifolds). We study the Floer cohomology defined in [FOOO3]. According to [FOOO1] and [FOOO3], we need extra data, the bounding cochain, to make the definition of Floer cohomology more flexible to allow a more general class of Lagrangian submanifolds. In the current context of Lagrangian torus fibers in toric manifolds, we use *weak bounding cochains*. Denote by $\mathscr{M}_{\mathrm{weak}}(L(u); \Lambda_0)$ the moduli space of (weak) bounding cochains for a weakly unobstructed Lagrangian submanifold $L(u)$ (see the end of Section 4).

In this situation, we first show that each element in $H^1(L(u); \Lambda_0)$ gives rise to a weak bounding cochain, that is, there is a natural embedding

$$H^1\big(L(u); \Lambda_0\big) \hookrightarrow \mathscr{M}_{\mathrm{weak}}\big(L(u); \Lambda_0\big) \tag{1.1}$$

(see Proposition 4.3). Here we use the universal Novikov ring

$$\Lambda = \Big\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \,\Big|\, a_i \in \mathbb{Q}, \lambda_i \in \mathbb{R}, \lim_{i\to\infty} \lambda_i = \infty \Big\}, \tag{1.2}$$

where T is a formal parameter. (We do not use the grading parameter e used in [FOOO3] since it does not play much of a role in this article.) Then Λ_0 is a subring of Λ defined by

$$\Lambda_0 = \Big\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \in \Lambda \,\Big|\, \lambda_i \geq 0 \Big\}. \tag{1.3}$$

We also use another subring

$$\Lambda_+ = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \in \Lambda \,\middle|\, \lambda_i > 0 \right\}. \tag{1.4}$$

We note that Λ is the field of fractions of Λ_0 and that Λ_0 is a local ring with maximal ideal Λ_+. Here we take the universal Novikov ring over $\mathbb{Q}$, but we also use the universal Novikov ring over $\mathbb{C}$ or other ring R, which we denote $\Lambda^{\mathbb{C}}$, Λ^R, respectively. (In case R does not contain $\mathbb{Q}$, Floer cohomology over Λ^R is defined only in the Fano case.)

Remark 1.1
If we strictly follow the way taken in [FOOO3], we get only the embedding $H^1(L(u); \Lambda_+) \hookrightarrow \mathcal{M}_{\text{weak}}(L(u))$, not (1.1). Here

$$\mathcal{M}_{\text{weak}}\big(L(u)\big) = \mathcal{M}_{\text{weak}}\big(L(u); \Lambda_+\big)$$

is defined in [FOOO3, Definitions 3.6.29, 4.3.21]. (We note that $\mathcal{M}_{\text{weak}}(L(u); \Lambda_+) \neq \mathcal{M}_{\text{weak}}(L(u); \Lambda_0)$, where the right-hand side is defined at the end of Section 4.)

However, we can modify the definition of weak unobstructedness so that (1.1) follows, using the idea of Cho [Cho, Section 2.1] (see Section 12).

For the rest of this article, we use the symbol b for an element of $\mathcal{M}_{\text{weak}}(L(u); \Lambda_+)$ and $\mathfrak{x}$ for an element of $\mathcal{M}_{\text{weak}}(L(u); \Lambda_0)$.

We next consider the quantum cohomology ring $QH(X; \Lambda)$ with the universal Novikov ring Λ as a coefficient ring (see Section 6). It is a commutative ring for the toric case, since $QH(X; \Lambda)$ is generated by cohomology classes of even degree.

Definition 1.2

(1) We define the set $\operatorname{Spec}(QH(X; \Lambda))(\Lambda^{\mathbb{C}})$ to be the set of Λ-algebra homomorphisms $\varphi : QH(X; \Lambda) \to \Lambda^{\mathbb{C}}$. (In other words, it is the set of all $\Lambda^{\mathbb{C}}$-valued points of the scheme $\operatorname{Spec}(QH(X; \Lambda))$.

(2) We next denote by $\mathfrak{M}(\mathfrak{Lag}(X))$ the set of all pairs $(\mathfrak{x}, u)$, $u \in \operatorname{Int} P$, $\mathfrak{x} \in H^1(L(u); \Lambda_0^{\mathbb{C}})/H^1(L(u); 2\pi\sqrt{-1}\mathbb{Z})$ such that

$$HF\big((L(u), \mathfrak{x}), (L(u), \mathfrak{x}); \Lambda^{\mathbb{C}}\big) \neq \{0\}.$$

THEOREM 1.3
If X is a Fano toric manifold, then

$$\operatorname{Spec}\big(QH(X; \Lambda)\big)(\Lambda^{\mathbb{C}}) \cong \mathfrak{M}\big(\mathfrak{Lag}(X)\big).$$

If $QH(X;\Lambda)$ is semisimple in addition, then we have

$$\sum_d \mathrm{rank}_{\mathbb{Q}} H_d(X;\mathbb{Q}) = \#\big(\mathfrak{M}(\mathfrak{Lag}(X))\big). \tag{1.5}$$

We remark that a commutative ring that is a finite-dimensional vector space over a field (e.g., Λ in our case) is semisimple if and only if it does not contain any nilpotent element. We also remark that a compact toric manifold is Fano if and only if every nontrivial holomorphic sphere has positive Chern number.

We believe that (1.5) still holds in the non-Fano case, but we have been unable to prove it to date. However, we can prove that there exists a fiber $L(u)$ whose Floer cohomology is nontrivial, by a method different from the proof of Theorem 1.3. Due to technical reasons, we can only prove the following slightly weaker statement.

THEOREM 1.4
Assume that the Kähler form ω of X is rational. Then there exists $u \in \mathrm{Int}\, P$ such that for any $\mathcal{N} \in \mathbb{R}_+$ there exists $\mathfrak{x} \in H^1(L(u);\Lambda_0^{\mathbb{R}})$ with

$$HF\big((L(u),\mathfrak{x}),(L(u),\mathfrak{x});\Lambda_0^{\mathbb{R}}/(T^{\mathcal{N}})\big) \cong H(T^n;\mathbb{R}) \otimes_{\mathbb{R}} \Lambda_0^{\mathbb{R}}/(T^{\mathcal{N}}).$$

We suspect that the rationality assumption in Theorem 1.4 can be removed. It is also likely that we can prove $\mathfrak{M}(\mathfrak{Lag}(X))$ is nonempty, but its proof at the moment is a bit cumbersome to write down. We can, however, derive the following theorem from Theorem 1.4, without rationality assumption.

THEOREM 1.5
Let X be an n-dimensional compact toric manifold. There exists $u_0 \in \mathrm{Int} P$ such that the following holds for any Hamiltonian diffeomorphism $\psi : X \to X$,

$$\psi\big(L(u_0)\big) \cap L(u_0) \neq \emptyset. \tag{1.6}$$

If in addition $\psi(L(u_0))$ is transversal to $L(u_0)$, then

$$\#\big(\psi(L(u_0)) \cap L(u_0)\big) \geq 2^n. \tag{1.7}$$

Theorem 1.5 is proved in Section 13.

We wish to point out that (1.6) can be derived from a more general intersection result, [EP1, Theorem 2.1], obtained by Entov and Polterovich with a different method using a very interesting notion of partial symplectic quasi-state constructed out of the spectral invariants defined in [Sc] and [O3] (see also [V] and [O2] for similar constructions in the context of exact Lagrangian submanifolds).

Remark 1.6
Strictly speaking, [EP1, Theorem 2.1] is stated under the assumption that X is semipositive and ω is rational because the theory of spectral invariant was developed in [O3] under these conditions. The rationality assumption was removed in [O4] and [Us], and the semipositivity assumption of ω was removed in [Us]. Thus the spectral invariant satisfying all the properties listed in [EP1, Section 5] is now established for an arbitrary compact symplectic manifold. By the argument of [EP1, Section 7], this implies the existence of a partial symplectic quasi-state. Therefore, the proof of [EP1, Theorem 2.1] goes through without these assumptions (semipositivity and rationality), and hence it implies (1.6) (see the introduction of [Us]). But the result (1.7) is new.

Our proof of Theorem 1.5 gives an explicit way of locating u_0, as we show in Section 9. (The method of [EP1] is indirect and does not provide a way of finding such u_0; see [EP2]. Below, we make some remarks concerning the Entov-Polterovich approach in the perspective of homological mirror symmetry.) In various explicit examples we can find more than one element u_0 that have the properties stated in this theorem. Following terminology employed in [CO], we call any such torus fiber $L(u_0)$ as in Theorem 1.4 a *balanced* Lagrangian torus fiber (see Definition 4.11 for its precise definition).

A criterion for $L(u_0)$ to be balanced, for the case $\mathfrak{x} = 0$, is provided by Cho and Oh [CO] and Cho [Cho] under the Fano condition. Our proofs of Theorems 1.4 and 1.5 are largely based on this criterion, and on the idea of Cho [Cho] of twisting *nonunitary* complex line bundles in the construction of Floer boundary operator. This criterion in turn specializes to the one predicted by physicists (see [HV], [H]), which relates the location of u_0 to the critical points of the Landau-Ginzburg superpotential.

A precise description of balanced Lagrangian fibers including the data of bounding cochains involves the notion of a *potential function*. In [FOOO3], the authors have introduced a function

$$\mathfrak{PO}^L : \mathscr{M}_{\mathrm{weak}}(L) \to \Lambda_0$$

for an arbitrary weakly unobstructed Lagrangian submanifold $L \subset (X, \omega)$. By varying the function $\mathfrak{PO}^L$ over $L \in \{\pi^{-1}(u) \mid u \in \mathrm{Int}\, P\}$, we obtain the potential function

$$\mathfrak{PO} : \bigcup_{L \in \{\pi^{-1}(u) \mid u \in \mathrm{Int}\, P\}} \mathscr{M}_{\mathrm{weak}}(L) \to \Lambda_0. \tag{1.8}$$

This function is constructed purely in terms of A-model data of the general symplectic manifold (X, ω) *without* using mirror symmetry.

For a toric (X, ω), the restriction of $\mathfrak{PO}$ to $H^1(L(u); \Lambda_+)$ (see (1.1)) can be made explicit when combined with the analysis of holomorphic discs attached to torus fibers of toric manifolds carried out in [CO], at least in the Fano case. (In the non-Fano case, we can make it explicit modulo "higher order terms.") This function extends to $H^1(L(u); \Lambda_0)$.

Remark 1.7
In [EP3], some relationships between quantum cohomology, quasi-state, spectral invariant and displacement of Lagrangian submanifolds are discussed. Consider an idempotent **i** of quantum cohomology. The (asymptotic) spectral invariants associated to **i** give rise to a partial symplectic quasi-state via the procedure concocted in [EP3], which in turn detects nondisplaceability of certain Lagrangian submanifolds. (The assumption of [EP1] is weaker than ours.)

In the current context of toric manifolds, we could also relate them to Floer cohomology and mirror symmetry in the following way. Quantum cohomology is decomposed into indecomposable factors (see Proposition 7.7). Let **i** be the idempotent corresponding to one of the indecomposable factors. Let $L = L(u(1, \mathbf{i}))$ be a Lagrangian torus fiber whose nondisplaceability is detected by the partial symplectic quasi-state obtained from **i**. We conjecture that Floer cohomology $HF(L(u(1, \mathbf{i}), \mathfrak{x}), (L(u(1, \mathbf{i}), \mathfrak{x})))$ is nontrivial for some $\mathfrak{x}$ (see Remark 5.8). This bounding cochain $\mathfrak{x}$ in turn is shown to be a critical point of the potential function $\mathfrak{PO}$ defined in [FOOO3].

On the other hand, **i** also determines a homomorphism $\varphi_{\mathbf{i}} : QH(X; \Lambda) \to \Lambda$. It corresponds to some Lagrangian fiber $L(u(2, \mathbf{i}))$ by Theorem 1.3. Then this implies via Theorem 4.10 that the fiber $L(u(2, \mathbf{i}))$ is nondisplaceable.

We conjecture that $u(1, \mathbf{i}) = u(2, \mathbf{i})$. We remark that $u(2, \mathbf{i})$ is explicitly calculable. Hence in view of the way $u(1, \mathbf{i})$ is found in [EP1], $u(1, \mathbf{i}) = u(2, \mathbf{i})$ gives some information on the asymptotic behavior of the spectral invariant associated with **i**.

We fix a basis of the Lie algebra $\mathfrak{t}$ of T^n which induces a basis of $\mathfrak{t}^*$ and hence a coordinate of the moment polytope $P \subset \mathfrak{t}^*$. This in turn induces a basis of $H^1(L(u); \Lambda_0)$ for each $u \in \operatorname{Int} P$ and so identification $H^1(L(u); \Lambda_0) \cong (\Lambda_0)^n$. We then regard the potential function as a function

$$\mathfrak{PO}(x_1, \dots, x_n; u_1, \dots, u_n) : (\Lambda_0)^n \times \operatorname{Int} P \to \Lambda_0$$

and prove in Theorem 4.10 that Floer cohomology $HF((L(u), \mathfrak{x}), (L(u), \mathfrak{x}); \Lambda)$ with $\mathfrak{x} = (\mathfrak{x}_1, \dots, \mathfrak{x}_n)$, $u = (u_1, \dots, u_n)$ is nontrivial if and only if $(\mathfrak{x}, u)$ satisfies

$$\frac{\partial \mathfrak{PO}}{\partial x_i}(\mathfrak{x}; u) = 0, \quad i = 1, \dots, n. \tag{1.9}$$

To study (1.9), it is useful to change the variables x_i to

$$y_i = e^{x_i}.$$

In these variables, we can write potential function as a sum

$$\mathfrak{PO}(x_1, \ldots, x_n; u_1, \ldots, u_n) = \sum T^{c_i(u)} P_i(y_1, \ldots, y_n),$$

where P_i are Laurent polynomials which do not depend on u, and $c_i(u)$ are positive real valued functions. When X is Fano, we can express the right-hand side as a finite sum (see Theorem 4.5).

We define a function $\mathfrak{PO}^u$ of y_i's by

$$\mathfrak{PO}^u(y_1, \ldots, y_n) = \mathfrak{PO}(x_1, \ldots, x_n; u_1, \ldots, u_n)$$

as a Laurent polynomial of n-variables with coefficient in Λ. We denote the set of Laurent polynomials by

$$\Lambda[y_1, \ldots, y_n, y_1^{-1}, \ldots, y_n^{-1}]$$

and consider its ideal generated by the partial derivatives of $\mathfrak{PO}^u$, namely,

$$\Big(\frac{\partial \mathfrak{PO}^u}{\partial y_i}; i = 1, \ldots, n\Big).$$

Definition 1.8
We call the quotient ring

$$\mathrm{Jac}(\mathfrak{PO}^u) = \frac{\Lambda[y_1, \ldots, y_n, y_1^{-1}, \ldots, y_n^{-1}]}{(\partial \mathfrak{PO}^u / \partial y_i; i = 1, \ldots, n)}$$

the *Jacobian ring* of $\mathfrak{PO}^u$.

We prove that the Jacobian ring is independent of the choice of u up to isomorphism (see the end of Section 6), and so we just write $\mathrm{Jac}(\mathfrak{PO})$ for $\mathrm{Jac}(\mathfrak{PO}^{\mathrm{u}})$ when there is no danger of confusion.

THEOREM 1.9
If X is Fano, then there exists a Λ-algebra isomorphism

$$\psi_u : QH(X; \Lambda) \to \mathrm{Jac}(\mathfrak{PO})$$

from quantum cohomology ring to the Jacobian ring such that

$$\psi_u\big(c_1(X)\big) = \mathfrak{PO}^u.$$

Theorem 1.9 (or Theorem 1.12 below) enables us to explicitly determine all the pairs $(\mathfrak{x}, u)$ with $HF((L(u), \mathfrak{x}), (L(u), \mathfrak{x}); \Lambda) \neq 0$ out of the quantum cohomology of X. More specifically, Batyrev's presentation of quantum cohomology in terms of the Jacobian ring plays an essential role for this purpose; we explain how this is done in Sections 7 and 8.

Remark 1.10
(1) The idea that the quantum cohomology ring coincides with the Jacobian ring begins with a celebrated article by Givental (see [G1, Theorem 5(1)]). There it was claimed also that the D-module defined by an oscillatory integral with the superpotential as its kernel is isomorphic to S^1-equivariant Floer cohomology of the periodic Hamiltonian system. When one takes its WKB limit, the former becomes the ring of functions on its characteristic variety, which is nothing but the Jacobian ring. The latter becomes the (small) quantum cohomology ring under the same limit. *Assuming* the Ansatz that quantum cohomology can be calculated by fixed-point localization, these claims are proved in a subsequent article [G2] for, at least, toric Fano manifolds. Then the required fixed-point localization is made rigorous later in [GP] (see also Iritani [I1]).

In physics literature, it has been advocated that the Landau-Ginzburg model of superpotential (that is, the potential function $\mathfrak{PO}$ in our situation) calculates quantum cohomology of X. A precise mathematical statement thereof is our Theorem 1.9 (see, e.g., [HKKP, page 473]).

Our main new idea in the proof of Theorem 1.3 (other than those already in [FOOO3]) is the way we combine them to extract information on Lagrangian submanifolds. In fact, Theorem 1.9 itself easily follows if we use the claim made by Batyrev that the quantum cohomology of a toric Fano manifold is a quotient of a polynomial ring by the ideal of relations, called the *quantum Stanley-Reisner relation* and the linear relation (this claim is now well established). We include this simple derivation in Section 6 for completeness, since it is essential to take the Novikov ring Λ as the coefficient ring in our applications. This version does not seem to have been proven in the literature in the form that we need.

(2) The proof of Theorem 1.9 given in this article does not contain a serious study of pseudoholomorphic spheres. The argument we outline in Remark 6.15 is based on open-closed Gromov-Witten theory, and it is different from other various methods that have been used to calculate Gromov-Witten invariants in the literature. In particular, this argument does not use the method of fixed-point localization. We will present this conceptual proof of Theorem 1.9 in a future sequel to this article.

(3) The isomorphism in Theorem 1.9 may be regarded as a particular case of the conjectural relation between quantum cohomology and Hochschild cohomology of Fukaya category (see Remark 6.15).

(4) In this article, we only involve small quantum cohomology rings but we can also include big quantum cohomology rings. Then, we expect, Theorem 1.9 can be enhanced to establish a relationship between the Frobenius structure of the deformation theory of quantum cohomology (see, e.g., [M]) and the Landau-Ginzburg model (which is due to K. Saito [S]). This statement (and Theorems 1.3 and 1.9) can be regarded as a version of mirror symmetry between the toric A-model and the Landau-Ginzburg B-model. In various literature on mirror symmetry, such as [Ab], [AKO], and [U], the B-model is dealt with for Fano or toric manifolds in which the derived category of coherent sheaves is studied, while the A-model is dealt with for Landau-Ginzburg A-models where the directed A_∞-category of Seidel [Se2] is studied.

(5) In [Ar], Auroux discussed a mirror symmetry between the A-model side of toric manifolds and the B-model side of Landau-Ginzburg models. The discussion of [Ar] uses Floer cohomology with $\mathbb{C}$-coefficients. In this article, we use Floer cohomology over the Novikov ring, which is more suitable for the applications to symplectic topology.

(6) Even when X is not necessarily Fano, we can still prove a similar isomorphism

$$\psi_u : QH^\omega(X;\Lambda) \cong \mathrm{Jac}(\mathfrak{PO}_0) \tag{1.10}$$

where the left-hand side is the Batyrev quantum cohomology ring (see Definition 6.4) and the right-hand side is the Jacobian ring of some function $\mathfrak{PO}_0$; it coincides with the actual potential function $\mathfrak{PO}$ "up to higher order terms" (see (4.9)). In the Fano case $\mathfrak{PO}_0 = \mathfrak{PO}$; (1.10) is Proposition 6.8.

(7) During the final stage of writing this article, another article, [CL] by Chan and Leung, appeared in which the above isomorphism was studied via SYZ transformations. Chan and Leung give a proof of this isomorphism for the case where X is a product of projective spaces and use the coefficient ring $\mathbb{C}$, not the Novikov ring. Leung presented their result [CL] in a conference held at Kyoto University in January 2008, where Fukaya also presented the content of this article.

From our definition, it follows that the leading-order potential function $\mathfrak{PO}_0$ (see (4.9)) can be extended to the whole product $(\Lambda_0^{\mathbb{C}})^n \times \mathbb{R}^n$ so that they are invariant under the translations by elements in $(2\pi\sqrt{-1}\mathbb{Z})^n \subseteq (\Lambda_0^{\mathbb{C}})^n$. Hence we may regard $\mathfrak{PO}_0$ as a function defined on

$$\big(\Lambda_0^{\mathbb{C}}/(2\pi\sqrt{-1}\mathbb{Z})\big)^n \times (\mathbb{R}^n)^* \cong \big(\Lambda_0^{\mathbb{C}}/(2\pi\sqrt{-1}\mathbb{Z})\big)^n \times \mathbb{R}^n.$$

In the non-Fano case, the function $\mathfrak{PO}$ is invariant under the translations by elements in $(2\pi\sqrt{-1}\mathbb{Z})^n \subseteq (\Lambda_0^{\mathbb{C}})^n$ but may not extend to $\Lambda_0^{\mathbb{C}}/(2\pi\sqrt{-1}\mathbb{Z}))^n \times \mathbb{R}^n$. This is because the infinite sum appearing in the right-hand side of (4.7) may not converge in non-Archimedean topology for $u \notin \operatorname{Int} P$.

Definition 1.11
We denote by

$$\operatorname{Crit}(\mathfrak{PO}_0), \quad (\text{respectively, } \operatorname{Crit}(\mathfrak{PO}))$$

the subset of pairs

$$(\mathfrak{x}, u) \in \big(\Lambda_0^{\mathbb{C}}/(2\pi\sqrt{-1}\mathbb{Z})\big)^n \times \mathbb{R}^n, \quad (\text{respectively, } (\mathfrak{x}, u) \in \big(\Lambda_0^{\mathbb{C}}/(2\pi\sqrt{-1}\mathbb{Z})\big)^n \times \operatorname{Int} P)$$

satisfying the equation

$$\frac{\partial \mathfrak{PO}_0}{\partial x_i}(\mathfrak{x}; u) = 0, \quad \left(\text{respectively, } \frac{\partial \mathfrak{PO}}{\partial x_i}(\mathfrak{x}; u) = 0\right),$$

$i = 1, \ldots, n$.

We define $\mathfrak{M}(\mathfrak{Lag}(X))$ in Definition 1.2. (We use the same definition in the non-Fano case.) In view of Theorem 1.12(2) below, we also introduce the subset

$$\mathfrak{M}_0\big(\mathfrak{Lag}(X)\big) = \{(\mathfrak{x}, u) \in \operatorname{Crit}(\mathfrak{PO}_0) \mid u \in \operatorname{Int} P\}.$$

We also note that $\mathfrak{PO}_0 = \mathfrak{PO}$ in case X is Fano. The following is a more precise form of Theorem 1.3.

THEOREM 1.12
(1) *There exists a bijection*

$$\operatorname{Spec}\big(QH^{\omega}(X; \Lambda)\big)(\Lambda^{\mathbb{C}}) \cong \operatorname{Crit}(\mathfrak{PO}_0).$$

(2) *There exists a bijection*

$$\mathfrak{M}\big(\mathfrak{Lag}(X)\big) \cong \operatorname{Crit}(\mathfrak{PO}).$$

(3) *If X is Fano and $\mathfrak{x}$ is a critical point of $\mathfrak{PO}_0^u$, then* $u \in \operatorname{Int} P$.
(4) *If $QH^{\omega}(X; \Lambda)$ is semisimple, then*

$$\sum_d \operatorname{rank}_\Lambda QH^{\omega}(X; \Lambda) = \#\big(\operatorname{Crit}(\mathfrak{PO}_0)\big).$$

Remark 1.13
(1) Theorem 1.12(3) does not hold in the non-Fano case. We give a counterexample (Example 8.2) in Section 8. In fact, in the case of Example 8.2 some of the critical points of $\mathfrak{PO}_0$ correspond to a point $u \in \mathbb{R}^n$ which lies outside the moment polytope.

(2) In the Fano case, Theorem 1.12(3) implies that

$$\mathfrak{M}\big(\mathfrak{Lag}(X)\big) = \mathrm{Crit}(\mathfrak{PO}) \cong \mathrm{Crit}(\mathfrak{PO}_0).$$

We note that $\mathfrak{PO}_0$ is explicitly computable. But we do not know the explicit form of $\mathfrak{PO}$. However, we can show that elements of $\mathrm{Crit}(\mathfrak{PO}_0)$ and of $\mathrm{Crit}(\mathfrak{PO})$ can be naturally related to each other under a mild nondegeneracy condition (see Theorem 10.4). So we can use $\mathfrak{PO}_0$ in place of $\mathfrak{PO}$ in most of the cases. For example, we can use it to prove the following.

THEOREM 1.14
For any k, there exists a Kähler form on $X(k)$, the k-points blow-up of $\mathbb{C}P^2$, that is toric and has exactly $k+1$ balanced fibers.

Balanced fiber satisfies conclusions (1.6), (1.7) of Theorem 1.5 (see Definition 4.11 for the definition of balanced fibers). We prove Theorem 1.14 in Section 10.

Remark 1.15
The cardinality of $\mathfrak{x} \in H^1(L(u); \Lambda_0^{\mathbb{C}})/H^1(L(u); 2\pi\sqrt{-1}\mathbb{Z})$ with nonvanishing Floer cohomology is an invariant of Lagrangian submanifold $L(u)$. This is a consequence of [FOOO3, Theorem G] (equivalent to [FOOO2, Theorem G]).

The organization of this article is now in order. In Section 2, we gather some basic facts on toric manifolds and fix our notation. Section 3 is a brief review of Lagrangian Floer theory in [FOOO1] and [FOOO3]. In Section 4, we describe our main results on the potential function $\mathfrak{PO}$ and on its relation to the Floer cohomology. We illustrate these theorems by several examples and derive their consequences in Sections 5–10. We postpone their proofs until Sections 11–13.

In Section 5, we illustrate explicit calculations involving the potential functions in such examples as $\mathbb{C}P^n$, $S^2 \times S^2$, and the 2-point blow-up of $\mathbb{C}P^2$. We also discuss a relationship between displacement energy of Lagrangian submanifolds (see Definition 5.9) and Floer cohomology. In Sections 6 and 7, we prove the results that mainly apply to the Fano case; in particular, we prove Theorems 1.3, 1.9, and 1.12 in those sections. Section 7 contains some applications of Theorem 1.9, especially to the case of monotone torus fibers and to the $\mathbb{Q}$-structure of the quantum cohomology ring. In Section 8, we first illustrate usage of (the proof of) Theorem 1.3 to locate balanced fibers by the example of the 1-point blow-up of $\mathbb{C}P^2$. We then turn to the study of non-Fano cases and discuss Hirzebruch surfaces. Section 8 also contains some discussion on the semisimplicity of quantum cohomology.

In Sections 9 and 10, we prove the results that can be used in all toric cases, whether they are Fano or not. In Section 9, using variational analysis, we prove existence of a critical point of the potential function, which is an important step toward the proof of Theorems 1.4 and 1.5. Using the arguments of this section, we can locate a balanced fiber in any compact toric manifold, explicitly solving simple linear equalities and inequalities finitely many times. In Section 10, we prove that we can find solutions of (1.9) by studying its reduction to $\mathbb{C} = \Lambda_0^{\mathbb{C}}/\Lambda_+^{\mathbb{C}}$, which we call the leading-term equation. This result is purely algebraic. It implies that our method of locating balanced fibers, which is used in the proof of Theorem 1.3, can also be used in the non-Fano case under certain nondegeneracy condition. We apply this method to prove Theorem 1.14. We also discuss an example of blow-up of $\mathbb{C}P^n$ along the high-dimensional blow-up center $\mathbb{C}P^m$ in Section 10, giving several other examples and demonstrating various interesting phenomena that occur in Lagrangian Floer theory. For example, we provide a sequence $((X, \omega_i), L_i)$ of pairs that have nonzero Floer cohomology for some choice of bounding cochains, while its limit $((X, \omega), L)$ has vanishing Floer cohomology for any choice of bounding cochain (Example 10.17). We also provide an example of Lagrangian submanifold L such that it has a nonzero Floer cohomology over $\Lambda^{\mathbb{C}}$ for some choice of bounding cochain, but vanishing Floer cohomology for any choice of bounding cochain over the field Λ^F with a field F of characteristic 3 (Example 10.19).

In Section 11, we review the results on the moduli space of holomorphic discs from [CO] which are used in the calculation of the potential function. We rewrite them in the form that can be used for the purpose of this article. We also discuss the non-Fano case in this section. (Our result is less explicit in the non-Fano case, but still can be used to explicitly locate balanced fibers in most of the cases.) In Section 12, we use the idea of Cho [Cho] to deform Floer cohomology by an element from $H^1(L(u); \Lambda_0)$ rather than from $H^1(L(u); \Lambda_+)$. This enhancement is crucial to obtain an optimal result about the nondisplacement of Lagrangian fibers. In Section 13, we use those results to calculate Floer cohomology and complete the proof of Theorems 1.4 and 1.5.

We attempted to make this article largely independent of our book (see [FOOO1], [FOOO3]) as much as possible and also to make the relationship of the contents of the article with the general story transparent. Here are a few examples.

(1) Our definition of the potential function for the fibers of toric manifolds in this article is given in a way independent of that of [FOOO3] except for the statement on the existence of compatible Kuranishi structures and multisections on the moduli space of pseudoholomorphic discs, which provides a rigorous definition of Floer

cohomology of single Lagrangian fiber. Such details are provided in [FOOO3, Section 7.1] (equivalent to [FOOO2, Section 29]).

(2) Similarly, the definition of A_∞-algebra in this article on the Lagrangian fiber of toric manifolds is also independent of the book except for the process going from $A_{n,K}$-structure to A_∞-structure, for which we refer to [FOOO3, Section 7.2] (equivalent to [FOOO2, Section 30]). However, for all the applications in this article, only the existence of $A_{n,K}$-structures is needed.

(3) The property of the Floer cohomology $HF(L, L)$ detecting the Lagrangian intersection of L with its Hamiltonian deformation relies on the fact that Floer cohomology of the pair is independent under the Hamiltonian isotopy. This independence is established in [FOOO3]. In the toric case, its alternative proof based on the de Rham version is given in [FOOO5] in a more general form than we need here.

2. Compact toric manifolds

In this section, we summarize basic facts on the toric manifolds and set up our notations to be consistent with those in [CO], which in turn closely follow those in Batyrev [B1] and Audin [Au].

2.1. Complex structure

In order to obtain an n-dimensional compact toric manifold X, we need a combinatorial object Σ, a *complete fan of regular cones*, in an n-dimensional vector space over $\mathbb{R}$.

Let N be the lattice $\mathbb{Z}^n$, and let $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the dual lattice of rank n. Let $N_{\mathbb{R}} = N \otimes \mathbb{R}$, and let $M_{\mathbb{R}} = M \otimes \mathbb{R}$.

Definition 2.1

A convex subset $\sigma \subset N_{\mathbb{R}}$ is called a *regular k-dimensional cone* ($k \geq 1$) if there exist k linearly independent elements $v_1, \ldots, v_k \in N$ such that

$$\sigma = \{a_1 v_1 + \cdots + a_k v_k \mid a_i \in \mathbb{R}, a_i \geq 0\},$$

and the set $\{v_1, \ldots, v_k\}$ is a subset of some $\mathbb{Z}$-basis of N. In this case, we call $v_1, \ldots, v_k \in N$ the *integral generators of* σ.

Definition 2.2

A regular cone σ' is called a *face* of a regular cone σ (we write $\sigma' \prec \sigma$) if the set of integral generators of σ' is a subset of the set of integral generators of σ.

Definition 2.3

A finite system $\Sigma = \sigma_1, \ldots, \sigma_s$ of regular cones in $N_{\mathbb{R}}$ is called a *complete n-dimensional fan* of regular cones if the following conditions are satisfied:

(1) if $\sigma \in \Sigma$ and $\sigma' \prec \sigma$, then $\sigma' \in \Sigma$;

(2) if σ, σ' are in Σ, then $\sigma' \cap \sigma \prec \sigma$ and $\sigma' \cap \sigma \prec \sigma'$;
(3) $N_{\mathbb{R}} = \sigma_1 \cup \cdots \cup \sigma_s$.

The set of all k-dimensional cones in Σ is denoted by $\Sigma^{(k)}$.

Definition 2.4
Let Σ be a complete n-dimensional fan of regular cones. Denote by $G(\Sigma) = \{v_1, \dots, v_m\}$ the set of all generators of 1-dimensional cones in Σ ($m = \text{Card}\, \Sigma^{(1)}$). We call a subset $\mathscr{P} = \{v_{i_1}, \dots, v_{i_p}\} \subset G(\Sigma)$ a *primitive collection* if $\{v_{i_1}, \dots, v_{i_p}\}$ does not generate p-dimensional cones in Σ, while for all $k\,(0 \le k < p)$ each k-element subset of $\mathscr{P}$ generates a k-dimensional cone in Σ.

Definition 2.5
Let $\mathbb{C}^m$ be an m-dimensional affine space over $\mathbb{C}$ with the set of coordinates $z_1, \dots, z_m$ which are in one-to-one correspondence $z_i \leftrightarrow v_i$ with elements of $G(\Sigma)$. Let $\mathscr{P} = \{v_{i_1}, \dots, v_{i_p}\}$ be a primitive collection in $G(\Sigma)$. Denote by $\mathbb{A}(\mathscr{P})$ the $(m-p)$-dimensional affine subspace in $\mathbb{C}^n$ defined by the equations

$$z_{i_1} = \cdots = z_{i_p} = 0.$$

Since every primitive collection $\mathscr{P}$ has at least two elements, the codimension of $\mathbb{A}(\mathscr{P})$ is at least 2.

Definition 2.6
Define the closed algebraic subset $Z(\Sigma)$ in $\mathbb{C}^m$ as follows:

$$Z(\Sigma) = \bigcup_{\mathscr{P}} \mathbb{A}(\mathscr{P}),$$

where $\mathscr{P}$ runs over all primitive collections in $G(\Sigma)$. Put

$$U(\Sigma) = \mathbb{C}^m \setminus Z(\Sigma).$$

Definition 2.7
Let $\mathbb{K}$ be the subgroup in $\mathbb{Z}^m$ consisting of all lattice vectors $\lambda = (\lambda_1, \dots, \lambda_m)$ such that

$$\lambda_1 v_1 + \cdots + \lambda_m v_m = 0.$$

Obviously, $\mathbb{K}$ is isomorphic to $\mathbb{Z}^{m-n}$, and we have the exact sequence

$$0 \to \mathbb{K} \to \mathbb{Z}^m \xrightarrow{\pi} \mathbb{Z}^n \to 0, \tag{2.1}$$

where the map π sends the basis vectors e_i to v_i for $i = 1, \dots, m$.

Definition 2.8
Let Σ be a complete n-dimensional fan of regular cones. Define $D(\Sigma)$ to be the connected commutative subgroup in $(\mathbb{C}^*)^m$ generated by all 1-parameter subgroups

$$a_\lambda : \mathbb{C}^* \to (\mathbb{C}^*)^m,$$

$$t \mapsto (t^{\lambda_1}, \ldots, t^{\lambda_m}),$$

where $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{K}$.

It is easy to see from the definition that $D(\Sigma)$ acts freely on $U(\Sigma)$. Now we are ready to give a definition of the compact toric manifold X_Σ associated with a complete n-dimensional fan of regular cones Σ.

Definition 2.9
Let Σ be a complete n-dimensional fan of regular cones. Then the quotient

$$X_\Sigma = U(\Sigma)/D(\Sigma)$$

is called the *compact toric manifold associated with* Σ.

There exists a simple open covering of $U(\Sigma)$ by affine algebraic varieties.

PROPOSITION 2.10
Let σ be a k-dimensional cone in Σ generated by $\{v_{i_1}, \ldots, v_{i_k}\}$. Define the open subset $U(\sigma) \subset \mathbb{C}^m$ as

$$U(\sigma) = \Big\{(z_1, \ldots, z_m) \in \mathbb{C}^m \,\Big|\, z_j \neq 0 \;\; \textit{for all } j \notin \{i_1, \ldots, i_k\}\Big\}.$$

Then the open sets $U(\sigma)$ have the properties
(1)

$$U(\Sigma) = \bigcup_{\sigma \in \Sigma} U(\sigma);$$

(2) *if $\sigma \prec \sigma'$, then $U(\sigma) \subset U(\sigma')$;*
(3) *for any two cone $\sigma_1, \sigma_2 \in \Sigma$, one has $U(\sigma_1) \cap U(\sigma_2) = U(\sigma_1 \cap \sigma_2)$; in particular,*

$$U(\Sigma) = \sum_{\sigma \in \Sigma^{(n)}} U(\sigma).$$

PROPOSITION 2.11

Let σ be an n-dimensional cone in $\Sigma^{(n)}$ generated by $\{v_{i_1}, \dots, v_{i_n}\}$, which spans the lattice M. We denote the dual $\mathbb{Z}$-basis of the lattice N by $\{u_{i_1}, \dots, u_{i_n}\}$, that is,

$$\langle v_{i_k}, u_{i_l}\rangle = \delta_{k,l}, \tag{2.2}$$

where $\langle\cdot,\cdot\rangle$ is the canonical pairing between lattices N and M.

Then the affine open subset $U(\sigma)$ is isomorphic to $\mathbb{C}^n \times (\mathbb{C}^*)^{m-n}$, the action of $D(\Sigma)$ on $U(\sigma)$ is free, and the space of $D(\Sigma)$-orbits is isomorphic to the affine space $U_\sigma = \mathbb{C}^n$ whose coordinate functions $y_1^\sigma, \dots, y_n^\sigma$ are n Laurent monomials in $z_1, \dots, z_m$:

$$\begin{cases} y_1^\sigma = z_1^{\langle v_1, u_{i_1}\rangle} \cdots z_m^{\langle v_m, u_{i_1}\rangle} \\ \qquad \vdots \\ y_n^\sigma = z_1^{\langle v_1, u_{i_n}\rangle} \cdots z_m^{\langle v_m, u_{i_n}\rangle} \end{cases} \tag{2.3}$$

The last statement yields a general formula for the local affine coordinates $y_1^\sigma, \dots, y_n^\sigma$ of a point $p \in U_\sigma$ as functions of its "homogeneous coordinates" $z_1, \dots, z_m$.

2.2. Symplectic structure

In Section 2.1, we associated a compact manifold X_Σ to a fan Σ. In this section, we review the construction of symplectic (Kähler) manifolds associated to a convex polytope P.

Let M be a dual lattice; we consider a convex polytope P in $M_\mathbb{R}$ defined by

$$\bigl\{u \in M_\mathbb{R} \bigm| \langle u, v_j\rangle \ge \lambda_j \text{ for } j = 1, \dots, m\bigr\}, \tag{2.4}$$

where $\langle\cdot,\cdot\rangle$ is a dot product of $M_\mathbb{R} \cong \mathbb{R}^n$; namely, v_j's are inward normal vectors to the codimension 1 faces of the polytope P. We associate to it a fan in the lattice N as follows. With any face Γ of P, fix a point u_0 in the (relative) interior of Γ and define

$$\sigma_\Gamma = \bigcup_{r \ge 0} r \cdot (P - u_0).$$

The associated fan is the family $\Sigma(P)$ of dual convex cones

$$\begin{aligned} \check{\sigma}_\Gamma &= \bigl\{x \in N_\mathbb{R} \bigm| \langle y, x\rangle \ge 0 \ \ \forall y \in \sigma_\Gamma\bigr\} &(2.5)\\ &= \bigl\{x \in N_\mathbb{R} \bigm| \langle u, x\rangle \le \langle p, x\rangle \ \ \forall p \in P, u \in \Gamma\bigr\}, &(2.6) \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is dual pairing $M_\mathbb{R}$ and $N_\mathbb{R}$. Hence we obtain a compact toric manifold $X_{\Sigma(P)}$ associated to a fan $\Sigma(P)$.

Now we define a symplectic (Kähler) form on $X_{\Sigma(P)}$ as follows. Recall the exact sequence

$$0 \to \mathbb{K} \xrightarrow{i} \mathbb{Z}^m \xrightarrow{\pi} \mathbb{Z}^n \to 0.$$

It induces another exact sequence:

$$0 \to K \to \mathbb{R}^m/\mathbb{Z}^m \to \mathbb{R}^n/\mathbb{Z}^n \to 0.$$

Denote by k the Lie algebra of the real torus K. Then we have the exact sequence of Lie algebras

$$0 \to k \to \mathbb{R}^m \xrightarrow{\pi} \mathbb{R}^n \to 0.$$

We also have the dual of the exact sequence above:

$$0 \to (\mathbb{R}^n)^* \to (\mathbb{R}^m)^* \xrightarrow{i^*} k^* \to 0.$$

Now, consider $\mathbb{C}^m$ with symplectic form $i/2 \sum dz_k \wedge d\bar{z}_k$. The standard action T^n on $\mathbb{C}^n$ is hamiltonian with the moment map

$$\mu(z_1, \ldots, z_m) = \frac{1}{2}(|z_1|^2, \ldots, |z_m|^2). \tag{2.7}$$

For the moment map μ_K of the K-action is then given by

$$\mu_K = i^* \circ \mu : \mathbb{C}^m \to k^*.$$

If we choose a $\mathbb{Z}$-basis of $\mathbb{K} \subset \mathbb{Z}^m$ as

$$Q_1 = (Q_{11}, \ldots, Q_{m1}), \ldots, Q_k = (Q_{1k}, \ldots, Q_{mk})$$

and $\{q^1, \ldots, q^k\}$ be its dual basis of $\mathbb{K}^*$. Then the map i^* is given by the matrix Q^t, and so we have

$$\mu_K(z_1, \ldots, z_m) = \frac{1}{2}\Big(\sum_{j=1}^m Q_{j1}|z_j|^2, \ldots, \sum_{j=1}^m Q_{jk}|z_j|^2\Big) \in \mathbb{R}^k \cong k^* \tag{2.8}$$

in the coordinates associated to the basis $\{q^1, \ldots, q^k\}$. We denote again by μ_K the restriction of μ_K on $U(\Sigma) \subset \mathbb{C}^m$.

PROPOSITION 2.12 ([Au, Chapter VII])
Then for any $r = (r_1, \ldots, r_{m-n}) \in \mu_K(U(\Sigma)) \subset k^*$, we have a diffeomorphism

$$\mu_K^{-1}(r)/K \cong U(\Sigma)/D(\Sigma) = X_\Sigma. \tag{2.9}$$

And for each (regular) value of $r \in k^*$*, we can associate a symplectic form* ω_P *on the manifold* X_Σ *by symplectic reduction (see [MW]).*

To obtain the original polytope P that we started with, we need to choose r as follows. Consider λ_j for $j = 1, \ldots, m$ which we used to define our polytope P by the set of inequalities $\langle u, v_j \rangle \geq \lambda_j$. Then, for each $a = 1, \ldots, m - n$, let

$$r_a = -\sum_{j=1}^{m} Q_{ja}\lambda_j.$$

Then we have

$$\mu_K^{-1}(r_1, \ldots, r_{m-n})/K \cong X_{\Sigma(P)},$$

and for the residual $T^n \cong T^m/K$-action on $X_{\Sigma(P)}$ and for its moment map π, we have

$$\pi(X_{\Sigma(P)}) = P.$$

Using Delzant's theorem [D], one can reconstruct the symplectic form out of the polytope P (up to T^n-equivariant symplectic diffeomorphisms). In fact, Guillemin [Gu] proved the following explicit closed formula for the T^n-invariant Kähler form associated to the canonical complex structure on $X = X_\Sigma(P)$.

THEOREM 2.13 ([Gu, Theorem 4.5])
Let P*,* $X_{\Sigma(P)}$*,* ω_P *be as above, and let*

$$\pi : X_{\Sigma(P)} \to (\mathbb{R}^m/k)^* \cong (\mathbb{R}^n)^*$$

be the associated moment map. Define the functions on $(\mathbb{R}^n)^*$

$$\ell_i(u) = \langle u, v_i \rangle - \lambda_i \text{ for } i = 1, \ldots, m \tag{2.10}$$

$$\ell_\infty(u) = \sum_{i=1}^{m} \langle u, v_i \rangle = \Big\langle u, \sum_{i=1}^{m} v_i \Big\rangle.$$

Then we have

$$\omega_P = \sqrt{-1}\partial\bar\partial\bigg(\pi^*\Big(\sum_{i=1}^{m} \lambda_i(\log \ell_i) + \ell_\infty\Big)\bigg) \tag{2.11}$$

on Int P.

The affine functions ℓ_i play an important role in our description of potential function as in [CO] since they also measure symplectic areas $\omega(\beta_i)$ of the canonical generators

β_i of $H_2(X, L(u); \mathbb{Z})$. More precisely, we have

$$\omega(\beta_i) = 2\pi \ell_i(u) \tag{2.12}$$

(see [CO, Theorem 8.1]). We also recall that

$$P = \{u \in M_{\mathbb{R}} \mid \ell_i(u) \geq 0,\ i = 1, \dots, m\} \tag{2.13}$$

by definition (2.4).

3. Deformation theory of filtered A_∞-algebras

In this section, we provide a quick summary of the deformation and obstruction theory of Lagrangian Floer cohomology developed in [FOOO1] and [FOOO3] for the reader's convenience. We also refer the reader to Ohta's survey article [Oh] for a more detailed review and to [FOOO3] for complete details of the proofs of the results described in this section.

We start our discussion with the classical unfiltered A_∞-algebra. Let C be a graded R-module where R is the coefficient ring. We denote by $C[1]$ its suspension defined by $C[1]^k = C^{k+1}$. Define the *bar complex* $B(C[1])$ by

$$B_k(C[1]) = \underbrace{C[1] \otimes \cdots \otimes C[1]}_{k}, \quad B(C[1]) = \bigoplus_{k=0}^{\infty} B_k(C[1]).$$

Here $B_0(C[1]) = R$ by definition. $B(C[1])$ has the structure of *graded coalgebra*.

Definition 3.1

The structure of *A_∞-algebra* is a sequence of R-module homomorphisms

$$\mathfrak{m}_k : B_k(C[1]) \to C[1], \quad k = 1, 2, \dots,$$

of degree +1 such that the coderivation $\widehat{d} = \sum_{k=1}^{\infty} \widehat{\mathfrak{m}}_k$ satisfies $\widehat{d}\,\widehat{d} = 0$, which is called the *$A_\infty$-relation*. Here we denote by $\widehat{\mathfrak{m}}_k : B(C[1]) \to B(C[1])$ the unique extension of $\mathfrak{m}_k$ as a coderivation on $B(C[1])$, that is

$$\widehat{\mathfrak{m}}_k(x_1 \otimes \cdots \otimes x_n) = \sum_{i=1}^{k-i+1} (-1)^* x_1 \otimes \cdots \otimes \mathfrak{m}_k(x_i, \dots, x_{i+k-1}) \otimes \cdots \otimes x_n, \tag{3.1}$$

where $* = \deg x_1 + \cdots + \deg x_{i-1} + i - 1$.

The relation $\widehat{d}\,\widehat{d} = 0$ can be written as

$$\sum_{k=1}^{n-1}\sum_{i=1}^{k-i+1} (-1)^{*}\mathfrak{m}_{n-k+1}(x_1 \otimes \cdots \otimes \mathfrak{m}_k(x_i, \ldots, x_{i+k-1}) \otimes \cdots \otimes x_n) = 0,$$

where $*$ is the same as above. In particular, we have $\mathfrak{m}_1\mathfrak{m}_1 = 0$ and so it defines a complex $(C, \mathfrak{m}_1)$.

A *weak* (or *curved*) A_∞-algebra is defined in the same way, except that it also includes the $\mathfrak{m}_0$-term $\mathfrak{m}_0 : R \to B(C[1])$. The first two terms of the A_∞-relation for a weak A_∞-algebra are given as

$$\mathfrak{m}_1\bigl(\mathfrak{m}_0(1)\bigr) = 0, \quad \mathfrak{m}_1\mathfrak{m}_1(x) + (-1)^{\deg x+1}\mathfrak{m}_2\bigl(x, \mathfrak{m}_0(1)\bigr) + \mathfrak{m}_2\bigl(\mathfrak{m}_0(1), x\bigr) = 0.$$

In particular, for the case of weak A_∞-algebras, $\mathfrak{m}_1$ does not satisfy boundary property, that is, $\mathfrak{m}_1\mathfrak{m}_1 \neq 0$ in general.

We now recall the notion of *unit* in A_∞-algebra.

Definition 3.2
An element $\mathbf{e} \in C^0 = C[1]^{-1}$ is called a *unit* if it satisfies
(1) $\mathfrak{m}_{k+1}(x_1, \ldots, \mathbf{e}, \ldots, x_k) = 0$ for $k \geq 2$ or $k = 0$,
(2) $\mathfrak{m}_2(\mathbf{e}, x) = (-1)^{\deg x}\mathfrak{m}_2(x, \mathbf{e}) = x$ for all x.

Combining this definition of unit and (3.1), we have the following immediate lemma.

LEMMA 3.3
Consider an A_∞-algebra $(C[1], \mathfrak{m})$ over a ground ring R for which $\mathfrak{m}_0(1) = \lambda\mathbf{e}$ for some $\lambda \in R$. Then $\mathfrak{m}_1\mathfrak{m}_1 = 0$.

Now we explain the notion of the *filtered A_∞-algebra*. We define the *universal Novikov ring* $\Lambda_{0,\mathrm{nov}}$ by

$$\Lambda_{0,\mathrm{nov}} = \Bigl\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} e^{n_i} \Bigm| a_i \in R, n_i \in \mathbb{Z}, \lambda_i \in \mathbb{R}_{\geq 0}, \lim_{i\to\infty} \lambda_i = \infty \Bigr\}.$$

This is a graded ring by defining $\deg T = 0$, $\deg e = 2$. Let $\Lambda_{0,\mathrm{nov}}^{+}$ be its maximal ideal, which consists of the elements $\sum_{i=1}^{\infty} a_i T^{\lambda_i} e^{n_i}$ with $\lambda_i > 0$.

Let $\bigoplus_{m\in\mathbb{Z}} C^m$ be the free graded $\Lambda_{0,\mathrm{nov}}$-module over the basis $\{\mathbf{v}_i\}$. We define a filtration $\bigoplus_{m\in\mathbb{Z}} F^\lambda C^m$ on it such that $\{T^\lambda \mathbf{v}_i\}$ is a free basis of $\bigoplus_{m\in\mathbb{Z}} F^\lambda C^m$. Here $\lambda \in \mathbb{R}$, $\lambda \geq 0$. We call this filtration the *energy filtration*. (Our algebra $\bigoplus_{m\in\mathbb{Z}} C^m$ may not be finitely generated, so we need to take completion.) We denote by C the completion of $\bigoplus_{m\in\mathbb{Z}} C^m$ with respect to the energy filtration. The filtration induces a natural non-Archimedean topology on C.

A *filtered* A_∞*-algebra* $(C, \mathfrak{m})$ is a weak A_∞-algebra such that A_∞-operators $\mathfrak{m}$ have these properties:

(1) $\mathfrak{m}_k$ respect the energy filtration;

(2) $\mathfrak{m}_0(1) \in F^\lambda C^1$ with $\lambda > 0$;

(3) the reduction $\mathfrak{m}_k$ mod $\Lambda^+_{0,\mathrm{nov}} : B_k\overline{C}[1] \otimes R[e, e^{-1}] \to \overline{C} \otimes R[e, e^{-1}]$ does not contain e; more precisely, it has the form $\overline{\mathfrak{m}}_k \otimes {}_R R[e, e^{-1}]$, where $\overline{\mathfrak{m}}_k : B_k\overline{C}[1] \to \overline{C}$ is an R-module homomorphism (here $\overline{C}$ is the free R-module over the basis $\mathbf{v}_i$).

For further details, see [FOOO3, Definition 3.2.20], which is equivalent to [FOOO2, Definition 7.20].

Remark 3.4

(1) In [FOOO3], we assume that $\mathfrak{m}_0 = 0$ for the (unfiltered) A_∞-algebra. On the other hand, $\mathfrak{m}_0 = 0$ is not assumed for the filtered A_∞-algebra. The filtered A_∞-algebra satisfying $\mathfrak{m}_0 = 0$ is said to be *strict*.

(2) In this section, to be consistent with the exposition given in [FOOO3], we use the Novikov ring $\Lambda_{0,\mathrm{nov}}$ which includes the variable e. In [FOOO3], the variable e is used so that the operations $\mathfrak{m}_k$ come to have degree 1 for all k (with respect to the shifted degree). But for the applications of this article, it is enough to use the $\mathbb{Z}_2$-grading and so encoding the degree with a formal parameter is not necessary. Therefore, we use the ring Λ_0 in other sections, which do not contain e. An advantage of using the ring Λ_0 is that it is a local ring while $\Lambda_{0,\mathrm{nov}}$ is not. This makes it easier to use some standard results from commutative algebra in later sections. We note that a $\mathbb{Z}$-graded complex over $\Lambda_{0,\mathrm{nov}}$ is equivalent to the $\mathbb{Z}_2$-graded complex over Λ_0.

Next we explain how one can deform the given filtered A_∞-algebra $(C, \mathfrak{m})$ by an element $b \in F^\lambda C[1]^0$ with $\lambda > 0$, by re-defining the A_∞-operators as

$$\mathfrak{m}^b_k(x_1, \dots, x_k) = \mathfrak{m}(e^b, x_1, e^b, x_2, e^b, x_3, \dots, x_k, e^b).$$

This defines a new weak A_∞-algebra for arbitrary b.

Here we simplify notation by writing $e^b = 1+b+b\otimes b+\cdots+b\otimes\cdots\otimes b+\cdots$. Note that each summand in this infinite sum has degree zero in $C[1]$. When the ground ring is $\Lambda_{0,\mathrm{nov}}$, the infinite sum converges in the non-Archimedean topology since $b \in F^\lambda C[1]^0$ with $\lambda > 0$.

PROPOSITION 3.5
For A_∞-algebra $(C, \mathfrak{m}_k^b)$, $\mathfrak{m}_0^b \equiv 0 \mod \Lambda_{0,\mathrm{nov}}\{\mathbf{e}\}$ if and only if b satisfies

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b, \ldots, b) \equiv 0 \mod \Lambda_{0,\mathrm{nov}}\{\mathbf{e}\}. \tag{3.2}$$

We call the equation (3.2) the *A_∞-Maurer-Cartan equation*.

Definition 3.6
Let $(C, \mathfrak{m})$ be a filtered A_∞-algebra in general, and let $BC[1]$ be its bar-complex. An element $b \in F^\lambda C[1]^0$ $(\lambda > 0)$ is called a *weak bounding cochain* if it satisfies the equation (3.2). If the b satisfies the strict equation

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b, \ldots, b) = 0,$$

then we call it a (strict) *bounding cochain*.

For the rest of this article, we also call a weak bounding cochain just a *bounding cochain* since we mainly focus on weak bounding cochains. In general, a given A_∞-algebra may or may not have a solution to (3.2).

Definition 3.7
A filtered A_∞-algebra is called *weakly unobstructed* if the equation (3.2) has a solution $b \in F^\lambda C[1]^0$ with $\lambda > 0$.

One can define a notion of gauge equivalence between two bounding cochains as described in [FOOO3, Section 4.3] (equivalent to [FOOO2, Section 16]).

The way a filtered A_∞-algebra is attached to a Lagrangian submanifold $L \subset (M, \omega)$ arises as an A_∞-deformation of the classical singular cochain complex including the instanton contributions. In particular, when there is no instanton contribution, as in the case $\pi_2(M, L) = 0$, it reduces to an A_∞-deformation of the singular cohomology in the chain level including all possible higher Massey products.

We now describe the basic A_∞-operators $\mathfrak{m}_k$ in the context of A_∞-algebra of Lagrangian submanifolds. For a given compatible almost complex structure J, consider the moduli space $\mathscr{M}_{k+1}(\beta; L)$ of stable maps of genus zero. It is a compactification of

$$\big\{\big(w, (z_0, z_1, \ldots, z_k)\big) \,\big|\, \overline{\partial}_J w = 0,\ z_i \in \partial D^2,\ [w] = \beta \ \text{ in}\, \pi_2(M, L)\big\} / \sim,$$

where $\sim$ is the conformal reparameterization of the disc D^2. The expected dimension of this space is given by $n + \mu(\beta) - 3 + (k + 1) = n + \mu(\beta) + k - 2$.

Now given k singular chains $[P_1, f_1], \ldots, [P_k, f_k] \in C_*(L)$ of L, we put the cohomological grading $\deg P_i = n - \dim P_i$, and we regard the chain complex $C_*(L)$ as the cochain complex $C^{\dim L - *}(L)$. We consider the fiber product

$$\mathrm{ev}_0 : \mathscr{M}_{k+1}(\beta; L) \times_{(\mathrm{ev}_1,\ldots,\mathrm{ev}_k)} (P_1 \times \cdots \times P_k) \to L,$$

where $\mathrm{ev}_i([w, (z_0, z_1, \ldots, z_k)]) = w(z_i)$.

A simple calculation shows that we have the expected degree

$$\deg\big[\mathscr{M}_{k+1}(\beta; L) \times_{(\mathrm{ev}_1,\ldots,\mathrm{ev}_k)} (P_1 \times \cdots \times P_k), \mathrm{ev}_0\big] = \sum_{j=1}^{n}(\deg P_j - 1) + 2 - \mu(\beta).$$

For each given $\beta \in \pi_2(M, L)$ and $k = 0, \ldots,$ we define

$$\mathfrak{m}_{k,\beta}(P_1, \ldots, P_k) = \big[\mathscr{M}_{k+1}(\beta; L) \times_{(\mathrm{ev}_1,\ldots,\mathrm{ev}_k)} (P_1 \times \cdots \times P_k), \mathrm{ev}_0\big] \tag{3.3}$$

and $\mathfrak{m}_k = \sum_{\beta\in\pi_2(M,L)} \mathfrak{m}_{k,\beta} \cdot T^{\omega(\beta)} e^{\mu(\beta)/2}$.

Now we denote by $C[1]$ the completion of a *suitably chosen countably generated* (singular) chain complex with $\Lambda_{0,\mathrm{nov}}$ as its coefficients with respect to the non-Archimedean topology. (We regard $C[1]$ as a *cochain* complex.) Then by choosing a system of multivalued perturbations of the right-hand side of (3.3) and a triangulation of its zero sets, the map $\mathfrak{m}_k : B_k(C[1]) \to C[1]$ is defined, has degree 1, and is continuous with respect to non-Archimedean topology. We extend $\mathfrak{m}_k$ as a coderivation $\widehat{\mathfrak{m}}_k : BC[1] \to BC[1]$ by (3.1). Finally, we take the sum

$$\widehat{d} = \sum_{k=0}^{\infty} \widehat{\mathfrak{m}}_k : BC[1] \to BC[1]. \tag{3.4}$$

A main theorem proven in [FOOO1] and [FOOO3] then is the following coboundary property.

THEOREM 3.8 ([FOOO3, Theorem 3.5.11], [FOOO2, Theorem 10.11])
Let L be an arbitrary compact relatively spin Lagrangian submanifold of an arbitrary tame symplectic manifold (M, ω). The coderivation $\widehat{d}$ is a continuous map that satisfies the A_∞-relation $\widehat{d}\widehat{d} = 0$.

The A_∞-algebra associated to L in this way has the *homotopy unit*, not a *unit*. In general, a filtered A_∞-algebra with homotopy unit canonically induces another filtered *unital* A_∞-algebra called a *canonical model* of the given filtered A_∞-algebra. In the geometric context of the A_∞-algebra associated to a Lagrangian submanifold $L \subset M$ of a general symplectic manifold (M, ω), the canonical model is defined on the cohomology group $H^*(L;\Lambda_{0,\mathrm{nov}})$. (See [FOOO4] for a quick explanation of construction by summing over trees of this canonical model.)

Once the A_∞-algebra is attached to each Lagrangian submanifold L, we then construct a filtered A_∞*-bimodule* $C(L,L')$ for the transversal pair of Lagrangian submanifolds L and L'. Here $C(L,L')$ is the free $\Lambda_{0,\mathrm{nov}}$-module such that its basis is identified with $L\cap L'$. The filtered A_∞-bimodule structure is by definition a family of operators

$$\mathfrak{n}_{k_1,k_2} : B_{k_1}\big(C(L)[1]\big)\ \widehat{\otimes}_{\Lambda_{0,\mathrm{nov}}}\ C(L,L')\ \widehat{\otimes}_{\Lambda_{0,\mathrm{nov}}}\ B_{k_2}\big(C(L')[1]\big) \to C(L,L')$$

for $k_1,k_2\geq 0$. (Here $\widehat{\otimes}_{\Lambda_{0,\mathrm{nov}}}$ is the completion of the algebraic tensor product.) Let us briefly describe the definition of $\mathfrak{n}_{k_1,k_2}$. A typical element of the tensor product $B_{k_1}(C(L)[1])\ \widehat{\otimes}_{\Lambda_{0,\mathrm{nov}}}\ C(L,L')\ \widehat{\otimes}_{\Lambda_{0,\mathrm{nov}}}\ B_{k_2}(C(L')[1])$ has the form

$$P_{1,1}\otimes\cdots\otimes P_{1,k_1}\otimes\langle p\rangle\otimes P_{2,1}\otimes\cdots\otimes P_{2,k_2}$$

with $p\in L\cap L'$. Then the image $\mathfrak{n}_{k_1,k_2}$ thereof is given by

$$\sum_{q,B} T^{\omega(B)}e^{\mu(B)/2}\#\big(\mathscr{M}(p,q;B;P_{1,1},\dots,P_{1,k_1};P_{2,1},\dots,P_{2,k_2})\big)\langle q\rangle.$$

Here B denotes homotopy class of Floer trajectories connecting p and q, the summation is taken over all (q,B) with

$$\mathrm{vir.dim}\,\mathscr{M}(p,q;B;P_{1,1},\dots,P_{1,k_1};P_{2,1},\dots,P_{2,k_2})=0,$$

and $\#(\mathscr{M}(p,q;B;P_{1,1},\dots,P_{1,k_1};P_{2,1},\dots,P_{2,k_2}))$ is the "number" of elements in the "zero"-dimensional moduli space $\mathscr{M}(p,q;B;P_{1,1},\dots,P_{1,k_1};P_{2,1},\dots,P_{2,k_2})$. Here the moduli space $\mathscr{M}(p,q;B;P_{1,1},\dots,P_{1,k_1};P_{2,1},\dots,P_{2,k_2})$ is the Floer moduli space $\mathscr{M}(p,q;B)$ cut down by intersecting with the given chains $P_{1,i}\subset L$ and $P_{2,j}\subset L'$.

THEOREM 3.9 ([FOOO3, Theorem 3.7.21], [FOOO2, Theorem 12.21])
Let (L,L') be an arbitrary relatively spin pair of compact Lagrangian submanifolds. Then the family $\{\mathfrak{n}_{k_1,k_2}\}$ defines a left $(C(L),\mathfrak{m})$ and right $(C(L'),\mathfrak{m}')$ filtered A_∞-bimodule structure on $C(L,L')$.

What this theorem means is explained below as Proposition 3.10.

Let $B(C(L)[1])\,\widehat{\otimes}_{\Lambda_{0,\mathrm{nov}}}\,C(L,L')\,\widehat{\otimes}_{\Lambda_{0,\mathrm{nov}}}\,B(C(L')[1])$ be the completion of the direct sum of $B_{k_1}(C(L)[1])\,\widehat{\otimes}_{\Lambda_{0,\mathrm{nov}}}\,C(L,L')\,\widehat{\otimes}_{\Lambda_{0,\mathrm{nov}}}\,B_{k_2}(C(L')[1])$ over $k_1\geq 0$, $k_2\geq 0$. We define the boundary operator $\widehat{d}$ on it by using the maps $\mathfrak{n}_{k_1,k_2}$ and $\mathfrak{m}_k$, $\mathfrak{m}'_k$, as in the following:

$$
\begin{aligned}
&\widehat{d}\big((x_1 \otimes \cdots \otimes x_n) \otimes x \otimes (x'_1 \otimes \cdots \otimes x'_m)\big) \\
&\quad = \widehat{d}(x_1 \otimes \cdots \otimes x_n) \otimes x \otimes (x'_1 \otimes \cdots \otimes x'_m) \\
&\qquad + (-1)^{\deg x_1 + \cdots + \deg x_n + \deg x + n + 1} (x_1 \otimes \cdots \otimes x_n) \otimes x \otimes \widehat{d}(x'_1 \otimes \cdots \otimes x'_m) \\
&\qquad + \sum_{k_1 \le n} \sum_{k_2 \le m} (-1)^{\deg x_1 + \cdots + \deg x_{n-k_1} + n - k_1} (x_1 \otimes \cdots \otimes x_{n-k_1}) \\
&\qquad\qquad\qquad \otimes \mathfrak{n}_{k_1,k_2}\big((x_{n-k_1+1} \otimes \cdots \otimes x_n) \otimes x \otimes (x'_1 \otimes \cdots \otimes x'_{k_2})\big) \\
&\qquad\qquad\qquad \otimes (x'_{k_2+1} \otimes \cdots \otimes x'_m).
\end{aligned}
$$

Here $\widehat{d}$ in the second and the third lines are induced from $\mathfrak{m}$ and $\mathfrak{m}'$ by Formula (3.4), respectively.

PROPOSITION 3.10
The map $\widehat{d}$ satisfies $\widehat{d}\,\widehat{d} = 0$.

The A_∞-bimodule structure, which defines a boundary operator on the bar complex, induces an operator $\delta = \mathfrak{n}_{0,0}$ on a much smaller, ordinary free $\Lambda_{0,\mathrm{nov}}$-module $C(L, L')$ generated by the intersections $L \cap L'$. However, the boundary property of this Floer's "boundary" map δ again meets obstruction coming from the obstructions cycles of either L, L', or of both. We need to deform δ using suitable bounding cochains of L, L'.

In the case where both L, L' are weakly unobstructed, we can carry out this deformation of $\mathfrak{n}$ using weak bounding chains b and b' of fibered A_∞-algebras associated to L and L', respectively, in a way similar to $\mathfrak{m}^b$; namely, we define $\delta_{b,b'} : C(L, L') \to C(L, L')$ by

$$\delta_{b,b'}(x) = \sum_{k_1,k_2} \mathfrak{n}_{k_1,k_2}(b^{\otimes k_1} \otimes x \otimes b'^{\otimes k_2}) = \widehat{\mathfrak{n}}(e^b, x, e^{b'}).$$

We can generalize the story to the case where L has clean intersection with L', especially to the case $L = L'$. In the case $L = L'$, we have $\mathfrak{n}_{k_1,k_2} = \mathfrak{m}_{k_1+k_2+1}$. So in this case, we have $\delta_{b,b'}(x) = \mathfrak{m}(e^b, x, e^{b'})$.

In general, $\delta_{b,b'}$ does not satisfy the equation $\delta_{b,b'}\delta_{b,b'} = 0$. It turns out that there is an elegant condition for $\delta_{b,b'}\delta_{b,b'} = 0$ to hold in terms of the *potential function* introduced in [FOOO3], which we explain in Section 4. In the case $\delta_{b,b'}\delta_{b,b'} = 0$, we define Floer cohomology by

$$HF\big((L, b), (L', b'); \Lambda_{0,\mathrm{nov}}\big) = \operatorname{Ker} \delta_{b,b'} / \operatorname{Im} \delta_{b,b'}.$$

Let Λ_{nov} be the field of fractions of $\Lambda_{0,\mathrm{nov}}$. We define $HF((L,b),(L',b');\Lambda_{\mathrm{nov}})$ by extending the coefficient ring $\Lambda_{0,\mathrm{nov}}$ to Λ_{nov}. Then $HF((L,b),(L',b');\Lambda_{\mathrm{nov}})$ is invariant under the Hamiltonian isotopies of L and L'. Therefore, we can use it to obtain the following result about nondisplacement of Lagrangian submanifolds.

THEOREM 3.11 ([FOOO3, Theorem G], [FOOO2, Theorem G])
Assume that $\delta_{b,b'}\delta_{b,b'} = 0$. Let $\psi : X \to X$ be a Hamiltonian diffeomorphism such that $\psi(L)$ is transversal to L'. Then we have

$$\#\big(\psi(L) \cap L'\big) \geq \mathrm{rank}_{\Lambda_{\mathrm{nov}}} HF\big((L,b),(L',b');\Lambda_{\mathrm{nov}}\big).$$

The Floer cohomology $HF((L,b),(L',b');\Lambda_{0,\mathrm{nov}})$ with coefficient $\Lambda_{0,\mathrm{nov}}$ is not invariant under the Hamiltonian isotopy. However, we can prove the following Theorem 3.12. There exists an integer a and positive numbers λ_i $(i = 1,\dots,b)$ such that

$$HF\big((L,b),(L',b');\Lambda_{0,\mathrm{nov}}\big) = \Lambda_{0,\mathrm{nov}}^{\oplus a} \oplus \bigoplus_{i=1}^{b} (\Lambda_{0,\mathrm{nov}}/T^{\lambda_i}\Lambda_{0,\mathrm{nov}})$$

(see [FOOO3, Theorem 6.1.20], [FOOO2, Theorem 24.20]). Let $\psi : X \to X$ be a Hamiltonian diffeomorphism; $\|\psi\|$ is its Hofer distance (see [Ho]) from the identity map.

THEOREM 3.12 ([FOOO3, Theorem J], [FOOO2, Theorem J])
If $\psi(L)$ is transversal to L', then we have an inequality:

$$\#\big(\psi(L) \cap L'\big) \geq a + 2\,\#\{i \mid \lambda_i \geq \|\psi\|\}.$$

In later sections, we apply Theorems 3.11, 3.12 to study nondisplacement of Lagrangian fibers of toric manifolds.

4. Potential function

The A_∞-structure defined on a countably generated chain complex $C(L,\Lambda_0)$ itself explained in the previous section is not suitable for explicit calculations as in our study of toric manifolds. For this computational purpose, we work with the filtered A_∞-structure on the canonical model defined on $H(L;\Lambda_0)$ which has a *finite* rank over Λ_0. Furthermore, this has a strict unit $\mathbf{e}$ given by the dual of the fundamental class $PD([L])$. Recall that $C(L,\Lambda_0)$ itself has only a homotopy-unit.

An element $b \in H^1(L;\Lambda_+)$ is called a *weak bounding cochain* if it satisfies the A_∞-Maurer-Cartan equation

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b,\dots,b) \equiv 0 \mod PD([L]) \tag{4.1}$$

where $\{\mathfrak{m}_k\}_{k=0}^{\infty}$ is the A_∞-structure associated to L, $[L] \in H_n(L)$ is the fundamental class, and $PD([L]) \in H^0(L)$ is its Poincaré dual. We denote by $\widehat{\mathscr{M}}_{\text{weak}}(L)$ the set of weak bounding cochains of L. We say L is *weakly unobstructed* if $\widehat{\mathscr{M}}_{\text{weak}}(L) \neq \emptyset$. The moduli space $\mathscr{M}_{\text{weak}}(L)$ is then defined to be the quotient space of $\widehat{\mathscr{M}}_{\text{weak}}(L)$ by suitable gauge equivalence (see [FOOO3, Chapters 3, 4] for more explanations).

LEMMA 4.1 ([FOOO3, Lemma 3.6.32], [FOOO2, Lemma 11.32])
If $b \in \widehat{\mathscr{M}}_{\text{weak}}(L)$, then $\delta_{b,b} \circ \delta_{b,b} = 0$, where $\delta_{b,b}$ is the deformed Floer operator defined by

$$\delta_{b,b}(x) = \mathfrak{m}_1^b(x) =: \sum_{k,\ell \geq 0} \mathfrak{m}_{k+\ell+1}(b^{\otimes k}, x, b^{\otimes \ell}).$$

For $b \in \widehat{\mathscr{M}}_{\text{weak}}(L)$, we define

$$HF\big((L;b),(L;b)\big) = \frac{\operatorname{Ker}(\delta_{b,b} : C \to C)}{\operatorname{Im}(\delta_{b,b} : C \to C)},$$

where C is an appropriate subcomplex of the singular chain complex of L. When L is weakly unobstructed (i.e., $\widehat{\mathscr{M}}_{\text{weak}}(C) \neq \emptyset$), we define a function

$$\mathfrak{PO} : \widehat{\mathscr{M}}_{\text{weak}}(C) \to \Lambda_+$$

by the equation

$$\mathfrak{m}(e^b) = \mathfrak{PO}(b) \cdot PD([L]).$$

This is the *potential function* introduced in [FOOO3].

THEOREM 4.2 ([FOOO3, Proposition 3.7.17], [FOOO2, Proposition 12.17])
For each $b \in \widehat{\mathscr{M}}_{\text{weak}}(L)$ and $b' \in \widehat{\mathscr{M}}_{weak}(L')$, the map $\delta_{b,b'}$ defines a continuous map $\delta_{b,b'} : CF(L,L') \to CF(L,L')$ that satisfies $\delta_{b,b'}\delta_{b,b'} = 0$, provided that

$$\mathfrak{PO}(b) = \mathfrak{PO}(b'). \tag{4.2}$$

Therefore, for each pair (b, b') of $b \in \widehat{\mathscr{M}}_{\text{weak}}(L)$ and for $b' \in \widehat{\mathscr{M}}_{\text{weak}}(L')$ that satisfy (4.2), we define the (b, b')-*Floer cohomology* of the pair (L, L') by

$$HF\big((L, b), (L', b'); \Lambda_{\text{nov}}\big) = \frac{\text{Ker}\delta_{b,b'}}{\text{Im }\delta_{b,b'}}.$$

In the rest of this section, we state the main results concerning the detailed structure of the potential function for the case of Lagrangian fibers of toric manifolds.

For the later analysis of examples, we recall from [FOOO1] and [FOOO3] that $\mathfrak{m}_k$ is further decomposed into

$$\mathfrak{m}_k = \sum_{\beta \in \pi_2(M,L)} \mathfrak{m}_{k,\beta} \otimes T^{\omega(\beta)} e^{\mu(\beta)/2}.$$

Here μ is the Maslov index.

First, we remove the grading parameter e from the ground ring. Second, to eliminate many appearances of 2π in front of the affine function ℓ_i in the exponents of the parameter T later in this article, we redefine T as $T^{2\pi}$. Under this arrangement, we get the formal power series expansion

$$\mathfrak{m}_k = \sum_{\beta \in \pi_2(M,L)} \mathfrak{m}_{k,\beta} \otimes T^{\omega(\beta)/2\pi}, \tag{4.3}$$

which we use throughout this article.

Now we restrict to the case of the toric manifold. Let $X = X_\Sigma$ be associated to a complete regular fan Σ (in other words, Σ is the normal fan of X), and let $\pi : X \to \mathfrak{t}^*$ be the moment map of the action of the torus $T^n \cong T^m/K$. We make the identifications

$$\mathfrak{t} = \text{Lie}(T^n) \cong N^n_{\mathbb{R}} \cong \mathbb{R}^n, \qquad \mathfrak{t}^* \cong M_{\mathbb{R}} \cong (\mathbb{R}^n)^*.$$

We use $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ exclusively instead of $\mathfrak{t}$ (or $\mathbb{R}^n$) and $\mathfrak{t}^*$ (or $(\mathbb{R}^n)^*$), as much as possible to be consistent with the standard notation in toric geometry.

Denote the image of $\pi : X \to M_{\mathbb{R}}$ by $P \subset M_{\mathbb{R}}$, which is the moment polytope of the T^n-action on X.

We prove the following in Section 11.

PROPOSITION 4.3
For any $u \in \text{Int}P$, the fiber $L(u)$ is weakly unobstructed. Moreover, we have the canonical inclusion

$$H^1\big(L(u); \Lambda_+\big) \hookrightarrow \mathscr{M}_{\text{weak}}\big(L(u)\big).$$

Choose an integral basis $\mathbf{e}_i^*$ of N, and let $\mathbf{e}_i$ be its dual basis on M. With this choice made, we identify $M_{\mathbb{R}}$ with $\mathbb{R}^n$ as long as its meaning is obvious from the context. Identifying $H_1(T^n;\mathbb{Z})$ with $N \cong \mathbb{Z}^n$ via $T^n = \mathbb{R}^n/\mathbb{Z}^n$, we regard $\mathbf{e}_i$ as a basis of $H^1(L(u);\mathbb{Z})$. The following immediately follows by definition.

LEMMA 4.4
We write $\pi = (\pi_1, \dots, \pi_n) : X \to M_{\mathbb{R}}$ using the coordinate of $M_{\mathbb{R}}$ associated to the basis $\mathbf{e}_i$. Let $S_i^1 \subset T^n$ be the subgroup whose orbit represents $\mathbf{e}_i^* \in H_1(T^n;\mathbb{Z})$. Then π_i is proportional to the moment map of S_i^1-action on X.

Let

$$b = \sum x_i \mathbf{e}_i \in H^1\big(L(u);\Lambda_+\big) \subset \mathcal{M}_{\mathrm{weak}}\big(L(u)\big).$$

We study the potential function

$$\mathfrak{PO} : H^1\big(L(u);\Lambda_+\big) \to \Lambda_+.$$

Once a choice of the family of bases $\{\mathbf{e}_i\}$ on $H^1(L(u);\mathbb{Z})$ for $u \in \operatorname{Int} P$ is made as above starting from a basis on N, then we can regard this function as a function of $(x_1, \dots, x_n) \in (\Lambda_+)^n$ and $(u_1, \dots, u_n) \in P \subset M_{\mathbb{R}}$. We denote its value by $\mathfrak{PO}(x;u) = \mathfrak{PO}(x_1, \dots, x_n; u_1, \dots, u_n)$. We put

$$y_i = e^{x_i} = \sum_{k=0}^{\infty} \frac{x_i^k}{k!} \in \Lambda_0.$$

Let

$$\partial P = \bigcup_{i=1}^{m} \partial_i P$$

be the decomposition of the boundary of the moment polytope into its faces of codimension one. ($\partial_i P$ is a polygon in an $(n-1)$-dimensional affine subspace of $M_{\mathbb{R}}$.)

Let ℓ_i be the affine functions

$$\ell_i(u) = \langle u, v_i\rangle - \lambda_i \ \text{ for } i = 1, \dots, m$$

appearing in Theorem 2.13. Then the following hold from construction:

(1) $\ell_i \equiv 0$ on $\partial_i P$;

(2) $P = \{u \in M_{\mathbb{R}} \mid \ell_i(u) \geq 0, i = 1, \dots, m\}$;

(3) the coordinates of the vectors $v_i = (v_{i,1}, \dots, v_{i,n})$ satisfy

$$v_{i,j} = \frac{\partial \ell_i}{\partial u_j} \tag{4.4}$$

and are all integers.

THEOREM 4.5
Let $L(u) \subset X$ be as in Theorem 1.5, and let ℓ_i be as above. Suppose that X is Fano. Then we can take the canonical model of A_∞-structure of $L(u)$ over $u \in \operatorname{Int} P$ so that the potential function restricted to

$$\bigcup_{u \in \operatorname{Int} P} H^1\big(L(u); \Lambda_+\big) \cong (\Lambda_+)^n \times \operatorname{Int} P$$

has the form

$$\mathfrak{PO}(x; u) = \sum_{i=1}^{m} y_1^{v_{i,1}} \cdots y_n^{v_{i,n}} T^{\ell_i(u)} \tag{4.5}$$

$$= \sum_{i=1}^{m} e^{\langle v_i, x\rangle} T^{\ell_i(u)}, \tag{4.6}$$

where $(x; u) = (x_1, \dots, x_n; u_1, \dots, u_n)$ and $v_{i,j}$ is as in (4.4).

Theorem 4.5 is a minor improvement of a result from [CO] (see [CO, (15.1)] and [Cho]): the case considered in [CO] corresponds to the case where $y_i \in U(1) \subset \{z \in \mathbb{C} \mid |z| = 1\}$, and the case in [Cho] corresponds to the one where $y_i \in \mathbb{C} \setminus \{0\}$. The difference of Theorem 4.5 from those is that y_i is allowed to contain T, the formal parameter of the universal Novikov ring encoding the energy.

For the non-Fano case, we prove the following slightly weaker statement. The proof is given in Section 11.

THEOREM 4.6
Let X be an arbitrary toric manifold, and let $L(u)$ be as above. Then there exist $c_j \in \mathbb{Q}$, $e^i_j \in \mathbb{Z}_{\geq 0}$, and $\rho_j > 0$, such that $\sum_{i=1}^m e^i_j > 0$ and

$$\mathfrak{PO}(x_1, \dots, x_n; u_1, \dots, u_n) - \sum_{i=1}^{m} y_1^{v_{i,1}} \cdots y_n^{v_{i,n}} T^{\ell_i(u)}$$

$$= \sum_{j=1} c_j y_1^{v'_{j,1}} \cdots y_n^{v'_{j,n}} T^{\ell'_j(u)+\rho_j}, \tag{4.7}$$

where

$$v'_{j,k} = \sum_{i=1}^{m} e^i_j v_{i,k}, \quad \ell'_j = \sum_{i=1}^{m} e^i_j \ell_i.$$

If there are infinitely many nonzero c_j's, then we have

$$\lim_{j\to\infty} \ell'_j(u) + \rho_j = \infty.$$

Moreover, $\rho_j = [\omega] \cap \alpha_j$ for some $\alpha_j \in \pi_2(X)$ with nonpositive first Chern number $c_1(X) \cap [\alpha_j]$.

We note that although $\mathfrak{PO}$ is defined originally on $(\Lambda_+^{\mathbb{C}})^n \times P$, Theorems 4.5 and 4.6 imply that $\mathfrak{PO}$ extends to a function on $(\Lambda_0^{\mathbb{C}})^n \times P$ in general, and to one on $(\Lambda_0^{\mathbb{C}})^n \times M_{\mathbb{R}}$ for the Fano case. Furthermore, these theorems also imply the periodicity of $\mathfrak{PO}$ in x_i's,

$$\mathfrak{PO}(x_1, \dots, x_i + 2\pi\sqrt{-1}, \dots, x_n; u) = \mathfrak{PO}(x_1, \dots, x_n; u). \tag{4.8}$$

We write

$$\mathfrak{PO}_0 = \sum_{i=1}^{m} y_1^{v_{i,1}} \cdots y_n^{v_{i,n}} T^{\ell_i(u)} \tag{4.9}$$

to distinguish it from $\mathfrak{PO}$. We call $\mathfrak{PO}_0$ the *leading-order potential function*.

We focus on the existence of the bounding cochain $\mathfrak{x}$ for which the Floer cohomology $HF((L(u), \mathfrak{x}), (L(u), \mathfrak{x}))$ is not zero, and prove that critical points of the $\mathfrak{PO}^u$ (as a function of $y_1, \dots, y_n$) have this property (see Theorem 4.10).

This leads us to study the equation

$$\frac{\partial \mathfrak{PO}^u}{\partial y_k}(\mathfrak{y}_1, \dots, \mathfrak{y}_n) = 0, \quad k = 1, \dots, n, \tag{4.10}$$

where $\mathfrak{y}_i \in \Lambda_0 \setminus \Lambda_+$.

We regard $\mathfrak{PO}^u$ as either a function of x_i or of y_i. Since the variable (x_i or y_i) is clear from the situation, we do not always mention it.

PROPOSITION 4.7
We assume that the coordinates of the vertices of P are rational. Then there exists $u_0 \in \mathrm{Int} P \cap \mathbb{Q}^n$ such that for each $\mathcal{N}$ there exists $\mathfrak{y}_1, \dots, \mathfrak{y}_n \in \Lambda_0 \setminus \Lambda_+$ satisfying

$$\frac{\partial \mathfrak{PO}^{u_0}}{\partial y_k}(\mathfrak{y}_1, \dots, \mathfrak{y}_n) \equiv 0, \qquad \mathrm{mod}\ T^{\mathcal{N}} \quad k = 1, \dots, n. \tag{4.11}$$

Moreover, there exists $\mathfrak{y}'_1, \dots, \mathfrak{y}'_n \in \Lambda_0 \setminus \Lambda_+$ *such that*

$$\frac{\partial \mathfrak{PO}_0^{u_0}}{\partial y_k}(\mathfrak{y}'_1, \dots, \mathfrak{y}'_n) = 0, \quad k = 1, \dots, n. \tag{4.12}$$

We prove Proposition 4.7 in Section 9.

Remark 4.8

(1) While u_0 is independent of $\mathcal{N}$, $\mathfrak{y}_i$ may depend on $\mathcal{N}$. (We believe it does not depend on $\mathcal{N}$ but are unable to prove it at this time.)

(2) If $[\omega] \in H^2(X;\mathbb{R})$ is contained in $H^2(X;\mathbb{Q})$, then we may choose P so that its vertices are rational.

(3) We believe that rationality of the vertices of P is superfluous. We also believe there exists not only a solution of (4.11) or (4.12) but also of (4.10). However, then the proof seems to become more cumbersome. Since we can reduce the general case to the rational case by approximation in most of the applications, we are content here to prove the above weaker statement.

We put

$$\mathfrak{x}_i = \log \mathfrak{y}_i \in \Lambda_0,$$

and we write

$$\mathfrak{x} = \sum_i \mathfrak{x}_i \mathbf{e}_i \in H^1\big(L(u_0); \Lambda_0\big). \tag{4.13}$$

Since $\mathfrak{y}_i \in \Lambda_0 \setminus \Lambda_+$, $\log \mathfrak{y}_i$ is well defined (by using non-Archimedean topology on Λ_0) and is contained in Λ_0.

We note that $\mathfrak{x}_i$ is determined from $\mathfrak{y}_i$ up to addition by an element of $2\pi\sqrt{-1}\mathbb{Z}$. It follows from (4.8) that changing $\mathfrak{x}_i$ by an element of $2\pi\sqrt{-1}\mathbb{Z}$ does not change corresponding Floer cohomology. So we take, for example, $\operatorname{Im} \mathfrak{x}_i \in [0, 2\pi) \mod \Lambda_+$ (see also Definition 1.2(2)).

Let $\mathfrak{y}_{i,0} \in \mathbb{C} \setminus \{0\}$ be the zero-order term of $\mathfrak{y}_i$, that is, the complex number such that

$$\mathfrak{y}_i - \mathfrak{y}_{i,0} \equiv 0 \mod \Lambda_+^{\mathbb{C}}.$$

If we make an additional assumption that $\mathfrak{y}_{i,0} = 1$ for $i = 1, \dots, n$, then $\mathfrak{x}$ lies in

$$H^1\big(L(u_0); \Lambda_+\big) \subset H^1\big(L(u_0); \Lambda_0\big).$$

Therefore, Proposition 4.3 implies that the Floer cohomology $HF((L(u_0), \mathfrak{x}), (L(u_0), \mathfrak{x}); \Lambda_0)$ is defined. Then (4.11), combined with the argument from [CO] (see Section 13), implies that

$$HF\big((L(u_0), \mathfrak{x}), (L(u_0), \mathfrak{x}); \Lambda_0^{\mathbb{C}}/(T^{\mathcal{N}})\big) \cong H\big(T^n; \Lambda_0^{\mathbb{C}}/(T^{\mathcal{N}})\big). \tag{4.14}$$

We now consider the case when $\mathfrak{y}_{i,0} \neq 1$ for some i. In this case, we follow the idea of Cho [Cho] of twisting the Floer cohomology of $L(u)$ by a *nonunitary* flat line bundle and proceed as follows.

We define $\rho : H_1\big(L(u); \mathbb{Z}\big) \to \mathbb{C} \setminus \{0\}$ by

$$\rho(\mathbf{e}_i^*) = \mathfrak{y}_{i,0}. \tag{4.15}$$

Let $\mathfrak{L}_\rho$ be the flat complex line bundle on $L(u)$ whose holonomy representation is ρ. We use ρ to twist the operator $\mathfrak{m}_k$ in the same way as in [F2], [Cho] to obtain a filtered A_∞-algebra, which we write $\big(H(L(u); \Lambda_0), \mathfrak{m}^\rho\big)$. It is weakly unobstructed and $\mathcal{M}_{\text{weak}}\big(H(L(u); \Lambda_0), \mathfrak{m}^\rho\big) \supseteq H^1(L(u); \Lambda_+)$ (see Section 12).

We deform the filtered A_∞-structure $\mathfrak{m}^\rho$ to $\mathfrak{m}^{\rho,b}$ using $b \in H^1(L(u); \Lambda_+)$ for which $\mathfrak{m}_1^{\rho,b}\mathfrak{m}_1^{\rho,b} = 0$ holds. Denote by $HF\big((L(u_0), \rho, b), (L(u_0), \rho, b); \Lambda_0^{\mathbb{C}}\big)$ the cohomology of $\mathfrak{m}_1^{\rho,b}$. We denote the potential function of $\big(H(L(u); \Lambda_0), \mathfrak{m}^\rho\big)$ by

$$\mathfrak{PO}_\rho^u : H^1\big(L(u); \Lambda_+\big) \to \Lambda_+,$$

which is defined in the same way as $\mathfrak{PO}^u$ by using $\mathfrak{m}^\rho$ instead of $\mathfrak{m}$.

Let $\mathfrak{x}$ be as in (4.13), and put

$$\mathfrak{x}_{i,0} = \log \mathfrak{y}_{i,0}, \qquad b = \sum (\mathfrak{x}_i - \mathfrak{x}_{i,0})\mathbf{e}_i \in H^1\big(L(u); \Lambda_+\big). \tag{4.16}$$

Based on the definition, we can easily prove the following.

LEMMA 4.9
We have

$$\mathfrak{PO}_\rho^u(b) = \mathfrak{PO}^u(\mathfrak{x}).$$

As we note in the paragraph following Theorem 4.6, $\mathfrak{PO}^u$ has been extended to a function on $(\Lambda_0^{\mathbb{C}})^n$ and hence the right-hand side of the identity in this lemma has a well-defined meaning. (Lemma 4.9 is proved in Section 13.)

Now we have the following.

THEOREM 4.10
Let $\mathfrak{x}_i$, $\mathfrak{y}_i = e^{\mathfrak{x}_i}$, *and let* ρ *satisfy* (4.10), (4.13), *and* (4.15). *Let* $\mathfrak{x}_{i,0}$ *and* b *be as in* (4.16). *Then we have*

$$HF\big((L(u_0),\rho,b),(L(u_0),\rho,b);\Lambda_0^{\mathbb{C}}\big) \cong H(T^n;\Lambda_0^{\mathbb{C}}). \tag{4.17}$$

If (4.11), (4.13), (4.15), *and* (4.16) *are satisfied instead, then we have*

$$HF\big((L(u_0),\rho,b),(L(u_0),\rho,b);\Lambda_0^{\mathbb{C}}/(T^{\mathcal{N}})\big) \cong H\big(T^n;\Lambda_0^{\mathbb{C}}/(T^{\mathcal{N}})\big). \tag{4.18}$$

Theorem 4.10 is proved in Section 13. Using this, we prove Theorem 1.5 in Section 13; more precisely, we also discuss there the following two points.
(1) We need to study the case where ω is not necessarily rational.
(2) We have only (4.18) instead of (4.17).

Definition 4.11
Let (X,ω) be a smooth compact toric manifold, and let P be its moment polytope. We say that a fiber $L(u_0)$ at $u_0 \in P$ is *balanced* if there exists a sequence ω_i, u_i satisfying the following:
(1) ω_i is a T^n-invariant Kähler structure on X such that $\lim_{i\to\infty}\omega_i = \omega$.
(2) u_i is in the interior of the moment polytope P_i of (X,ω_i), and we make an appropriate choice of moment polytope P_i so that they converge to P and then $\lim_{i\to\infty} u_i = u_0$.
(3) For each $\mathcal{N}$, there exist a sufficiently large i and $\mathfrak{x}_{i,\mathcal{N}} \in H^1(L(u_i);\Lambda_0^{\mathbb{C}})$ such that

$$HF\big((L(u_i),\mathfrak{x}_{i,\mathcal{N}}),(L(u_i),\mathfrak{x}_{i,\mathcal{N}});\Lambda^{\mathbb{C}}/(T^{\mathcal{N}})\big) \cong H(T^n;\mathbb{C})\otimes\Lambda^{\mathbb{C}}/(T^{\mathcal{N}}).$$

We say that $L(u_0)$ is *strongly balanced* if there exists $\mathfrak{x} \in H^1(L;\Lambda_0^{\mathbb{C}})$ such that $HF\big((L(u_0),\mathfrak{x}),(L(u_0),\mathfrak{x});\Lambda_0^{\mathbb{C}}\big) \cong H(T^n;\mathbb{Q})\otimes\Lambda_0^{\mathbb{C}}$.

Obviously "strongly balanced" implies "balanced." The converse is not true, in general (see Example 10.17). We also refer the reader to Remark 13.9 for other characterizations of being balanced (or strongly balanced).

Theorem 4.10 implies that $L(u_0)$ in Proposition 4.7 is balanced (see Proposition 13.2). We next prove the following intersection result in Section 13. Theorem 1.5 then is a consequence of Propositions 4.7 and 4.12.

PROPOSITION 4.12
If $L(u_0)$ *is a balanced Lagrangian fiber, then the following holds for any Hamiltonian diffeomorphism* $\psi : X \to X$:

$$\psi\big(L(u_0)\big)\cap L(u_0) \neq \emptyset. \tag{4.19}$$

If, in addition, $\psi(L(u_0))$ is transversal to $L(u_0)$, then

$$\#\big(\psi(L(u_0)) \cap L(u_0)\big) \geq 2^n. \tag{4.20}$$

Denoting $\mathfrak{x} = b + \sum \mathfrak{x}_{i,0}\mathbf{e}_i$, we sometimes write $HF\big((L(u_0), \mathfrak{x}), (L(u_0), \mathfrak{x}); \Lambda_0\big)$ for $HF\big((L(u_0), \rho, b), (L(u_0), \rho, b); \Lambda_0\big)$ from now on. We also define

$$\mathscr{M}_{\text{weak}}\big(L(u); \Lambda_0\big) := \big\{(\rho, b) \,\big|\, \rho : \pi_1 L(u) \to \mathbb{C} \setminus \{0\}, b \in \mathscr{M}_{\text{weak}}\big(H(L(u)), \mathfrak{m}^\rho\big)\big\}.$$

Namely, it is the set of pairs (ρ, b) where ρ is a holonomy of a flat $\mathbb{C}$-bundle over $L(u)$ and $b \in H(L(u); \Lambda_+)$ is a weak bounding cochain of the filtered A_∞-algebra associated to $L(u)$ and twisted by ρ. With this definition of $\mathscr{M}_{\text{weak}}(L(u); \Lambda_0)$, we have the following:

$$H^1\big(L(u); \Lambda_0\big) \subseteq \mathscr{M}_{\text{weak}}\big(L(u); \Lambda_0\big).$$

5. Examples

In this section, we discuss various examples of toric manifolds that illustrate the results of Section 4.

Example 5.1

Consider $X = S^2$ with standard symplectic form with area 2π. The moment polytope of the standard S^1-action by rotations along an axis becomes $P = [0, 1]$ after a suitable translation. We have $\ell_1(u) = u$, $\ell_2(u) = 1 - u$, and

$$\mathfrak{PO}(x; u) = e^x T^u + e^{-x} T^{1-u} = yT^u + y^{-1}T^{1-u}.$$

The zero of

$$\frac{\partial \mathfrak{PO}^u}{\partial y} = T^u - y^{-2}T^{1-u}$$

is $\mathfrak{y} = \pm T^{(1-2u)/2}$. If $u \neq 1/2$, then

$$\log \mathfrak{y} = \frac{1 - 2u}{2} \log(\pm T)$$

is not an element of the universal Novikov ring. In particular, there is no critical point in $\Lambda_0^{\mathbb{C}} \setminus \Lambda_+^{\mathbb{C}}$.

If $u = 1/2$, then $\mathfrak{y} = \pm 1$. The case $\mathfrak{y} = 1$ corresponds to $\mathfrak{x} = 0$. We have

$$HF\big((L(1/2), 0), (L(1/2), 0); \Lambda_0\big) \cong H(S^1; \Lambda_0^{\mathbb{C}}).$$

The other case, $\mathfrak{y} = -1$, corresponds to a nontrivial flat bundle on S^1.

Example 5.2
We consider $X = \mathbb{C}P^n$. Then

$$P = \left\{(u_1, \ldots, u_n) \,\middle|\, 0 \le u_i, u_1 + \cdots + u_n \le 1\right\},$$

is a simplex. We have

$$\mathfrak{PO}(x_1, \ldots, x_n; u_1, \ldots, u_n) = \sum_{i=1}^{n} e^{x_i} T^{u_i} + e^{-\sum x_i} T^{1-\sum u_i}.$$

We put $u = u_0 = (1/(n+1), \ldots, 1/(n+1))$. Then

$$\mathfrak{PO}^{u_0} = (y_1 + \cdots + y_n + y_1^{-1} y_2^{-1} \cdots y_n^{-1}) T^{1/(n+1)}.$$

Solutions of the equation (4.10) are given by

$$\mathfrak{y}_1 = \cdots = \mathfrak{y}_n = e^{2\pi k \sqrt{-1}/(n+1)}, \quad k = 0, \ldots, n.$$

Hence the conclusion of Theorem 1.5 holds for our torus. The case $k = 0$ corresponds to $b = 0$. The other cases correspond to appropriate flat bundles on T^n.

Remark 5.3
The critical values of the potential function is $(n+1)e^{2\pi\sqrt{-1}k/(n+1)}$, $k = 0, \ldots, n$.

We consider the quantum cohomology ring

$$QH(\mathbb{C}P^n; \Lambda_0) \cong \Lambda_0[z, T]/(z^{n+1} - T).$$

The first Chern class c_1 is $(n+1)z$. The eigenvalues of the operator

$$c : QH(\mathbb{C}P^n) \to QH(\mathbb{C}P^n), \alpha \mapsto c_1 \cup_Q \alpha$$

are $(n+1)e^{2\pi\sqrt{-1}k/(n+1)}$, $k = 0, \ldots, n$. It coincides with the set of critical values.

Kontsevich announced this statement at the Vienna conference on homological mirror symmetry in 2006. (According to some physicists, this statement had been known to them before; see [Ar].) In our situation of Lagrangian fiber of compact toric manifolds, we can prove it by using Theorem 1.9.

In the rest of this section, we discuss 2-dimensional examples.

Let $\mathbf{e}_1$, $\mathbf{e}_2$ be the basis of $H^1(T^2; \mathbb{Z})$ as in Lemma 4.4. We put $\mathbf{e}_{12} = \mathbf{e}_1 \cup \mathbf{e}_2 \in H^2(T^2; \mathbb{Z})$. Let $\mathbf{e}_\emptyset$ be the standard basis of $H^0(T^2; \mathbb{Z}) \cong \mathbb{Z}$. (The proof of Proposition 5.4 is found in Section 13.)

PROPOSITION 5.4
Let $\mathfrak{x} = \sum \mathfrak{x}_i \mathbf{e}_i \in H^1(L(u); \Lambda_0)$, $\mathfrak{y}_i = e^{\mathfrak{x}_i}$. *Then the boundary operator* $\mathfrak{m}_1^{\mathfrak{x}}$ *is given as*

$$\begin{cases} \mathfrak{m}_1^{\mathfrak{x}}(\mathbf{e}_i) = \dfrac{\partial \mathfrak{PO}^u}{\partial y_i}(\mathfrak{y})\mathbf{e}_\emptyset, \\ \mathfrak{m}_1^{\mathfrak{x}}(\mathbf{e}_{12}) = \dfrac{\partial \mathfrak{PO}^u}{\partial y_1}(\mathfrak{y})\mathbf{e}_2 - \dfrac{\partial \mathfrak{PO}^u}{\partial y_2}(\mathfrak{y})\mathbf{e}_1, \\ \mathfrak{m}_1^{\mathfrak{x}}(\mathbf{e}_\emptyset) = 0. \end{cases} \tag{5.1}$$

We note that we do not use the grading parameter e, which was introduced in [FOOO3]. So the boundary operator $\mathfrak{m}_1^{\mathfrak{x}}$ is of degree -1 rather than $+1$. (Note that we are using cohomology notation.) In other words, our Floer cohomology is only $\mathbb{Z}_2$-graded.

With (5.1) at our disposal, we examine various examples.

Example 5.5
We consider $M = \mathbb{C}P^2$ again. We put $u_1 = \epsilon + 1/3$, $u_2 = 1/3$ ($\epsilon > 0$). Using (5.1), we can easily find the following isomorphism for the Floer cohomology with Λ_0-coefficients:

$$HF^{\mathrm{odd}}\big((L(u), 0), (L(u), 0)\big) \cong HF^{\mathrm{even}}\big((L(u), 0), (L(u), 0)\big) \cong \Lambda_0/(T^{1/3-\epsilon}).$$

Let us apply [FOOO3, Theorem J] (equivalent to Theorem 3.12 above) in this situation (see also Theorem 5.11 below). We consider a Hamiltonian diffeomorphism $\psi : \mathbb{C}P^2 \to \mathbb{C}P^2$. We denote by $\|\psi\|$ the Hofer distance of ψ from identity. Then we have

$$\#\big(\psi(L(u)) \cap L(u)\big) \geq 4$$

if $\|\psi\| < 2\pi(1/3 - \epsilon)$ and $\psi(L(u))$ is transversal to $L(u)$. We note that $\omega \cap [\mathbb{C}P^1] = 2\pi$ by (2.12).

We note that this fact was already proved by Chekanov [Che, main theorem]. (Actually, the basic geometric idea behind our proof is the same as Chekanov's.)

Example 5.6
Let $M = S^2(a/2) \times S^2(b/2)$, where $S^2(r)$ denotes the 2-sphere with radius r. We assume that $a < b$. Then $B = [0, a] \times [0, b]$, and we have

$$\mathfrak{PO}(x_1, x_2; u_1, u_2) = y_1 T^{u_1} + y_2 T^{u_2} + y_1^{-1} T^{a-u_1} + y_2^{-1} T^{b-u_2}.$$

Let us take $u_1 = a/2, u_2 = b/2$. Then

$$\frac{\partial \mathfrak{PO}^u}{\partial y_1} = (1 - y_1^{-2})T^{a/2}, \qquad \frac{\partial \mathfrak{PO}^u}{\partial y_2} = (1 - y_2^{-2})T^{b/2}.$$

Therefore, $y_1 = \pm 1$, $y_2 = \pm 1$ are solutions of (4.10). Hence, we can apply Theorem 4.10 to our torus.

We next put $u_1 = a/2$, $a < 2u_2 < b$. Then

$$\frac{\partial \mathfrak{PO}^u}{\partial y_1} = (1 - y_1^{-2})T^{a/2}, \qquad \frac{\partial \mathfrak{PO}^u}{\partial y_2} = T^{u_2} - y_2^{-2}T^{b-u_2}.$$

We put $y_1 = y_2 = 1$. Then $\frac{\partial \mathfrak{PO}^u}{\partial y_1} = 0$, $\frac{\partial \mathfrak{PO}^u}{\partial y_2} \neq 0$. We find that

$$HF^{\text{odd}}\big((L(u), 0), (L(u), 0)\big) \cong HF^{\text{even}}\big((L(u), 0), (L(u), 0)\big) \cong \Lambda_0/(T^{u_2}).$$

Let $\psi : \mathbb{C}P^2 \to \mathbb{C}P^2$ be a Hamiltonian diffeomorphism. Then, [FOOO3, Theorem J] (equivalent to Theorem 3.12 above) implies that

$$\#\big(\psi(L(u)) \cap L(u)\big) \geq 4$$

if $\|\psi\| < 2\pi u_2$ and $\psi(L(u))$ is transversal to $L(u)$. Note that there exists a pseudo-holomorphic disc with symplectic area πa ($< 2\pi u_2$). Hence, our result improves a result from [Che, main theorem].

Example 5.7

Let X be the 2-point blow-up of $\mathbb{C}P^2$. We may take its Kähler form so that the moment polytope is given by

$$P = \big\{(u_1, u_2) \,\big|\, -1 \leq u_1 \leq 1, -1 \leq u_2 \leq 1, u_1 + u_2 \leq 1 + \alpha\big\},$$

where $-1 < \alpha < 1$ depends on the choice of Kähler form. We have

$$\begin{aligned}\mathfrak{PO}(x_1, x_2; u_1, u_2) = y_1 T^{1+u_1} + y_2 T^{1+u_2} + y_1^{-1} T^{1-u_1} \\ + y_2^{-1} T^{1-u_2} + y_1^{-1} y_2^{-1} T^{1+\alpha-u_1-u_2}.\end{aligned} \tag{5.2}$$

Note X is Fano in our case.

Case 1: $\alpha = 0$. In this case, X is monotone. We put $u_0 = (0, 0)$. $L(u_0)$ is a monotone Lagrangian submanifold. We have

$$\frac{\partial \mathfrak{PO}^{u_0}}{\partial y_1} = (1 - y_1^{-2} - y_1^{-2}y_2^{-1})T, \qquad \frac{\partial \mathfrak{PO}^{u_0}}{\partial y_2} = (1 - y_2^{-2} - y_1^{-1}y_2^{-2})T.$$

The solutions of (4.10) are given by $y_2 = 1/(y_1^2 - 1)$, $y_1^5 + y_1^4 - 2y_1^3 - 2y_1^2 + 1 = 0$ in $\mathbb{C}$. (There are 5 solutions.)

Case 2: $\alpha > 0$. We put $u_0 = (0, 0)$. Then

$$\frac{\partial \mathfrak{PO}^{u_0}}{\partial y_1} = (1 - y_1^{-2})T - y_1^{-2} y_2^{-1} T^{1+\alpha}, \qquad \frac{\partial \mathfrak{PO}^{u_0}}{\partial y_2} = (1 - y_2^{-2})T - y_1^{-1} y_2^{-2} T^{1+\alpha}.$$

We consider, for example, the case $y_1 = y_2 = \tau$. Then (4.10) becomes

$$\tau^3 - \tau - T^\alpha = 0. \tag{5.3}$$

The solution of (5.3) with $\tau \equiv 1 \mod \Lambda_+$ is given by

$$\tau = 1 + \frac{1}{2}T^\alpha - \frac{3}{8}T^{2\alpha} + \frac{1}{2}T^{3\alpha} + \sum_{k=4}^{\infty} c_k T^{k\alpha}.$$

Let us put $\mathfrak{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2$ with

$$x_1 = x_2 = \log\Big(1 + \frac{1}{2}T^\alpha - \frac{3}{8}T^{2\alpha} + \frac{1}{2}T^{3\alpha} + \cdots\Big) \in \Lambda_+.$$

Then by Theorem 4.10 we have

$$HF\big((L(u_0), \mathfrak{x}), (L(u_0), \mathfrak{x}); \Lambda_0\big) \cong H(T^2; \Lambda_0).$$

We point out that in this example it is essential to deform Floer cohomology using an element $\mathfrak{x}$ of $H^1(L(u_0); \Lambda_+)$ containing the formal parameter T to obtain nonzero Floer cohomology.

At u_0, there are actually four solutions such that

$$(y_1, y_2) \equiv (1, 1), (1, -1), (-1, 1), (-1, -1) \mod \Lambda_+,$$

respectively.

In the current case, there is another point $u_0' = (\alpha, \alpha) \in P$ at which $L(u_0')$ is balanced.* In fact at $u_0' = (\alpha, \alpha)$, the equation (5.3) becomes

$$0 = -(y_1^{-2} y_2^{-1} + y_1^{-2})T^{1-\alpha} + T^{1+\alpha}, \quad 0 = -(y_1^{-1} y_2^{-2} + y_2^{-2})T^{1-\alpha} + T^{1+\alpha}.$$

*Using the method of spectral invariants and symplectic quasi-states, Entov and Polterovich discovered some nondisplaceable Lagrangian fiber which was not covered by the criterion given in [CO] (see [EP1, Section 9]). Recently this example, among others, was explained by Cho [Cho] via Lagrangian Floer cohomology twisted by nonunitary line bundles.

We put $\tau = y_1 = y_2$ to obtain

$$\tau^3 T^{2\alpha} - \tau - 1 = 0.$$

This equation has a unique solution with $\tau \equiv -1 \mod \Lambda_+$ (the other solution is $T^{2\alpha}\tau^3 \equiv 1 \mod \Lambda_+$, for which Theorem 4.10 is not applicable).

The total number of the solutions $(\mathfrak{x}, u)$ is 5.

Case 3: $\alpha < 0$. We first consider $u_0 = (0, 0)$. Then

$$\frac{\partial \mathfrak{PO}^{u_0}}{\partial y_1} = -y_1^{-2}y_2^{-1}T^{1+\alpha} + (1 - y_1^{-2})T, \quad \frac{\partial \mathfrak{PO}^{u_0}}{\partial y_2} = -y_1^{-1}y_2^{-2}T^{1+\alpha} + (1 - y_2^{-2})T.$$

We assume that y_i satisfies (4.10). It is then easy to see that $y_1^{-1} \equiv 0$ or that $y_2^{-1} \equiv 0 \mod \Lambda_+$. In other words, there is no (y_1, y_2) to which we can apply Theorem 4.10. Actually, it is easy to find a Hamiltonian diffeomorphism $\psi : X \to X$ such that $\psi(L(u_0)) \cap L(u_0) = \emptyset$.

We next take $u_0' = (\alpha/3, \alpha/3)$. Then

$$\frac{\partial \mathfrak{PO}^{u_0'}}{\partial y_1} = (1 - y_1^{-2}y_2^{-1})T^{1+\alpha/3} - y_1^{-2}T^{1-\alpha/3},$$
$$\frac{\partial \mathfrak{PO}^{u_0'}}{\partial y_2} = (1 - y_1^{-1}y_2^{-2})T^{1+\alpha/3} - y_2^{-2}T^{1-\alpha/3}.$$

By putting $y_1 = y_2 = \tau$, for example, (4.10) becomes

$$\tau^3 - T^{-2\alpha/3}\tau - 1 = 0. \tag{5.4}$$

Let us put $\mathfrak{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ with

$$x_1 = x_2 = \log\tau = \log\Big(1 + \frac{1}{3}T^{-2\alpha/3} - \frac{1}{81}T^{-6\alpha/3} + \cdots\Big) \in \Lambda_+,$$

where τ solves (5.4). Theorem 4.10 is applicable. (There are actually three solutions of (4.10) corresponding to the three solutions of (5.4).)

There are two more points $u = (-\alpha, \alpha), (\alpha, -\alpha)$ where (4.10) has a solution in $(\Lambda_0 \setminus \Lambda_+)$. Each u has one solution b.

Thus, the total number of the pair $(\mathfrak{x}, u)$ is again 5. We note that

$$5 = \sum \operatorname{rank} H^k(X; \mathbb{Q}).$$

This is not just a coincidence but an example of general phenomenon stated as in Theorem 1.3.

We note that, as $\alpha \to 1$, our X blows down to $S^2(1) \times S^2(1)$. On the other hand, as $\alpha \to -1$, our X blows down to $\mathbb{C}P^2$. The situation of the case $\alpha > 0$ can be regarded as a perturbation of the situation of $S^2(1) \times S^2(1)$, by the effect of exceptional curve corresponding to the segment $u_1 + u_2 = 1 + \alpha$. The situation of the case $\alpha < 0$ can be regarded as a perturbation of the situation of $\mathbb{C}P^2$ by the effect of the two exceptional curves corresponding to the segments $u_1 = 1$ and $u_2 = 1$. An interesting phase change occurs at $\alpha = 0$.

We note that $H^2(X;\mathbb{R})$ is 2-dimensional. So there is actually a 2-parameter family of symplectic structures. We study the 2-point blow-ups of $\mathbb{C}P^2$ more in Example 10.17.

The discussion of this section strongly suggests that Lagrangian Floer theory ([FOOO3, Theorems G, J] (equivalent to Theorems 3.11, 3.12 above) gives the optimal result for the study of nondisplacement of Lagrangian fibers in toric manifolds.

Remark 5.8

Let X be a compact toric manifold, and let $L(u) = \pi^{-1}(u)$, $u \in \mathrm{Int}P$. We consider the following two conditions.

(1) There exists no Hamiltonian diffeomorphism $\psi : X \to X$ such that $\psi(L(u)) \cap L(u) = \emptyset$.

(2) $L(u)$ is balanced.

Here, (2) $\Rightarrow$ (1) follows from Proposition 4.12. In many cases, (1) $\Rightarrow$ (2) can be proved by the method of [Mc]. However, there is a case in which (1) $\Rightarrow$ (2) does not follow, as we explain in Remark 10.18 and will prove in a future article using the bulk deformation of Lagrangian Floer cohomology. We conjecture that after including this wider class of Floer cohomology, we can detect all the nondisplaceable Lagrangian fibers in toric manifolds, by Floer cohomology.

Using the argument employed in Example 5.6, we can discuss the relationship between the Hofer distance and displacement. First, we introduce some notation for this purpose. We denote by $\mathrm{Ham}(X,\omega)$ the group of Hamiltonian diffeomorphisms of (X,ω). For a time-dependent Hamiltonian $H : [0,1] \times X \to \mathbb{R}$, we denote by ϕ_H^t the time t-map of Hamilton's equation $\dot{x} = X_H(t,x)$. The Hofer norm of $\psi \in \mathrm{Ham}(X,\omega)$ is defined to be

$$\|\psi\| = \inf_{H;\phi_H^1=\psi} \int_0^1 (\max H_t - \min H_t)\, dt$$

(see [Ho]).

Definition 5.9
Let $Y \subset X$. We define the *displacement energy* $e(Y) \in [0, \infty]$ by

$$e(Y) := \inf \left\{ \|\psi\| \;\middle|\; \psi \in \mathrm{Ham}(X, \omega),\ \psi(Y) \cap \overline{Y} = \emptyset \right\}.$$

We put $e(Y) = \infty$ if there exists no $\psi \in \mathrm{Ham}(X, \omega)$ with $\psi(Y) \cap \overline{Y} = \emptyset$.

Let us consider $\mathfrak{PO}(y_1, \dots, y_n; u_1, \dots, u_n) : (\Lambda_0 \setminus \Lambda_+)^n \times P \to \Lambda_+$ as in Theorem 4.5.

Definition 5.10
We define the number $\mathfrak{E}(u) \in (0, \infty]$ as the supremum of all λ such that there exists $\mathfrak{y}_1, \dots, \mathfrak{y}_n \in (\Lambda_0 \setminus \Lambda_+)^n$ satisfying

$$\frac{\partial \mathfrak{PO}}{\partial y_i}(\mathfrak{y}_1, \dots, \mathfrak{y}_n; u) \equiv 0 \mod T^{\lambda} \tag{5.5}$$

for $i = 1, \dots, n$ (here we consider universal Novikov ring with $\mathbb{C}$-coefficients). We call $\mathfrak{E}(u)$ the $\mathfrak{PO}$-*threshold* of the fiber $L(u)$. We put

$$\overline{\mathfrak{E}}(u) = \limsup_{\omega_i \to \omega, u_i \to u} \mathfrak{E}(u_i).$$

Here limsup is taken over all sequences ω_i and u_i such that ω_i is a sequence of T^n-invariant symplectic structures on X with $\lim_{i\to\infty} \omega_i = \omega$ and u_i is a sequence of points of moment polytopes P_i of (X, ω_i) such that P_i converges to P and u_i converges to u.

Clearly, $\overline{\mathfrak{E}}(u) \ge \mathfrak{E}(u)$. We give an example where $\overline{\mathfrak{E}}(u) \ne \mathfrak{E}(u)$ in Example 10.17 (10.16).

THEOREM 5.11
For any compact toric manifold X and $L(u) = \pi^{-1}(u)$, $u \in \mathrm{Int}P$, we have

$$e\big(L(u)\big) \ge 2\pi \overline{\mathfrak{E}}(u). \tag{5.6}$$

We prove Theorem 5.11 in Section 13.

Remark 5.12
The equality in (5.6) holds in various examples. However, there are cases in which the equality in (5.6) does not hold. The situation is the same as in Remark 5.8.

We note that $\mathfrak{E}(u)$, $\overline{\mathfrak{E}}(u)$ can be calculated in most of the cases once the toric manifold X is given explicitly. In fact, the leading-order potential function $\mathfrak{PO}_0$ is explicitly calculated by Theorem 4.5. We can then find the maximal value λ for which the polynomial equation

$$\frac{\partial \mathfrak{PO}_0}{\partial y_i}(u; \mathfrak{y}_1, \dots, \mathfrak{y}_n) \equiv 0 \mod T^{\lambda}$$

has a solution $\mathfrak{y}_i \in \Lambda_0 \setminus \Lambda_+$. In a weakly nondegenerate case, this value of λ for $\mathfrak{PO}(u; \dots)$ is the same as the value for $\mathfrak{PO}_0(u; \dots)$ (see Section 10).

Remark 5.13
The appearance of a new family of pseudoholomorphic discs with Maslov index 2 after blow-up, which we observed in Examples 5.7, can be related to the operator $\mathfrak{q}$ that we introduced in [FOOO3, Section 3.8] and [FOOO2, Section 13] in the following way.

We denote by $\mathscr{M}^{\text{main}}_{k+1,l}(\beta)$ the moduli space of stable maps $f : (\Sigma, \partial\Sigma) \to (X, L)$ from bordered Riemann surface Σ of genus zero with l interior and $k+1$ boundary marked points and in homology class β (see [FOOO1, Section 3] and [FOOO3, Section 2.1.2]). The symbol main means that we require the boundary marked points to respect the cyclic order of $\partial\Sigma$. Let us consider the case when the Maslov index of β is $2n$. More precisely, we take the following class β. We use the notation introduced at the beginning of Section 11. We put $\beta = \beta_{i_1} + \cdots + \beta_{i_n}$, where $\partial_{i_1} P \cap \cdots \cap \partial_{i_n} P = \overline{p}$ is a vertex of P. We assume that $[f] \in \mathscr{M}^{\text{main}}_{0+1,1}(\beta)$ and that $\mathscr{M}^{\text{main}}_{0+1,1}(\beta)$ is Fredholm-regular at f. The virtual dimension of $\mathscr{M}^{\text{main}}_{0+1,1}(\beta)$ is $3n$. Let us take the unique point $p \in X$ such that $\pi(p) = \overline{p}$. p is a T^n-fixed point. Moreover, we assume that $f(0) = p$. We blow up X at a point $p = f(0) \in X$ and obtain $\widehat{X}$. Let $[E] \in H_{2n-2}(\widehat{X})$ be the homology class of the exceptional divisor $E = \pi^{-1}(p)$. Now f induces a map $\widehat{f} : (\Sigma; \partial\Sigma) \to (\widehat{X}, L)$. The Maslov index of the homology class $[\widehat{f}] \in H_2(\widehat{X}, L)$ becomes 2. We put $\widehat{\beta} = [\widehat{f}]$.

Since p is a fixed point of T^n-action, a T^n-invariant perturbation lifts to a perturbation of the moduli space $\mathscr{M}^{\text{main}}_{0+1,0}(\widehat{\beta})$. Then any T^n-orbit of the moduli space $\mathscr{M}^{\text{main}}_{0+1,0}(X; \beta)$ of holomorphic discs passing through p corresponds to a T^n-orbit of $\mathscr{M}^{\text{main}}_{0+1,0}(\widehat{X}; \widehat{\beta})$ and vice versa. Namely, we have an isomorphism:

$$\mathscr{M}^{\text{main}}_{0+1,1}(\beta)\ {}_{\text{ev}}\times_X \{p\} \cong \mathscr{M}^{\text{main}}_{0+1,0}(\widehat{\beta}). \tag{5.7}$$

Here ev in the left-hand side is the evaluation map at the interior marked point. (Actually we need to work out the analytic details of gluing construction and so forth. It seems very likely that we can do it in the same way as the argument of [FOOO2, Chapter 10]; see [LR].)

Using (5.7) we may prove that

$$\mathfrak{q}_{1,k;\beta}(PD([p]); b, \dots, b) = \mathfrak{m}_{k,\widehat{\beta}}(b, \dots, b),$$

where

$$\mathfrak{q}_{1,k;\beta}(Q; P_1, \dots, P_k) = \mathrm{ev}_{0*}\big(\mathscr{M}^{\mathrm{main}}_{k+1,1}(\beta) \times_{(X \times L^k)} (Q \times P_1 \times \cdots \times P_k)\big)$$

is defined in [FOOO3, Section 3.8] and [FOOO2, Section 13]. (Here Q is a chain in X and P_i are chains in $L(u)$, and $\mathrm{ev}_0 : \mathscr{M}^{\mathrm{main}}_{k+1,1}(\beta) \to X$ is the evaluation map at the zeroth boundary marked point. In the right-hand side, we take fiber product over $X \times L^k$.) This is an example of a blow-up formula in Lagrangian Floer theory.

6. Quantum cohomology and Jacobian ring

In this section, we prove Theorem 1.9. Let $\mathfrak{PO}_0$ be the leading-order potential function. (Recall that if X is Fano, we have $\mathfrak{PO}_0 = \mathfrak{PO}$.) We define the monomial

$$\overline{z}_i(u) = y_1^{v_{i,1}} \cdots y_n^{v_{i,n}} T^{\ell_i(u)} \in \Lambda_0[y_1, \dots, y_n, y_1^{-1}, \dots, y_n^{-1}]. \tag{6.1}$$

Compare this with (2.3). It is also suggestive to write $\overline{z}_i$ also as

$$\overline{z}_i(u) = e^{\langle x, v_i \rangle} T^{\ell_i(u)}, \; x = (x_1, \dots, x_n), \; y_i = e^{x_i}. \tag{6.2}$$

By definition, we have

$$\mathfrak{PO}_0^u = \sum_{i=1}^m \overline{z}_i(u), \tag{6.3}$$

$$y_j \frac{\partial \overline{z}_i}{\partial y_j} = v_{i,j} \overline{z}_i(u). \tag{6.4}$$

The following is a restatement of Theorem 1.9. Let $z_i \in H^2(X; \mathbb{Z})$ be the Poincaré dual of the divisor $\pi^{-1}(\partial_i P)$.

THEOREM 6.1
If X is Fano, there exists an isomorphism

$$\psi_u : QH(X; \Lambda) \cong \mathrm{Jac}(\mathfrak{PO})$$

such that $\psi_u(z_i) = \overline{z}_i(u)$.

Since $c_1(X) = \sum_{i=1}^m z_i$ (see [Fu]) and since $\mathfrak{PO}_0^u = \sum_{i=1}^m \overline{z}_i(u)$ by definition, Theorem 1.9 follows from Theorem 6.1.

In the rest of this section, we prove Theorem 6.1. We note that z_i ($i = 1, \dots, m$) generates the quantum cohomology ring $QH(X; \Lambda)$ as a Λ-algebra (see Theorem 6.6). Therefore, it is enough to prove that the assignment $\tilde{\psi}_u(z_i) = \overline{z}_i(u)$ extends to a homomorphism $\tilde{\psi}_u : \Lambda[z_1, \dots, z_m] \to \mathrm{Jac}(\mathfrak{PO}_0^u)$ that induces an isomorphism in

$QH(X;\Lambda)$. In other words, it suffices to show that the relations among the generators in $\Lambda[z_1, \dots, z_m]$ and in $\mathrm{Jac}(\mathfrak{PO}_0^u)$ are mapped to each other under the assignment $\tilde{\psi}_u(z_i) = \overline{z}_i(u)$. To establish this correspondence, we now review Batyrev's description of the relations among z_i's.

We first clarify the definition of quantum cohomology ring over the universal Novikov rings Λ_0 and Λ. Let (X, ω) be a symplectic manifold, and let $\alpha \in \pi_2(X)$. Let $\mathscr{M}_3(\alpha)$ be the moduli space of stable map with homology class α of genus 0 with 3 marked points. Let $\mathrm{ev} : \mathscr{M}_3(\alpha) \to X^3$ be the evaluation map. We can define the virtual fundamental class $\mathrm{ev}_*[\mathscr{M}_3(\alpha)] \in H_d(X^3;\mathbb{Q})$, where $d = 2(\dim_{\mathbb{C}} X + c_1(X) \cap \alpha)$. Let $a_i \in H^*(X;\mathbb{Q})$. We define $a_1 \cup_Q a_2 \in H^*(X;\Lambda_0)$ by the following formula.

$$\langle a_1 \cup_Q a_2, a_3 \rangle = \sum_{\alpha} T^{\omega\cap\alpha/2\pi} \mathrm{ev}_*[\mathscr{M}_3(\alpha)] \cap (a_1 \times a_2 \times a_3). \tag{6.5}$$

Here $\langle \cdot, \cdot \rangle$ is the Poincaré duality. Extending this linearly, we obtain the quantum product

$$\cup_Q : H(X;\Lambda_0) \otimes H(X;\Lambda_0) \to H(X;\Lambda_0).$$

Extending the coefficient ring further to Λ, we obtain the (small) quantum cohomology ring $QH(X;\Lambda)$.

Now we specialize to the case of compact toric manifolds and review Batyrev's presentation of quantum cohomology ring. We consider the exact sequence

$$0 \longrightarrow \pi_2(X) \longrightarrow \pi_2\big(X;L(u)\big) \longrightarrow \pi_1\big(L(u)\big) \longrightarrow 0. \tag{6.6}$$

We note that $\pi_2(X;L(u)) \cong \mathbb{Z}^m$, and we choose its basis adapted to this exact sequence as follows. Consider the divisor $\pi^{-1}(\partial_i P)$, and take a small disc transversal to it. Each such disc gives rise to an element

$$[\beta_i] \in H_2\big(X;\pi^{-1}(\mathrm{Int}P)\big) \cong H_2\big(X;L(u)\big) \cong \pi_2\big(X, L(u)\big). \tag{6.7}$$

The set of $[\beta_i]$ with $i = 1, \dots, m$ forms a basis of $\pi_2(X;L(u)) \cong \mathbb{Z}^m$. The boundary map $[\beta] \mapsto [\partial\beta] : \pi_2(X;L(u)) \to \pi_1(L(u))$ is identified with the corresponding map $H_2(X;L(u)) \to H_1(L(u))$. Using the basis chosen in Lemma 4.4 on $H_1(L(u))$, we identify $H_1(L(u)) \cong \mathbb{Z}^n$. Then this homomorphism maps $[\beta_i]$ to

$$[\partial\beta_i] \cong v_i = (v_{i,1}, \dots, v_{i,n}), \tag{6.8}$$

where $v_{i,j}$ is as in (4.4). By the exactness of (6.6), we have an isomorphism:

$$H_2(X) \cong \big\{\beta \in H_2(X;L(u)) \,\big|\, [\partial\beta] = 0\big\}. \tag{6.9}$$

LEMMA 6.2
We have

$$\omega \cap \Big[\sum k_i \beta_i \Big] = 2\pi \sum k_i \ell_i(u). \tag{6.10}$$

If $\big[\sum k_i \partial\beta_i \big] = 0$*, then*

$$\sum k_i \frac{d\ell_i}{du_j} = 0. \tag{6.11}$$

In particular, the right-hand side of (6.10) *is independent of* u.

Proof
Here (6.10) follows from the area formula (2.12), $\omega(\beta_i) = 2\pi \ell_i(u)$. On the other hand, if $\big[\sum k_i \partial\beta_i \big] = 0$, we have

$$\sum_{i=1}^{m} k_i v_i = 0.$$

By the definition of ℓ_i, $\ell_i(u) = \langle u, v_i \rangle - \lambda_i$, from Theorem 2.13, this equation is precisely (6.11), and hence the proof. □

Let $\mathscr{P} \subset \{1, \dots, m\}$ be a primitive collection (see Definition 2.4). There exists a unique subset $\mathscr{P}' \subset \{1, \dots, m\}$ such that $\sum_{i\in\mathscr{P}} v_i$ lies in the interior of the cone spanned by $\{v_{i'} \mid i' \in \mathscr{P}'\}$, which is a member of the fan Σ. (Since X is compact, we can choose such $\mathscr{P}'$; see [Fu, Section 2.4].) We write

$$\sum_{i\in\mathscr{P}} v_i = \sum_{i'\in\mathscr{P}'} k_{i'} v_{i'}. \tag{6.12}$$

Since X is assumed to be nonsingular, $k_{i'}$ are all positive integers (see [Fu, page 29]). We put

$$\omega(\mathscr{P}) = \sum_{i\in\mathscr{P}} \ell_i(u) - \sum_{i'\in\mathscr{P}'} k_{i'} \ell_{i'}(u). \tag{6.13}$$

It follows from (6.10) that $2\pi\omega(\mathscr{P})$ is the symplectic area of the homotopy class

$$\beta(\mathscr{P}) = \sum_{i\in\mathscr{P}} \beta_i - \sum_{i'\in\mathscr{P}'} k_{i'} \beta_{i'} \in \pi_2(X). \tag{6.14}$$

LEMMA 6.3
Let $\mathscr{P}$ *be any primitive collection. Then* $\omega(\mathscr{P}) > 0$.

Proof
Since the cone spanned by $\{v_{i'} | i' \in \mathscr{P}'\}$ belongs to the fan Σ, we have

$$\bigcap_{i' \in \mathscr{P}'} \pi^{-1}(\partial_{i'} P) \neq \emptyset.$$

Then for any $u' \in \bigcap_{i' \in \mathscr{P}'} \partial_{i'} P$, we have $\ell_{i'}(u') = 0$. By the choice of $k_{i'}$ in (6.12), we have $\partial \beta(\mathscr{P}) = 0$. Therefore, by Lemma 6.2 and by the continuity of the right-hand side of (6.13), we can evaluate $\omega(\mathscr{P})$ at a point $u' \in \bigcap_{i' \in \mathscr{P}'} \pi^{-1}(\partial_{i'} P)$. Then we obtain

$$\omega(\mathscr{P}) = \sum_{i \in \mathscr{P}} \ell_i(u').$$

Since $\mathscr{P}$ is a primitive collection, and in particular does not form a member of the fan, there must be an element $v_i \in \mathscr{P}$ such that $\ell_i(u') > 0$, and so $\omega(\mathscr{P}) > 0$. This finishes the proof. □

Now we associate the formal variables, $z_1, \ldots, z_m$, to $v_1, \ldots, v_m$, respectively.

Definition 6.4 ([B1, Definition 5.1, Theorem 5.3])
(1) The *quantum Stanley-Reisner ideal* $SR_\omega(X)$ is the ideal generated by

$$z(\mathscr{P}) = \prod_{i \in \mathscr{P}} z_i - T^{\omega(\mathscr{P})} \prod_{i' \in \mathscr{P}'} z_{i'}^{k_{i'}} \tag{6.15}$$

in the polynomial ring $\Lambda[z_1, \ldots, z_m]$. Here $\mathscr{P}$ runs over all primitive collections.
(2) We denote by $P(X)$ the ideal generated by

$$\sum_{i=1}^{m} v_{i,j} z_i \tag{6.16}$$

for $j = 1, \ldots, n$. In this article, we call $P(X)$ the *linear relation ideal*.
(3) We call the quotient

$$QH^\omega(X; \Lambda) = \frac{\Lambda[z_1, \ldots, z_m]}{(P(X) + SR_\omega(X))} \tag{6.17}$$

the *Batyrev quantum cohomology ring*.

Remark 6.5
We do not take closure of our ideal $P(X) + SR_\omega(X)$ here (see Proposition 8.6).

THEOREM 6.6 (see Batyrev [B1], Givental [G2])
If X is Fano, then there exists a ring isomorphism from $QH^\omega(X; \Lambda)$ to the quantum cohomology ring $QH(X; \Lambda)$ of X such that z_i is sent to the Poincaré dual to $\pi^{-1}(\partial_i P)$.

The main geometric part of the proof of Theorem 6.6 is the following.

PROPOSITION 6.7
The Poincaré dual to $\pi^{-1}(\partial_i P)$ satisfies the quantum Stanley-Reisner relation.

(We do not prove Proposition 6.7 in this article; see Remarks 6.15 and 6.16.) However, since our choice of the coefficient ring is different from other literature, we explain here for the reader's convenience how Theorem 6.6 follows from Proposition 6.7.

Proposition 6.7 implies that we can define a ring homomorphism $h : QH^{\omega}(X;\Lambda) \to QH(X;\Lambda)$ by sending z_i to $PD(\pi^{-1}(\partial_i P))$. Let $F^k QH(X;\Lambda)$ be the direct sum of elements of degree $\le k$. Let $F^k QH^{\omega}(X;\Lambda)$ be the submodule generated by the polynomial of degree at most $k/2$ on z_i. Clearly, $h(F^k QH^{\omega}(X;\Lambda)) \subset F^k QH(X;\Lambda)$.

Since X is Fano, it follows that

$$x \cup_Q y - x \cup y \in F^{\deg x + \deg y - 2} QH(X;\Lambda).$$

We also recall the cohomology ring $H(X;\mathbb{Q})$ is obtained by putting $T = 0$ in a quantum Stanley-Reisner relation. Moreover, we find that the second product of the right-hand side of (6.15) has degree strictly smaller than the first since X is Fano.

Therefore, the graded ring

$$\operatorname{gr}\bigl(QH(X;\Lambda)\bigr) = \bigoplus_k F^k\bigl(QH(X;\Lambda)\bigr)/F^{k-1}\bigl(QH(X;\Lambda)\bigr)$$

is isomorphic to the (usual) cohomology ring (with Λ-coefficient) as a ring. The same holds for $QH^{\omega}(X;\Lambda)$. It follows that h is an isomorphism. □

In the rest of this section, we prove Proposition 6.8 (Theorem 6.1 follows immediately from Proposition 6.8 and Theorem 6.6).

PROPOSITION 6.8
There exists an isomorphism

$$\psi_u : QH^{\omega}(X;\Lambda) \cong \operatorname{Jac}(\mathfrak{PO}_0^u)$$

such that $\psi_u(z_i) = \overline{z}_i(u)$.

Note that we do *not* assume that X is Fano in Proposition 6.8. Note also that, for our main purpose of calculating $\mathfrak{M}_0(\mathfrak{Lag}(X))$, Proposition 6.8 suffices. Proposition 6.8 is a rather simple algebraic result, and its proof does not require the study of pseudoholomorphic discs or spheres.

Proof of Proposition 6.8
We start with the following proposition.

PROPOSITION 6.9
The assignment

$$\widehat{\psi}_u(z_i) = \overline{z}_i(u) \tag{6.18}$$

induces a well-defined ring isomorphism

$$\widehat{\psi}_u : \frac{\Lambda[z_1, \dots, z_m]}{SR_\omega(X)} \to \Lambda[y_1, \dots, y_n, y_1^{-1}, \dots, y_n^{-1}]. \tag{6.19}$$

Proof
Let $\mathscr{P}$ be a primitive collection and $\mathscr{P}'$, $k_{i'}$ be as in (6.12). We calculate

$$\prod_{i \in \mathscr{P}} \overline{z}_i(u) = \prod_{i \in \mathscr{P}} y_1^{v_{i,1}} \cdots y_n^{v_{i,n}} T^{\ell_i(u)} \tag{6.20}$$

by (6.1). On the other hand,

$$\begin{aligned}\prod_{i' \in \mathscr{P}'} \overline{z}_{i'}^{k_{i'}}(u) &= \prod_{i' \in \mathscr{P}'} y_1^{k_{i'} v_{i',1}} \cdots y_n^{k_{i'} v_{i',n}} T^{k_{i'} \ell_{i'}(u)} \\ &= \prod_{i \in \mathscr{P}} y_1^{v_{i,1}} \cdots y_n^{v_{i,n}} \prod_{i' \in \mathscr{P}'} T^{k_{i'} \ell_{i'}(u)}\end{aligned}$$

by (6.12). Moreover,

$$\sum_{i \in \mathscr{P}} \ell_i(u) - \sum_{i \in \mathscr{P}'} k_{i'} \ell_{i'}(u) = \omega(\mathscr{P})$$

by (6.13). Therefore,

$$\prod_{i \in \mathscr{P}} \overline{z}_i(u) = T^{\omega(\mathscr{P})} \prod_{i' \in \mathscr{P}'} \overline{z}_{i'}^{k_{i'}}(u)$$

in $\Lambda[y_1, \dots, y_n, y_1^{-1}, \dots, y_n^{-1}]$. In other words, (6.18) defines a well-defined ring homomorphism (6.19).

We now prove that $\widehat{\psi}_u$ is an isomorphism. Let

$$\mathrm{pr} : \mathbb{Z}^m \cong \pi_2\big(X; L(u)\big) \longrightarrow \mathbb{Z}^n \cong \pi_1\big(L(u)\big)$$

be the homomorphism induced by the boundary map $\mathrm{pr}([\beta]) = [\partial\beta]$ (see (2.1)). We note that $\mathrm{pr}(c_1, \dots, c_m) = (d_1, \dots, d_n)$ with $d_j = \sum_i c_i v_{i,j}$. Let $A = \sum_i c_i \beta_i$ be an element in the kernel of pr. We write it as

$$\sum_{i\in I} a_i\beta_i - \sum_{j\in J} b_j\beta_j,$$

where a_i, b_j are positive and $I \cap J = \emptyset$. We define

$$r(A) = \prod_{i\in I} z_i^{a_i} - T^{\sum_i a_i\ell_i(u) - \sum_j b_j\ell_j(u)} \prod_{j\in J} z_j^{b_j}. \tag{6.21}$$

We note that a generator of quantum Stanley-Reisner ideal corresponds to $r(A)$, for which I is a primitive collection $\mathscr{P}$ and $J = \mathscr{P}'$. We also note that the case $I = \emptyset$ or $J = \emptyset$ is included.

LEMMA 6.10
We have

$$r(A) \in SR_\omega(X).$$

Proof
This lemma is proved in [B1, Theorem 5.3]. We include its proof (which is different from the one in [B1]) here for the reader's convenience. We prove the lemma by an induction over the values

$$E(A) = \sum_{i\in I} a_i\ell_i(u_0) + \sum_{j\in J} b_j\ell_j(u_0).$$

Here we fix a point $u_0 \in \mathrm{Int}P$ during the proof of Lemma 6.10.

Since $I \cap J = \emptyset$, at least one of $\{v_i \mid i \in I\}$, $\{v_j \mid j \in J\}$ cannot span a cone that is a member of the fan Σ. Without loss of generality, we assume that $\{v_i \mid i \in I\}$ does not span such a cone. Then it contains a subset $\mathscr{P} \subset I$ that is a primitive collection. We take $\mathscr{P}'$, $k_{i'}$ as in (6.12), and we define

$$Z = \prod_{i\in I} z_i^{a_i} - T^{\omega(\mathscr{P})} \prod_{i\in I\setminus\mathscr{P}} z_i^{a_i} \prod_{i\in\mathscr{P}} z_i^{a_i-1} \prod_{i''\in\mathscr{P}'} z_{i''}^{k_{i''}}. \tag{6.22}$$

Then Z lies in $SR_\omega(X)$ by construction. We recall from Lemma 6.2 that the values

$$\sum_{i\in\mathscr{P}} \ell_i(u) - \sum_{i\in\mathscr{P}'} k_i\ell_i(u) = \omega(\mathscr{P})$$

are independent of u and positive. By the definitions (6.21) and (6.22) of $r(A)$ and Z, we can express

$$r(A) - Z = T^{\omega(\mathscr{P})+c}\Big(\prod_{h\in K} z_h^{n_h}\Big) r(B)$$

for an appropriate B in the kernel of pr and a constant c. Moreover, we have

$$E(B) + 2\sum_{h\in K} n_h \ell_h(u_0) + \omega(\mathscr{P}) = E(A).$$

Since $u_0 \in \operatorname{Int} P$, it follows that $\ell_h(u_0) > 0$, which in turn gives rise to $E(B) < E(A)$. The induction hypothesis then implies that $r(B) \in SR_\omega(X)$. The proof of the lemma is now complete. □

COROLLARY 6.11
The element z_i is invertible in

$$\frac{\Lambda[z_1, \dots, z_m]}{SR_\omega(X)}.$$

Proof
Since X is compact, the vector $-v_i$ is in some cone spanned by v_j ($j \in I$). Namely,

$$-v_i = \sum_{j\in I} k_j v_j,$$

where k_j are nonnegative integers. Then

$$T^{\ell_i(u)+\sum_j k_j \ell_j(u)} = z_i \prod_{j\in I} z_j^{k_j} \mod SR_\omega(X)$$

by Lemma 6.10. Since $T^{\ell_i(u)+\sum_j k_j \ell_j(u)}$ is invertible in the field Λ, it follows that $\prod_{j\in I} z_j^{k_j}$ defines the inverse of z_i in the quotient ring. □

We recall from Lemma 6.2 that $\ell_i(u) + \sum_j k_j \ell_j(u)$ is independent of u. We define

$$z_i^{-1} = T^{-\ell_i(u)-\sum_j k_j \ell_j(u)} \prod_{j\in I} z_j^{k_j}. \tag{6.23}$$

(Note that we have not yet proved that $\Lambda[z_1, \dots, z_m]/SR_\omega(X)$ is an integral domain; this comes later in the proof of Proposition 6.9.)

Since $v_1, \dots, v_m$ generate the lattice $\mathbb{Z}^n$, we can always assume the following by changing the order of v_i, if necessary.

Condition 6.12
The determinant of the $(n \times n)$-matrix $(v_{i,j})_{i,j=1,\dots,n}$ is ± 1.

Let $(v^{i,j})$ be the inverse matrix of $(v_{i,j})$, namely, $\sum_j v^{i,j} v_{j,k} = \delta_{i,k}$. Condition 6.12 implies that each $v^{i,j}$ is an integer. Inverting the matrix $(v_{i,j})$, we obtain

$$y_i = T^{-c_i(u)} \prod_{j=1}^{n} \overline{z}_i^{v^{i,j}} \tag{6.24}$$

from (6.20), where $c_i(u) = \sum v^{i,j} \ell_j(u)$. Using Corollary 6.11, we define

$$\widehat{\phi}_u(y_i^{\pm 1}) = T^{-\pm c_i(u)} \prod_{j=1}^{n} z_j^{\pm v^{i,j}} \in \frac{\Lambda[z_1, \dots, z_m]}{SR_\omega(X)}.$$

More precisely, we plug (6.23) here if $\pm v^{i,j}$ is negative.

The identity $\widehat{\psi}_u \circ \widehat{\phi}_u = \mathrm{id}$ is a consequence of (6.24). We next calculate $(\widehat{\phi}_u \circ \widehat{\psi}_u)(z_h) = \widehat{\phi}_u(\overline{z}_h(u))$, and we prove that

$$(\widehat{\phi}_u \circ \widehat{\psi}_u)(z_h) = T^{\ell_h(u)} \widehat{\phi}_u(y_1^{v_{h,1}} \cdots y_n^{v_{h,n}}) = T^{e(h;u)} \prod_{j=1}^{n} z_j^{m_j},$$

where $m_j \geq 0$ and

$$v_h = \sum m_j v_j, \qquad \ell_h(u) = e(h;u) + \sum m_j \ell_j(u). \tag{6.25}$$

To see (6.25), we consider any *monomial* Z of $y_i, z_i, \overline{z}_i, T^\alpha$. We define its multiplicative valuation $\mathfrak{v}_u(Z) \in \mathbb{R}$ by putting

$$\mathfrak{v}_u(y_i) = 0, \qquad \mathfrak{v}_u(z_i) = \mathfrak{v}_u(\overline{z}_i) = \ell_i(u), \qquad \mathfrak{v}_u(T^\alpha) = \alpha.$$

We also define a (multiplicative) grading $\rho(Z) \in \mathbb{Z}^n$ by

$$\rho(y_i) = \mathbf{e}_i, \qquad \rho(z_i) = \rho(\overline{z}_i) = v_i, \qquad \rho(T^\alpha) = 0$$

and by $\rho(ZZ') = \rho(Z) + \rho(Z')$. We remark that $\mathfrak{v}_u$ and ρ are consistent with (6.1). We next observe that both $\mathfrak{v}_u$ and ρ are preserved by $\widehat{\psi}_u, \widehat{\phi}_u$ and by (6.23). This implies (6.25).

Now we use Lemma 6.10 and (6.25) to conclude that

$$z_h - T^{e(h;u)} \prod_{j=1}^{n} z_j^{m_j} \in SR_\omega(X).$$

The proof of Proposition 6.9 is now complete. □

Next we prove the following.

LEMMA 6.13
Let $P(X)$ be the linear relation ideal defined in Definition 6.4. Then

$$\widehat{\psi}_u\big(P(X)\big) = \Big(\frac{\partial \mathfrak{PO}_0^u}{\partial y_i}; i = 1, \ldots, n\Big).$$

Proof
Let $\sum_{i=1}^m v_{i,j} z_i$ be in $P(X)$. Then we have

$$\widehat{\psi}_u\Big(\sum_{i=1}^m v_{i,j} z_i\Big) = \sum_{i=1}^m v_{i,j} \overline{z}_i = \sum_{i=1}^m y_j \frac{\partial \overline{z}_i}{\partial y_j} = y_j \frac{\partial \mathfrak{PO}_0^u}{\partial y_j}$$

by (6.1) and (6.4). Since y_j's are invertible in $\Lambda[y_1, \ldots, y_n, y_1^{-1}, \ldots, y_n^{-1}]$, this identity implies the lemma. □

The proofs of Theorem 6.1 and of Proposition 6.8 are now complete. □

Remark 6.14
Proposition 6.8 holds over Λ^R-coefficient for arbitrary commutative ring R with unit. The proof is the same.

We define

$$\psi_{u',u} : \mathrm{Jac}(\mathfrak{PO}_0^u) \to \mathrm{Jac}(\mathfrak{PO}_0^{u'})$$

by

$$\psi_{u',u}\big(\overline{z}_i(u)\big) = \overline{z}_i(u') = T^{\ell_i(u')-\ell_i(u)} \overline{z}_i(u). \tag{6.26}$$

It is an isomorphism. We have

$$\psi_{u',u} \circ \psi_u = \psi_{u'}.$$

The well-definedness of $\psi_{u',u}$ is proved from this formula or by checking directly.

As long as no confusion occurs, we identify $\mathrm{Jac}(\mathfrak{PO}_0^u)$, $\mathrm{Jac}(\mathfrak{PO}_0^{u'})$ by $\psi_{u',u}$ and denote them by $\mathrm{Jac}(\mathfrak{PO}_0)$. Since $\psi_{u',u}(\overline{z}_i(u)) = \overline{z}_i(u')$, we write them $\overline{z}_i$ when we regard it as an element of $\mathrm{Jac}(\mathfrak{PO}_0)$. Note that $\psi_{u',u}(y_i) \neq y_i$. In case we regard $y_i \in \mathrm{Jac}(\mathfrak{PO}_0^u)$ as an element of $\mathrm{Jac}(\mathfrak{PO}_0)$, we write it as $y_i(u) := \psi_u(y_i)$.

Remark 6.15
The above proof of Theorem 6.1 uses Batyrev's presentation of the quantum cohomology ring, and it is not likely generalized beyond the case of compact toric manifolds. (In fact, the proof is purely algebraic and does *not* contain a serious study of pseudoholomorphic curve, except Proposition 6.7, which we quote without proof, and Theorem 4.5, which is a minor improvement of an earlier result of [CO].) There is an alternative way of constructing the ring homomorphism ψ_u which is less computational. (This will give a new proof of Proposition 6.7.) We will give this conceptual proof in a future article.

We use the operation

$$\mathfrak{q}_{1,k;\beta} : H(X;\mathbb{Q})[2] \otimes B_k H\big(L(u);\mathbb{Q}\big)[1] \to H\big(L(u);\mathbb{Q}\big)[1],$$

which was introduced by the authors in [FOOO3, Section 3.8] and [FOOO2, Section 13]. Using the class $z_i \in H^2(X;\mathbb{Z})$ the Poincaré dual to $\pi^{-1}(\partial_i P)$, we put

$$\psi_u(z_i) = \sum_k \sum_{i=1}^{m} T^{\beta_i \cap \omega / 2\pi} \int_{L(u)} \mathfrak{q}_{1,k;\beta_i}(z_i \otimes x^{\otimes k}). \tag{6.27}$$

Here we put $x = \sum x_i \mathbf{e}_i$, and the right-hand side is a formal power series of x_i with coefficients in Λ.

Using the description of the moduli space defining the operators $\mathfrak{q}_{1,k;\beta}$ (see Section 11), it is easy to see that the right-hand side of (6.27) coincides with the definition of $\overline{z}_i$ in the current case, when X is Fano toric. Extending the expression (6.27) to an arbitrary homology class z of arbitrary degree, we obtain

$$\psi_u(z) = \sum_k \sum_{\beta;\mu(\beta)=\deg z} T^{\beta\cap\omega/2\pi} \int_{L(u)} \mathfrak{q}_{1,k;\beta}(z \otimes x^{\otimes k}). \tag{6.28}$$

Since $\mu(\beta) = \deg z$, then $\mathfrak{q}_{1,k;\beta}(z \otimes x^{\otimes k}) \in H^n(L(u);\mathbb{Q})$. One can prove that (6.28) defines a ring homomorphism from the quantum cohomology to the Jacobian ring $\mathrm{Jac}(\mathfrak{PO}^u)$. We may regard $\mathrm{Jac}(\mathfrak{PO}^u)$ as the moduli space of deformations of Floer theories of Lagrangian fibers of X. (Note that the Jacobian ring parameterizes deformations of a holomorphic function up to an appropriate equivalence. In our case, the equivalence is the right equivalence, that is, the coordinate change of the *domain*.)

Thus (6.28) is a particular case of the ring homomorphism

$$QH(X) \to HH\big(\mathfrak{Lag}(X)\big),$$

where $HH(\mathfrak{Lag}(X))$ is the Hochschild cohomology of Fukaya category of X. (We remark that Hochschild cohomology parameterizes deformations of A_∞-category.) Existence of such a homomorphism is a folk theorem (see [K]), which has been verified by various people in various favorable situations (see, e.g., [Ar]). It has been conjectured to be an isomorphism under certain conditions by various people, including Seidel and Kontsevich.

This point of view is suitable for generalizing our story to more general X (to non-Fano toric manifolds, for example) and also for including the big quantum cohomology into our story. (We also need to use the operators $\mathfrak{q}_{\ell,k}$ mentioned above for $\ell \geq 2$.)

These points will be discussed in future articles. In this article, we follow a more elementary but less conceptual approach exploiting the known calculation of the quantum cohomology of toric manifolds.

Remark 6.16
There are two other approaches to a proof of Proposition 6.7 besides the fixed-point localization. One, written by Cieliebak and Salamon [CS], uses vortex equations (gauged sigma model), and the other, written by McDuff and Tolman [McT], uses Seidel's result [Se1].

7. Localization of the quantum cohomology ring at the moment polytope

In this section, we discuss applications of Theorem 1.9. In particular, we prove Theorem 1.12. (Note that Theorem 1.3 is a consequence of Theorem 1.12.) Theorem 7.1 and Theorem 1.9 immediately imply Theorem 1.12(1).

THEOREM 7.1
There exists a bijection

$$\mathrm{Crit}(\mathfrak{PO}_0) \cong \mathrm{Hom}(\mathrm{Jac}(\mathfrak{PO}_0); \Lambda^{\mathbb{C}}).$$

Here the right-hand side is the set of unital $\Lambda^{\mathbb{C}}$-algebra homomorphisms.

We start with the following definition.

Definition 7.2
For an element $x \in \Lambda \setminus \{0\}$, we define its valuation $\mathfrak{v}_T(x)$ as the unique number $\lambda \in \mathbb{R}$ such that $T^{-\lambda}x \in \Lambda_0 \setminus \Lambda_+$.

We note that $\mathfrak{v}_T$ is a multiplicative non-Archimedean valuation; that is, it satisfies

$$\mathfrak{v}_T(x+y) \geq \min\big(\mathfrak{v}_T(x), \mathfrak{v}_T(y)\big),$$
$$\mathfrak{v}_T(xy) = \mathfrak{v}_T(x) + \mathfrak{v}_T(y).$$

LEMMA 7.3
For any $\varphi \in \mathrm{Hom}(\mathrm{Jac}(\mathfrak{PO}_0); \Lambda^{\mathbb{C}})$ *there exists a unique* $u \in M_{\mathbb{R}}$ *such that*

$$\mathfrak{v}_T\big(\varphi(y_j(u))\big) = 0 \tag{7.1}$$

for all $j = 1, \ldots, n$.

Proof
We still assume Condition 6.12. By definition (6.1) of $\overline{z}_i$, the homomorphism property of φ, and the multiplicative property of valuation, we obtain

$$\mathfrak{v}_T\big(\varphi(\overline{z}_i)\big) = \ell_i(u) + \sum_{j=1}^{n} v_{i,j}\,\mathfrak{v}_T\big(\varphi(y_j(u))\big), \tag{7.2}$$

for $i = 1, \ldots, m$. On the other hand, since $\ell_i(u) = \langle u, v_i\rangle - \lambda_i$ and $(v_{i,j})_{i,j=1,\ldots,n}$ is invertible, there is a unique u that satisfies

$$\mathfrak{v}_T\big(\varphi(\overline{z}_i)\big) = \ell_i(u) \tag{7.3}$$

for $i = 1, \ldots, n$. But by the invertibility of $(v_{i,j})_{i,j=1,\ldots,n}$ and (7.2), this is equivalent to (7.1), and hence we have the proof. □

We note that obviously by the above proof the formula (7.3) automatically holds for $i = n+1, \ldots, m$ and u in Lemma 7.3 as well.

Proof of Theorem 7.1
Consider the maps

$$\Psi_1(\varphi) = \sum_{i=1}^{n} \big(\log \varphi(y_i(u))\big)\mathbf{e}_i \in H^1\big(L(u); \Lambda_0\big), \qquad \Psi_2(\varphi) = u \in M_{\mathbb{R}},$$

where u is obtained as in Lemma 7.3. Since $y_i(u) \in \Lambda_0 \setminus \Lambda_+$, it follows that we can define its logarithm on Λ_0 as a convergent power series with respect to the non-Archimedean norm.

Set $(\mathfrak{x}, u) = (\Psi_1(\varphi), \Psi_2(\varphi))$. Since φ is a ring homomorphism from $\mathrm{Jac}(\mathfrak{PO}_0) \cong \mathrm{Jac}(\mathfrak{PO}_0^u)$, it follows from the definition of the Jacobian ring that

$$\frac{\partial \mathfrak{PO}_0^u}{\partial y_i}(\mathfrak{x}) = 0.$$

Therefore by Theorem 4.10, $HF((L(u), \mathfrak{x}), (L(u), \mathfrak{x}); \Lambda) \neq 0$. We have thus defined

$$\Psi : \mathrm{Hom}\big(\mathrm{Jac}(\mathfrak{PO}_0); \Lambda^{\mathbb{C}}\big) \to \mathrm{Crit}(\mathfrak{PO}_0).$$

Let $(\mathfrak{x}, u) \in \mathrm{Crit}(\mathfrak{PO}_0)$. We put $\mathfrak{x} = \sum \mathfrak{x}_i \mathbf{e}_i$. We define a homomorphism $\varphi : \mathrm{Jac}(\mathfrak{PO}_0) \to \Lambda$ by assigning

$$\varphi\big(y_i(u)\big) = e^{\mathfrak{x}_i}.$$

It is straightforward to check that φ is well defined. Then we define $\Phi(\mathfrak{x}, u) := \varphi$. It easily follows by definition that Φ is an inverse to Ψ. The proof of Theorem 7.1 is complete. □

We next work with the (Batyrev) quantum cohomology side.

Definition 7.4
For each z_i, we define a Λ-linear map $\widehat{z_i} : QH^\omega(X; \Lambda^{\mathbb{C}}) \to QH^\omega(X; \Lambda^{\mathbb{C}})$ by $\widehat{z_i}(z) = z_i \cup_Q z$, where $\cup_Q$ is the product in $QH^\omega(X; \Lambda^{\mathbb{C}})$.

Since $QH^\omega(X; \Lambda)$ is generated by even-degree elements, it follows that it is commutative. Therefore, we have

$$\widehat{z_i} \circ \widehat{z_j} = \widehat{z_j} \circ \widehat{z_i}. \tag{7.4}$$

Definition 7.5
For $\mathfrak{w} = (\mathfrak{w}_1, \dots, \mathfrak{w}_n) \in (\Lambda^{\mathbb{C}})^n$, we put

$$QH^\omega(X; \mathfrak{w}) = \big\{x \in QH^\omega(X; \Lambda^{\mathbb{C}}) \,\big|\, (\widehat{z_i} - \mathfrak{w}_i)^N x = 0 \ \text{ for } i = 1, \dots, n \text{ and large } N.\big\}$$

We say that $\mathfrak{w}$ is a *weight* of $QH^\omega(X)$ if $QH^\omega(X; \mathfrak{w})$ is nonzero. We denote by $W(X; \omega)$ the set of weights of $QH^\omega(X)$.

We remark that $\mathfrak{w}_i \neq 0$ since z_i is invertible (see Corollary 6.11).

Remark 7.6
Since z_i, $i = 1, \dots, n$ generates $QH^\omega(X; \Lambda)$ by Condition 6.12, we have the following. For each $\mathfrak{w} = (\mathfrak{w}_1, \dots, \mathfrak{w}_n)$ there exist $\mathfrak{w}_{n+1}, \dots, \mathfrak{w}_m$ depending only on $\mathfrak{w}$ such that $(\widehat{z_i} - \mathfrak{w}_i)^N x = 0$ also holds for $i = n + 1, \dots, m$ if N is sufficiently large and $x \in QH^\omega(X; \mathfrak{w})$.

PROPOSITION 7.7
(1) *There exists a factorization of the ring*

$$QH^\omega(X; \Lambda^{\mathbb{C}}) \cong \prod_{\mathfrak{w} \in W(X;\omega)} QH^\omega(X; \mathfrak{w}).$$

(2) *There exists a bijection*

$$W(X;\omega) \cong \mathrm{Hom}\big(QH^{\omega}(X;\Lambda);\Lambda^{\mathbb{C}}\big).$$

(3) *In addition,* $QH^{\omega}(X;\mathfrak{w})$ *is a local ring and* (1) *is the factorization to indecomposables.*

Proof
Existence of decomposition (1) as a $\Lambda^{\mathbb{C}}$-vector space is a standard linear algebra, using the fact that $\Lambda^{\mathbb{C}}$ is an algebraically closed field. (We prove this fact in Lemma A.1.) If $z \in QH^{\omega}(X;\mathfrak{w})$ and $z' \in QH^{\omega}(X;\mathfrak{w}')$, then

$$(z_i - \mathfrak{w}_i)^N \cup_Q (z \cup_Q z') = \big((z_i - \mathfrak{w}_i)^N \cup_Q z\big) \cup_Q z' = 0,$$
$$(z_i - \mathfrak{w}'_i)^N \cup_Q (z \cup_Q z') = \big((z_i - \mathfrak{w}'_i)^N \cup_Q z'\big) \cup_Q z = 0.$$

Therefore, $z \cup_Q z' \in QH^{\omega}(X;\mathfrak{w}) \cap QH^{\omega}(X;\mathfrak{w}')$. This implies that the decomposition (1) is a ring factorization.

Let $\varphi : QH^{\omega}(X;\mathfrak{w}) \to \Lambda^{\mathbb{C}}$ be a unital $\Lambda^{\mathbb{C}}$-algebra homomorphism. It induces a homomorphism $QH^{\omega}(X;\Lambda) \to \Lambda^{\mathbb{C}}$ by (1). We denote this ring homomorphism by the same letter φ. Let $z \in QH^{\omega}(X;\mathfrak{w})$ be an element such that $\varphi(z) \neq 0$. Then we have

$$\big(\varphi(z_i) - \mathfrak{w}_i\big)^N \varphi(z) = \varphi\big((z_i - \mathfrak{w}_i)^N \cup_Q z\big) = 0.$$

Therefore,

$$\mathfrak{w}_i = \varphi(z_i). \tag{7.5}$$

Since z_i generates $QH^{\omega}(X;\Lambda)$, it follows from (7.5) that there is a unique $\Lambda^{\mathbb{C}}$-algebra homomorphism : $QH^{\omega}(X;\mathfrak{w}) \to \Lambda^{\mathbb{C}}$. Then (2) follows.

Since $QH^{\omega}(X;\mathfrak{w})$ is a finite-dimensional $\Lambda^{\mathbb{C}}$-algebra and since $\Lambda^{\mathbb{C}}$ is algebraically closed, we have an isomorphism

$$\frac{QH^{\omega}(X;\mathfrak{w})}{\mathrm{rad}} \cong (\Lambda^{\mathbb{C}})^k \tag{7.6}$$

for some k. (Here, rad equals $\{z \in QH^{\omega}(X;\mathfrak{w}) \mid z^N = 0 \text{ for some } N\}$.) Since there is a unique unital $\Lambda^{\mathbb{C}}$-algebra homomorphism $QH^{\omega}(X;\mathfrak{w}) \to \Lambda^{\mathbb{C}}$, it follows that $k = 1$. Specifically, $QH^{\omega}(X;\mathfrak{w})$ is a local ring.

It also implies that $QH^{\omega}(X;\mathfrak{w})$ is indecomposable. □

The result up to here also works for the non-Fano case. But Theorem 7.8 requires the fact that X is Fano since we use the equality $QH^{\omega}(X;\Lambda) \cong QH(X;\Lambda)$.

THEOREM 7.8
If X is Fano, then $\mathrm{Crit}(\mathfrak{PO}_0) = \mathfrak{M}(\mathfrak{Lag}(X))$.

Proof
Let $\mathfrak{w}$ be a weight. We take $z \in QH^{\omega}(X;\mathfrak{w}) \subset H(X;\Lambda^{\mathbb{C}}) \cong H(X;\mathbb{C}) \otimes \Lambda^{\mathbb{C}}$. We may take an eigen vector z so that

$$z \in \big(H(X;\mathbb{C}) \otimes \Lambda_0^{\mathbb{C}}\big) \setminus \big(H(X;\mathbb{C}) \otimes \Lambda_+^{\mathbb{C}}\big).$$

Since

$$z_i \cup_Q z \equiv z_i \cup z \mod \Lambda_+^{\mathbb{C}},$$

where $\cup$ is the classical cup product (we use $QH^{\omega}(X;\Lambda) = QH(X;\Lambda)$ here), it follows that

$$\mathfrak{w}_i^n z = (\widehat{z}_i)^n(z) = (z_i)^n \cup_Q z \equiv (z_i)^n \cup z \mod \Lambda_+^{\mathbb{C}}.$$

Therefore, $\mathfrak{w}_i \in \Lambda_+^{\mathbb{C}}$ as $(z_i)^n \cup z = 0$. Then (7.3) and (7.5) imply that

$$\ell_i(u) = \mathfrak{v}_T(\mathfrak{w}_i) > 0.$$

In particular, $u \in \mathrm{Int} P$. □

We are now ready to complete the proof of Theorem 1.12. Here (1) is Theorem 7.1; (2) is a consequence of Theorem 4.10; (3) is Theorem 7.8. If $QH^{\omega}(X;\Lambda^{\mathbb{C}})$ is semisimple, then (7.6) and $k = 1$ imply that

$$QH^{\omega}(X;\Lambda^{\mathbb{C}}) \cong (\Lambda^{\mathbb{C}})^{\#W(X;\omega)} \tag{7.7}$$

as a $\Lambda^{\mathbb{C}}$-algebra. Thus (4) follows from (7.7), Proposition 7.7(2), and Theorem 7.1. The proof of Theorem 1.12 is complete. □

We next explain the factorization in Proposition 7.7(1) from the point of view of the Jacobian ring. Let $(\mathfrak{x}, u) \in \mathrm{Crit}(\mathfrak{PO}_0)$.

Definition 7.9
We consider the ideal generated by

$$\frac{\partial}{\partial w_i}\mathfrak{PO}_0^u(\mathfrak{y}_1 + w_1, \ldots, \mathfrak{y}_n + w_n) \tag{7.8}$$

$i = 1, \ldots, n$, in the ring $\Lambda[[w_1, \ldots, w_n]]$ of formal power series, where $\mathfrak{x} = \sum \mathfrak{x}_i \mathbf{e}_i$ and $\mathfrak{y}_i = e^{\mathfrak{x}_i}$. We denote its quotient ring by $\mathrm{Jac}(\mathfrak{PO}_0; \mathfrak{x}, u)$.

PROPOSITION 7.10

(1) *There is a direct product decomposition*

$$\mathrm{Jac}(\mathfrak{PO}_0) \cong \prod_{(\mathfrak{x},u)\in \mathrm{Crit}(\mathfrak{PO}_0)} \mathrm{Jac}(\mathfrak{PO}_0;\mathfrak{x},u)$$

as a ring.

(2) *If* $(\mathfrak{x},u) \in \mathrm{Crit}(\mathfrak{PO}_0)$ *corresponds to* $\mathfrak{w} \in W(X;\omega)$ *via the isomorphism given in Proposition 7.7(2) and Theorem 7.1, then* ψ_u *induces an isomorphism*

$$\psi_u : QH^{\omega}(X;\mathfrak{w}) \cong \mathrm{Jac}(\mathfrak{PO}_0;\mathfrak{x},u).$$

(3) $\mathrm{Jac}(\mathfrak{PO}_0;\mathfrak{x},u)$ *is* 1*-dimensional (over* Λ*) if and only if the Hessian*

$$\Big(\frac{\partial^2 \mathfrak{PO}_0^u}{\partial y_i \partial y_j}\Big)_{i,j=1,\dots,n}$$

is invertible over Λ *at* $\mathfrak{x}$.

Proof

We put $\mathfrak{x} = \sum \mathfrak{x}_i \mathbf{e}_i$ and $\mathfrak{y}_i = e^{\mathfrak{x}_i}$. Let $\mathfrak{m}(\mathfrak{x},u)$ be the ideal generated by $y_i - \mathfrak{y}_i$ in the ring

$$\mathrm{Jac}(\mathfrak{PO}_0) = \frac{\Lambda[y_1^{\pm},\dots,y_n^{\pm}]}{\big(y_i(\partial \mathfrak{PO}_0^u/\partial y_i)\big)}.$$

Since $\mathrm{Jac}(\mathfrak{PO}_0)$ is finite-dimensional over Λ, it follows that

$$\mathrm{Jac}(\mathfrak{PO}_0) \cong \prod_{(\mathfrak{x},u)\in \mathrm{Crit}(\mathfrak{PO}_0)} \mathrm{Jac}(\mathfrak{PO}_0)_{\mathfrak{m}(\mathfrak{x},u)},$$

where $\mathrm{Jac}(\mathfrak{PO}_0)_{\mathfrak{m}(\mathfrak{x},u)}$ is the localization of the ring $\mathrm{Jac}(\mathfrak{PO}_0)$ at $\mathfrak{m}(\mathfrak{x},u)$. Using finite-dimensionality of $\mathrm{Jac}(\mathfrak{PO}_0)$ again, we have an isomorphism $\mathrm{Jac}(\mathfrak{PO}_0)_{\mathfrak{m}(\mathfrak{x},u)} \cong \mathrm{Jac}(\mathfrak{PO}_0;\mathfrak{x},u)$, which sends $y_i - \mathfrak{y}_i$ to w_i. Here $\mathfrak{x} = \sum_i \mathfrak{x}_i \mathbf{e}_i$ and $\mathfrak{y}_i = e^{\mathfrak{x}_i}$, and (1) follows.

Now we prove (2). If $z \in QH^{\omega}(X;\mathfrak{w})$, then $(\widehat{z_i} - \mathfrak{w}_i)^N z = 0$. Let $\pi_{\mathfrak{x},u} : \mathrm{Jac}(\mathfrak{PO}_0) \to \mathrm{Jac}(\mathfrak{PO}_0;\mathfrak{x},u)$ be the projection. We then have

$$(T^{\ell_i(u)} \widehat{y}_1^{v_{i,1}} \cdots \widehat{y}_n^{v_{i,n}} - \mathfrak{w}_i)^N \pi_{\mathfrak{x},u}\big(\psi_u(z)\big) = 0. \tag{7.9}$$

We note that

$$\mathfrak{w}_i = T^{\ell_i(u')} \mathfrak{y}_1'^{v_{i,1}} \cdots \mathfrak{y}_n'^{v_{i,n}} \tag{7.10}$$

if $\mathfrak{w}_i$ corresponds $(\mathfrak{x}', u')$ and $\mathfrak{y}'_i$ are exponential of the coordinates of $\mathfrak{x}'$. We define the operator $\widehat{y_i} : \mathrm{Jac}(\mathfrak{PO}_0; \mathfrak{x}, u) \to \mathrm{Jac}(\mathfrak{PO}_0; \mathfrak{x}, u)$ by

$$\widehat{y_i}(c) = \mathfrak{y}_i c.$$

By definition of $\mathrm{Jac}(\mathfrak{PO}_0; \mathfrak{x}, u)$, the eigenvalue of $\widehat{y_i}$ is $\mathfrak{y}_i$. Therefore, (7.9) and (7.10) imply that $\pi_{\mathfrak{x},u}(\psi_u(z)) = 0$ unless $(\mathfrak{x}, u) = (\mathfrak{x}', u')$. Thus (2) follows.

Let us prove (3). We first note that $\Lambda = \Lambda^{\mathbb{C}}$ is an algebraically closed field (Lemma A.1). Therefore, $\dim_\Lambda \mathrm{Jac}(\mathfrak{PO}_0; \mathfrak{x}, u) = 1$ if and only if the ideal generated by (7.8) (for $i = 1, \dots, n$) is the maximal ideal $\mathfrak{m} = (w_1, \dots, w_n)$. We note that $\mathfrak{m}/\mathfrak{m}^2 = \Lambda^n$ and that element (7.8) reduces to

$$\Big(\frac{\partial^2 \mathfrak{PO}_0^u}{\partial y_i \partial y_j}\Big)_{j=1,\dots,n} \in \Lambda^n,$$

modulo $\mathfrak{m}^2$. So (3) follows easily. □

We recall that a symplectic manifold (X, ω) is said to be (spherically) monotone if there exists $\lambda > 0$ such that $c_1(X) \cap \alpha = \lambda\, [\omega] \cap \alpha$ for all $\alpha \in \pi_2(X)$. The Lagrangian submanifold L of (X, ω) is said to be monotone if there exists $\lambda > 0$ such that $\mu(\beta) = \lambda \omega(\beta)$ for any $\beta \in \pi_2(X, L)$. (Here μ is the Maslov index.) In the monotone case we have the following.

THEOREM 7.11
If X is a monotone compact toric manifold, then there exists a unique u_0 such that

$$\mathfrak{M}\big(\mathfrak{Lag}(X)\big) \subset \Lambda \times \{u_0\}$$

(i.e., whenever $(\mathfrak{x}, u) \in \mathfrak{M}(\mathfrak{Lag}(X))$, $u = u_0$). Moreover, $L(u_0)$ is monotone.

Remark 7.12
Related results are discussed in [EP1].

Proof
Since X is Fano, we have $QH^\omega(X; \Lambda) = QH(X; \Lambda)$. We assume that $c_1(X) \cap \alpha = \lambda\, [\omega] \cap \alpha$ with $\lambda > 0$. Let $\cup_\alpha$ be the contribution to the moduli space of pseudo-holomorphic curve of homology class $\alpha \in H_2(X; \mathbb{Z})$ in the quantum cup product (see (6.5)). We have a decomposition:

$$x \cup_Q y = x \cup y + \sum_{\alpha \in \pi_2(X) \setminus \{0\}} T^{\alpha \cap [\omega]/2\pi} x \cup_\alpha y.$$

Then

$$\deg(x \cup_\alpha y) = \deg x + \deg y - 2c_1(X) \cap \alpha = \deg x + \deg y - 2\lambda\alpha \cap [\omega]. \quad (7.11)$$

We define

$$\mathfrak{v}_{\deg}(T^{1/2\pi}) = 2\lambda, \qquad \mathfrak{v}_{\deg}(x) = \deg x \quad (\text{for } x \in H(X;\mathbb{Q})).$$

$\mathfrak{v}_{\deg}$ is a multiplicative non-Archimedean valuation on $QH(X;\Lambda)$ such that $\mathfrak{v}_{\deg}(a \cup_Q b) = \mathfrak{v}_{\deg}(a) + \mathfrak{v}_{\deg}(b)$, by virtue of (7.11). Moreover, for $c \in \Lambda$ and $a \in QH(X;\Lambda)$, we have $\mathfrak{v}_{\deg}(ca) = 2\lambda\mathfrak{v}_T(c) + \mathfrak{v}_{\deg}(a)$. Now let $\mathfrak{w}$ be a weight and $x \in QH^\omega(X;\mathfrak{w})$. Since $\mathfrak{v}_{\deg}(z_i) = 2$, it follows that

$$2\lambda\mathfrak{v}_T(\mathfrak{w}_i) + \mathfrak{v}_{\deg}(x) = \mathfrak{v}_{\deg}(z_i x) = 2 + \mathfrak{v}_{\deg}(x).$$

Therefore, if $(\mathfrak{x}, u)$ corresponds to $\mathfrak{w}$, then $\ell_i(u) = \mathfrak{v}_T(\mathfrak{w}_i) = 1/\lambda$, and thus u is independent of $\mathfrak{w}$. We denote it by u_0.

For $\beta_i \in H_2(X, L(u_0))$ $(i = 1, \dots, m)$ given by (6.7), we have $\omega(\beta_i) = 2\pi\ell_i(u_0) = 2\pi/\lambda$. Hence $\mu(\beta_i) = \lambda\omega(\beta_i)/\pi$. Since β_i generates $H_2(X, L(u_0))$, it follows that $L(u_0)$ is monotone, as required. □

So far we have studied Floer cohomology with $\Lambda^{\mathbb{C}}$-coefficients. We next consider the case of Λ^F-coefficients where F is a finite Galois extension of $\mathbb{Q}$. We choose F so that all of the weights $\mathfrak{w}$ lie in $(\Lambda_0^F)^n$. (Since every finite extension of $\Lambda^{\mathbb{Q}}$ is contained in such Λ^F, we can always find such an F; see appendix.) Then we have a decomposition

$$QH^\omega(X;\Lambda^F) \cong \prod_{\mathfrak{w} \in W(X;\omega)} QH^\omega(X;\mathfrak{w};F). \quad (7.12)$$

It follows that the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ acts on $W(X;\omega)$; it induces a $\mathrm{Gal}(F/\mathbb{Q})$-action on $\mathrm{Crit}(\mathfrak{PO}_0)$ (we use Remark 6.14 here). We write it as $(\mathfrak{x}, u) \mapsto (\sigma(\mathfrak{x}), \sigma(u))$. We note the following.

PROPOSITION 7.13
(1) *We have* $\sigma(u) = u$.
(2) *We write by* $y_i(\mathfrak{x})$ *the exponential of the coordinates of* $\mathfrak{x}$. *Then* $y_i(\mathfrak{x}) \in \Lambda^F$ *and* $y_i(\sigma(\mathfrak{x})) = \sigma(y_i(\mathfrak{x}))$.
(3) *If* $QH^\omega(X;\Lambda^{\mathbb{Q}})$ *is indecomposable, there exists* u_0 *such that whenever* $(\mathfrak{x}, u) \in \mathrm{Crit}(\mathfrak{PO}_0))$, *we have* $u = u_0$.

Proof
Let $\mathfrak{w}_i(\mathfrak{x})$ correspond to $(\mathfrak{x}, u)$. Then

$$\ell_i\big(\sigma(u)\big) = \mathfrak{v}_T\big(\mathfrak{w}_i(\sigma(\mathfrak{x}, u))\big) = \mathfrak{v}_T\big(\sigma\mathfrak{w}_i(\mathfrak{x}, u)\big) = \mathfrak{v}_T\big(\mathfrak{w}_i(\mathfrak{x}, u)\big) = \ell_i(u).$$

Thus (1) follows; (2) follows from the definition and (1); and (3) is a consequence of (1). □

A monotone blow-up of $\mathbb{C}P^2$ (at one point) gives an example where the assumption of Proposition 7.13(3) is satisfied.

It seems interesting to observe that the ring $QH(X; \Lambda^{\mathbb{Q}})$ jumps sometimes when we deform the symplectic structure of X. The point where this jump occurs is closely related to the point where the number of balanced Lagrangian fibers jumps. In the case of Example 5.7, we have

$$QH(X; \Lambda^{\mathbb{Q}}) \cong \begin{cases} (\Lambda^{\mathbb{Q}})^5 & \alpha > 0, \\ (\Lambda^{\mathbb{Q}})^3 \times \Lambda^{\mathbb{Q}(\sqrt{-3})} & \alpha < 0, \\ \Lambda^{\mathbb{Q}(\sqrt{5})} \times \Lambda^F & \alpha = 0, \end{cases}$$

where $F = \mathbb{Q}[x]/(x^3 - x - 1)$. We observe that $x^5 + x^4 - 2x^3 - 2x^2 + 1 = (x^2 + x - 1)(x^3 - x - 1)$ (we also refer the reader to Example 10.10 for further examples).

Remark 7.14
In Sections 11 – 13, we use de Rham cohomology of the Lagrangian submanifold to define and study Floer cohomology. As a consequence, our results on Floer cohomology are proved over $\Lambda_0^{\mathbb{R}}$ or $\Lambda_0^{\mathbb{C}}$ but not over $\Lambda_0^{\mathbb{Q}}$ or Λ_0^F. (The authors believe that those results can be also proved over $\Lambda_0^{\mathbb{Q}}$ by using the singular cohomology version developed in [FOOO3], although the detail of their proofs could be more complicated.)

On the other hand, Proposition 6.8 and Theorem 6.1 are proved over $\Lambda_0^{\mathbb{Q}}$. Therefore, the discussion on quantum cohomology here works over Λ_0^F.

We also note that, though Proposition 7.13(3) is related to Floer cohomology, its proof given above does *not* use Floer cohomology over $\Lambda_0^{\mathbb{Q}}$ but only Floer cohomology over $\Lambda_0^{\mathbb{C}}$ and quantum cohomology over $\Lambda_0^{\mathbb{Q}}$. In fact, the proof above implies the following. If $u \in \mathrm{Int}P$ and $\mathfrak{x} \in H^1(L(u); \Lambda_0^{\mathbb{C}})$ satisfy

$$\frac{\partial \mathfrak{PO}_0^u}{\partial x_i}(\mathfrak{x}) = 0,$$

then $u = u_0$. This is because

$$\mathrm{Jac}(\mathfrak{PO}_0^u; \Lambda^{\mathbb{C}}) = \mathrm{Jac}(\mathfrak{PO}_0^u; \Lambda^{\mathbb{Q}}) \otimes_{\Lambda^{\mathbb{Q}}} \Lambda^{\mathbb{C}}$$

and

$$\mathrm{Jac}(\mathfrak{PO}_0^u; \Lambda^{\mathbb{Q}}) \cong QH(X; \Lambda^{\mathbb{Q}}).$$

8. Further examples and remarks

In this section, we show how we can use the arguments of Sections 6 and 7 to illustrate calculations of $\mathfrak{M}(\mathfrak{Lag}(X))$ in examples.

Example 8.1

We consider the 1-point blow-up X of $\mathbb{C}P^2$. We choose its Kähler form so that the moment polytope is

$$P = \left\{(u_1, u_2) \,\middle|\, 0 \le u_1, u_2,\ u_1 + u_2 \le 1,\ u_2 \le 1 - \alpha\right\},$$

$0 < \alpha < 1$. The potential function is

$$\mathfrak{PO} = y_1 T^{u_1} + y_2 T^{u_2} + (y_1 y_2)^{-1} T^{1-u_1-u_2} + y_2^{-1} T^{1-\alpha-u_2}.$$

We put $\overline{z}_1 = y_1 T^{u_1}$, $\overline{z}_2 = y_2 T^{u_2}$, $\overline{z}_3 = (y_1 y_2)^{-1} T^{1-u_1-u_2}$, $\overline{z}_4 = y_2^{-1} T^{1-\alpha-u_2}$.

The quantum Stanley-Reisner relation is

$$\overline{z}_1 \overline{z}_3 = \overline{z}_4 T^{\alpha}, \qquad \overline{z}_2 \overline{z}_4 = T^{1-\alpha}, \tag{8.1}$$

and the linear relation is

$$\overline{z}_1 - \overline{z}_3 = 0, \qquad \overline{z}_2 - \overline{z}_3 - \overline{z}_4 = 0. \tag{8.2}$$

We put $X = \overline{z}_1$ and $Y = \overline{z}_2$, and we solve (8.1) and (8.2). We obtain

$$X^3(T^{\alpha} + X) = T^{1+\alpha}, \tag{8.3}$$

with $Y = X + T^{-\alpha} X^2$. We consider valuations of both sides of (8.3). There are three different cases to consider.

Case 1: $\mathfrak{v}_T(X) > \alpha$. Here (8.3) implies that $3\mathfrak{v}_T(X) + \alpha = 1 + \alpha$, namely, $\mathfrak{v}_T(X) = 1/3$. So $\alpha < 1/3$. Moreover, $\mathfrak{v}_T(Y) = 1/3$. We have $u_1 = \mathfrak{v}_T(X) = 1/3$, $u_2 = \mathfrak{v}_T(Y) = 1/3$ (see Lemma 7.3). Writing $X = a_1 T^{1/3} + a_2 T^{\lambda} +$ higher-order terms with $\lambda > 1/3$ and substituting this into (8.3), we obtain $a_1^3 = 1$, which has three simple roots. Each of them corresponds to the solution for $\mathfrak{x}$ by Hensel's lemma (see, e.g., [BGR, Proposition 3]). (It also follows from Theorem 10.4 in Section 10.)

Case 2: $\mathfrak{v}_T(X) < \alpha$. By taking the valuation of (8.3), we obtain $u_1 = \mathfrak{v}_T(X) = (1+\alpha)/4$. Hence $\alpha > 1/3$. Moreover, $u_2 = \mathfrak{v}_T(Y) = (1-\alpha)/2$. In the same way as in Case 1, we can check that there are four solutions.

Case 3: $\mathfrak{v}_T(X) = \alpha$. We put $X = a_1 T^{\alpha} + a_2 T^{\lambda} +$ higher-order terms where $\lambda > \alpha$.

Case 3(1): $a_1 \neq -1$. By taking valuation of (8.3), we obtain $u_1 = \mathfrak{v}_T(X) = 1/3$. Then $\alpha = 1/3$ and $u_2 = \mathfrak{v}_T(Y) = 1/3$. So (8.3) becomes

$$a_1^4 + a_1^3 - 1 = 0. \tag{8.4}$$

(In this case, $X = a_1 T^{\alpha}$ has no higher term.) There are four solutions. We note that (8.4) is irreducible over $\mathbb{Q}$, since it is also irreducible over $\mathbb{Z}_2$. Thus the assumption of Proposition 7.13(3) is satisfied. Actually, X is monotone in the case $\alpha = 1/3$. Hence the same conclusion (the uniqueness of u) follows from Theorem 7.11 also.

Case 3(2): $a_1 = -1$. By taking valuation of (8.3), we obtain $\lambda = 1 - 2\alpha$. Here $\lambda > \alpha$ implies that $\alpha < 1/3$; $u_2 = \mathfrak{v}_T(Y) = 1 - 2\alpha$ and $u_1 = \mathfrak{v}_T(X) = \alpha$. There is one solution.

In summary, if $\alpha < 1/3$, there are two choices of $u = (\alpha, 1-2\alpha), (1/3, 1/3)$. On the other hand, the numbers of choices of $\mathfrak{x}$ are 1 and 3, respectively.

If $\alpha \geq 1/3$, there is the unique choice $u = ((1+\alpha)/4, (1-\alpha)/2)$. The number of choices of $\mathfrak{x}$ is 4.

We next study a non-Fano case, the Hirzebruch surface F_n. Note that F_1 is a 1-point blow-up of $\mathbb{C}P^2$, which we have already studied. We leave the case F_2 to the reader.

Example 8.2

We consider the Hirzebruch surface F_n, $n \geq 3$. We take its Kähler form so that the moment polytope is

$$P = \left\{(u_1, u_2) \,\middle|\, 0 \leq u_1, u_2,\ u_1 + n u_2 \leq n,\ u_2 \leq 1 - \alpha\right\},$$

$0 < \alpha < 1$. The leading-order potential function is

$$\mathfrak{PO}_0 = y_1 T^{u_1} + y_2 T^{u_2} + y_1^{-1} y_2^{-n} T^{n-u_1-nu_2} + y_2^{-1} T^{1-\alpha-u_2}.$$

We put $\overline{z}_1 = y_1 T^{u_1}$, $\overline{z}_2 = y_2 T^{u_2}$, $\overline{z}_3 = y_1^{-1} y_2^{-n} T^{n-u_1-nu_2}$, $\overline{z}_4 = y_2^{-1} T^{1-\alpha-u_2}$.

The quantum Stanley-Reisner relation and the linear relation give

$$\overline{z}_1 \overline{z}_3 = \overline{z}_4^n T^{n\alpha}, \qquad \overline{z}_2 \overline{z}_4 = T^{1-\alpha}, \tag{8.5}$$

$$\overline{z}_1 - \overline{z}_3 = 0, \qquad \overline{z}_2 - n\overline{z}_3 - \overline{z}_4 = 0. \tag{8.6}$$

Let us assume that n is odd. We put

$$\overline{z}_1 = Z^n, \qquad \overline{z}_4 = Z^2 T^{-\alpha}.$$

(In case $n = 2n'$ is even, we put $\overline{z}_1 = Z^{n'}$, $\overline{z}_4 = \pm Z T^{-\alpha}$. The rest of the arguments are similar and so are omitted.) Then $\overline{z}_2 = T^{-\alpha} Z^2 + n Z^n$ and

$$Z^4(nZ^{n-2} + T^{-\alpha}) = T. \tag{8.7}$$

Case 1: $(n-2)\mathfrak{v}_T(Z) > -\alpha$. In the first case, we have $\mathfrak{v}_T(Z) = (\alpha+1)/4$. (Then $(n-2)\mathfrak{v}_T(Z) > -\alpha$ is automatically satisfied.) Therefore, $u_1 = \mathfrak{v}_T(z_1) = n(\alpha+1)/4$, $u_2 = \mathfrak{v}_T(z_2) = (1-\alpha)/2$. We also can check that there are four solutions. We note that we are using $\mathfrak{PO}_0$ in place of $\mathfrak{PO}$. However, we can use Corollary 10.6 to prove Lemma 8.3. This lemma in particular implies that $L(n(\alpha+1)/4, (1-\alpha)/2)$ is balanced, which was already shown above in Example 8.1 for the case $n = 1$.

LEMMA 8.3
Let $y^{(i)} \in \Lambda_0 \times \Lambda_0$ *(*$i = 1, \dots, 4$*) be critical points of* $\mathfrak{PO}_0^u$ *for* $u = (n(\alpha + 1)/4, (1-\alpha)/2)$. *Then there exists* $y^{(i)\prime} \in \Lambda_0 \times \Lambda_0$, *which is a critical point of* $\mathfrak{PO}^u$ *and* $y^{(i)} \equiv y^{(i)\prime} \mod \Lambda_+$.

We prove Lemma 8.3 in Section 10.

Case 2: $(n-2)\mathfrak{v}_T(Z) < -\alpha$. We have $v_T(Z) = 1/(n+2)$. This can never occur since $1/(n+2) > 0 > -\alpha/(n-2)$.

Case 3: $(n-2)\mathfrak{v}_T(Z) = -\alpha$. We put $Z = a_1 T^{-\alpha/(n-2)} + a_2 T^{\lambda} +$ higher-order term.

Case 3(1): $na_1^{n-2} \neq -1$. Then $\mathfrak{v}_T(Z) = (\alpha+1)/4$. Since $(\alpha+1)/4 \neq -\alpha/(n-2)$, this case never occurs.

Case 3(2): $na_1^{n-2} = -1$. We have $4v_T(Z) + (n-3)v_T(Z) + \lambda = 1$. Therefore,

$$\lambda = \frac{n-2+(n+1)\alpha}{n-2}.$$

We have

$$u_1 = v_T(z_1) = -\frac{n\alpha}{n-2}, \qquad u_2 = v_T(z_2) = 1 - \alpha - v_T(z_4) = \frac{n-2+2\alpha}{n-2}.$$

Thus (u_1, u_2) is *not* in the moment polytope.

In Example 8.2, we have

$$\mathfrak{M}\big(\mathfrak{Lag}(X)\big) = \mathfrak{M}_0\big(\mathfrak{Lag}(X)\big) \neq \mathrm{Crit}(\mathfrak{PO}_0).$$

On the other hand, the order of $\mathfrak{M}(\mathfrak{Lag}(X))$ is 4 and is equal to the sum of Betti numbers.

Remark 8.4
In a future article, we will prove the equality

$$\sum_d \mathrm{rank}\, H_d(X;\mathbb{Q}) = \#\big(\mathfrak{M}(\mathfrak{Lag}(X))\big)$$

for any compact toric manifold X (which is not necessarily Fano) such that $QH(X;\Lambda)$ is semisimple. If we count the right-hand side with multiplicity, the same equality holds without assuming semisimplicity.

We next discuss a version of the above in which we substitute some explicit numbers into the formal variable T. Let $u \in \mathrm{Int}P$. We define a Laurent polynomial

$$\mathfrak{PO}^u_{0,T=t} \in \mathbb{C}[y_1, \dots, y_n, y_1^{-1}, \dots, y_n^{-1}]$$

by substituting a complex number $t \in \mathbb{C} \setminus \{0\}$. In the same way, we define the algebra $QH^\omega(X; T = t; \mathbb{C})$ over $\mathbb{C}$ by substituting $T = t$ in the quantum Stanley-Reisner relation. The argument of Section 6 goes through to show that

$$QH^\omega(X; T = t; \mathbb{C}) \cong \mathrm{Jac}(\mathfrak{PO}^u_{0,T=t}). \tag{8.8}$$

In particular, the right-hand side is independent of u up to an isomorphism. Here the $\mathbb{C}$-algebra in the right-hand side of (8.8) is the quotient of the polynomial ring $\mathbb{C}[y_1, \dots, y_n, y_1^{-1}, \dots, y_n^{-1}]$ by the ideal generated by $\partial\mathfrak{PO}_{0,T=t}/\partial y_i$ ($i = 1, \dots, n$).

We remark that the right-hand side of (8.8) is always nonzero for small t by Proposition 4.7. It follows that the equation

$$\frac{\partial\mathfrak{PO}^u_{0,T=t}}{\partial y_i} = 0 \tag{8.9}$$

has a solution $y_i \neq 0$ for *any* u: namely, as far as the Floer cohomology after $T = t$ substituted, there *always* exists $b \in H^1(X;\mathbb{C})$ with nonvanishing Floer cohomology $HF\big((L(u), b), (L(u), b); \mathbb{C}\big)$ for any $u \in \mathrm{Int}P$. Since the version of Floer cohomology after substituting $T = t$ is *not* invariant under the Hamiltonian isotopy, this is not useful for the application to symplectic topology (cf. [CO, Section 14.2]).

The relation between the set of solutions of (8.9) and that of (4.10) is stated as follows. Let $(y_1^{(c)}(t;u), \dots, y_n^{(c)}(t;u))$ be a branch of the solutions of (8.9) for $t \neq 0$, where c is an integer with $1 \le c \le l$ for some $l \in \mathbb{N}$. We can easily show that it is a holomorphic function of t on $\mathbb{C} \setminus \mathbb{R}_-$. We consider its behavior as $t \to 0$. For generic u, the limit either diverges or converges to zero. However, if $(\mathfrak{x}, u) \in \mathfrak{M}_0(\mathfrak{Lag}(X))$ and $\mathfrak{x} = \sum x_i \mathbf{e}_i$, then there is some c such that

$$\lim_{t \to 0} y_i^{(c)}(t;u) \in \mathbb{C} \setminus \{0\} \qquad \text{and} \qquad y_i^{(c)}(t;u) = e^{x_i(t)}.$$

To prove this claim, it suffices to show that if $(\mathfrak{x}, u) \in \mathrm{Crit}(\mathfrak{PO}_0)$ and $\mathfrak{x} = \sum x_i \mathbf{e}_i$, $y_i = e^{x_i}$, $y_i = \sum_j y_{ij} T^{\lambda_{ij}}$, then $\sum_j y_{ij} t^{\lambda_{ij}}$ converges for $0 < |t| < \epsilon$ (here ϵ is a sufficiently small positive number). This follows from Lemma 8.5 below. Let $\Lambda_0^{\mathrm{conv}}$ be the ring

$$\Big\{ \sum_i a_i T^{\lambda_i} \in \Lambda_0^{\mathbb{C}} \,\Big|\, \exists \epsilon > 0 \text{ such that } \sum_i |a_i||t|^{\lambda_i} \text{ converges for } |t| < \epsilon \Big\},$$

and let Λ^{conv} be its field of fractions. We put $\Lambda_+^{\mathrm{conv}} = \Lambda_0^{\mathrm{conv}} \cap \Lambda_+$.

LEMMA 8.5
The field Λ^{conv} is algebraically closed.

We prove Lemma 8.5 in Section A.

We go back to the discussion on the difference between two sets $\mathfrak{M}_0(\mathfrak{Lag}(X))$ and $\mathrm{Crit}(\mathfrak{PO}_0)$ (see Definition 1.11). The rest of this section owes much to the discussion with Iritani and also to his articles [I1], [I2]. The results we describe below are not used elsewhere in this article.

We recall that we did *not* take closure of the ideal $(P(X) + SR_\omega(X))$ in Section 6. This is actually the reason why we have $\mathfrak{M}_0(\mathfrak{Lag}(X)) \neq \mathrm{Crit}(\mathfrak{PO}_0)$; more precisely, we have Proposition 8.6 below.

We consider the polynomial ring $\Lambda[z_1, \dots, z_m]$. We define its norm $\|\cdot\|$ so that

$$\Big\| \sum_{\vec{i}} a_{\vec{i}} z_1^{i_1} \cdots z_m^{i_m} \Big\| = \exp\big(- \inf_{\vec{i}} \mathfrak{v}_T(a_{\vec{i}}) \big).$$

We take the closure of the ideal $(P(X) + SR_\omega(X))$ with respect to this norm and denote it by $\mathrm{Clos}(P(X) + SR_\omega(X))$. We put

$$\overline{QH}^{\omega}(X;\Lambda) = \frac{\Lambda[z_1, \dots, z_m]}{\mathrm{Clos}(P(X) + SR_\omega(X))}. \tag{8.10}$$

Let $W^{\mathrm{geo}}(X;\omega)$ be the set of all weights such that the corresponding $(\mathfrak{x}, u)$ satisfy $u \in \mathrm{Int}\, P$. We note that $\mathfrak{w} \in W^{\mathrm{geo}}(X;\omega)$ if and only if $\mathfrak{v}_T(\mathfrak{w}_i) > 0$ for all i.

PROPOSITION 8.6 (Iritani)
There exists an isomorphism:

$$\overline{QH}^{\omega}(X;\Lambda^{\mathbb{C}}) \cong \prod_{\mathfrak{w}\in W^{\mathrm{geo}}(X;\omega)} QH^{\omega}(X;\mathfrak{w}).$$

Proof
Let $\mathfrak{w} \in W(X;\omega) \setminus W^{\mathrm{geo}}(X;\omega)$. We first assume that $\mathfrak{v}_T(\mathfrak{w}_i) = -\lambda < 0$. (The case $\mathfrak{v}_T(\mathfrak{w}_i) = 0$ is discussed at the end of the proof.)

Then there exists $f \in \Lambda_0 \setminus \Lambda_+$ such that $T^{\lambda} f \mathfrak{w}_i = 1$. Let $x \in QH^{\omega}(X;\mathfrak{w})$. We assume that $x \neq 0$. We take k such that $(\widehat{z}_i - \mathfrak{w}_i)^k x \neq 0$, $(\widehat{z}_i - \mathfrak{w}_i)^{k+1} x = 0$ and replace x by $(\widehat{z}_i - \mathfrak{w}_i)^k x$. We then have $T^{\lambda} f \widehat{z}_i x = x$. Since $\lim_{N\to\infty} \|(f z_i T^{\lambda})^N\| = 0$, it follows that $x = 0$ in $\overline{QH}^{\omega}(X;\Lambda^{\mathbb{C}})$. This is a contradiction.

We next assume that $\mathfrak{v}_T(\mathfrak{w}_i) > 0$ for all i. We consider the homomorphism

$$\varphi : \Lambda[z_1,\ldots,z_m] \to \mathrm{Hom}_{\Lambda}\big(QH^{\omega}(X;\mathfrak{w}), QH^{\omega}(X;\mathfrak{w})\big),$$

defined by

$$\varphi(z_i)(x) = z_i \cup_Q x.$$

We have $\varphi(P(X) + SR_{\omega}(X)) = 0$. We may choose the basis of $QH^{\omega}(X;\mathfrak{w})$ so that $\varphi(z_i)$ is an upper-triangular matrix whose diagonal entries are all $\mathfrak{w}_i$ and whose off-diagonal entries are all zero or 1. We use it and $\mathfrak{v}_T(\mathfrak{w}_i) > 0$ to show that $\varphi\big(\mathrm{Clos}(P(X) + SR_{\omega}(X))\big) = 0$, and specifically, that φ induces a homomorphism from $\overline{QH}^{\omega}(X;\Lambda)$. It follows easily that the restriction of the projection $QH^{\omega}(X;\Lambda^{\mathbb{C}}) \to \overline{QH}^{\omega}(X;\Lambda^{\mathbb{C}})$ to $QH^{\omega}(X;\mathfrak{w})$ is an isomorphism to its image.

We finally show that for $u \in \partial P$, there is no critical point of $\mathfrak{PO}_0$ on $(\Lambda_0 \setminus \Lambda_+)^n$. Let

$$u \in \bigcup_{i\in I} \partial_i P \setminus \bigcup_{i\notin I} \partial_i P_i.$$

Then

$$\mathfrak{PO}_0^u \equiv \sum_{i\in I} y_1^{v_{i,1}} \cdots y_n^{v_{i,n}} \mod \Lambda_+.$$

We note that v_i $(i \in I)$ is a part of the $\mathbb{Z}$ basis of $\mathbb{Z}^n$, since X is nonsingular toric. Hence by changing the variables to appropriate y_i', it is easy to see that there is no nonzero critical point of $\sum_{i\in I} y_1^{v_{i,1}} \cdots y_n^{v_{i,n}} = \sum_{i\in I'} y_i'$. The proof of Proposition 8.6 is now complete. □

To further discuss the relationship between the contents of Sections 6 and 7 and those in [I2], we compare the coefficient rings used here and in [I2]. In [I2] (like many of the literatures on quantum cohomology such as [G1]), the formal power series ring $\mathbb{Q}[[q_1, \dots, q_{m-n}]]$ is taken as the coefficient ring ($m-n$ is the rank of $H^2(X;\mathbb{Q})$ and we choose a basis of it). The superpotential in [I2] (which is the same as the one used in [G1]) is given as*

$$F_q = \sum_{i=1}^{m} \Big(\prod_{a=1}^{m-n} q_a^{l_{a,i}} \prod_{j=1}^{n} s_j^{v_{i,j}} \Big). \tag{8.11}$$

Here $l_{a,i}$ is a matrix element of a splitting of $H_2(X;\mathbb{Z}) \to H_2(X, T^n;\mathbb{Z})$. We show that (8.11) pulls back to our potential function $\mathfrak{PO}_0^u$ after a simple change of variables. Let $\alpha_a \in H_2(X;\mathbb{Z})$ be the basis we have chosen (we choose it so that $[\omega] \cap \alpha_a$ is positive).

LEMMA 8.7
There exists $f_j(u) \in \mathbb{R}$ ($j = 1, \dots, n$) such that

$$\frac{1}{2\pi} \sum_a l_{a,i}[\omega] \cap \alpha_a = \ell_i(u) - \sum_{j=1}^{n} v_{i,j} f_j(u).$$

Proof
We consider the exact sequence

$$0 \longrightarrow H_2(X;\mathbb{Z}) \xrightarrow{i_*} H_2\big(X, L(u);\mathbb{Z}\big) \longrightarrow H_1\big(L(u);\mathbb{Z}\big) \longrightarrow 0.$$

So $(c_1, \dots, c_m) \in H_2(X, L(u);\mathbb{Z})$ is in the image of $H_2(X;\mathbb{Z})$ if and only if $\sum_i c_i v_i = 0$ (here $v_i = (v_{i,1}, \dots, v_{i,n}) \in \mathbb{Z}^n$). For a given $\alpha \in H_2(X, \mathbb{Z})$, denote $i_*(\alpha) = (c_1, \dots, c_m)$. Then we have

$$\sum_a [\omega] \cap c_i l_{a,i} \alpha_a = [\omega] \cap \alpha = 2\pi \sum c_i \ell_i(u).$$

This implies the lemma. □

We now put

$$q_a = T^{[\omega]\cap\alpha_a/2\pi}, \qquad s_j(u) = T^{f_j(u)} y_j. \tag{8.12}$$

*We change the notation so that it is consistent to ours; $m, n, v_{i,j}$ here corresponds to $r+N$, r, $x_{i,b}$ in [I2], respectively.

We obtain the identity

$$F_q\big(s_1(u), \ldots, s_n(u)\big) = \mathfrak{PO}_0^u(y_1, \ldots, y_n). \tag{8.13}$$

We note that if we change the choice of Kähler form, then the identification (8.12) changes. In other words, the study of quantum cohomology over $\mathbb{Q}[[q_1, \ldots, q_{m-n}]]$ corresponds to studying all the symplectic structures simultaneously, while the study of quantum cohomology over Λ focuses on one particular symplectic structure.

In [I2, Corollary 5.12], Iritani proved the semisimplicity of a quantum cohomology ring of a toric manifold with a coefficient ring $\mathbb{Q}[[q_1, \ldots, q_{m-n}]]$; it does not imply the semisimplicity of our $QH^{\omega}(X; \Lambda)$ since the semisimplicity in general is not preserved by the pullback. (On the other way around, semisimplicity follows from semisimplicity of the pullback.) However, it is preserved by the pullback at a generic point. Specifically, we have the following.

PROPOSITION 8.8
The set of T^n-invariant symplectic structures on X for which $\mathrm{Jac}(\mathfrak{PO}_0^u)$ *is semisimple is open and dense.*

Proof
Here we give a proof for completeness, following the argument in [I2, proof of Proposition 5.11]. Consider the polynomial

$$F_{w_1,\ldots,w_m} = \sum_{i=1}^{m} w_i y_1^{v_{i,1}} \cdots y_n^{v_{i,n}},$$

where $w_i \in \mathbb{C} \setminus \{0\}$. By Kushnirenko's theorem [K], the Jacobian ring of $F_{w_1,\ldots,w_m}$ is semisimple for a generic $w_1, \ldots, w_m$. We put

$$w_i = \exp\Big(\frac{1}{2\pi} \sum_a l_{a,i}[\omega] \cap \alpha + \sum_j v_{i,j} f_j(u)\Big).$$

It is easy to see that when we move $[\omega] \cap \alpha_a$ and u (there are $(m-n)$ and n parameters, respectively), w_i moves in an arbitrary way. Therefore, for generic choice of ω and u, the Jacobian ring $\mathrm{Jac}(\mathfrak{PO}_0^u)$ is semisimple. Since $\mathrm{Jac}(\mathfrak{PO}_0^u)$ is independent of u up to isomorphism, the proposition follows. □

Remark 8.9
Combined with Theorem 1.9, this proposition gives a partial answer to [EP2, Section 3, Question].

9. Variational analysis of potential function

In this section, we prove Proposition 4.7. Let $\mathfrak{PO}$ be defined as in (4.7).

We define

$$s_1(u) = \inf\left\{\ell_i(u) \,\middle|\, i = 1, \dots, m\right\}.$$

Here s_1 is a continuous, piecewise affine and convex function, and $s_1 \equiv 0$ on ∂P. Recall that if $u \in \partial_i P$, then $\ell_i(u) = 0$ by definition.

We put

$$S_1 = \sup\left\{s_1(u) \,\middle|\, u \in P\right\},$$
$$P_1 = \left\{u \in P \,\middle|\, s_1(u) = S_1\right\}.$$

PROPOSITION 9.1

There exist s_k, S_k, and P_k with these properties:

(1) P_{k+1} *is a convex polyhedron in* $M_{\mathbb{R}}$. $\dim P_{k+1} \le \dim P_k$*;*

(2) $s_{k+1} : P_k \to \mathbb{R}$ *is a continuous, convex piecewise affine function;*

(3) $s_{k+1}(u) = \inf\{\ell_i(u) \mid \ell_i(u) > S_k\}$ *for* $u \in \operatorname{Int} P_k$*;*

(4) $s_{k+1}(u) = S_k$ *for* $u \in \partial P_k$*;*

(5) $S_{k+1} = \sup\{s_{k+1}(u) \mid u \in P_k\}$*;*

(6) $P_{k+1} = \{u \in P_k \mid s_{k+1}(u) = S_{k+1}\}$*;*

(7) $P_{k+1} \subset \operatorname{Int} P_k$*;*

(8) s_k, S_k, P_k *are defined for* $k = 1, 2, \dots, K$ *for some* $K \in \mathbb{Z}_+$, *and* P_K *consists of a single point.*

Example 9.2

Let $P = [0, a] \times [0, b]$ $(a < b)$. Then $s_1(u_1, u_2) = \inf\{u_1, u_2, a - u_1, b - u_2\}$ *and* $S_1 = a/2$, $P_1 = \{(a/2, u_2) \mid a/2 \le u_2 \le b - a/2\}$, $s_2(1/2, u_2) = \inf\{u_2, b - u_2\}$, $S_2 = b/2$, $P_2 = \{(a/2, b/2)\}$.

Proof

We define s_k, S_k, and P_k inductively over k. We assume that s_k, S_k, and P_k are defined for $k = 1, \dots, k_0$ so that items (1)–(7) of Proposition 9.1 are satisfied for $k = 1, \dots, k_0 - 1$.

We define s_{k_0+1} by (3) and (4). We prove that it satisfies (2), using Lemma 9.3 for this purpose.

LEMMA 9.3

Let $u_j \in \operatorname{Int} P_{k_0}$, *and let* $\lim_{j\to\infty} u_j = u_\infty \in \partial P_{k_0}$. *Then*

$$\lim_{j\to\infty} s_{k_0+1}(u_j) = S_{k_0}.$$

Proof
We put

$$I'_{k_0} = \{\ell_i \mid \ell_i(u_\infty) = S_{k_0}\}. \tag{9.1}$$

By (6) for $k = k_0 - 1$, we find that $s_{k_0}(u_\infty) = S_{k_0}$. Then (3) for $k = k_0 - 1$ implies that there is ℓ_i such that $\ell_i(u_\infty) = S_{k_0}$. Thus I'_{k_0} is nonempty. We take the affine space $A_{k_0} \subset M_{\mathbb{R}}$ such that $\mathrm{Int} P_{k_0}$ is relatively open in A_{k_0}.

Now since $u_\infty \in \partial P_{k_0}$, we can take $\vec{u} \in T_{u_\infty} A_{k_0}$ such that $u_\infty + \epsilon\vec{u} \notin P_{k_0}$ for any sufficiently small $\epsilon > 0$. It follows from (7) for $k = k_0 - 1$ that $u_\infty + \epsilon\vec{u} \in \mathrm{Int}\, P_{k_0-1}$, and hence $u + \epsilon\vec{u} \in \mathrm{Int}\, P_{k_0-1} \setminus P_{k_0}$.

By definition, we also have $s_{k_0}(u) \le S_{k_0}$ for all $u \in P_{k_0-1}$. Therefore, we have

$$s_{k_0}(u_\infty + \epsilon\vec{u}) < S_{k_0}.$$

It follows that there exists $\ell_i \in I'_{k_0}$ such that

$$\ell_i(u_\infty + \epsilon\vec{u}) < \ell_i(u_\infty) = S_{k_0} < \ell_i(u_\infty - \epsilon\vec{u}). \tag{9.2}$$

Since (9.2) holds for any $\vec{u} \in T_{u_\infty} A_{k_0}$ with $u_\infty + \epsilon\vec{u} \notin P_{k_0}$, it follows that $\epsilon\vec{u} := u_j - u_\infty$ for any sufficiently large j. We note that since $u_j \in \mathrm{Int}\, P_{k_0} \subset A_{k_0}$, $\vec{u}_j = u_\infty - u_j$ is an "outward" vector as a tangent vector in $T_{u_\infty} A_{k_0}$ at $u_\infty \in \partial P_{k_0}$. Therefore, we have $u_\infty + \vec{u}_j \notin P_{k_0}$. Because $u_j = u_\infty - \vec{u}_j$, it follows from (9.2) that

$$\ell_i(u_j) > S_{k_0} \tag{9.3}$$

for any sufficiently large j. Therefore, we have

$$s_{k_0+1}(u_j) = \inf\{\ell_i(u_j) \mid \ell_i \in I'_{k_0},\ \ell_i(u_j) > S_{k_0}\} \tag{9.4}$$

and $\lim_{j\to\infty} s_{k_0+1}(u_j) = \lim_{j\to\infty} \ell_i(u_j) = \ell_i(u_\infty) = S_{k_0}$. This finishes the proof of the lemma. □

Lemma 9.3 implies that s_{k_0+1} is continuous and piecewise linear in a neighborhood of ∂P_{k_0}. We can then check (2) easily.

We define S_{k_0+1} by (5). Then we can define P_{k_0+1} by (6) (in other words, the right-hand side of (6) is nonempty); (7) is a consequence of Lemma 9.3. We can easily check that P_{k_0+1} satisfies (1).

We finally prove that P_K becomes a point for some K. Let $\mathbf{u}_k \in \mathrm{Int}\, P_k$, and put

$$I_k = \{\ell_i \mid \ell_i(\mathbf{u}_k) = S_k\}. \tag{9.5}$$

Here $k = 1, \ldots, K$. We remark that I_k is independent of the choice of $\mathbf{u}_k \in \mathrm{Int}\, P_k$.

Note that we defined I'_{k_0} by the formula (9.1). We have $I_{k_0} \subseteq I'_{k_0}$. But the equality may not hold in general. In fact, u_∞ is in the boundary of P_{k_0}, but $\mathbf{u}_{k_0}$ is an interior point of P_{k_0}. Therefore, if $\ell_i \in I_{k_0}$, then ℓ_i is constant on P_{k_0}. But the element of I'_{k_0} may not have this property.

In case some $\ell_i \in I_{k_0+1}$ is not constant on P_{k_0}, it is easy to see that $\dim P_{k_0+1} < \dim P_{k_0}$. There exists some $\ell_j \notin \bigcup_{k \le k_0} I_{k_0}$ which is not constant on S_{k_0} unless S_{k_0} is a point. Therefore, if $\dim S_{k_0} \neq 0$, there exists $k' > k_0$ such that $\dim P_{k'} < \dim P_{k_0}$. Therefore there exists K such that P_K becomes zero-dimensional (namely, a point). Hence we have achieved (8). The proof of Proposition 9.1 is now complete. □

Remark 9.4
In [Mc], McDuff points out an error in statement (1) of Proposition 9.1 in the previous version of this article. We have corrected the statement and have modified the last paragraph of its proof, following the corresponding argument in [Mc, Section 2.2], and we thank her for pointing out this error.

The next lemma easily follows from construction.

LEMMA 9.5
If all the vertices of P lie in $\mathbb{Q}^n$, then $u_0 \in \mathbb{Q}^n$. Here $\{u_0\} = P_K$.

By parallel translation of the polytope, we may assume, without loss of generality, that $u_0 = \mathbf{0}$, the origin. In the rest of this section, we prove that $\mathfrak{PO}^{\mathbf{0}}$ has a critical point in $(\Lambda_0 \setminus \Lambda_+)^n$. More precisely, we prove Proposition 4.7 for $u_0 = \mathbf{0}$ (we note that if P and ℓ_i are given, we can easily locate u_0).

Example 9.6
Let us consider Example 8.1 in the case $\alpha > 1/3$. At $u_0 = ((1+\alpha)/4, (1-\alpha)/2)$, we have

$$\mathfrak{PO}^{u_0} = (y_2 + y_2^{-1})T^{(1-\alpha)/2} + \big(y_1 + (y_1 y_2)^{-1}\big)T^{(1+\alpha)/4}.$$

Therefore, the constant term $\mathfrak{y}_{i;0}$ of the coordinate y_i of the critical point is given by

$$1 - \mathfrak{y}_{2;0}^{-2} = 0, \qquad 1 - \mathfrak{y}_{1;0}^{-2}\mathfrak{y}_{2;0}^{-1} = 0. \tag{9.6}$$

Note that the first equation comes from the term of the smallest exponent and contains only $\mathfrak{y}_{2;0}$. The second equation comes from the term which has the second smallest exponent and contains both $\mathfrak{y}_{1;0}$ and $\mathfrak{y}_{2;0}$. So we need to solve the equation inductively according to the order of the exponent. This is the situation we want to work out in general.

We observe that the affine space A_i defined above in the proof of Lemma 9.3,

$$M_{\mathbb{R}} = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_{K-1} \supseteq A_K = \{\mathbf{0}\},$$

is a nonincreasing sequence of linear subspaces such that Int P_k is an open subset of A_k. Let

$$A_l^{\perp} \subset (M_{\mathbb{R}})^* \cong N_{\mathbb{R}}$$

be the annihilator of $A_l \subset M_{\mathbb{R}}$. Then we have

$$\{\mathbf{0}\} = A_0^{\perp} \subseteq A_1^{\perp} \subseteq \cdots \subseteq A_{K-1}^{\perp} \subseteq A_K^{\perp} = N_{\mathbb{R}}.$$

We recall that

$$I_k = \left\{\ell_i \,\middle|\, \ell_i(\mathbf{0}) = S_k\right\}, \tag{9.7}$$

for $k = 1, \ldots, K$. In fact, $\mathbf{0} \in P_{k+1} \subseteq \operatorname{Int} P_k$ for $k < K$.

We renumber each of I_k in (9.7) so that

$$\left\{\ell_{k,j} \,\middle|\, j = 1, \ldots, a(k)\right\} = I_k. \tag{9.8}$$

By construction,

$$s_k(u) = \inf_j \ell_{k,j}(u) \tag{9.9}$$

in a neighborhood of $\mathbf{0}$ in P_{k-1}. In fact, $s_{k-1}(\mathbf{0}) = S_{k-1} < S_k = s_k(\mathbf{0})$ and

$$\left\{\ell_i(\mathbf{0}) \,\middle|\, i = 1, \ldots, m\right\} \cap (S_{k-1}, S_k) = \emptyset.$$

LEMMA 9.7
If $u \in A_k$, then $\ell_{k,j}(u) = S_k$.

Proof
We may assume that $k < K$. Hence $\mathbf{0} \in \operatorname{Int} P_k$. We regard $u \in A_k = T_{\mathbf{0}} A_k$. By (9.9), we have

$$s_k(\varepsilon u) = \inf\left\{\ell_{k,j}(\epsilon u) \,\middle|\, j = 1, \ldots, a(k)\right\}.$$

Since $s_k(\varepsilon u) = S_k$ for $\varepsilon u \in P_k$, it follows that $\ell_{k,j}(u) = S_k$. □

Lemma 9.7 implies that the linear part $d\ell_{k,j}$ of $\ell_{k,j}$ is an element of $A_k^{\perp} \subset \mathfrak{t} = N_{\mathbb{R}}$. In fact, if $\ell_{k,j} = \ell_i$, we have $d\ell_{k,j} = v_i$ from the definition of ℓ_i, $\ell_i(u) = \langle u, v_i \rangle - \lambda_i$ given in Theorem 2.13.

LEMMA 9.8
For any $v \in A_k^\perp$, *there exist nonnegative real numbers* $c_j \geq 0$, $j = 1, \dots, a(k)$ *such that*

$$v - \sum_{j=1}^{a(k)} c_j d\ell_{k,j} \in A_{k-1}^\perp.$$

Proof
Suppose to the contrary that

$$\left\{ v - \sum_{j=1}^{a(k)} c_j d\ell_{k,j} \,\middle|\, c_j \geq 0, \;\; j = 1, \dots, a(k) \right\} \bigcap A_{k-1}^\perp = \emptyset.$$

Then we can find $u \in A_{k-1} \setminus A_k$ such that

$$d\ell_{k,j}(u) \geq 0 \tag{9.10}$$

for all $j = 1, \dots, a(k)$.

Since $\varepsilon u \in A_{k-1} \setminus A_k$, it follows that

$$s_k(\varepsilon u) < S_k$$

for a sufficiently small ε. On the other hand, (9.10) implies that $d\ell_{k,j}(\varepsilon u) \geq 0$ for all $\varepsilon > 0$, and so $\ell_{k,j}(\varepsilon u) \geq \ell_{k,j}(\mathbf{0}) = S_k$. Therefore, by definition of s_k in Proposition 9.1, we have

$$\begin{aligned} s_k(\varepsilon u) &\geq \inf\left\{ \ell_{k,j}(\varepsilon u) \,\middle|\, j = 1, \dots, a(k) \right\} \\ &\geq \inf\left\{ \ell_{k,j}(\mathbf{0}) \,\middle|\, j = 1, \dots, a(k) \right\} = S_k. \end{aligned}$$

This is a contradiction. □

Applying Lemma 9.8 inductively downward starting from $\ell = k$ and ending at $\ell = 1$, we immediately obtain the following.

COROLLARY 9.9
For any $v \in A_k^\perp$, *there exist* $c_{l,j} \geq 0$ *for* $l = 1, \dots, k$, $j = 1, \dots, a(l)$ *such that*

$$v = \sum_{l=1}^{k} \sum_{j=1}^{a(l)} c_{l,j} d\ell_{l,j}.$$

We denote

$$\mathfrak{I} = \left\{\ell_i \,\middle|\, i = 1, \ldots, m\right\} \setminus \bigcup_{k=1}^{K} I_k. \tag{9.11}$$

It is easy to see that

$$\ell \in \mathfrak{I} \Rightarrow \ell(\mathbf{0}) > S_K. \tag{9.12}$$

Now we go back to the situation of (4.7) and we use the notation of (4.7). In this case, for each $k = 1, \ldots, K$ we also associate, in Definition 9.10, a set $\mathfrak{I}_k$ consisting of pairs (ℓ, ρ) with an affine map $\ell : M_{\mathbb{R}} \to \mathbb{R}$ and $\rho \in \mathbb{R}_+$.

Definition 9.10
We say that a pair $(\ell, \rho) = (\ell'_j, \rho_j)$ is an element of $\mathfrak{I}_k$ if the following holds:
(1) If $e^i_j \neq 0$, then $\ell_i \in \bigcup_{l=1}^{k} I_l$ (note that $\ell'_j = \sum_i e^i_j \ell_i$).
(2) Item (1) does not hold for some i, j if we replace k by $k - 1$.
A pair $(\ell, \rho) = (\ell'_j, \rho_j)$ as in (4.7) is, by definition, an element of $\mathfrak{I}_{K+1}$ if it is not contained in any of $\mathfrak{I}_k$, $k = 1, \ldots, K$.

LEMMA 9.11
(1) *If $(\ell, \rho) \in \mathfrak{I}_k$, then $d\ell \in A_k^{\perp}$.*
(2) *If $(\ell, \rho) \in \mathfrak{I}_k$, then $\ell(\mathbf{0}) + \rho > S_k$.*
(3) *If $(\ell, \rho) \in \mathfrak{I}_{K+1}$, then $\ell(\mathbf{0}) + \rho > S_K$.*

Proof
Item (1) follows from Definition 9.10(1) and Lemma 9.7.

If $(\ell, \rho) = (\ell'_j, \rho_j) \in \mathfrak{I}_k$, then there exists $e^i_j \neq 0$, $\ell_i = \ell_{k,j'}$. Then

$$\ell(\mathbf{0}) + \rho \geq e^i_j \ell_i(\mathbf{0}) + \rho_j > \ell_i(\mathbf{0}) = S_k.$$

So (2) follows. The proof of (3) is the same. □

LEMMA 9.12
The vector space A_k is defined over $\mathbb{Q}$.

Proof
The vector spaceA_k is defined by equalities of the type $\ell_i = S_k$ on A_{k-1}. Since the linear part of ℓ_i has integer coefficients, the lemma follows by induction on k. □

We put $d(k) = \dim A_{k-1} - \dim A_k = \dim A_k^{\perp} - \dim A_{k-1}^{\perp}$. We choose $\mathbf{e}^*_{i,j} \in \mathrm{Hom}(M_{\mathbb{Q}}, \mathbb{Q}) \cong N_{\mathbb{Q}}$ ($i = 1, \dots, K$, $j = 1, \dots, d(k)$) such that the following condition holds. Here $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$ and $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$.

Condition 9.13

We have

(1) $\mathbf{e}^*_{1,1}, \dots, \mathbf{e}^*_{k,d(k)}$ is a $\mathbb{Q}$ basis of $A_k^{\perp} \cap N_{\mathbb{Q}}$;

(2) $d\ell_{k,j} = \sum_{k',j'} v_{(k,j),(k',j')} \mathbf{e}^*_{k',j'}$ with $v_{(k,j),(k',j')} \in \mathbb{Z}$;

(3) if $(\ell, \rho) \in \mathfrak{I}_k$ or $\ell \in \mathfrak{I}$, then $d\ell = \sum_{k',j'} v_{\ell,(k',j')} \mathbf{e}^*_{k',j'}$ with $v_{\ell,(k',j')} \in \mathbb{Z}$.

Note that $d(k) = 0$ if $A_k = A_{k-1}$.

We identify $\mathbb{R}^n$ with $H^1(L(u); \mathbb{R})$ in the same way as in Lemma 4.4, and we let $x_{k,j} \in \mathrm{Hom}(H^1(L(u); \mathbb{R}), \mathbb{R})$ be the element corresponding to $\mathbf{e}^*_{k,j}$ by this identification. In other words, if

$$\mathbf{e}^*_{k,j} = \sum_i a_{(k,j);i} \mathbf{e}^*_i,$$

where $\mathbf{e}^*_i$ is as in Lemma 4.4, then we have

$$x_{k,j} = \sum_i a_{(k,j);i} x_i.$$

We put $y_{k,j} = e^{x_{k,j}}$. We define

$$Y(k, j) = \prod_{k'=1}^{K} \prod_{j'=1}^{d(k')} y_{k',j'}^{v_{(k,j),(k',j')}}. \tag{9.13}$$

And for $(\ell, \rho) \in \mathfrak{I}_k$ or $\ell \in \mathfrak{I}$, we define

$$Y(\ell) = \prod_{k=1}^{K} \prod_{j=1}^{d(k)} y_{k,j}^{v_{\ell,(k,j)}}. \tag{9.14}$$

By Theorem 4.6 there exists $c_{(\ell,\rho)} \in \mathbb{Q}$ such that

$$\begin{aligned} \mathfrak{PO}^{\mathbf{0}} = &\sum_{k=1}^{K} \Big(\sum_{j=1}^{a(k)} Y(k, j) \Big) T^{S_k} + \sum_{\ell \in \mathfrak{I}} Y(\ell) T^{\ell(\mathbf{0})} \\ &+ \sum_{k=1}^{K+1} \sum_{(\ell,\rho) \in \mathfrak{I}_k} c_{(\ell,\rho)} Y(\ell) T^{\ell(\mathbf{0})+\rho}, \end{aligned} \tag{9.15}$$

where $\mathfrak{PO}^{\mathbf{0}}$ is $\mathfrak{PO}^u$ with $u = \mathbf{0}$.

LEMMA 9.14

(1) *If* $k' < k$, *then*

$$\frac{\partial Y(k',j')}{\partial y_{k,j}} = 0. \tag{9.16}$$

(2) *If* $(\ell,\rho) \in \mathfrak{I}_{k'}$, $k' < k$, *then*

$$\frac{\partial Y(\ell)}{\partial y_{k,j}} = 0. \tag{9.17}$$

(3) *If* $(\ell,\rho) \in \mathfrak{I}_k$, *then* $\ell(\mathbf{0}) + \rho > S_k$.

(4) *If* $(\ell,\rho) \in \mathfrak{I}_{K+1}$, *then* $\ell(\mathbf{0}) + \rho > S_K$.

(5) *If* $\ell \in \mathfrak{I}$, *then* $\ell(\mathbf{0}) > S_K$.

Proof

Since $d\ell_{k',j'} \in A_{k'}^{\perp}$ by Lemma 9.7, it follows that $v_{(k',j'),(k,j)} = 0$ for $k > k'$. Then (1) follows; (2) follows from Lemma 9.11(1) in the same way; (3) follows from Lemma 9.11(2); (4) follows from Lemma 9.11(3); and (5) follows from (9.12). □

Now equation (4.10) becomes

$$0 = \frac{\partial \mathfrak{PO}^{\mathbf{0}}}{\partial y_{k,j}}.$$

We calculate this equation using Lemma 9.14 to find that it is equivalent to

$$\begin{aligned} 0 = &\sum_{j'=1}^{a(k)} \frac{\partial Y(k,j')}{\partial y_{k,j}} + \sum_{k'>k}\sum_{j'=1}^{a(k')} \frac{\partial Y(k',j')}{\partial y_{k,j}} T^{S_{k'}-S_k} \\ &+ \sum_{k'=k}^{K+1} \sum_{(\ell,\rho)\in\mathfrak{I}_{k'}} c_{(\ell,\rho)} \frac{\partial Y(\ell)}{\partial y_{k,j}} T^{\ell(\mathbf{0})+\rho-S_k} + \sum_{\ell\in\mathfrak{I}} \frac{\partial Y(\ell)}{\partial y_{k,j}} T^{\ell(\mathbf{0})-S_k}. \end{aligned} \tag{9.18}$$

Note that the exponents of T in the second, third, and fourth terms of (9.18) are all strictly positive. So after putting $T = 0$, we have

$$0 = \sum_{j'=1}^{a(k)} \frac{\partial Y(k,j')}{\partial y_{k,j}}. \tag{9.19}$$

Note that the equation (9.19) does not involve T but becomes a numerical equation. We call (9.19) the *leading-term equation*.

LEMMA 9.15
There exist positive real numbers $\mathfrak{y}_{k,j;0}$, $k = 1, \dots, K$, $j = 1, \dots, d(k)$, *solving the leading-term equations for* $k = 1, \dots, K$.

Proof
We remark that the leading-term equations for k, j contain the monomials involving only $y_{k',j}$ for $k' \le k$. We first solve the leading-term equation for $k = 1$. Denote

$$f_1(x_{1,1}, \dots, x_{1,d(1)}) = \sum_{j=1}^{a(1)} Y(1, j).$$

It follows from Corollary 9.9 that for any $(x_{1,1}, \dots, x_{1,d(1)}) \neq 0$, there exists j such that

$$d\ell_{1,j}(x_{1,1}, \dots, x_{1,d(1)}) > 0.$$

Therefore, we have

$$\lim_{t\to\infty} f_1(tx_{1,1}, \dots, tx_{1,d(1)}) \ge \lim_{t\to\infty} C \exp\bigl(td\ell_{1,j}(x_{1,1}, \dots, x_{1,d(1)})\bigr) = +\infty.$$

Hence $f_1(x_{1,1}, \dots, x_{1,d(1)})$ attains its minimum at some point of $\mathbb{R}^{d(1)}$. Taking its exponential, we obtain $\mathfrak{y}_{1,j;0} \in \mathbb{R}_+$.

Suppose that we have already found $\mathfrak{y}_{k',j;0}$ for $k' < k$. Then we put

$$F_k(x_{1,1}, \dots, x_{k,1}, \dots, x_{k,d(k)}) = \sum_{j=1}^{a(k)} Y(k, j)$$

and

$$f_k(x_{k,1}, \dots, x_{k,d(k)}) = F_k(\mathfrak{x}_{1,1;0}, \dots, \mathfrak{x}_{k-1,d(k-1);0}, x_{k,1}, \dots, x_{k,d(k)}),$$

where $\mathfrak{x}_{k',j;0} = \log \mathfrak{y}_{k',j;0}$. Again using Corollary 9.9, we find that

$$\lim_{t\to\infty} f_k(tx_{k,1}, \dots, tx_{k,d(k)}) = +\infty$$

for any $(x_{k,1}, \dots, x_{k,d(k)}) \neq 0$. Hence $f_k(x_{k,1}, \dots, x_{k,d(k)})$ attains a minimum and we obtain $\mathfrak{y}_{k,j;0}$. Lemma 9.15 now follows by induction. □

We next find the solution of our equation (4.11) or (4.12). We take a sufficiently large $\mathcal{N}$ and put

$$\begin{aligned}\mathfrak{PO}^{\mathbf{0}}_{k,\mathcal{N}} = &\sum_{j=1}^{a(k)} Y(k,j) + \sum_{k'>k}\sum_{j'=1}^{a(k')} Y(k',j')T^{S_{k'}-S_k} \\ &+ \sum_{\ell\in\mathfrak{I},\ \ell(\mathbf{0})\le\mathcal{N}} Y(\ell)T^{\ell(\mathbf{0})-S_k} \\ &+ \sum_{k'=k+1}^{K+1}\ \sum_{(\ell,\rho)\in\mathfrak{I}_{k'},\ell(\mathbf{0})+\rho\le\mathcal{N}} c_{(\ell,\rho)}Y(\ell)T^{\ell(\mathbf{0})+\rho-S_k}.\end{aligned} \tag{9.20}$$

We remark that (4.11) is equivalent to

$$\frac{\partial\mathfrak{PO}^{\mathbf{0}}_{k,\mathcal{N}}}{\partial y_{k,j}}(\mathfrak{y}_1,\dots,\mathfrak{y}_n) \equiv 0 \mod T^{\mathcal{N}-S_k} \qquad k=1,\dots,K,\ j=1,\dots,a(k). \tag{9.21}$$

We also put

$$\overline{\mathfrak{PO}}^{\mathbf{0}}_k = \sum_{j=1}^{a(k)} Y(k,j).$$

It satisfies

$$\overline{\mathfrak{PO}}^{\mathbf{0}}_k \equiv \mathfrak{PO}^{\mathbf{0}}_{k,\mathcal{N}} \mod \Lambda_+. \tag{9.22}$$

For given positive numbers $R(1),\dots,R(K)$, we define the discs

$$D\big(R(k)\big) = \big\{(x_{k,1},\dots,x_{k,d(k)}) \,\big|\, x_{k,1}^2+\cdots+x_{k,d(k)}^2 \le R(k)\big\} \subset \mathbb{R}^{d(k)}$$

and the polydiscs

$$\begin{aligned}D\big(R(\cdot)\big) &= \prod_{k=1}^{K} D\big(R(k)\big) \\ &= \big\{(x_{1,1},\dots,x_{K,d(K)}) \,\big|\, x_{k,1}^2+\cdots+x_{k,d(k)}^2 \le R(k),\ k=1,\dots,K\big\}.\end{aligned}$$

We factorize

$$\mathbb{R}^n = \prod_{k=1}^{K} \mathbb{R}^{d(k)}.$$

Then we consider the Jacobian of $\overline{\mathfrak{PO}}^{\mathbf{0}}_k$ or

$$\nabla\overline{\mathfrak{PO}}^{\mathbf{0}}_k : \mathbb{R}^n \to \mathbb{R}^{d(k)},$$

that is, the map

$$(\mathfrak{x}_{1,1}, \ldots, \mathfrak{x}_{K,d(K)}) \mapsto \left(\frac{\partial\overline{\mathfrak{PO}}^{\mathbf{0}}_k}{\partial x_{k,j}}(\mathfrak{x}_{1,1}, \ldots, \mathfrak{x}_{K,d(K)})\right)_{j=1,\ldots,d(k)}. \tag{9.23}$$

We remark that $\nabla\overline{\mathfrak{PO}}^{\mathbf{0}}_k$ depends only on $\mathbb{R}^{d(1)} \times \cdots \times \mathbb{R}^{d(k)}$ components.

Combining all $\nabla\overline{\mathfrak{PO}}^{\mathbf{0}}_k$, $k = 1, \ldots, K$, (9.23) induces a map

$$\nabla\overline{\mathfrak{PO}}^{\mathbf{0}} : \mathbb{R}^n \to \mathbb{R}^n$$

defined by

$$\nabla\overline{\mathfrak{PO}}^{\mathbf{0}} = (\nabla\overline{\mathfrak{PO}}^{\mathbf{0}}_1, \ldots, \nabla\overline{\mathfrak{PO}}^{\mathbf{0}}_K).$$

Lemma 9.16 is closely related to Lemma 9.15.

LEMMA 9.16
We may choose the positive numbers $R(k)$ *for* $k = 1, \ldots, K$ *such that the following hold:*

(1) $\nabla\overline{\mathfrak{PO}}^{\mathbf{0}}$ *is nonzero on* $\partial\big(D(R(\cdot))\big)$.

(2) *The map* $\partial\big(D(R(\cdot))\big) \to S^{n-1}$

$$\mathfrak{x} \mapsto \frac{\nabla\overline{\mathfrak{PO}}^{\mathbf{0}}}{\|\nabla\overline{\mathfrak{PO}}^{\mathbf{0}}\|}$$

has degree 1.

Proof
We first prove sublemma 9.17 by an upward induction on k_0.

SUBLEMMA 9.17
There exist $R(k)$*'s for* $1 \le k \le K$ *such that, for any given* $1 \le k_0 \le K$*, we have*

$$\sum_{j=1}^{d(k_0)} x_{k_0,j}\frac{\partial\overline{\mathfrak{PO}}^{\mathbf{0}}_{k_0}}{\partial x_{k_0,j}}(x_{1,1}, \ldots, x_{k_0,d(k_0)}) > 0 \tag{9.24}$$

if $(x_{k,1}, \ldots, x_{k,d(k)}) \in D(R(k))$ *for all* $1 \le k \le k_0 - 1$ *and* $(x_{k_0,1}, \ldots, x_{k_0,d(k_0)}) \in \partial D(R(k_0))$.

Proof

In the case $k_0 = 1$, the existence of $R(1)$ satisfying (9.24) is a consequence of Corollary 9.9. We assume that the sublemma is proved for $1, \ldots, k_0 - 1$.

For each fixed $\mathbf{x} = (x_{1,1}, \ldots, x_{k_0-1,d(k_0-1)})$, we can find $R(k_0)_{\mathbf{x}}$ such that (9.24) holds for $(x_{k_0,1}, \ldots, x_{k_0,d(k_0)}) \in \mathbb{R}^{d(k_0)} \setminus D(R(k_0)_{\mathbf{x}}/2)$. This is also a consequence of Corollary 9.9.

We take the supremum of $R(k_0)_{\mathbf{x}}$ over the compact set $\mathbf{x} \in \prod_{k=1}^{k_0-1} D(R(k))$ and obtain $R(k_0)$. The proof of Sublemma 9.17 is complete. □

It is easy to see that Lemma 9.16 follows from Sublemma 9.17. □

We now use our assumption that the vertices of P lie in $M_{\mathbb{Q}} = \mathbb{Q}^n$ and that $\rho_j \in \mathbb{Q}$. Replacing T by $T^{1/\mathscr{C}}$ if necessary, we may assume that all the exponents of $y_{k,j}$ and T appearing in (9.20) are integers. Then

$$\mathfrak{PO}^{\mathbf{0}}_{k,\mathscr{N}} = \mathfrak{PO}^{\mathbf{0}}_{k,\mathscr{N}}(y_{1,1}, \ldots, y_{K,d(K)}; T)$$

are polynomials of $y_{k,j}$, $y_{k,j}^{-1}$ and T. Define the set $\mathfrak{X}$ by the set consisting of

$$(\mathfrak{y}_{1,1}, \ldots, \mathfrak{y}_{K,d(K)}; q) \in (\mathbb{R}_+)^n \times \mathbb{R}$$

that satisfy

$$\frac{\partial \mathfrak{PO}^{\mathbf{0}}_{k,\mathscr{N}}}{\partial y_{k,j}}(\mathfrak{y}_{1,1}, \ldots, \mathfrak{y}_{K,d(K)}; q) = 0, \tag{9.25}$$

for $k = 1, \ldots, K$, $j = 1, \ldots, d(k)$. Clearly, $\mathfrak{X}$ is a real affine algebraic variety. (Note that the equations for y_i are polynomials. So we need to regard y_i (not x_i) as variables to regard $\mathfrak{X}$ as a real affine algebraic variety.)

Consider the projection

$$\pi : \mathfrak{X} \to \mathbb{R}, \quad \pi(\mathfrak{y}_{1,1}, \ldots, \mathfrak{y}_{K,d(K)}; q) = q,$$

which is a morphism of algebraic varieties.

LEMMA 9.18

There exists a sufficiently small $\epsilon > 0$ *such that, if* $|q| < \epsilon$, *then*

$$\pi^{-1}(q) \cap \left\{ (e^{x_1}, \ldots, e^{x_n}) \,\middle|\, (x_1, \ldots, x_n) \in D(R(\cdot)) \right\} \neq \emptyset.$$

Proof
We consider the real analytic q-family of polynomials

$$\mathfrak{PO}^{\mathbf{0}}_{k,\mathcal{N},q}(y_{1,1},\dots,y_{K,d(K)}) = \mathfrak{PO}^{\mathbf{0}}_{k,\mathcal{N}}(y_{1,1},\dots,y_{K,d(K)};q).$$

Then

$$\mathfrak{PO}^{\mathbf{0}}_{k,\mathcal{N},0} = \overline{\mathfrak{PO}}^{\mathbf{0}}_{k,\mathcal{N}}. \tag{9.26}$$

Replacing $\overline{\mathfrak{PO}}^{\mathbf{0}}_{k,\mathcal{N}}$ by $\mathfrak{PO}^{\mathbf{0}}_{k,\mathcal{N},q}$, we can repeat construction of the map

$$\nabla\mathfrak{PO}^{\mathbf{0}}_{\mathcal{N},q} : \mathbb{R}^n \to \mathbb{R}^n$$

for each fixed $q \in \mathbb{R}$ in the same way as we defined $\nabla\overline{\mathfrak{PO}}^{\mathbf{0}}$. Then the conclusion of Lemma 9.16 holds for $\nabla\mathfrak{PO}^{\mathbf{0}}_{\mathcal{N},q}$ if $|q|$ is sufficiently small (this is a consequence of Lemma 9.16 and (9.26)). Lemma 9.18 follows from elementary algebraic topology. □

Lemma 9.18 implies that we can find

$$\mathfrak{y}_0 = (\mathfrak{y}_{1,1;0},\dots,\mathfrak{y}_{K,d(K);0}) \in \mathbb{R}^n_+$$

and a sequence

$$(\mathfrak{y}_h, q_h) = (\mathfrak{y}^h_{1,1;0},\dots,\mathfrak{y}^h_{K,d(K);0}; q_h) \in \mathfrak{X} \subset \mathbb{R}^{n+1},$$

$h = 1,2,\dots,$ such that $q_h > 0$ and $\lim_{h\to\infty}(\mathfrak{y}_h, q_h) = (\mathfrak{y}_0, 0)$. Therefore, by the curve selection lemma [Mi, Lemma 3.1] there exists a real analytic map

$$\gamma : [0,\epsilon) \to \mathfrak{X}$$

such that $\gamma(0) = (\mathfrak{y}_0, 0)$ and $\pi(\gamma(t)) > 0$ for $t > 0$. We reparameterize $\gamma(t)$, so that its q-component is $t^{a/b}$, where a and b are relatively prime integers. We put $T = t^{a/b}$ (i.e., $t = T^{b/a}$), and we denote the $y_{k,j}$-components of $\gamma(t)$ by

$$\mathfrak{y}_{k,j} = \mathfrak{y}_{k,j;0} + \sum_{\ell=1}^{\infty} \mathfrak{y}_{k,j;\ell} T^{b\ell/a}.$$

Since $\gamma(t) \in \mathfrak{X}$, the element $(\mathfrak{y}_{k,j})_{k,j} \in (\Lambda_0^{\mathbb{R}} \setminus \Lambda_+^{\mathbb{R}})^n$ is the required solution of (4.11).

Since $\mathfrak{PO}_0$ contains only a finite number of summands, we can take $\mathfrak{PO}_{0,\mathcal{N}} = \mathfrak{PO}_0$. Therefore, we can find a solution of (4.12) for $\mathfrak{PO}_0$.

The proof of Proposition 4.7 is now complete. □

10. Elimination of higher-order term in nondegenerate cases

In this section, we prove a rather technical (but useful) result, which shows that solutions of the leading-term equation (9.19) correspond to actual critical points under certain nondegeneracy condition. For this purpose, we slightly modify the argument of the last part of Section 9. This result is useful to determine $u \in \operatorname{Int} P$ such that $HF((L(u);\mathfrak{x}),(L(u);\mathfrak{x});\Lambda_0) \neq 0$ for some $\mathfrak{x}$ in non-Fano cases. (In other words, we study the image of $\mathfrak{M}(\mathfrak{Lag}(X)) \to \operatorname{Int} P$ by the map $(\mathfrak{x},u) \mapsto u$.) In fact, it shows that we can use $\mathfrak{PO}_0$ in place of $\mathfrak{PO}$ in most practical cases. Note that we explicitly calculate $\mathfrak{PO}_0$, but we do not know the precise form of $\mathfrak{PO}$ in non-Fano cases.

In order to state the result in a general form, we prepare some notation. Let $u_0 \in \operatorname{Int} P$. (In Section 9, u_0 is determined as the unique element of P_K defined in Proposition 9.1. The present situation is more general.)

We define positive real numbers $S_1 < S_2 < \cdots$ by

$$\left\{\ell_i(u_0) \,\middle|\, i=1,\dots,m\right\} = \{S_1, S_2, \dots, S_{m'}\} \tag{10.1}$$

and the sets

$$I_k = \left\{\ell_i \,\middle|\, \ell_i(u_0) = S_k\right\}, \tag{10.2}$$

for $k = 1, \dots$. We renumber each of I_k so that

$$\left\{\ell_{k,j} \,\middle|\, j = 1,\dots,a(k)\right\} = I_k. \tag{10.3}$$

Definition 10.1
Let $A_l^{\perp}$ be the linear subspace of $N_{\mathbb{R}}$ spanned by $d\ell_{k,j}$ $k \le l$, $j \le a(k)$. We define K to be the smallest number such that $A_K^{\perp} = N_{\mathbb{R}}$.

Our notation here is consistent with that in Section 9 in the case $\{u_0\} = P_K$. We define $\mathfrak{I}$ and $\mathfrak{I}_k$ by (9.11) and Definition 9.10. Then Lemma 9.11 and (9.12) hold. We choose $\mathbf{e}^*_{i,j} \in \operatorname{Hom}(M_{\mathbb{Q}}, \mathbb{Q})$ such that Condition 9.13 is satisfied (note that $A_l^{\perp}$ is defined over $\mathbb{Q}$). Then $x_{i,j}$ and $y_{i,j}$ are defined in the same way as in Section 9. We define $Y(k,j)$ by (9.13) and $Y(\ell)$ by (9.14). Then (9.15) and Lemma 9.14 hold.

We note that Corollary 9.9 does *not* hold in general in the current situation. In fact, we can write

$$v = \sum_{l=1}^{k} \sum_{j=1}^{a(l)} c_{l,j} d\ell_{l,j}$$

under the assumption of Corollary 9.9, but we may not be able to ensure that $c_{l,j} \ge 0$.

Definition 10.2

(1) We call

$$0 = \sum_{j'=1}^{a(k)} \frac{\partial Y(k, j')}{\partial y_{k,j}}, \quad k = 1, \dots, K, \;\; j = 1, \dots, d(k)$$

the *leading-term equation* at u_0. We regard it as a polynomial equation for $\mathfrak{y}_{k,j} \in \mathbb{C} \setminus \{0\}$, $k = 1, \dots, K$, $j = 1, \dots, d(k)$.

(2) A solution $\mathfrak{y}^0 = (\mathfrak{y}_{k,j;0})_{k=1,\dots,K,\, j=1,\dots,d(k)}$ of leading-term equation is said to be *weakly nondegenerate* if it is isolated in the set of solutions.

(3) A solution $\mathfrak{y}^0 = (\mathfrak{y}_{k,j;0})_{k=1,\dots,K,\, j=1,\dots,d(k)}$ of leading-term equation is said to be *strongly nondegenerate* if the matrices

$$\Big(\sum_{j'=1}^{a(k)} \frac{\partial^2 Y(k, j')}{\partial y_{k,j_1} \partial y_{k,j_2}} \Big)_{j_1, j_2 = 1, \dots, a(k)}$$

are invertible for $k = 1, \dots, K$ at $\mathfrak{y}^0$.

(4) We define the multiplicity of leading-term equation in the standard way of algebraic geometry, in the weakly nondegenerate case.

Example 10.3

In Example 9.6, the equation (9.6) is the leading-term equation.

Let $\mathfrak{PO}^{u_0}_*$ be either $\mathfrak{PO}^{u_0}_0$ or $\mathfrak{PO}^{u_0}$.

THEOREM 10.4

For any strongly nondegenerate solution $\mathfrak{y}^0 = (\mathfrak{y}_{k,j;0})$ *of leading-term equation, there exists a solution* $\mathfrak{y} = (\mathfrak{y}_{k,j}) \in (\Lambda_0^{\mathbb{C}} \setminus \Lambda_+^{\mathbb{C}})^n$ *of*

$$\frac{\partial \mathfrak{PO}^{u_0}_*}{\partial y_{k,j}}(\mathfrak{y}) = 0 \tag{10.4}$$

such that $\mathfrak{y}_{k,j} \equiv \mathfrak{y}_{k,j;0} \mod \Lambda_+^{\mathbb{C}}$.

If all the vertices of P *and* u_0 *are rational, the same conclusion holds for weakly nondegenerate* $\mathfrak{y}^0$.

We prove the following at the end of Section 13.

LEMMA 10.5

We assume that $[\omega] \in H^2(X; \mathbb{Q})$, *and we choose the moment polytope* P *such that its vertices are all rational. Let* $u_0 \in \operatorname{Int} P$ *such that* $\mathfrak{PO}^{u_0}_0$ *has weakly nondegenerate critical point in* $(\Lambda_0 \setminus \Lambda_+)^n$. *Then* u_0 *is rational.*

The following corollary is an immediate consequence of Theorem 10.4 and Lemma 10.5.

COROLLARY 10.6
Let $(\mathfrak{x}, u) \in \mathfrak{M}_0(\mathfrak{Lag}(X))$, *and let* $u \in \operatorname{Int} P$. *Assume one of the following conditions:*
(1) *The corresponding solution of the leading-term equation is strongly nondegenerate.*
(2) *The cohomology class* $[\omega] \in H^2(X; \mathbb{R})$ *is rational, and the corresponding solution of leading-term equation is weakly nondegenerate.*
Then there exists $\mathfrak{x}'$ *such that* $(\mathfrak{x}', u) \in \mathfrak{M}(\mathfrak{Lag}(X))$ *and* $\mathfrak{x}' \equiv \mathfrak{x} \mod \Lambda_+^{\mathbb{C}}$.

Remark 10.7
(1) Using Proposition 10.8 below, we can also apply Theorem 10.4 and Corollary 10.6 to study nondisplacement of Lagrangian fibers for the weakly nondegenerate case, without assuming rationality (see the last step of the proof of Theorem 1.5 given at the end of Section 13).
(2) The conclusion of Theorem 10.4 does not hold in general without weakly nondegenerate assumption. We give an example (Example 10.17) where both the assumption of weak nondegeneracy and the conclusion of Theorem 10.4 fail to hold.
(3) In this section, we work with $\Lambda^{\mathbb{C}}$-coefficients, while in the last section we work with $\Lambda^{\mathbb{R}}$-coefficients. We also remark that in the last section, we did *not* assume the weak nondegeneracy condition.
(4) If we define the multiplicity of the element of $\mathfrak{M}_0(\mathfrak{Lag}(X))$ as the dimension of the Jacobian ring $\operatorname{Jac}(\mathfrak{PO}_0; \mathfrak{x}, u_0)$ in Definition 7.9 (namely, as the Milnor number), then the sum of the multiplicities of the solutions of (10.4) converging to $\mathfrak{y}^0$ as $T \to 0$ is equal to the multiplicity of $\mathfrak{y}^0$.
(5) In the strongly nondegenerate case, the solution of (10.4) with the given leading term is unique.

PROPOSITION 10.8
Let (X, ω) *be a compact toric manifold with moment polytope* P, *and let* $u_0 \in \operatorname{Int} P$. *Then there exist* (X, ω^h) *with moment polytope* P^h *and* $u_0^h \in \operatorname{Int} P^h$ *such that*
(1) $\lim_{h\to\infty} \omega^h = \omega$, $\lim_{h\to\infty} u_0^h = u_0$;
(2) *the vertices of* P^h *and* u_0^h *are rational;*
(3) *the leading-term equation at* u_0^h *is the same as the leading-term equation at* u_0.

We prove Proposition 10.8 at the end of Section 13.

We first derive Theorem 1.14 and Lemma 8.3 from Theorem 10.4 before proving Theorem 10.4.

Proof of Theorem 1.14
We start with $\mathbb{C}P^2$ and blow up a T^2-fixed point to obtain $\mathbb{C}P^2\#(-\mathbb{C}P^2)$. We take a Kähler form so that the volume of the exceptional $\mathbb{C}P^1$ is $2\pi\epsilon_1$, which is small. We next blow up again at one of the fixed points so that the volume of the exceptional $\mathbb{C}P^1$ is $2\pi\epsilon_2$ and is small compared with ϵ_1. We continue k times to obtain $X(k)$, whose Kähler structure depends on $\epsilon_1, \dots, \epsilon_k$. Note that $X(k)$ is non-Fano for $k > 3$.

Let $P(k)$ be the moment polytope of $X(k)$, and let $\mathfrak{PO}_{0,k}$ be the leading-order potential function of $X(k)$. We remark that $P(k)$ is obtained by cutting out a vertex of $P(k-1)$ (see [Fu]).

LEMMA 10.9
We may choose ϵ_i $(i = 1, \dots, k)$ so that for $l \le k$,
(1) *the number of balanced fibers of $P(l)$ is $l + 1$, and we write them as $L(u^{(l,i)})$ $i = 0, \dots, l$;*
(2) *$u^{(l-1,i)} = u^{(l,i)}$ for $i \le l - 1$, $u^{(l,0)} = (1/3, 1/3)$;*
(3) *$u^{(l,l)}$ is in an $o(\epsilon_l)$-neighborhood of the vertices corresponding to the point we blow up;*
(4) *the leading-term equation of $\mathfrak{PO}_{0,l-1}$ at $u^{(l-1,i)}$ is the same as the leading-term equation of $\mathfrak{PO}_{0,l}$ at $u^{(l,i)}$ for $i \le l - 1$;*
(5) *the leading-term equations are all strongly nondegenerate.*

Proof
The proof is by induction on k. There is nothing to show for $k = 0$. Suppose that we have proved Lemma 10.9 up to $k - 1$. Let w be the vertex of the polytope we cut out which corresponds to the blow-up of $X(k - 1)$. Let ℓ_i, $\ell_{i'}$ be the affine functions associated to the two edges containing w. It is easy to see that

$$P(k) = \{u \in P(k-1) \,|\, \ell_i(u) + \ell_{i'}(u) \ge \epsilon_k\}.$$

We also have

$$\mathfrak{PO}_{0,k} = \mathfrak{PO}_{0,k-1} + T^{\ell_i(u)+\ell_{i'}(u)-\epsilon_k} y_1^{v_{i,1}+v_{i',1}} y_2^{v_{i,2}+v_{i',2}}.$$

Therefore, if we choose ϵ_k sufficiently small, the leading-term equation at $u^{(k-1,i)}$ does not change.

We take $u^{(k,k)}$ such that

$$\ell_i(u^{(k,k)}) = \ell_{i'}(u^{(k,k)}) = \epsilon_k.$$

It is easy to see that there exists such $u^{(k,k)}$ uniquely if ϵ_k is sufficiently small. We put

$$y_1' = y_1^{v_{i,1}} y_2^{v_{i,2}}, \qquad y_2' = y_1^{v_{i',1}} y_2^{v_{i',2}}.$$

(We observe that v_i and $v_{i'}$ are $\mathbb{Z}$ basis of $\mathbb{Z}^2$, since $X(k-1)$ is smooth toric.) Then we have

$$\mathfrak{PO}_{0,k}^{u^{(k,k)}} \equiv (y_1' + y_2' + y_1' y_2') T^{\epsilon_k} \mod T^{\epsilon_k} \Lambda_+.$$

Therefore, the leading-term equation is

$$1 + y_1' = 1 + y_2' = 0,$$

and hence it has a unique solution $(-1,-1)$. In particular, it is strongly nondegenerate. We can also easily check that there is no other solution of the leading-term equation. The proof of Lemma 10.9 now follows by Theorem 10.4. □

Theorem 1.14 immediately follows from Lemma 10.9. □

Note that Theorem 1.14 can be generalized to $\mathbb{C}P^n$ by the same proof.

Proof of Lemma 8.3
Let $u_0 = (n(\alpha+1)/4, (1-\alpha)/2)$. We calculate

$$\mathfrak{PO}_0^{u_0} = (y_2 + y_2^{-1}) T^{(1-\alpha)/2} + (y_1 + y_1^{-1} y_2^{-1}) T^{n(\alpha+1)/4}.$$

The leading-term equation is

$$1 - y_2^{-2} = 0, \qquad 1 - y_1^{-2} y_2^{-1} = 0.$$

Its solutions are $(1,1)$, $(-1,1)$, $(\sqrt{-1},-1)$, $(-\sqrt{-1},-1)$, all of which are strongly nondegenerate. The lemma then follows from Corollary 10.6. □

We give another example demonstrating the way one can use the leading-term equation and Theorem 10.4 to locate balanced fibers.

Example 10.10
Let us consider $\mathbb{C}P^n$ with moment polytope $P = \{(u_1,\ldots,u_n) \mid u_i \ge 0, \sum u_i \le 1\}$. We take $\mathbb{C}P^{n-\ell} \subset \mathbb{C}P^n$ corresponding to $u_1 = \cdots = u_\ell = 0$ ($\ell \ge 2$). We blow up $\mathbb{C}P^n$ along the center $\mathbb{C}P^{n-\ell}$ and denote the blow-up by X. (The case $\ell = n = 2$ is Example 8.1.) We take $\alpha \in (0,1)$ so that the moment polytope of X is

$$P_\alpha = \left\{ (u_1,\ldots,u_n) \in P \;\middle|\; \sum_{i=1}^{\ell} u_i \ge \alpha \right\}.$$

Below we use $\mathfrak{PO}_0$ in place of the potential function $\mathfrak{PO}$. Since all the critical points of $\mathfrak{PO}_0$ are weakly nondegenerate, they correspond to the critical points of $\mathfrak{PO}$. (We thank D. McDuff for pointing out that this example is not Fano.) The function $\mathfrak{PO}_0$ is given by

$$\mathfrak{PO}_0 = \sum_{i=1}^{n} T^{u_i} y_i + T^{1-\sum_{i=1}^{n} u_i}(y_1 \cdots y_n)^{-1} + T^{\sum_{i=1}^{\ell} u_i - \alpha} y_1 \cdots y_\ell.$$

We denote

$$z_i = T^{u_i} y_i, \qquad z_0 = T^{1-\sum_{i=1}^{n} u_i}(y_1 \cdots y_n)^{-1}, \qquad z = T^{\sum_{i=1}^{\ell} u_i - \alpha} y_1 \cdots y_\ell.$$

Then the quantum Stanley-Reisner relations are

$$z_1 \cdots z_n z_0 = T, \qquad z_1 \cdots z_\ell = z T^{\alpha}.$$

By computing the derivatives $y_i \frac{\partial \mathfrak{PO}_0}{\partial y_i}$, we obtain the linear relations, which can be written as

$$\begin{aligned} z_i - z_0 + z &= 0 \quad \text{for } i \le \ell, \\ z_i - z_0 &= 0 \quad \text{for } i > \ell. \end{aligned}$$

Putting $X = z_0 - z$, $Y = z$, we obtain

$$z_i = \begin{cases} X & \text{for } 1 \le i \le \ell, \\ X + Y & \text{for } i > \ell \text{ or } i = 0. \end{cases}$$

We also have $X^\ell = Y T^\alpha$ and

$$X^{n+1}(X^{\ell-1}T^{-\alpha} + 1)^{n-\ell+1} = T. \tag{10.5}$$

Case 1: $(\ell - 1)\mathfrak{v}_T(X) < \alpha$. In this case, we obtain

$$\mathfrak{v}_T(X) = \frac{1 + \alpha(n - \ell + 1)}{n + 1 + (n - \ell + 1)(\ell - 1)}.$$

The condition $(\ell - 1)\mathfrak{v}_T(X) < \alpha$ then is equivalent to

$$\alpha > \frac{\ell - 1}{n + 1}.$$

We have

$$\mathfrak{v}_T(X + Y) = \mathfrak{v}_T(Y) = \ell \mathfrak{v}_T(X) - \alpha < \mathfrak{v}_T(X)$$

for $i > \ell$. At the point $\mathbf{u} = (u_1, \dots, u_n)$ with $u_i = \mathfrak{v}_T(X)$ for $1 \le i \le \ell$ and $u_i = \mathfrak{v}_T(X+Y)$ for $i \ge \ell$, we have

$$\mathfrak{PO}_0^{\mathbf{u}} = T^{\mathfrak{v}_T(Y)}(y_{\ell+1} + \cdots + y_n + (y_1 \cdots y_n)^{-1} + y_1 \cdots y_\ell) + T^{\mathfrak{v}_T(X)}(y_1 + \cdots + y_\ell).$$

To obtain the leading-term equation, it is useful to make a change of variables from $y_1, \dots, y_n$ to $y_2, \dots, y_n, y$ with $y = y_1 \cdots y_\ell$.

In fact, using the notation which we introduced at the beginning of this section, we have

$$S_1 = \mathfrak{v}_T(Y), \qquad S_2 = \mathfrak{v}_T(X),$$

and $A_1^\perp$ is the vector space generated by $\partial/\partial u_i$, $(i = \ell+1, \dots, n)$ and $\partial/\partial u_1 + \cdots + \partial/\partial u_\ell$. We also have $A_2^\perp = \mathbb{R}^n$. Therefore, a basis satisfying Condition 9.13 is

$$\frac{\partial}{\partial u_2}, \dots, \frac{\partial}{\partial u_n}, \qquad \frac{\partial}{\partial u_1} + \cdots + \frac{\partial}{\partial u_\ell}.$$

The variables corresponding to this basis is $y_2, \dots, y_n, y$. In these variables, $\mathfrak{PO}_0^{\mathbf{u}}$ has the form

$$\begin{aligned}\mathfrak{PO}_0^{\mathbf{u}} = T^{\mathfrak{v}_T(Y)}&(y_{\ell+1} + \cdots + y_n + (y y_{\ell+1} \cdots y_n)^{-1} + y), \\ &+ T^{\mathfrak{v}_T(X)}(y(y_2 \cdots y_\ell)^{-1} + y_2 + \cdots + y_\ell).\end{aligned}$$

Then the leading-term equation becomes

$$\begin{cases} 0 = 1 - y^{-1}(y y_{\ell+1} \cdots y_n)^{-1}, & \\ 0 = 1 - y_i^{-1}(y y_{\ell+1} \cdots y_n)^{-1} & \text{for } i > \ell, \\ 0 = 1 - y_i^{-1} y (y_2 \cdots y_\ell)^{-1} & \text{for } 2 \le i \le \ell. \end{cases}$$

Its solutions are

$$\begin{aligned} y_{\ell+1} = \cdots = y_n = y = \theta, &\qquad \theta^{n-\ell+2} = 1 \\ y_2 = \cdots = y_\ell = \rho, &\qquad \rho^\ell = \theta. \end{aligned}$$

It follows that this leading-term equation has $\ell(n-\ell+2)$ solutions, all of which are strongly nondegenerate.

Case 2: $(\ell - 1)\mathfrak{v}_T(X) > \alpha$. We have

$$\mathfrak{v}_T(X) = \frac{1}{n+1}.$$

We also have $\mathfrak{v}_T(Y) = \ell\mathfrak{v}_T(X) - \alpha > \mathfrak{v}_T(X)$ and hence

$$\mathfrak{v}_T(X+Y) = \mathfrak{v}_T(X) = \frac{1}{n+1}.$$

The condition $(\ell-1)\mathfrak{v}_T(X) > \alpha$ becomes $\alpha < (\ell-1)/(n+1)$. If we consider the point $\mathbf{u} = (u_1, \dots, u_n)$ with $u_i = \mathfrak{v}_T(X) = 1/(n+1)$ for $i = 1, \dots, n$, $\mathfrak{PO}_0^{\mathbf{u}}$ has the form

$$\mathfrak{PO}_0^{\mathbf{u}} = T^{1/n+1}(y_1 + \cdots + y_n + (y_1 \cdots y_n)^{-1}) + T^{\ell/(n+1)-\alpha} y_1 \cdots y_\ell,$$

and so the leading-term equation is obtained by differentiating

$$y_1 + \cdots + y_n + (y_1 \cdots y_n)^{-1}.$$

Its solutions are

$$y_1 = \cdots = y_n = \theta, \qquad \theta^{n+1} = 1.$$

There are $n+1$ solutions, all of which are strongly nondegenerate.

Case 3: $(\ell-1)\mathfrak{v}_T(X) = \alpha$.

Case 3(1): $-X^{\ell-1} \not\equiv T^\alpha \mod T^\alpha\Lambda_+$. We have $u_i = \mathfrak{v}_T(X) = \mathfrak{v}_T(Y) = 1/(n+1)$ $(i = 1, \dots, n)$, $\alpha = (\ell-1)/(n+1)$. In this case,

$$\mathfrak{PO}_0^{\mathbf{u}} = T^{1/(n+1)}(y_1 + \cdots + y_n + (y_1 \cdots y_n)^{-1} + y_1 \cdots y_\ell).$$

This formula implies that the symplectic area of all discs with Maslov index 2 are $2\pi/(n+1)$.

The leading-term equation is

$$\begin{cases} 1 - \frac{1}{y_i}\big((y_1 \cdots y_n)^{-1} - y_1 \cdots y_\ell\big) = 0, & i = 1, \dots, \ell, \\ 1 - \frac{1}{y_i}\big((y_1 \cdots y_n)^{-1}\big) = 0, & i = \ell+1, \dots, n. \end{cases}$$

Its solutions are $y_1 = \cdots = y_\ell = \rho$, $y_{\ell+1} = \cdots = y_n = \theta$ with

$$\rho^\ell(\rho + \rho^\ell)^{n-\ell+1} = 1, \tag{10.6}$$

and $\theta = \rho + \rho^\ell$. It looks rather cumbersome to check whether (10.6) has multiple root. Certainly, all the solutions are weakly nondegenerate. The number of solutions counted with multiplicity is $\ell(n-\ell+2)$.

Case 3(2): $-X^{\ell-1} \equiv T^{\alpha} \mod T^{\alpha}\Lambda_+$. We have

$$z_0 = X + Y = T^{-\alpha}X(T^{\alpha} + X^{\ell-1}).$$

We put

$$\mathfrak{v}_T(z_0) = \lambda > \mathfrak{v}_T(X) = \frac{\alpha}{\ell - 1}.$$

Using (10.5), we obtain

$$\lambda = \frac{\ell - 1 - \ell\alpha}{(\ell - 1)(n - \ell + 1)}.$$

The condition $\lambda > \alpha/(\ell - 1)$ becomes $\alpha < (\ell - 1)/(n + 1)$. We have $u_i = \mathfrak{v}_T(X) = \alpha/(\ell - 1)$, $i \le \ell$ and $u_i = \mathfrak{v}_T(Y) = \lambda$, $i > \ell$. We have

$$\mathfrak{PO}_0^{\mathbf{u}} = T^{\mathfrak{v}_T(X)}(y_1 + \cdots + y_\ell + y_1 \cdots y_\ell) + T^{\mathfrak{v}_T(Y)}(y_{\ell+1} + \cdots + y_n + (y_1 \cdots y_n)^{-1}).$$

The first term gives the leading-term equation

$$1 + y_1 \cdots \widehat{y_i} \cdots y_\ell = 0, \quad i = 1, \ldots, \ell.$$

Its solution is $y_1 = \cdots = y_\ell = \rho$ and $\rho^{\ell-1} = -1$. The second term of $\mathfrak{PO}_0^{\mathbf{u}}$ gives the leading-term equation

$$1 - y_i^{-1}\rho^{-\ell}(y_{\ell+1} \cdots y_n)^{-1} = 0, \quad i = \ell + 1, \ldots, n.$$

Its solutions are $y_{\ell+1} = \cdots = y_n = \theta$, with

$$\rho^{\ell}\theta^{n-\ell+1} = 1.$$

Thus the leading-term equation has $(\ell - 1)(n + 1 - \ell)$ solutions, all of which are strongly nondegenerate.

We note that $(\ell - 1)(n + 1 - \ell) + (n + 1) = \ell(n - \ell + 2)$. Hence the number of solutions is always $\ell(n - \ell + 2)$, which coincides with the Betti number of X. There are two balanced fibers for $\alpha < (\ell - 1)/(n + 1)$ and one balanced fiber for $\alpha \ge (\ell - 1)/(n + 1)$.

By the above discussion and Theorem 6.1 (see also Remark 6.14), we can calculate $QH(X; \Lambda^{\mathbb{Q}})$ as

$$QH(X; \Lambda^{\mathbb{Q}}) = \begin{cases} \Lambda^{R_1} & \alpha > (\ell - 1)/(n + 1), \\ \Lambda^{R_2} & \alpha = (\ell - 1)/(n + 1), \\ \Lambda^{R_3} \times \Lambda^{R_4} & \alpha < (\ell - 1)/(n + 1), \end{cases} \tag{10.7}$$

where

$$R_1 = \mathbb{Q}[Z]/(Z^{\ell(n-\ell+2)} - 1),$$
$$R_2 = \mathbb{Q}[Z]/(Z^{\ell}(Z + Z^{\ell})^{n-\ell+1} - 1),$$
$$R_3 = \mathbb{Q}[Z]/(Z^{n+1} - 1),$$
$$R_4 = \mathbb{Q}[Z_1, Z_2]/(Z_1^{\ell-1} + 1, Z_1^{\ell} Z_2^{n-\ell+1} - 1).$$

Here we assume that (10.6) has only a simple root in case $\alpha = (\ell - 1)/(n + 1)$. We use Lemma 10.11 to show (10.7).

LEMMA 10.11
Let $\mathfrak{x} = \sum \mathfrak{x}_i \mathbf{e}_i$ be a critical point of $\mathfrak{PO}^{\mathbf{u}}_*$. We assume that $\mathfrak{y}_{i;0} = e^{\mathfrak{x}_{i;0}}$ (where $\mathfrak{x}_i = \mathfrak{x}_{i;0}$ mod Λ_+*) is a strongly nondegenerate solution of the leading-term equation. We also assume that $\mathfrak{y}_{i;0} \in F$, where $F \subset \mathbb{C}$ is a field.*

Then $\mathfrak{y}_i = e^{\mathfrak{x}_i}$ is an element of Λ_0^F.

We prove Lemma 10.11 later in this section.

We are now ready to give the proof of Theorem 10.4.

Proof of Theorem 10.4
We first consider the weakly nondegenerate case. Let $\mathfrak{m}$ be the multiplicity of $\mathfrak{y}^0$. We choose δ such that the ball $B_\delta(\mathfrak{y}^0)$ centered at $\mathfrak{y}^0$ and of radius δ does not contain a solution of the leading-term equation other than $\mathfrak{y}^0$. For $y \in \partial B_\delta(\mathfrak{y}^0)$, we define

$$\nabla\overline{\mathfrak{PO}}(y) = \Big(\sum_{j'=1}^{a(k)} \frac{\partial Y(k, j')}{\partial y_{k,j}}(y) \Big)_{k=1,\dots,K,\ j=1,\dots,d(k)} \in \mathbb{C}^n.$$

The map

$$y \mapsto \frac{\nabla\overline{\mathfrak{PO}}(y)}{\|\nabla\overline{\mathfrak{PO}}(y)\|} \in S^{2n-1}$$

is well defined and of degree $\mathfrak{m} \neq 0$ by the definition of multiplicity.

We define $\mathfrak{PO}^{u_0}_{*,k,\mathcal{N}}$ in the same way as (9.20). For $q \in \mathbb{C}$, we define $\mathfrak{PO}^{u_0}_{*,k,\mathcal{N}}(\cdots ; q)$ by substituting q to T. Then, in the same way as the proof of Lemma 9.18, we can prove the following.

LEMMA 10.12
There exists $\epsilon > 0$ such that if $|q| < \epsilon$, the equation

$$0 = \frac{\partial}{\partial y_{k,j}} \mathfrak{PO}^{u_0}_{*,k,\mathcal{N}}(\cdots ; q) \tag{10.8}$$

has a solution in $B_\delta(\mathfrak{y}^0)$. The sum of multiplicities of the solutions of (10.8) converging to $\mathfrak{y}_{k,j;0}$ is $\mathfrak{m}$.

Equation (10.8) is a polynomial equation. Hence multiplicity of its solution is well defined in the standard sense of algebraic geometry.

Now we assume that all the vertices of P and u_0 are contained in $\mathbb{Q}^n$. Equation (10.8) also depends polynomially on $q' = T'$, where $T' = T^{1/\mathscr{C}!}$ for a sufficiently large integer $\mathscr{C}$. (We observe that $\mathscr{C}$ is determined by the denominators of the coordinates of the vertices of P and of u_0. In particular, it can be taken to be independent of $\mathcal{N}$.)

We denote $y = (y_1, \ldots, y_n)$ and put

$$\mathfrak{X} = \{(y, q') \mid y \in B_\delta(\mathfrak{y}^0),\ q' \text{ with } |q'| < \epsilon \text{ and } q = (q')^{\mathscr{C}!} \text{ satisfying (10.8)}\}.$$

We consider the projection

$$\pi_{q'} : \mathfrak{X} \to \{q' \in \mathbb{C} \mid |q'| < \epsilon\}. \tag{10.9}$$

By choosing a sufficiently small $\epsilon > 0$, we may assume that (10.9) is a local isomorphism on the punctured disc $\{q' \mid 0 < |q'| < \epsilon\}$; namely, $\pi_{q'}$ is an étale covering over the punctured disc.

We note that for each q', the fiber consists of at most $\mathfrak{m}$ points, since the multiplicity of the leading-term equation is $\mathfrak{m}$. We put $q'' = (q')^{1/\mathfrak{m}!}$. Then the pullback

$$\pi_{q''} : \mathfrak{X}' \to \{q'' \in \mathbb{C} \mid 0 < |q''| < \epsilon\} \tag{10.10}$$

of (10.9) becomes a trivial covering space; specifically, there exists a single valued section of $\pi_{q''}$ on $\{q'' \mid 0 < |q''| < \epsilon\}$. It extends to a holomorphic section of $\{q'' \mid |q''| < \epsilon\}$.

In other words, there exists a holomorphic family of solutions of (10.8) which is parameterized by $q'' \in \{q'' \mid |q''| < \epsilon\}$. We put $T'' = (T')^{1/\mathfrak{m}!}$. Then by taking the Taylor series of the q''-parameterized family of solutions at zero, we obtain the following.

LEMMA 10.13
If all the vertices of P and u_0 are rational, then for each $\mathcal{N}$ there exists $\mathfrak{y}^{(\mathcal{N})} = (\mathfrak{y}_{k,j}^{(\mathcal{N})})$

$$\mathfrak{y}_{k,j}^{(\mathcal{N})} = \sum_{l=0}^{\mathcal{N}} \mathfrak{y}_{k,j;l}^{(\mathcal{N})} (T'')^l$$

($\mathfrak{y}_{k,j;l}^{(\mathcal{N})} \in \mathbb{C}$) *such that*

$$\frac{\partial \mathfrak{PO}_*^{u_0}}{\partial y_{k,j}} (\mathfrak{y}_{k,j}^{(\mathcal{N})}) \equiv 0 \mod (T'')^{\mathcal{N}+1} \tag{10.11}$$

and such that $\mathfrak{y}_{k,j;0}^{(\mathcal{N})} \equiv \mathfrak{y}_{k,j;0}$.

We note that Lemma 10.13 is sufficient for most of the applications. In fact, it implies that $L(u_0)$ is balanced if there exists a weakly nondegenerate solution of leading-term equation at u_0. Hence we can apply Lemma 4.12.

For completeness, we prove the slightly stronger statement made for the weakly nondegenerate case in Theorem 10.4. The argument is similar to one in [FOOO3, Section 7.2.11] (equivalent to [FOOO2, Section 30.11]).

For each $\mathcal{N}$, we denote by $\widetilde{\mathfrak{M}}((\mathfrak{y}_{k,j;0}); \mathcal{N})$ the set of all $(\mathfrak{y}_{k,j;l}^{(\mathcal{N})})_{k,j;l} \in \mathbb{C}^{n\mathcal{N}}$, where $k = 1, \dots, K$, $j = 1, \dots, a(k)$, $l = 1, \dots, \mathcal{N}$ such that

$$\mathfrak{y}_{k,j}^{(\mathcal{N})} = \mathfrak{y}_{k,j;0} + \sum_{l=1}^{\mathcal{N}} \mathfrak{y}_{k,j;l}^{(\mathcal{N})} (T'')^l$$

satisfies (10.11).

By definition, $\widetilde{\mathfrak{M}}((\mathfrak{y}_{k,j;0}); \mathcal{N})$ is the set of $\mathbb{C}$-valued points of certain complex affine algebraic variety (of finite dimension). Lemma 10.13 implies that $\widetilde{\mathfrak{M}}((\mathfrak{y}_{k,j;0}); \mathcal{N})$ is nonempty. For $\mathcal{N}_1 > \mathcal{N}_2$ there exists an obvious morphism

$$I_{\mathcal{N}_1, \mathcal{N}_2} : \widetilde{\mathfrak{M}}\big((\mathfrak{y}_{k,j;0}); \mathcal{N}_1\big) \to \widetilde{\mathfrak{M}}\big((\mathfrak{y}_{k,j;0}); \mathcal{N}_2\big)$$

of complex algebraic variety.

To complete the proof of Theorem 10.4 in the weakly nondegenerate case, it suffices to show that the projective limit

$$\varprojlim \big(\widetilde{\mathfrak{M}}(\mathfrak{y}_{k,j;0}; \mathcal{N})\big) \tag{10.12}$$

is nonempty.

LEMMA 10.14
We have

$$\bigcap_{\mathcal{N}>1} \mathrm{Im} I_{\mathcal{N},1} \neq \emptyset.$$

Proof
By a theorem of Chevalley (see [Mt, Chapter 6]), each $\mathrm{Im} I_{\mathcal{N},1}$ is a constructible set. It is nonempty and its dimension $\dim \mathrm{Im} I_{\mathcal{N},1}$ is nonincreasing as $\mathcal{N} \to \infty$. Therefore, we may assume that $\dim \mathrm{Im} I_{\mathcal{N},1} = d$ for $\mathcal{N} \geq \mathcal{N}_1$.

We consider the number of d-dimensional irreducible components of $\mathrm{Im} I_{\mathcal{N},1}$. This number is nonincreasing for $\mathcal{N} \geq \mathcal{N}_1$. Therefore, there exists $\mathcal{N}_2$ such that, for $\mathcal{N} \geq \mathcal{N}_2$, the number of d-dimensional irreducible components of $\mathrm{Im} I_{\mathcal{N},1}$ is independent of $\mathcal{N}$. It follows that there exists $X_{\mathcal{N}}$ a sequence of d-dimensional irreducible components of $\mathrm{Im} I_{\mathcal{N},1}$ such that $X_{\mathcal{N}+1} \subset X_{\mathcal{N}}$. Since $\dim(X_{\mathcal{N}} \setminus X_{\mathcal{N}+1}) < d$, it follows from Baire's category theorem that $\bigcap_{\mathcal{N}} X_{\mathcal{N}} \neq \emptyset$. Hence the lemma. □

LEMMA 10.15
There exists a sequence $(\mathfrak{y}^{(n)}_{k,j;l})_{k,j;l}$ $n = 1, 2, 3, \ldots, m$ *such that*

$$I_{n,n-1}\big((\mathfrak{y}^{(n)}_{k,j;l})_{k,j;l}\big) = (\mathfrak{y}^{(n-1)}_{k,j;l})_{k,j;l}$$

for $n = 2, \ldots, m$ *and that*

$$(\mathfrak{y}^{(m)}_{k,j;l})_{k,j;l} \in \bigcap_{\mathcal{N}>m} \mathrm{Im} I_{\mathcal{N},m}.$$

Proof
The proof is by induction on m. The case $m = 1$ is Lemma 10.14. The inductive steps are similar to the proof of Lemma 10.14 and so are omitted. □

Lemma 10.15 implies that the projective limit (10.12) is nonempty. The proof of the weakly nondegenerate case of Theorem 10.4 is complete.

We next consider the strongly nondegenerate case. We prove Lemma 10.16 by induction on $\mathcal{N}$. Let G be a submonoid of $(\mathbb{R}_{\geq 0}, +)$ generated by the numbers appearing in the exponent of (9.18). In other words, it is generated by

$$S_{k'} - S_k \ (k' > k), \quad \ell(u_0) + \rho - S_k \ ((\ell, \rho) \in \mathfrak{I}_{k'},\ k' \geq k), \\ \ell(u_0) - S_k \ (\ell \in \mathfrak{I}). \tag{10.13}$$

We define $0 < \lambda_1 < \lambda_2 < \cdots$ by

$$\{\lambda_i \mid i = 1, 2, \ldots\} = G.$$

LEMMA 10.16

We assume that $\mathfrak{y}^0 = (\mathfrak{y}_{k,j;0})_{k=1,\ldots,K,\ j=1,\ldots,d(k)}$ *is a strongly nondegenerate solution of the leading-term equation. Then there exists*

$$\mathfrak{y}_{k,j}^{(\mathcal{N})} = \mathfrak{y}_{k,j;0} + \sum_{l=1}^{\mathcal{N}} \mathfrak{y}_{k,j;l} T^{\lambda_l}$$

such that

$$\sum_{j'=1}^{a(k)} \frac{\partial Y(k, j')}{\partial y_{k,j}} (\mathfrak{y}_{k,1}^{(\mathcal{N})}, \ldots, \mathfrak{y}_{K,d(K)}^{(\mathcal{N})}) \equiv 0 \mod T^{\lambda_{\mathcal{N}+1}}. \tag{10.14}$$

Moreover, we may choose $\mathfrak{y}_{k,j}^{(\mathcal{N})}$ *so that*

$$\mathfrak{y}_{k,j}^{(\mathcal{N})} \equiv \mathfrak{y}_{k,j}^{(\mathcal{N}+1)} \mod T^{\lambda_{\mathcal{N}+1}}.$$

Proof

The proof is by induction on $\mathcal{N}$. There is nothing to show in the case $\mathcal{N} = 0$. Assume that we have proved the lemma up to $\mathcal{N} - 1$. Then we have

$$\sum_{j'=1}^{a(k)} \frac{\partial Y(k, j')}{\partial y_{k,j}} (\mathfrak{y}_{k,1}^{(\mathcal{N}-1)}, \ldots, \mathfrak{y}_{K,d(K)}^{(\mathcal{N}-1)}) \equiv c_{k,j,M} T^{\lambda_{\mathcal{N}}} \mod T^{\lambda_{\mathcal{N}+1}}.$$

Consider $\mathfrak{y}_{k,j}^{(\mathcal{N})}$ of the form

$$\mathfrak{y}_{k,j}^{(\mathcal{N})} = \mathfrak{y}_{k,j}^{(\mathcal{N}-1)} + \Delta_{k,j,\mathcal{N}} T^{\lambda_{\mathcal{N}}}$$

for some $\Delta_{k,j,\mathcal{N}}$. Then we can write

$$\begin{aligned}&\sum_{j'=1}^{a(k)} \frac{\partial Y(k, j')}{\partial y_{k,j}} (\mathfrak{y}_{k,1}^{(M-1)} + \Delta_{k,1,\mathcal{N}} T^{\lambda_{\mathcal{N}}}, \ldots, \mathfrak{y}_{K,d(K)}^{(\mathcal{N}-1)} + \Delta_{K,d(K),\mathcal{N}} T^{\lambda_{\mathcal{N}}}) \\ &\quad \equiv \Big(c_{k,j,\mathcal{N}} + \sum_{j',j''=1}^{a(k)} \frac{\partial^2 Y(k, j')}{\partial y_{k,j} \partial y_{k,j''}} \Delta_{k,j'',\mathcal{N}}\Big) T^{\lambda_{\mathcal{N}}} \mod T^{\lambda_{\mathcal{N}+1}}.\end{aligned} \tag{10.15}$$

Since $\mathfrak{y}^0 = (\mathfrak{y}_{k,j;0})_{k=1,\dots,K,\, j=1,\dots,d(k)}$ is strongly nondegenerate, we can find $\Delta_{k,j'',\mathcal{N}} \in \mathbb{C}$ so that the right-hand side becomes zero module $T^{\lambda_{\mathcal{N}+1}}$. The proof of Lemma 10.16 is complete. □

By Lemma 10.16, the limit $\lim_{\mathcal{N}\to\infty} \mathfrak{y}^{(\mathcal{N})}_{k,j}$ exists. We set

$$\mathfrak{y}_{k,j} := \lim_{\mathcal{N}\to\infty} \mathfrak{y}^{(\mathcal{N})}_{k,j}.$$

This is the required solution of (10.4). The proof of Theorem 10.4 is complete. □

Proof of Lemma 10.11
We put

$$\mathfrak{y}_{k,j} = \mathfrak{y}_{k,j;0} + \sum_{l=1}^{\infty} \mathfrak{y}_{k,j;l} T^{\lambda_l}.$$

By assumption $\mathfrak{y}_{k,j;0} \in F$. We note that (10.15) gives a *linear* equation which determines $\mathfrak{y}_{k,j;l}$ inductively on l. We use it to show $\mathfrak{y}_{k,j;l} \in F$ inductively on l. □

We next give an example where weak nondegeneracy condition is not satisfied.

Example 10.17
Consider the 2-point blow-up $X(\alpha, \beta)$ of $\mathbb{C}P^2$ with its moment polytope given by

$$P = \left\{(u_1, u_2) \,\middle|\, 0 \le u_1 \le 1,\ 0 \le u_2 \le 1-\alpha,\ \beta \le u_1 + u_2 \le 1\right\}.$$

We consider the case when $1-\alpha$ is sufficiently small. The potential function is

$$\mathfrak{PO} = T^{u_1} y_1 + T^{u_2} y_2 + T^{1-\alpha-u_2} y_2^{-1} + T^{1-u_1-u_2} y_1^{-1} y_2^{-1} + T^{u_1+u_2-\beta} y_1 y_2.$$

(We remark that X is Fano.) We fix α and move β starting from zero. When β is small compared to $1-\alpha$, there are two balanced fibers.

One is located at $((1+\alpha)/4, (1-\alpha)/2)$. This corresponds to the location of the balanced fiber of the 1-point blow-up, which is nothing but the case $\beta = 0$. The other appears near the origin and is (β, β). The leading-term equation at the first point is

$$1 - y_2^{-2} = 0, \qquad 1 - y_1^{-2} y_2^{-1} = 0.$$

The solutions are $(y_1, y_2) = (\pm 1, 1), (\pm\sqrt{-1}, -1)$, all of which are strongly nondegenerate. The leading-term equation at the second point is

$$1 + y_1 = 1 + y_2 = 0$$

$((-1,-1)$ is the nondegenerate solution). Thus we have five solutions.

The situation jumps when $\beta = (1-\alpha)/2$. Denote $\beta_0 = (1-\alpha)/2$ for the simplicity of notation. In that case, the potential function at (β_0, β_0) becomes

$$T^{\beta_0}(y_1 + y_2 + y_1 y_2 + y_2^{-1}) + T^{1-2\beta_0} y_1^{-1} y_2^{-1}.$$

The leading-term equation is

$$1 + y_2 = 0, \qquad 1 + y_1 - y_2^{-2} = 0.$$

Its solution is $(0, -1)$. Since $y_1 = 0$, it follows that there is no solution in $(\Lambda_0 \setminus \Lambda_+)^2$. Hence there is no weak bounding cochain $\mathfrak{x}$ for which the Floer cohomology $HF\big((L(\beta_0, \beta_0), \mathfrak{x}), (L(\beta_0, \beta_0), \mathfrak{x}); \Lambda\big)$ is nontrivial. In other words, the fiber $L(\beta_0, \beta_0)$ in $X(\alpha, \beta_0)$ is not strongly balanced.

On the other hand, this fiber $L(\beta_0, \beta_0)$ in $X(\alpha, \beta_0)$ is balanced because by choosing β arbitrarily close to β_0 and $\beta < \beta_0$ we can approximate it by the fibers

$$L(\beta, \beta) \subset X(\alpha, \beta)$$

for which the Floer cohomology $HF\big((L(\beta, \beta), \mathfrak{x}), (L(\beta, \beta), \mathfrak{x})\big)$ is nontrivial. In particular, $L(\beta_0, \beta_0)$ in $X(\alpha, \beta_0)$ is not displaceable.

We can also verify that

$$\overline{\mathfrak{E}}(\beta_0, \beta_0) = \infty \quad \text{in } X(\alpha, \beta_0), \tag{10.16}$$

while $\mathfrak{E}(\beta_0, \beta_0) = \beta_0$ in $X(\alpha_0, \beta_0)$.

Now we examine where the missing solutions at $\beta = \beta_0$ have gone. We consider $(u_1, (1-\alpha)/2)$ where $\beta_0 = (1-\alpha)/2 < u_1 < (1+\alpha)/4$. The potential function is

$$T^{\beta_0}(y_2 + y_2^{-1}) + T^{\beta_0+\lambda_1}(y_1 + y_1 y_2) + T^{\beta_0+\lambda_2} y_1^{-1} y_2^{-1}. \tag{10.17}$$

Here

$$\lambda_1 = u_1 - \beta_0 < \lambda_2 = (1+\alpha)/2 - u_1 - \beta_0.$$

The leading-term equation is

$$1 - y_2^{-2} = 0, \qquad 1 + y_2 = 0. \tag{10.18}$$

The solution is $y_2 = -1$, and y_1 is arbitrary. Thus there are infinitely many solutions of the leading-term equation. Therefore, these solutions of (10.18) are not weakly nondegenerate.

So we need to study the critical point of (10.17) more carefully. The condition that (y_1, y_2) is a critical point of (10.17) is written as

$$\begin{cases} 1 - y_2^{-2} + T^{\lambda_1} y_1 - T^{\lambda_2} y_1^{-1} y_2^{-2} = 0, \\ 1 + y_2 - T^{\lambda_2-\lambda_1} y_1^{-2} y_2^{-1} = 0. \end{cases} \tag{10.19}$$

The leading-order term of y_2 should be -1. We need to study also the second-order term. We can write

$$y_2 = -1 + cT^{\mu}, \qquad y_1 = d,$$

where $c, d \in \Lambda_0 \setminus \Lambda_+$. Then we have

$$-2cT^{\mu} + dT^{\lambda_1} \equiv 0 \mod T^{\min\{\mu,\lambda_1\}}\Lambda_+, \tag{10.20}$$

$$cT^{\mu} + d^{-2}T^{\lambda_2-\lambda_1} \equiv 0 \mod T^{\min\{\mu,\lambda_2-\lambda_1\}}\Lambda_+. \tag{10.21}$$

Here (10.20) implies that $\mu = \lambda_1$; (10.21) then implies that $\lambda_2 - \lambda_1 = \lambda_1$. It implies that $u_1 = 1/3$. Furthermore,

$$c^3 \equiv -1/4 \mod \Lambda_+, \qquad d \equiv 2c \mod \Lambda_+. \tag{10.22}$$

Since the three solutions of the $\mathbb{C}$-reduction of (10.22) are all simple, we can show, by the same way as that of the proof of Theorem 10.4, that all solutions correspond to solutions of the equation (10.19). Therefore, $L(1/3, \beta_0)$ is a strongly balanced fiber.

We observe that solutions of the leading-term equation (10.18) do not lift to solutions of (10.19) unless $u_1 = 1/3$ and $y_1 = -1$. This shows that the weakly nondegeneracy assumption in Theorem 10.4 is essential.

We note that at $((1 + \alpha)/4, (1 - \alpha)/2)$ the leading-term equation becomes

$$1 - y_2^{-2} = 0, \qquad 1 + y_2 - y_1^{-2} y_2^{-1} = 0.$$

Its solutions in $(\mathbb{C} \setminus \{0\})^2$ are $(\pm 1/\sqrt{2}, 1)$. The number of solutions jumps from four to two here $(2 + 3 = 5)$. So this is consistent with Theorem 1.3.

In summary, for the case of (α, β_0) with $\beta_0 = (1 - \alpha)/2$, there are three balanced fibers $(1/3, \beta_0)$, $((1 + \alpha)/4, \beta_0)$, and (β_0, β_0). The first two of them are strongly balanced, and the last is not strongly balanced.

The balanced fiber $L(1/3, \beta_0) \subset X(\alpha, \beta_0)$ disappears as we deform $X(\alpha, \beta_0)$ to $X(\alpha, \beta)$ as β_0 moves to nearby β. To see this, let us take β that is slightly bigger than $\beta_0 = (1 - \alpha)/2$. Then $((\alpha + \beta)/2, (1 - \alpha)/2)$, and $(1 - \alpha - \beta, \beta)$ are the balanced fibers. The leading-term equation at the first point is

$$1 - y_2^{-2} = 0, \qquad y_2 - y_1^{-2} y_2^{-1} = 0.$$

Hence there are four solutions $(\pm 1, \pm 1)$. The leading-term equation at the second point is

$$1 + y_2 = 0, \qquad y_1 - y_2^{-2} = 0.$$

Hence the solution is $(1, -1)$, and the total number is again five.

We remark that the $\mathbb{Q}$-structure of quantum cohomology also jumps at $\beta = (1-\alpha)/2$. In particular,

$$QH\big(X(\alpha,\beta);\Lambda^{\mathbb{Q}}\big) = \begin{cases} \Lambda^{\mathbb{Q}(\sqrt{-1})} \times (\Lambda^{\mathbb{Q}})^3 & \beta \text{ is slightly smaller than } (1-\alpha)/2, \\ \Lambda^{\mathbb{Q}(\sqrt{2})} \times \Lambda^{\mathbb{Q}((-2)^{1/3})} & \beta = (1-\alpha)/2, \\ (\Lambda^{\mathbb{Q}})^5 & \beta \text{ is slightly larger than } (1-\alpha)/2. \end{cases}$$

Remark 10.18
In [FOOO5], we prove that $L(u_1, (1-\alpha)/2)$ is not displaceable for any $u_1 \in ((1-\alpha)/2, 1/3) \cup (1/3, (1+\alpha)/4)$ in the case $\beta = \beta_0$. We use the bulk deformation introduced by [FOOO3, Section 3.8] (equivalent to [FOOO2, Section 13]) to prove it.

The next example shows that Theorems 1.3, 1.4, and 10.4 cannot be generalized to the case of a positive characteristic.

Example 10.19
Consider the 2-point blow-up X of $\mathbb{C}P^2$ with moment polytope

$$P = \Big\{(u_1, u_2) \,\Big|\, 0 \le u_i \le 1-\epsilon, \sum u_i \le 1\Big\}.$$

Since X is monotone for $\epsilon = 1/3$, it follows that X is Fano. We assume that $\epsilon > 0$ is sufficiently small. Then the fiber at $u_0 = (1/3, 1/3)$ is balanced.

Now we consider the Novikov ring Λ^F with $F = \mathbb{F}_3$ a field of characteristic 3. We prove that there exists *no* element $\mathfrak{x} \in H(L(u); \Lambda_0^F)$ such that $HF\big((L(u_0), \mathfrak{x}), (L(u_0), \mathfrak{x}); \Lambda^F\big) \ne 0$.

The potential function at u_0 is

$$\mathfrak{PO}^{u_0} = T^{1/3}\big(y_1 + y_2 + 1/(y_1 y_2)\big) + T^{2/3-\epsilon}(y_1^{-1} + y_2^{-1}).$$

Therefore, the critical-point equation is given by

$$1 - 1/(y_i y_1 y_2) - t y_i^{-2} = 0 \quad i = 1, 2, \tag{10.23}$$

where $t = T^{1/3-\epsilon}$. From this it follows that $y_i \equiv 1 \mod \Lambda_+$. In fact, the leading-term equation is $a_i a_1 a_2 = 1$ for $i = 1, 2$, which is reduced to

$$a_1 = a_2 = a, \qquad a^3 = 1.$$

Obviously, this equation has the unique solution $a_1 = a_2 = 1$ in $\mathbb{F}_3$.

Going back to the study of solutions of the critical-point equation (10.23), we first prove that $y_1 = y_2$. We put $z_i = y_i^{-1}$. We assume that $z_i - z_j \neq 0$, and we put $z_i - z_j \equiv t^\lambda c \mod t^\lambda \Lambda_+$ with $c \in \mathbb{F}_3 \setminus \{0\}$. Then by (10.23), we have

$$(z_i - z_j)z_1 z_2 + t(z_i - z_j)(z_i + z_j) = 0.$$

This is a contradiction since the left-hand side is congruent to ct^λ modulo $t^\lambda \Lambda_+^{\mathbb{F}_3}$.

We now put $x = y_i$ and obtain

$$x^3 - tx - 1 = 0. \tag{10.24}$$

We prove the following.

LEMMA 10.20
Equation (10.24) has no solution in $\Lambda_0^{\mathbb{F}_3}$.

Proof
We put $x = 1 + t^{1/3}x'$ and obtain

$$(x')^3 - t^{1/3}x' - 1 = 0.$$

This equation resembles (10.24) except that t is replaced by $t^{1/3}$. We now put

$$x_N \equiv 1 + \sum_{k=1}^{N} t^{\sum_{i=1}^{k} 3^{-i}}.$$

Then $x_N^3 = 1 + tx_{N-1}$. Therefore,

$$(x_N)^3 - tx_N - 1 = -t^{1+\sum_{i=1}^{N} 3^{-i}}.$$

Therefore, x_N is a solution of (10.24) modulo $t^{1+\sum_{i=1}^{N} 3^{-i}}$. It is easy to see that there are no other solutions of (10.24) modulo $t^{1+\sum_{i=1}^{N} 3^{-i}}$.

However, since

$$1 + \sum_{i=1}^{\infty} 3^{-i} = \frac{3}{2} < \infty,$$

it follows that

$$\lim_{N\to\infty} x_N$$

does *not* converge in $\Lambda_0^{\mathbb{F}_3}$. Thus there is no solution of (10.24) over a field of characteristic 3. □

Lemma 10.20 implies that the field of fraction of the Puiseux series ring with coefficients in an algebraically closed field of *positive* characteristic is *not* algebraically closed. It is well known that this phenomenon does not occur in the case of characteristic zero (see, e.g., [Ei, Corollary 13.15]). Since we could not find a proof of a similar result for universal Novikov ring in the literature, we provide its proof in the appendix for completeness (we used it in the proof of Theorem 1.12 in Section 7).

11. Calculation of potential function

In this section, we prove Theorems 4.5 and 4.6. We begin with a review of [CO]. Let $\pi : X \to P$ be the moment map, and let $\partial P = \bigcup_{i=1}^m \partial_i P$ be the decomposition of the boundary of P into $(n-1)$-dimensional faces. Let $\beta_i \in H_2(X, L(u); \mathbb{Z})$ be elements such that

$$\beta_i \cap [\pi^{-1}(\partial P_j)] = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The Maslov index $\mu(\beta_i)$ is 2 (see [CO, Theorem 5.1]).

Let $\beta \in \pi_2(X, L(u))$, and let $\mathscr{M}_{k+1}^{\text{main}}(L(u), \beta)$ be the moduli space of stable maps from bordered Riemann surfaces of genus zero with $k+1$ boundary marked points in homology class β (see [FOOO1, Section 3], which is equivalent to [FOOO3, Section 2.1.2]). We require the boundary marked points to respect the cyclic order of $S^1 = \partial D^2$. (In other words, we consider the main component in the sense of [FOOO1, Section 3].) Let $\mathscr{M}_{k+1}^{\text{main,reg}}(L(u), \beta)$ be its subset consisting of all maps from a disc (namely, the stable map without disc or sphere bubble). Theorem 11.1 easily follows from the results of [CO]. In Theorem 11.1(3), we use the spin structure of $L(u)$ which is induced by the diffeomorphism of $L(u) \cong T^n$ by the T^n-action and the standard trivialization of the tangent bundle of T^n.

THEOREM 11.1

(1) *If* $\mu(\beta) < 0$, *or* $\mu(\beta) = 0$, $\beta \neq 0$, *then* $\mathscr{M}_{k+1}^{\text{main,reg}}(L(u), \beta)$ *is empty.*

(2) *If* $\mu(\beta) = 2$, $\beta \neq \beta_1, \ldots, \beta_m$, *then* $\mathscr{M}_{k+1}^{\text{main,reg}}(L(u), \beta)$ *is empty.*

(3) *For $i = 1, \ldots, m$, we have*

$$\mathscr{M}_1^{\mathrm{main,reg}}\big(L(u), \beta_i\big) = \mathscr{M}_1^{\mathrm{main}}\big(L(u), \beta_i\big). \tag{11.1}$$

Moreover, $\mathscr{M}_1^{\mathrm{main}}(L(u), \beta_i)$ is Fredholm regular. Furthermore, the evaluation map

$$\mathrm{ev} : \mathscr{M}_1^{\mathrm{main}}\big(L(u), \beta_i\big) \to L(u)$$

is an orientation-preserving diffeomorphism.

(4) *For any β, the moduli space $\mathscr{M}_1^{\mathrm{main,reg}}\big(L(u), \beta\big)$ is Fredholm regular. Moreover,*

$$\mathrm{ev} : \mathscr{M}_1^{\mathrm{main,reg}}\big(L(u), \beta\big) \to L(u)$$

is a submersion.

(5) *If $\mathscr{M}_1^{\mathrm{main}}(L(u), \beta)$ is not empty, then there exists $k_i \in \mathbb{Z}_{\geq 0}$ and $\alpha_j \in H_2(X; \mathbb{Z})$ such that*

$$\beta = \sum_i k_i \beta_i + \sum_j \alpha_j$$

and α_j is realized by holomorphic sphere. There is at least one nonzero k_i.

Proof

For the reader's convenience and for completeness, we explain how to deduce Theorem 11.1 from the results in [CO].

By [CO, Theorems 5.5, 6.1], $\mathscr{M}_{k+1}^{\mathrm{main,reg}}(L(u), \beta)$ is Fredholm regular for any β. Since the complex structure is invariant under the T^n-action and since $L(u)$ is T^n-invariant, it follows that T^n acts on $\mathscr{M}_{k+1}^{\mathrm{main,reg}}(L(u), \beta)$ and

$$\mathrm{ev} : \mathscr{M}_{k+1}^{\mathrm{main,reg}}\big(L(u), \beta\big) \to L(u)$$

is T^n-equivariant. Since the T^n-action on $L(u)$ is free and transitive, it follows that ev is a submersion if $\mathscr{M}_{k+1}^{\mathrm{main,reg}}(L(u), \beta)$ is nonempty. Item (4) follows.

We assume that $\mathscr{M}_{k+1}^{\mathrm{main,reg}}(L(u), \beta)$ is nonempty. Since ev is a submersion, it follows that

$$n = \dim L(u) \leq \dim \mathscr{M}_{k+1}^{\mathrm{main,reg}}\big(L(u), \beta\big) = n + \mu(\beta) - 2$$

if $\beta \neq 0$. Therefore, $\mu(\beta) \geq 2$, and (1) follows.

We next assume that $\mu(\beta) = 2$ and that $\mathscr{M}_{k+1}^{\mathrm{main,reg}}(L(u), \beta)$ is nonempty. Then by [CO, Theorem 5.3], we find that $\beta = \beta_i$ for some i, and (2) follows.

We next prove (5). It suffices to consider a map f such that

$$[f] \in \mathscr{M}_1^{\text{main}}\big(L(u), \beta\big) \setminus \mathscr{M}_1^{\text{main,reg}}\big(L(u), \beta\big).$$

We decompose the domain of f into irreducible components and restrict f there. Let $f_j : D^2 \to M$ and $g_k : S^2 \to M$ be the restriction of f to disc or sphere components, respectively. We have

$$\beta = \sum [f_j] + \sum [g_k].$$

Theorem 5.3 in [CO] implies that each of f_j is homologous to the sum of the elements of β_i. It implies (5).

We finally prove (3). The fact that ev is a diffeomorphism for $\beta = \beta_i$ follows directly from [CO, Theorem 5.3]. We next prove that ev is orientation-preserving. Since $L(u)$, $u \in \text{Int}\, P$, is a principal homogeneous space of T^n, the tangent bundle $TL(u)$ is trivialized once we fix an isomorphism, $T^n \equiv S^1 \times \cdots \times S^1$. Using the orientation and the spin structure on $L(u)$ induced by such a trivialization, we orient the moduli space $\mathscr{M}_1(\beta)$ of holomorphic discs. If we change the identification $T^n \equiv S^1 \times \cdots \times S^1$ by an orientation-preserving (resp., reversing) isomorphism, then the corresponding orientations on $L(u)$ and $\mathscr{M}_1(\beta)$ are preserved (resp., reversed). Therefore, whether ev : $\mathscr{M}(\beta_i) \to L(u)$ is orientation-preserving or not does not depend on the choice of the identification T^n and $S^1 \times \cdots \times S^1$.

For each $i = 1, \ldots, m$, we can find an automorphism ϕ of $(\mathbb{C}^*)^n$ and a biholomorphic map $f : X \setminus \cup_{j \neq i} \pi^{-1}(\partial_j P) \to \mathbb{C} \times (\mathbb{C}^*)^{n-1}$ such that

(1) f is ϕ-equivariant;

(2) $f(L(u)) = L_{\text{std}}$, where $L_{\text{std}} = \{(w_1, \ldots, w_n) \in (\mathbb{C})^n \mid |w_1| = \cdots = |w_n| = 1\}$.

Under this identification, $\mathscr{M}_1(\beta_i)$ is identified with the space of holomorphic discs

$$z \in D^2 \mapsto (\zeta \cdot z, w_2, \ldots, w_n) \in \mathbb{C} \times (\mathbb{C}^*)^{n-1}, \quad \zeta \in S^1 \subset \mathbb{C}^*,$$

where $w_k \in \mathbb{C}, k = 2, \ldots, n$ with $|w_k| = 1$. Therefore, it is enough to check the statement that ev is orientation-preserving in a single example. Cho [Cho] proved it in the case of the Clifford torus in $\mathbb{C}P^n$, and hence the proof.

To prove (11.1) and complete the proof of Theorem 11.1, it remains to prove $\mathscr{M}_1^{\text{main,reg}}(L(u), \beta_{i_0}) = \mathscr{M}_1^{\text{main}}(L(u), \beta_{i_0})$. (Here $i_0 \in \{1, \ldots, m\}$.) Let $[f] \in \mathscr{M}_1^{\text{main}}(L(u), \beta_{i_0})$. We take k_i and α_j as in (5). (Here $\beta = \beta_{i_0}$). We have

$$\partial \beta_{i_0} = \sum_i k_i \partial \beta_i.$$

Using the convexity of P, (5), and $k_i \geq 0$, we show that the inequality

$$\beta_{i_0} \cap \omega \leq \sum_i k_i \beta_i \cap \omega \tag{11.2}$$

holds and that the equality holds only if $k_i = 0$ $(i \neq i_0)$, $k_{i_0} = 1$, as follows. By (5) we have

$$\ell_{i_0} = \sum_{i=1}^{m} k_i \ell_i + c,$$

where c is a constant. Since $k_i \geq 0$ and $\ell_{i_0}(u') = 0$ for $u' \in \partial_{i_0} P$, it follows that $c \leq 0$. (Note that $\ell_i \geq 0$ on P.) Since $\beta_i \cap \omega = \ell_i(u)$, we have the inequality (11.2). Let us assume that the equality holds. If there exists $i \neq j$ with $k_i, k_j > 0$, then

$$\partial_{i_0} P = \left\{u' \in P \,\middle|\, \ell_{i_0}(u') = 0\right\} \subseteq \left\{u' \in P \,\middle|\, \ell_i(u') = \ell_j(u') = 0\right\} \subseteq \partial_i P \cap \partial_j P.$$

This is a contradiction since $\partial_{i_0} P$ is codimension 1. Therefore, there is only one nonzero k_i. It is easy to see that $i = i_0$ and that $k_{i_0} = 1$.

On the other hand, since $\alpha_j \cap \omega > 0$, it follows that

$$\beta_{i_0} \cap \omega \geq \sum_i k_i \beta_i \cap \omega.$$

Therefore, there is no sphere bubble (that is α_j). Moreover, the equality holds in (11.2). Hence the domain of our stable map is irreducible; namely,

$$\mathscr{M}_1^{\text{main,reg}}\big(L(u), \beta_{i_0}\big) = \mathscr{M}_1^{\text{main}}\big(L(u), \beta_{i_0}\big).$$

The proof of Theorem 11.1 is now complete. □

Next we discuss one delicate point to apply Theorem 11.1 to the proofs of Theorems 4.5 and 4.6 (this point was already mentioned in [CO, Section 16]). Let us consider the case where there exists a holomorphic sphere $g : S^2 \to X$ with

$$c_1(X) \cap g_*[S^2] < 0.$$

We assume, moreover, that there exists a holomorphic disc $f : (D^2, \partial D^2) \to (X, L(u))$ such that

$$f(0) = g(1).$$

We glue D^2 and S^2 at $0 \in D^2$ and $1 \in S^2$ to obtain Σ; f and g induce a stable map $h : (\Sigma, \partial\Sigma) \to (X, L(u))$.

In general, h is *not* Fredholm regular since g may not be Fredholm regular or the evaluation is not transversal at the interior nodes. In other words, elements of $\mathscr{M}_1^{\text{main}}(L(u),\beta) \setminus \mathscr{M}_1^{\text{main,reg}}(L(u),\beta)$ may not be Fredholm regular in general. Moreover, replacing g by its multiple cover, we obtain an element of $\mathscr{M}_1^{\text{main}}(L(u),\beta) \setminus \mathscr{M}_1^{\text{main,reg}}(L(u),\beta)$ such that $\mu(\beta)$ is negative. Theorem 11.1 says that all the holomorphic discs without any bubble are Fredholm regular. However, we cannot expect that all stable maps in $\mathscr{M}_1^{\text{main}}(L(u),\beta)$ are Fredholm regular.

In order to prove Theorem 4.6, we need to find appropriate perturbations of those stable maps. For this purpose, we use the T^n-action and proceed as follows. (Note that many of the arguments below are much simplified in the Fano case, where there exists no holomorphic sphere g with $c_1(M) \cap g_*[S^2] \leq 0$.)

We equip each of $\mathscr{M}_1(L(u),\beta)$ with Kuranishi structure (see [FO] for the general theory of Kuranishi structure and [FOOO1, Sections 17–18], [FOOO3, Section 7.1], and [FOOO2, Section 29] for its construction in the context we currently deal with). We may construct Kuranishi neighborhoods and obstruction bundles that carry T^n-actions induced by the T^n-action on X, and choose T^n-equivariant Kuranishi maps (see Definition B.4). We note that the evaluation map

$$\mathrm{ev} : \mathscr{M}_1\big(L(u),\beta\big) \to L(u)$$

is T^n-equivariant. We use the fact that the complex structure of X is T^n-invariant and that $L(u)$ is a free T^n-orbit to find such a Kuranishi structure (see Proposition B.7 for details).

We remark that the T^n-action on the Kuranishi neighborhood is free since the T^n-action on $L(u)$ is free and ev is T^n-equivariant. We take a perturbation (that is, a multisection) of the Kuranishi map that is T^n-equivariant. We can find such a multisection which is also transversal to zero as follows. Since the T^n-action is free, we can take the quotient of Kuranishi neighborhood, obstruction bundle, and so forth, to obtain a space with Kuranishi structure. Then we take a transversal multisection of the quotient Kuranishi structure and lift it to a multisection of the Kuranishi neighborhood of $\mathscr{M}_1(L(u),\beta)$ (see Corollary B.15 for details). Let $\mathfrak{s}_\beta$ be such a multisection, and let $\mathscr{M}_1(L(u),\beta)^{\mathfrak{s}_\beta}$ be its zero set. We note that the evaluation map

$$\mathrm{ev} : \mathscr{M}_1\big(L(u),\beta\big)^{\mathfrak{s}_\beta} \to L(u) \tag{11.3}$$

is a submersion. This follows from the T^n-equivariance. This makes our construction of systems of multisections much simpler than the general one in [FOOO3, Section 7.2] (equivalent to [FOOO2, Section 30]) since the fiber product appearing in the inductive construction is automatically transversal (see [FOOO3, Section 7.2.2], [FOOO2, Section 30.2] for the reason why this is crucial). More precisely, we prove

Lemma 11.2 below. Let

$$\mathfrak{forget}_0 : \mathcal{M}^{\mathrm{main}}_{k+1}(L(u), \beta) \to \mathcal{M}^{\mathrm{main}}_{1}(L(u), \beta) \tag{11.4}$$

be the forgetful map which forgets the (first, ..., kth) marked points. (In other words, only the zeroth marked point remains.) We can construct our Kuranishi structure so that it is compatible with $\mathfrak{forget}_0$ in the same sense as in [FOOO3, Lemma 7.3.8] or [FOOO2, Lemma 31.8].

LEMMA 11.2
For each given $E > 0$, there exists a system of multisections $\mathfrak{s}_{\beta,k+1}$ on $\mathcal{M}^{\mathrm{main}}_{k+1}(L(u), \beta)$ for $\beta \cap \omega < E$ with the following properties:
(1) *They are transversal to* 0.
(2) *They are invariant under the T^n-action.*
(3) *The multisection $\mathfrak{s}_{\beta,k+1}$ is the pullback of the multisection $\mathfrak{s}_{\beta,1}$ by the forgetful map* (11.4).
(4) *The restriction of $\mathfrak{s}_{\beta,1}$ to the boundary of $\mathcal{M}^{\mathrm{main}}_{1}(L(u), \beta)$ is the fiber product of the multisections $\mathfrak{s}_{\beta',k'}$ with respect to the identification of the boundary*

$$\partial \mathcal{M}^{\mathrm{main}}_{1}\bigl(L(u), \beta\bigr) = \bigcup_{\beta_1+\beta_2=\beta} \mathcal{M}^{\mathrm{main}}_{1}\bigl(L(u), \beta_1\bigr)_{\mathrm{ev}_0} \times_{\mathrm{ev}_1} \mathcal{M}^{\mathrm{main}}_{2}\bigl(L(u), \beta_2\bigr).$$

(5) *We do not perturb $\mathcal{M}^{\mathrm{main}}_{1}(L(u), \beta_i)$ for $i = 1, \dots, m$.*

Proof
We construct multisections inductively over $\omega \cap \beta$. Since (2) implies that fiber products of the perturbed moduli spaces which we have already constructed in the earlier stage of induction are automatically transversal, we can extend them so that (1), (2), (3), (4) are satisfied by the method we already explained above. We recall from Theorem 11.1 (3) that

$$\mathcal{M}^{\mathrm{main}}_{1}\bigl(L(u), \beta_i\bigr) = \mathcal{M}^{\mathrm{main,reg}}_{1}\bigl(L(u), \beta_i\bigr),$$

and it is Fredholm regular and its evaluation map is surjective to $L(u)$. Therefore, when we perturb the multisection we do not need to worry about compatibility of it with other multisections we have already constructed in the earlier stage of induction. This enables us to leave the moduli space $\mathcal{M}^{\mathrm{main}}_{1}(L(u), \beta_i)$ unperturbed for all β_i. The proof of Lemma 11.2 is complete. □

Remark 11.3
We need to fix E and stop the inductive construction of multisections at some finite stage. Specifically, we define $s_{\beta,k+1}$ only for β with $\beta \cap \omega < E$. The reason is explained

in [FOOO3, Section 7.2.3] or [FOOO2, Section 30.3]. We can get around this trouble in the same way as explained in [FOOO3, Section 7.2] or [FOOO2, Section 30] (see Remark 11.11).

Remark 11.4
We explain one delicate point of the proof of Lemma 11.2. Let $\alpha \in \pi_2(X)$ be represented by a holomorphic sphere with $c_1(X) \cap \alpha < 0$. We consider the moduli space $\mathscr{M}_1(\alpha)$ of the holomorphic sphere with one marked point and in homology class α. Let us consider $\beta \in \pi_2(X; L(u))$ and the moduli space $\mathscr{M}^{\text{main}}_{k+1,1}(\beta)$ of holomorphic discs with one interior and $(k+1)$-boundary marked points and of homotopy class β. The fiber product

$$\mathscr{M}_1(\alpha) \times_X \mathscr{M}^{\text{main}}_{k+1,1}(\beta)$$

taken by the evaluation maps at interior marked points are contained in $\mathscr{M}^{\text{main}}_{k+1,1}(\beta+\alpha)$. If we want to define a multisection compatible with the embedding

$$\mathscr{M}_1(\alpha) \times_X \mathscr{M}^{\text{main}}_{k+1,1}(\beta) \subset \mathscr{M}^{\text{main}}_{k+1}(\beta+\alpha) \tag{11.5}$$

then it is impossible to make it both transversal and T^n-equivariant in general. This is because the nodal point of such a singular curve could be contained in the part of X with nontrivial isotropy group.

Our perturbation constructed above satisfies items (1) and (2) of Lemma 11.2 and so may *not* be compatible with the embedding (11.5). Our construction of the perturbation given in Lemma 11.2 exploits the fact that the T^n-action acts freely on the Lagrangian fiber $L(u)$ and is carried out by induction on the number of *disc* components (and of energy) only, regardless of the number of sphere components.

The following corollary is an immediate consequence of Lemma 11.2.

COROLLARY 11.5
If $\mu(\beta) < 0$ or $\mu(\beta) = 0$, $\beta \neq 0$, then $\mathscr{M}^{\text{main}}_1(L(u), \beta)^{\mathfrak{s}_\beta}$ is empty.

Now we consider $\beta \in \pi_2(X; L)$ with $\mu(\beta) = 2$ and $\beta \cap \omega < E$, where E is as in Lemma 11.2. One immediate consequence of Corollary 11.5 is that the virtual fundamental chain of $\mathscr{M}^{\text{main}}_1(L(u), \beta)$ becomes a *cycle*. More precisely, we introduce the following.

Definition 11.6
Let $\beta \in \pi_2(X; L)$ with $\mu(\beta) = 2$ and $\beta \cap \omega < E$, where E is as in Lemma 11.2. We define a homology class $c_\beta \in H_n(L(u); \mathbb{Q}) \cong \mathbb{Q}$ by the pushforward

$$c_\beta = \mathrm{ev}_*\big([\mathscr{M}^{\text{main}}_1(L(u), \beta)^{\mathfrak{s}_\beta}]\big).$$

LEMMA 11.7
The number c_β is independent of the choice of the system of multisections $\mathfrak{s}_{\beta,k+1}$ satisfying items (1)–(5) of Proposition 11.2.

Proof
If there are two such systems, we can find a T^n-invariant homotopy between them which is also transversal to zero. By a dimension counting argument applied to the parameterized version of $\mathcal{M}_1^{\text{main}}(L(u), \beta)$ and its perturbation, we have the parameterized version of Corollary 11.5. This in turn implies that the perturbed (parameterized) moduli space defines a compact cobordism between the perturbed moduli spaces of $\mathcal{M}_1^{\text{main}}(\beta)$ associated to the two such systems. The lemma follows. □

We note that $c_{\beta_i} = 1$ where β_i $(i = 1, \ldots, m)$ are the classes corresponding to each of the irreducible components of the divisor $\pi^{-1}(\partial P)$. If X is Fano, then $c_\beta = 0$ for $\beta \neq \beta_i$. But this may not be the case if X is not Fano.

We now use our perturbed moduli space to define a structure of filtered A_∞-algebra on the de Rham cohomology $H(L(u); \Lambda_0^{\mathbb{R}}) \cong (H(L(u), \mathbb{R})) \otimes \Lambda_0$. We write it as $\mathfrak{m}^{\text{can}}$.

We take a T^n-equivariant Riemannian metric on $L(u)$. We observe that a differential form ρ on $L(u)$ is harmonic if and only if ρ is T^n-equivariant. So we identify $H(L(u), \mathbb{R})$ with the set of T^n-equivariant forms from now on.

We consider the evaluation map

$$\text{ev} = (\text{ev}_1, \ldots, \text{ev}_k, \text{ev}_0) : \mathcal{M}_{k+1}^{\text{main}}\big(L(u), \beta\big)^{\mathfrak{s}_\beta} \to L(u)^{k+1}.$$

Let $\rho_1, \ldots, \rho_k$ be T^n-equivariant differential forms on $L(u)$. We define

$$\mathfrak{m}_{k,\beta}^{\text{can}}(\rho_1, \ldots, \rho_k) = (\text{ev}_0)_!(\text{ev}_1, \ldots, \text{ev}_k)^*(\rho_1 \wedge \cdots \wedge \rho_k). \tag{11.6}$$

We remark that integration along fiber $(\text{ev}_0)_!$ is well defined and gives a smooth form since ev_0 is a submersion (this is a consequence of T^n-equivariance). More precisely, we apply Definition C.7 as follows. We put $\mathcal{M} = \mathcal{M}_{k+1}^{\text{main}}(L(u), \beta)$, $L_s = L^k$, $L_t = L$. Thus $\text{ev}_s = (\text{ev}_1, \ldots, \text{ev}_k) : \mathcal{M} \to L_s$, $\text{ev}_t = \text{ev}_0 : \mathcal{M} \to L_t$. Thus we are in the situation that we formulate at the beginning of Section C. Then using Lemma C.9 and Remark C.8 (1), we put

$$\mathfrak{m}_{k,\beta}^{\text{can}}(\rho_1, \ldots, \rho_k) = (\mathcal{M}; \text{ev}_s, \text{ev}_t, \mathfrak{s}_\beta)_*(\rho_1 \times \cdots \times \rho_k). \tag{11.7}$$

We remark that the right-hand side of (11.7) is again T^n-equivariant since $\mathfrak{s}_\beta$ and so forth are T^n-equivariant.

Lemma 11.2(4) implies that

$$\begin{aligned}&\partial \mathscr{M}^{\text{main}}_{k+1}\big(L(u),\beta\big)\\ &\qquad = \bigcup_{k_1+k_2=k+1}\ \bigcup_{\beta_1+\beta_2=\beta}\ \bigcup_{l=1}^{k_2} \mathscr{M}^{\text{main}}_{k_1+1}\big(L(u),\beta_1\big)_{\text{ev}_0}\times_{\text{ev}_l}\mathscr{M}^{\text{main}}_{k_2+1}\big(L(u),\beta_2\big).\end{aligned}$$

Therefore, using Lemmas C.9 and C.10, we have the following formula:

$$\sum_{\beta_1+\beta_2=\beta}\ \sum_{k_1+k_2=k+1}\ \sum_{l=1}^{k_1}(-1)^{*}\mathfrak{m}^{\text{can}}_{k_1,\beta_1}\big(\rho_1,\dots,\mathfrak{m}^{\text{can}}_{k_2,\beta_2}(\rho_l,\dots,),\dots,\rho_k\big)=0. \tag{11.8}$$

Here $* = \sum_{i=1}^{l}(\deg\rho_i + 1)$ (see the end of Section C for sign). We now put

$$\mathfrak{m}^{\text{can}}_{k}(\rho_1,\dots,\rho_k)=\sum_{\beta}T^{\beta\cap\omega/2\pi}\mathfrak{m}^{\text{can}}_{k,\beta}(\rho_1,\dots,\rho_k). \tag{11.9}$$

We extend (11.9) to $\rho\in H(L(u);\Lambda_0^{\mathbb{R}})$ such that it is Λ_0-multilinear. Then (11.8) implies that it defines a structure of a filtered A_∞-structure on $H(L(u);\Lambda_0^{\mathbb{R}})$ in the sense of Section 3.

We also observe that our filtered A_∞-algebra is unital and that the constant zero form $1\in H^0(L;\mathbb{R})$ is a unit. This is a consequence of Lemma 11.2(3).

We next calculate our filtered A_∞-structure in the case when ρ_i are degree 1-forms.

LEMMA 11.8
For $\mathfrak{x}\in H^1(L(u),\Lambda_0)$ and $\beta\in\pi_2(X,L)$ with $\mu(\beta)=2$, we have

$$\mathfrak{m}^{\text{can}}_{k,\beta}(\mathfrak{x},\dots,\mathfrak{x})=\frac{c_\beta}{k!}(\partial\beta\cap\mathfrak{x})^k\cdot PD\big([L(u)]\big).$$

Here $PD([L(u)])$ is the Poincaré dual to the fundamental class. In other words, it is the n-form with $\int_{L(u)}PD([L(u)])=1$.

Proof
It suffices to consider the case $\mathfrak{x}=\rho\in H^1(L(u);\mathbb{R})$ and to show that

$$\int_{L(u)}\mathfrak{m}^{\text{can}}_{k,\beta}(\rho,\dots,\rho)=\frac{c_\beta}{k!}(\partial\beta\cap\mathfrak{x})^k. \tag{11.10}$$

Let

$$C_k=\big\{(t_1,\dots,t_k)\,\big|\,0\le t_1\le\cdots\le t_k\le 1\big\}. \tag{11.11}$$

We define an iterated blow-up, denoted by $\widehat{C}_k$, of C_k in the following way. Let $S = \partial D$ be the boundary of the unit disc $D = D^2 \subset \mathbb{C}$, and let $\beta_D \in H_2(\mathbb{C}, S)$ be the homology class of the unit disc. We consider the moduli space $\mathscr{M}_{k+1}(\mathbb{C}, S; \beta_D)$ and the evaluation map $\vec{\mathrm{ev}} = (\mathrm{ev}_0, \dots, \mathrm{ev}_k) : \mathscr{M}_{k+1}(\mathbb{C}, S; \beta_D) \to (S^1)^{k+1}$. We fix a point $p_0 \in S \subset \mathbb{C}$, and we put

$$\widehat{C}_k := \mathrm{ev}_0^{-1}(p_0) \subset \mathscr{M}_{k+1}(\mathbb{C}, S; \beta_D).$$

We make the identification $S^1 \setminus \{p_0\} \cong (0, 1)$. Then $\vec{\mathrm{ev}}$ induces a diffeomorphism

$$\widehat{C}_k \cap \mathscr{M}^{\mathrm{reg}}_{k+1}(\mathbb{C}, S; \beta_D) \to \mathrm{Int}\, C_k$$

given by

$$[w, z_0, \dots, z_k] \mapsto \big(w(z_1) - w(z_0), \dots, w(z_k) - w(z_0)\big),$$

where

$$\mathrm{Int}\, C_k = \big\{(t_1, \dots, t_k) \,\big|\, 0 < t_1 < \dots < t_k < 1\big\} \subset C_k.$$

In this sense, $\widehat{C}_k$ is regarded as an iterated blow-up of C_k along the diagonal (that is, the set of points where $t_i = t_{i+1}$ for some i). We identify $\partial D = S \cong \mathbb{R}/\mathbb{Z} \cong S^1$. We have

$$\mathscr{M}^{\mathrm{main}}_{k+1}\big(L(u), \beta\big)^{\mathfrak{s}} \cong \mathscr{M}^{\mathrm{main}}_{1}\big(L(u), \beta\big)^{\mathfrak{s}} \times \widehat{C}_k. \tag{11.12}$$

In fact, Corollary 11.5 implies that $\mathscr{M}^{\mathrm{main}}_1(L(u), \beta)^{\mathfrak{s}}$ consists of finitely many free T^n-orbits (with multiplicity $\in \mathbb{Q}$) and $\mathscr{M}^{\mathrm{main}}_1(L(u), \beta)^{\mathfrak{s}} = \mathscr{M}^{\mathrm{main,reg}}_1(L(u), \beta)^{\mathfrak{s}}$. By Lemma 11.2(3), we have a map $\mathscr{M}^{\mathrm{main,reg}}_{k+1}(L(u), \beta)^{\mathfrak{s}} \to \mathscr{M}^{\mathrm{main,reg}}_1(L(u), \beta)^{\mathfrak{s}}$. It is easy to see that the fiber can be identified with $\widehat{C}_k$.

Under this identification, the evaluation map $\vec{\mathrm{ev}}$ is induced by

$$\mathrm{ev}_i(\mathfrak{u}; t_1, \dots, t_k) = [t_i \partial\beta] \cdot \mathrm{ev}(\mathfrak{u}) \tag{11.13}$$

for $(\mathfrak{u}; t_1, \dots, t_k) \in \mathscr{M}^{\mathrm{main}}_1(L(u), \beta) \times \mathrm{Int}\, C_k \subset \mathscr{M}^{\mathrm{main}}_1(L(u), \beta) \times \widehat{C}_k$.

Here $\partial\beta \in H_1(L(u); \mathbb{Z})$ is identified to an element of the universal cover $\widetilde{L}(u) \cong \mathbb{R}^n$ of $L(u)$, and $[t_i \partial\beta] \in L(u)$ acts as a multiplication on the torus; $\mathrm{ev}(\mathfrak{u})$ is defined by the evaluation map $\mathrm{ev} : \mathscr{M}^{\mathrm{main}}_1(L(u), \beta) \to L(u)$. We also have

$$\mathrm{ev}_0(\mathfrak{u}; t_1, \dots, t_k) = \mathrm{ev}(\mathfrak{u}). \tag{11.14}$$

We note that ev : $\mathscr{M}_1^{\text{main}}(L(u), \beta_i) \to L(u)$ is a diffeomorphism (see Theorem 11.1(3)). Now we have

$$\int_{L(u)} \mathfrak{m}^{dR}_{k,\beta}(\rho, \ldots, \rho) = c_\beta \mathrm{Vol}(C_k)\Big(\int_{\partial\beta} \rho\Big)^k = \frac{c_\beta}{k!}(\partial\beta \cap \mathfrak{x})^k.$$

The proof of Lemma 11.8 is now complete. □

Remark 11.9
We can prove that our filtered A_∞-algebra $\big(H(L(u); \Lambda_0^{\mathbb{R}}), \mathfrak{m}_*^{\text{can}}\big)$ is homotopy-equivalent to the one in [FOOO3, Theorem A] and [FOOO2, Theorem A]. The proof is a straightforward generalization of [FOOO3, Section 7.5] and [FOOO2, Section 33]; it is omitted here. In fact, we do not need to use this fact to prove Theorem 1.5 if we use the de Rham version in all the steps of the proof of Theorem 1.5 without involving the singular homology version.

Remark 11.10
We constructed our filtered A_∞-structure directly on de Rham cohomology group $H(L(u); \Lambda_0^{\mathbb{R}})$. The above construction uses the fact that the wedge products of harmonic forms are again harmonic. This is a special feature of our situation, where our Lagrangian submanifold L is a torus. (In other words, we use the fact that the rational homotopy type of L is formal.)

Alternatively, we can construct filtered A_∞-structure on the de Rham complex $\Omega(L(u))\widehat{\otimes}_{\mathbb{R}}\Lambda_0^{\mathbb{R}}$ and reduce it to the de Rham cohomology by homological algebra; namely, we consider smooth forms ρ_i which are not necessarily harmonic, and we use (11.6) and (11.9) to define $\mathfrak{m}_k^{dR}(\rho_1, \ldots, \rho_k)$. (The proof of the A_∞-formula is the same.) Using the formality of T^n, we can show that the canonical model of $(\Omega(L(u))\widehat{\otimes}_{\mathbb{R}}\Lambda_0^{\mathbb{R}}, \mathfrak{m}_*^{dR})$ is the same as $(H(L(u); \Lambda_0^{\mathbb{R}}), \mathfrak{m}^{\text{can}})$. We omit its proof since we do not use it here.

Using the continuous family of perturbations, this construction can be generalized to the case of arbitrary relatively spin Lagrangian submanifold in a symplectic manifold (see [F5]).

Proof of Proposition 4.3
Proposition 4.3 immediately follows from Corollary 11.5, Lemma 11.8, and Lemma 11.9. We just take the sum

$$\begin{aligned}\sum_{k=0}^{\infty} \mathfrak{m}_k^{\mathrm{can}}(b,\dots,b) &= \sum_{k=0}^{\infty} \sum_{\beta\in\pi_2(X,L(u))} T^{\omega\cap\beta/2\pi}\mathfrak{m}_{k,\beta}^{\mathrm{can}}(b,\dots,b)\\ &= \sum_{k=0}^{\infty}\sum_{\beta} T^{\omega\cap\beta/2\pi}\mathfrak{m}_{k,\beta}^{\mathrm{can}}(b,\dots,b)\\ &= \sum_{\beta}\sum_{k=0}^{\infty} \frac{c_\beta}{k!}(\partial\beta\cap b)^k T^{\beta\cap\omega/2\pi}\cdot PD\big([L(u)]\big). \quad (11.15)\end{aligned}$$

Note, by the degree reason, that we need to take sum over β with $\mu(\beta) = 2$.

Since b is assumed to lie in $H^1(L(u), \Lambda_+)$ and not just in $H^1(L(u), \Lambda_0)$, the series appearing as the scalar factor in (11.15) converges in non-Archimedean topology of Λ_0 and so the sum $\sum_{k=0}^{\infty} \mathfrak{m}_k^{\mathrm{can}}(b,\dots,b)$ is a multiple of $PD([L(u)])$. Hence $b \in \widehat{\mathscr{M}}_{\mathrm{weak}}(L(u))$ by definition (4.1). We remark that the gauge equivalence relation in [FOOO3, Chapter 4] is trivial on $H^1(L(u);\Lambda_0)$, and so $H^1(L(u);\Lambda_+) \hookrightarrow \mathscr{M}_{\mathrm{weak}}(L(u))$. We omit the proof of this fact since we do not use it here. □

Proof of Theorem 4.5
Suppose that there is no nontrivial holomorphic sphere whose Maslov index is nonpositive. Then Theorem 11.1(5) implies that if $\mu(\beta) \le 2$, $\beta \ne \beta_i$, $\beta \ne 0$, then $\mathscr{M}_1^{\mathrm{main}}(L(u),\beta)$ is empty. Therefore, again by dimension counting as in Corollary 11.5, we obtain

$$\sum_{k=0}^{\infty} \mathfrak{m}_k^{\mathrm{can}}(x,\dots,x) = \sum_{i=1}^{m}\sum_{k=0}^{\infty} T^{\omega\cap\beta_i/2\pi}\mathfrak{m}_{k,\beta_i}^{\mathrm{can}}(x,\dots,x)$$

for $x \in H^1(L(u),\Lambda_+)$. On the other hand, we obtain

$$\begin{aligned}\mathfrak{PO}(x;u) &= \sum_{i=1}^{m}\sum_{k=0}^{\infty}\frac{1}{k!}(\partial\beta_i\cap x)^k T^{\ell_i(u)}\\ &= \sum_{i=1}^{m}\sum_{k=0}^{\infty}\frac{1}{k!}\langle v_i, x\rangle^k T^{\ell_i(u)} = \sum_{i=1}^{m} e^{\langle v_i,x\rangle} T^{\ell_i(u)}\end{aligned}$$

from (6.8), (11.15), and the definition of $\mathfrak{PO}$. Writing $x = \sum_{i=1}^{n} x_i\mathbf{e}_i$ and recalling $y_i = e^{x_i}$, we obtain $e^{\langle v_i,x\rangle} = y_1^{v_{i,1}}\cdots y_n^{v_{i,n}}$ and hence the proof of Theorem 4.5. □

Proof of Theorem 4.6
Let $\beta \in \pi_2(X)$, let $\mu(\beta) = 2$, and let $\mathscr{M}_{\text{weak}}(\beta) \neq \emptyset$. Theorem 11.1(5) implies that

$$\partial\beta = \sum k_i \partial\beta_i, \quad \beta = \sum_i k_i \beta_i + \sum_j \alpha_j.$$

Hence

$$\sum_k T^{\beta\cap\omega/2\pi} \mathfrak{m}^{\text{can}}_{k,\beta}(b, \dots, b)$$

becomes one of the terms of the right-hand side of (4.7). We remark that class β with $\mu(\beta) \geq 4$ does not contribute to $\mathfrak{m}^{\text{can}}_k(b, \dots, b)$ by the degree reason.

When all the vertices of P lie in $\mathbb{Q}^n$, then the symplectic volume of all α_j are in $2\pi\mathbb{Q}$. Moreover, $\omega \cap \beta_i \in 2\pi\mathbb{Q}$. Therefore, the exponents $\beta \cap \omega/2\pi$ are rational.

The proof of Theorem 4.6 is complete. □

Remark 11.11
We remark that in Lemma 11.2 we constructed a system of multisections only for $\mathscr{M}^{\text{main}}_{k+1}(L(u), \beta)$ with $\beta \cap \omega < E$. So we obtain only an $A_{n,K}$-structure instead of a filtered A_∞-structure. Here $(n, K) = (n(E), K(E))$ depends on E and $\lim_{E\to\infty}(n(E), K(E)) = (\infty, \infty)$. It induces an $A_{n,K}$-structure $\mathfrak{m}^{(E)}$ on $H(L; \Lambda_0)$ (see [FOOO3, Section 7.2.7] or [FOOO2, Section 30.7]). In the same way as in [FOOO3, Section 7.2] and [FOOO2, Section 30], we can find $(n'(E), K'(E))$ such that $(n'(E), K'(E)) \to (\infty, \infty)$ as $E \to \infty$ and the following holds.

If $E_1 < E_2$, then the $A_{n(E_1),K(E_1)}$-structure $\mathfrak{m}^{(E_1)}$ is $(n'(E), K'(E))$-homotopy equivalent to $\mathfrak{m}^{(E_2)}$.

This implies that we can extend $\mathfrak{m}^{(E_1)}$ (regarded as an $A_{n'(E_1),K'(E_1)}$-structure) to a filtered A_∞-structure by [FOOO3, Theorem 7.2.72] and [FOOO2, Theorem 30.72]. (We also note that for all the applications in this article, we can use filtered $A_{n,K}$-structure for sufficiently large n, K in place of filtered A_∞-structure.)

Moreover, we can use Lemma 11.7 to show the following. If $\mathfrak{x}_i \in H^1(L(u); \mathbb{Q})$, then $\mathfrak{m}^{(E)}_{k,\beta}(\mathfrak{x}_1, \dots, \mathfrak{x}_k)$ is independent of E. So in particular, it coincides to one of filtered A_∞-structure we define as above.

In other words, since the number c_β is independent of the choice of the system of T^n-invariant multisections, it follows that the potential function in Theorem 4.6 is independent of it. However, we do not know how to calculate it.

Remark 11.12
We used de Rham cohomology to go around the problem of transversality among chains in the classical cup product. One drawback of this approach is that we lose control of the rational homotopy type. Specifically, we do *not* prove here that the filtered

A_∞-algebra (partially) calculated above is homotopy-equivalent to the one in [FOOO3, Theorem A] or [FOOO2, Theorem A] over $\mathbb{Q}$. (Note that all the operations we obtain are defined over $\mathbb{Q}$, however.) Nevertheless, we believe that they are indeed homotopy-equivalent over $\mathbb{Q}$. There may be several possible ways to prove this statement, one of which is to use the rational de Rham forms used by Sullivan.

Moreover, since the number c_β is independent of the choices we made, the structure of filtered A_∞-algebra on $H(L(u), \Lambda^{\mathbb{Q}})$ is well defined (i.e., independent of the choices involved). The $\mathbb{Q}$-structure is actually interesting in our situation (see, e.g., Proposition 7.13). However, homotopy equivalence of the $\mathbb{Q}$-version of Lemma 11.9 is not used in the statement of Proposition 7.13 or in its proof.

12. Nonunitary flat connection on $L(u)$

In this section, we explain how we can include (not necessarily unitary) flat bundles on Lagrangian submanifolds in Lagrangian Floer theory following [F2] and [Cho].

Remark 12.1

For our purposes, we need to use a flat complex line bundle due to the following reason. In [FOOO3], we assumed that our bounding cochain b is an element of $H(L;\Lambda_+)$ since we want the series

$$\mathfrak{m}_1^b(x) = \sum_{k,\ell} \mathfrak{m}_{k+\ell+1}(b^{\otimes k}, x, b^{\otimes \ell})$$

to converge. There we used convergence with respect to the non-Archimedean norm. For the case of Lagrangian fibers in toric manifold, the above series converges for $b \in H^1(L;\Lambda_0)$. The convergence is the usual (classical Archimedean) topology on $\mathbb{C}$ on each coefficient of T^λ.

This is not an accident, and in general, it was expected to happen (see [FOOO3, Conjecture 3.6.46] and [FOOO2, Conjecture 11.46]). However, for this convergence to occur, we need to choose the perturbations on $\mathscr{M}_{k+1}^{\mathrm{main}}(L,\beta)$ so that it is consistent with $\mathscr{M}_{k'+1}^{\mathrm{main}}(L,\beta)$ ($k' \neq k$) via the forgetful map. We can make this choice for the current toric situation by Lemma 11.2(3). In a more general situation, we need to regard $\mathscr{M}_1^{\mathrm{main}}(L,\beta)$ as a chain in the free loop space (see [F4]).

On the other hand, if we use a complex structure other than the standard one, we do not know if Lemma 11.2(3) holds. So in the proof of independence of Floer cohomology under the various choices made, there is trouble in using a bounding cochain b lying in $H^1(L;\Lambda_0)$. The idea, originally due to [Cho], is to change the leading-order term of $\mathfrak{x}$ by twisting the construction using *nonunitary* flat bundles on L.

Let X be a symplectic manifold, and let L be its relatively spin Lagrangian submanifold. Let $\rho : H_1(L;\mathbb{Z}) \to \mathbb{C} \setminus \{0\}$ be a representation, and let $\mathfrak{L}_\rho$ be the flat $\mathbb{C}$ bundle induced by ρ.

We replace the formula (11.9) by

$$\mathfrak{m}_k^{\rho,\mathrm{can}} = \sum_{\beta \in H_2(M,L)} \rho(\partial\beta)\ \mathfrak{m}_{k,\beta}^{\mathrm{can}} \otimes T^{\omega(\beta)/2\pi}.$$

(Compare this with (4.3).)

PROPOSITION 12.2
$(H(L(u);\Lambda_0^{\mathbb{R}}), \mathfrak{m}_k^{\rho,\mathrm{can}})$ *is a filtered* A_∞*-algebra.*

Proof
Suppose that $[f] \in \mathcal{M}_{k+1}^{\mathrm{main}}(L,\beta)$ is a fiber product of $[f_1] \in \mathcal{M}_{\ell+1}^{\mathrm{main}}(L,\beta_1)$ and $[f_2] \in \mathcal{M}_{k-\ell}^{\mathrm{main}}(L,\beta_2)$. In other words, $\beta_1 + \beta_2 = \beta$ and $\mathrm{ev}_0(f_2) = \mathrm{ev}_i(f_1)$ for some i. Then it is easy to see that

$$\rho(\partial\beta) = \rho(\partial\beta_1)\rho(\partial\beta_2). \tag{12.1}$$

Therefore, (12.1) and (11.8) imply the filtered A_∞-relation. □

The unitality can also be proved in the same way. The well-definedness (that is, the independence of various choices up to homotopy equivalence) can also be proved in the same way.

Remark 12.3
We have obtained our twisted filtered A_∞-structure on the (untwisted) cohomology group $H^*(L;\Lambda_0)$. This is because the flat bundle $\mathrm{Hom}(\mathfrak{L}_\rho,\mathfrak{L}_\rho)$ is trivial. In more general situations where we consider a flat bundle $\mathfrak{L}$ of higher rank, we obtain a filtered A_∞-structure on cohomology group with local coefficients with values in $\mathrm{Hom}(\mathfrak{L},\mathfrak{L})$.

The filtered A_∞-structure $\mathfrak{m}_k^{\rho,\mathrm{can}}$ is different from $\mathfrak{m}_k^{\mathrm{can}}$ in general, as we can see from the expression of the potential function given in Lemma 4.9.

In the rest of this section, we explain how Floer cohomology detects the Lagrangian intersection; namely, we sketch the proof of Theorem 3.11 in our case and its generalization to the case where we include the nonunital flat connection ρ.

Let $\psi_t : X \to X$ be a Hamiltonian isotopy with $\psi_0 =$ identity. We put $\psi_1 = \psi$. We consider the pair

$$L^{(0)} = L(u), \qquad L^{(1)} = \psi\big(L(u)\big)$$

such that $L^{(1)}$ is transversal to $L^{(0)}$. We then take a 1-parameter family $\{J_t\}_{t\in[0,1]}$ of compatible almost complex structures such that $J_0 = J$, which is the standard complex structure of X and $J_1 = \psi_*(J)$.

Let $p, q \in L^{(0)} \cap L^{(1)}$. We consider the homotopy class of maps

$$\varphi : \mathbb{R} \times [0,1] \to X \tag{12.2}$$

such that

(1) $\lim_{\tau\to-\infty} \varphi(\tau,t) = p$, $\lim_{\tau\to+\infty} \varphi(\tau,t) = q$;

(2) $\varphi(\tau,0) \in L^{(0)}$, $\varphi(\tau,1) \in L^{(1)}$.

We denote by $\pi_2(L^{(1)}, L^{(0)}; p, q)$ the set of all such homotopy classes. There are obvious maps

$$\begin{aligned}
&\pi_2(L^{(1)}, L^{(0)}; p, r) \times \pi_2(L^{(1)}, L^{(0)}; r, q) \to \pi_2(L^{(1)}, L^{(0)}; p, q),\\
&\pi_2(X; L^{(1)}) \times \pi_2(L^{(1)}, L^{(0)}; p, q) \to \pi_2(L^{(1)}, L^{(0)}; p, q),\\
&\pi_2(L^{(1)}, L^{(0)}; p, q) \times \pi_2(X; L^{(0)}) \to \pi_2(L^{(1)}, L^{(0)}; p, q).
\end{aligned} \tag{12.3}$$

We denote them by #.

Definition 12.4

We consider the moduli space of maps (12.2) satisfying (1), (2) above, in homotopy class $B \in \pi_2(L^{(1)}, L^{(0)}; p, q)$, and that satisfies the equation

$$\frac{\partial\varphi}{\partial\tau} + J_t\Big(\frac{\partial\varphi}{\partial t}\Big) = 0. \tag{12.4}$$

We denote it by $\widehat{\mathscr{M}}^{\mathrm{reg}}(L^{(1)}, L^{(0)}; p, q; B)$. We take its stable map compactification and denote it by $\widehat{\mathscr{M}}(L^{(1)}, L^{(0)}; p, q; B)$. We divide this space by the $\mathbb{R}$-action induced by the translation of τ direction to obtain $\mathscr{M}(L^{(1)}, L^{(0)}; p, q; B)$. We define evaluation maps $\mathrm{ev}_{L^{(i)}} : \widehat{\mathscr{M}}(L^{(1)}, L^{(0)}; p, q; B) \to L^{(i)}$ by $\mathrm{ev}_{L^{(i)}}(\varphi) = \varphi(0, i)$, for $i = 0, 1$.

LEMMA 12.5

The space $\mathscr{M}(L^{(1)}, L^{(0)}; p, q; B)$ has an oriented Kuranishi structure with corners. Its boundary is isomorphic to the union of the following three kinds of fiber products as spaces with Kuranishi structure:

(1)

$$\mathscr{M}(L^{(1)}, L^{(0)}; p, r; B') \times \mathscr{M}(L^{(1)}, L^{(0)}; r, q; B'').$$

Here $B' \in \pi_2(L^{(1)}, L^{(0)}; p, r)$, $B'' \in \pi_2(L^{(1)}, L^{(0)}; r, q)$.

(2)

$$\mathscr{M}_1(L(u);\beta') \;{}_{\mathrm{ev}_0}\times_{\mathrm{ev}_{L^{(1)}}}\; \widehat{\mathscr{M}}(L^{(1)}, L^{(0)}; p, q; B'').$$

Here $\beta' \in \pi_2(X; L^{(1)}) \cong \pi_2(X; L(u))$*,* $\beta' \# B'' = B$*. The fiber product is taken over* $L^{(1)} \cong L(u)$ *by using* $\mathrm{ev}_0 : \mathscr{M}(L(u);\beta') \to L(u)$ *and* $\mathrm{ev}_{L^{(1)}} : \widehat{\mathscr{M}}(L^{(1)}, L^{(0)}; p, q; B'') \to L^{(1)}$*.*

(3)

$$\widehat{\mathscr{M}}(L^{(1)}, L^{(0)}; p, q; B') \;{}_{\mathrm{ev}_{L^{(0)}}}\times_{\mathrm{ev}_0}\; \mathscr{M}\big(L(u);\beta''\big).$$

Here $\beta' \in \pi_2(X; L^{(1)}) \cong \pi_2(X; L(u))$*,* $B' \# \beta'' = B$*. The fiber product is taken over* $L^{(0)} \cong L(u)$ *by using* $\mathrm{ev}_0 : \mathscr{M}(L(u);\beta'') \to L(u)$ *and* $\mathrm{ev}_{L^{(0)}} : \widehat{\mathscr{M}}(L^{(1)}, L^{(0)}; p, q; B') \to L^{(0)}$*.*

We have

$$\dim \mathscr{M}(L^{(1)}, L^{(0)}; p, q; B) = \mu(B) - 1,$$

where

$$\mu(B_1 \# B_2) = \mu(B_1) + \mu(B_2),$$
$$\mu(\beta' \# B'') = \mu(\beta') + \mu(B''),$$
$$\mu(B' \# \beta'') = \mu(B') + \mu(\beta'').$$

Here the notation is as in items (1)*,* (2)*,* (3) *above.*

Lemma 12.5 is proved in [FOOO3, Section 7.1.4] and [FOOO2, Section 29.4].

LEMMA 12.6

There exists a system of multisections on $\mathscr{M}(L^{(1)}, L^{(0)}; p, q; B)$ *such that*

(1) *it is transversal to zero;*

(2) *it is compatible at the boundaries described in Lemma* 12.5 *(here we pull back the multisection of* $\mathscr{M}(L^{(1)}, L^{(0)}; p, q; B)$ *to one on* $\widehat{\mathscr{M}}(L^{(1)}, L^{(0)}; p, q; B)$ *and use the multisection in Lemma* 11.2 *on* $\mathscr{M}(L(u);\beta)$*).*

Proof

We can find such a system of multisections inductively over $\int_B \omega$ by using the facts that $\mathrm{ev}_0 : \mathscr{M}(L(u);\beta)^{\mathfrak{s}} \to L(u)$ is a submersion and Lemma 12.5(1) is a direct product. $\square$

In the case when $\mu(B) = 1$, we define

$$n(B) = \#\mathscr{M}(L^{(1)}, L^{(0)}; p, q; B)^{\mathfrak{s}} \in \mathbb{Q},$$

that is, the "number" of zeros of our multisection counted with sign and multiplicity. We use it to define Floer cohomology.

Let $CF(L^{(1)}, L^{(0)})$ be the free $\Lambda_{0,nov}$-module generated by $L^{(0)} \cap L^{(1)}$. We define boundary operator on it.

Let $\rho_i : \pi_1(L^{(i)}) \to \mathbb{C}^*$ be the representation. We take harmonic 1-form $h_i \in H^1(L^{(i)}; \mathbb{C})$ such that

$$\rho_i(\gamma) = \exp\Big(\int_\gamma h_i\Big).$$

Let $b_{i,+} \in H^1(L^{(0)}; \Lambda_+) \subset \mathscr{M}(L^{(0)})$. An element $B \in \pi_2(L^{(1)}, L^{(0)}; p, q)$ induces a path $\partial_i : [0, 1] \to L^{(i)}$ joining p to q in $L^{(i)}$ for $i = 0, 1$. We define

$$C\big(B; (h_0, h_1), (b_{0,+}, b_{1,+})\big) = \exp\Big(\int_{\partial_0 B} (h_0 + b_{0,+})\Big) \exp\Big(-\int_{\partial_1 B} (h_1 + b_{1,+})\Big).$$

This is an element of $\Lambda_0 \setminus \Lambda_+$. It is easy to see that

$$\begin{aligned} &C\big(B_1\#B_2; (h_0, h_1), (b_{0,+}, b_{1,+})\big) \\ &\quad = C\big(B_1; (h_0, h_1), (b_{0,+}, b_{1,+})\big)\, C\big(B_2; (h_0, h_1), (b_{0,+}, b_{1,+})\big), \end{aligned} \tag{12.5}$$

and

$$\begin{aligned} &C\big(\beta'\#B''; (h_0, h_1), (b_{0,+}, b_{1,+})\big) \\ &\quad = C\big(B''; (h_0, h_1), (b_{0,+}, b_{1,+})\big) \exp\Big(\int_{\partial\beta'} h_1 + b_{0,+}\Big). \end{aligned} \tag{12.6}$$

(Here B' and β'' are as in Lemma 12.5(2).) Now we define

$$\begin{aligned} &\langle \delta_{(h_0,h_1),(b_{0,+},b_{1,+})}(p), q \rangle \\ &\quad = \sum_{B \in \pi_2(L^{(1)}, L^{(0)}; p,q); \mu(B)=1} n(B) C\big(B; (h_0, h_1), (b_{0,+}, b_{1,+})\big). \end{aligned} \tag{12.7}$$

For the case $h_1 = \psi_*(h_0)$, $b_{1,+} = \psi_*(b_+)$, $b_{0,+} = b_+$, we write $C(B; h_0, b_+)$ and δ^{h_0,b_+}, in place of $C(B; (h_0, h_1), (b_{0,+}, b_{1,+}))$ and $\delta_{(h_0,h_1),(b_{0,+},b_{1,+})}$, respectively.

LEMMA 12.7
We have

$$\delta_{(h_0,h_1),(b_{0,+},b_{1,+})} \circ \delta_{(h_0,h_1),(b_{0,+},b_{1,+})} = \big(\mathfrak{PO}(b(1)) - \mathfrak{PO}(b(0))\big) \cdot \mathrm{id}$$

where $b(i) = h_i + b_{i,+} \in H^1(L(u); \Lambda_0)$.

Proof
Let $p, q \in L^{(1)} \cap L^{(0)}$. We consider $B \in \pi_2(L^{(1)}, L^{(0)}; p, q)$ with $\mu(B) = 2$. We consider the boundary of $\mathscr{M}(L^{(1)}, L^{(0)}; p, q; B)$. We put

$$\delta_B = \begin{cases} 1 & B \in \pi_2(L^{(1)}, L^{(0)}; p, q),\ p = q, \text{ and } B \text{ is the class of constant map,} \\ 0 & \text{otherwise.} \end{cases}$$

Then, using the classification of the boundary of $\mathscr{M}(L^{(1)}, L^{(0)}; p, q; B)$ in Lemma 12.5, we have the equality

$$0 = \sum_{r,B,B''} n(B')n(B'') + \sum_{\beta',B''} c_{\beta'}\delta_{B''} - \sum_{\beta'',B'} c_{\beta''}\delta_{B'}, \tag{12.8}$$

where the summention in the first, second, and third terms of the right-hand side is taken over the set described in items (1), (2), and (3) of Lemma 12.5, respectively, and where c_β is defined in Definition 11.6.

We multiply (12.8) by $C(B; (h_0, h_1), (b_{0,+}, b_{1,+}))$ and calculate the right-hand side by using Formulas (12.5) and (12.6) and Lemma 11.8. It is easy to see that the first term gives

$$\langle \delta_{(h_0,h_1),(h_{0,+},b_{1,+})} \circ \delta_{(h_0,h_1),(b_{0,+},b_{1,+})}(p), q\rangle.$$

The second term is 0 if $p \neq q$, and is

$$\sum_\beta c_\beta \exp\Big(\int_{\partial\beta} h_0 + b_{0,+}\Big) = \mathfrak{PO}\big(b(0)\big)$$

if $p = q$. The third term gives $\mathfrak{PO}(b(1)) \cdot \mathrm{id}$ in a similar way. □

Definition 12.8
Let $b(0) = h_0 + b_+$ and $b(1) = \psi_*(b(0))$. We define

$$HF\big((L^{(0)}, b(0)), (L^{(1)}, b(1)); \Lambda_0\big) \cong \frac{\mathrm{Ker}\ \delta_{h_0,h_+}}{\mathrm{Im}\ \delta_{h_0,h_+}}.$$

This is well defined by Lemma 12.7 and by the identity $\mathfrak{PO}\big(\psi_*(b(0))\big) = \mathfrak{PO}\big((b(0))\big)$.

Now we have the following.

LEMMA 12.9
We have

$$HF\big((L^{(0)}, b(0)), (L^{(1)}, b(1)); \Lambda_0\big) \otimes_{\Lambda_0} \Lambda \cong \frac{\operatorname{Ker} \mathfrak{m}_1^{\rho,\mathrm{can},b_+}}{\operatorname{Im} \mathfrak{m}_1^{\rho,\mathrm{can},b_+}} \otimes_{\Lambda_0} \Lambda.$$

Here $\rho(\gamma) = \exp\Big(\int_\gamma h_0\Big)$ *and*

$$\mathfrak{m}_1^{\rho,\mathrm{can},b_+}(x) = \sum_{k_1,k_2=0}^{\infty} \mathfrak{m}_{k_1+k_2+1}^{\rho,\mathrm{can}}(b_+^{\otimes k_1}, x, b_+^{\otimes k_2}).$$

Lemma 12.9 implies (the ρ twisted version of) Theorem 3.11 in our case. We omit the proof of Lemma 12.9 and refer the reader to [FOOO3, Section 5.3], [FOOO2, Section 22], or [FOOO5, Section 8].

13. Floer cohomology at a critical point of potential function

In this section, we prove Theorem 4.10 and so forth and complete the proof of Theorem 1.5.

Proof of Lemma 4.9
Let $\beta \in H_2(X, L(u_0))$ with $\mu(\beta) = 2$, and let $\mathscr{M}_1^{\mathrm{main}}(L(u_0), \beta)$ be nonempty. We have $\beta = \sum_{i=1}^m c_i \beta_i + \sum_j \alpha_j$ by Theorem 11.1(5). Let ρ be as in (4.15). We have $\rho(\partial\beta) = \prod \rho(\partial\beta_i)^{c_i}$. Note that $\partial\beta_i = \sum_j v_{i,j}\mathbf{e}_j^*$. Thus we have

$$\begin{cases} \rho(\partial\beta_i) = \mathfrak{y}_{1,0}^{v_{i,1}} \cdots \mathfrak{y}_{n,0}^{v_{i,n}}, \\ \rho(\partial\beta) = \displaystyle\prod_i \prod_j \mathfrak{y}_{j,0}^{c_i v_{i,j}}. \end{cases} \tag{13.1}$$

Therefore, for $b \in H^1(L(u_0); \Lambda_+^{\mathbb{C}})$, we have

$$\begin{aligned}\sum_{k=0}^{\infty} \mathfrak{m}_{k,\beta}^{\rho,\mathrm{can}}(b, \dots, b) &= \sum_{k=0}^{\infty} \mathfrak{y}_{1,0}^{v_{i,1}} \cdots \mathfrak{y}_{n,0}^{v_{i,n}} \mathfrak{m}_{k,\beta}^{\mathrm{can}}(b, \dots, b) \\ &= \sum_{k=0}^{\infty} e^{\mathfrak{x}_{1,0}v_{i,1}} \cdots e^{\mathfrak{x}_{n,0}v_{i,n}} \frac{c_\beta}{k!}(b \cap \partial\beta)^k \cdot [PD(L)] \\ &= \sum_{k=0}^{\infty} \mathfrak{m}_{k,\beta}^{\mathrm{can}}\Big(b + \sum_{j=1}^n \mathfrak{x}_{j,0}\mathbf{e}_j, \dots, b + \sum_{j=1}^n \mathfrak{x}_{j,0}\mathbf{e}_j\Big).\end{aligned}$$

On the other hand, it follows from Theorems 4.5 and 4.6 that the left and the right sides of this identity correspond to those in Lemma 4.9, respectively. This finishes the proof of Lemma 4.9. □

Proof of Theorem 4.10
Let $x^+ = (x_1^+, \dots, x_n^+)$, $x_1^+, \dots, x_n^+ \in \Lambda_+$. We put

$$x(x^+) = \sum_i (\mathfrak{x}_{i,0} + x_i^+)\mathbf{e}_i, \qquad b(x^+) = \sum_i x_i^+ \mathbf{e}_i.$$

From Lemma 4.9 we derive

$$\begin{aligned}\mathfrak{PO}_\rho^{u_0}\big(b(x^+)\big) &= \sum \mathfrak{m}_k^{\rho,\mathrm{can}}\big(b(x^+), \dots, b(x^+)\big) \cap [L(u_0)] \\ &= \sum \mathfrak{m}_k^{\mathrm{can}}\big(x(x^+), \dots, x(x^+)\big) \cap [L(u_0)] = \mathfrak{PO}^{u_0}\big(x(x^+)\big).\end{aligned}$$

Let $\mathfrak{x}$ be as in (4.13). Then we have

$$\frac{\partial}{\partial x_i^+}\mathfrak{PO}_\rho^{u_0}\big(b(x^+)\big)\Big|_{x(x^+)=\mathfrak{x}} = \frac{\partial}{\partial x_i^+}\mathfrak{PO}^{u_0}\big(x(x^+)\big)\Big|_{x(x^+)=\mathfrak{x}} = \frac{\partial \mathfrak{PO}^{u_0}}{\partial x_i}(\mathfrak{x}) = 0,$$

where the last equality follows from the assumption (4.12). (In the case when (4.11) is assumed, we have $\equiv 0 \mod T^{\mathcal{N}}$ in place of $= 0$.) On the other hand, we have

$$\begin{aligned}&\frac{\partial}{\partial x_i^+}\mathfrak{PO}_\rho^{u_0}\big(b(x^+)\big)\Big|_{x(x^+)=\mathfrak{x}} \\ &\quad = \sum_k \sum_\ell \mathfrak{m}_k^{\rho,\mathrm{can}}(b^{\otimes \ell}, \mathbf{e}_i, b^{\otimes(k-\ell-1)}) \cap [L(u_0)] \qquad (13.2) \\ &\quad = \mathfrak{m}_1^{\rho,\mathrm{can},b}(\mathbf{e}_i) \cap [L(u_0)].\end{aligned}$$

Note that here and hereafter we write b in place of $b(x^+)$ with $x(x^+) = \mathfrak{x}$; namely, $b = \mathfrak{x} - \sum \mathfrak{x}_{i,0}\mathbf{e}_i$.

Hence we obtain

$$\mathfrak{m}_1^{\rho,\mathrm{can},b}(\mathbf{e}_i) \begin{cases} = 0 & \text{if (4.12) is satisfied,} \\ \equiv 0 \mod T^{\mathcal{N}} & \text{if (4.11) is satisfied.} \end{cases} \qquad (13.3)$$

We also note that, by the degree reason, $\mathfrak{m}_1^{\rho,\mathrm{can},b}(\mathbf{e}_i)$ is proportional to $PD[L(u_0)]$.

We next prove the vanishing of $\mathfrak{m}_1^{\rho,\mathrm{can},b}(\mathbf{f})$ for the classes $\mathbf{f}$ of higher degree. We prove the following.

LEMMA 13.1
For $\mathbf{f} \in H^*(L(u_0); \Lambda_0^{\mathbb{C}})$, *we have*

$$\mathfrak{m}_{1,\beta}^{\rho,\mathrm{can},b}(\mathbf{f}) \begin{cases} = 0 & \text{if (4.12) is satisfied,} \\ \equiv 0 \mod T^{\mathcal{N}} & \text{if (4.11) is satisfied.} \end{cases}$$

Proof
Let $d = \deg \mathbf{f}$, and let $2\ell = \mu(\beta)$. We say $(d, \ell) < (d', \ell')$ if $\ell < \ell'$ or $\ell = \ell', d < d'$. We prove the lemma by an upward induction on (d, ℓ). The case $d = 1$ is (13.3). We note that $\mathfrak{m}_{k,\beta} = 0$ if $\mu(\beta) \le 0$.

We assume that the lemma is proved for (d', ℓ') smaller than (d, ℓ), and we prove the case of (d, ℓ). Since the case $d = 1$ is already proved, we may assume that $d \ge 2$. Let $\mathbf{f} = \mathbf{f}_1 \cup \mathbf{f}_2$, where $\deg \mathbf{f}_i \ge 1$. By the A_∞-relation, we have

$$\begin{aligned} \mathfrak{m}_{1,\beta}^{\rho,\mathrm{can},b}(\mathbf{f}_1 \cup \mathbf{f}_2) = & \sum_{\beta_1+\beta_2=\beta} \pm \mathfrak{m}_{2,\beta_1}^{\rho,\mathrm{can},b}\big(\mathfrak{m}_{1,\beta_2}^{\rho,\mathrm{can},b}(\mathbf{f}_1), \mathbf{f}_2\big) \\ & + \sum_{\beta_1+\beta_2=\beta} \pm \mathfrak{m}_{2,\beta_1}^{\rho,\mathrm{can},b}\big(\mathbf{f}_1, \mathfrak{m}_{1,\beta_2}(\mathbf{f}_2)\big) \\ & + \sum_{\beta_1+\beta_2=\beta, \beta_2 \ne 0} \pm \mathfrak{m}_{1,\beta_1}^{\rho,\mathrm{can},b}\big(\mathfrak{m}_{2,\beta_2}(\mathbf{f}_1, \mathbf{f}_2)\big). \end{aligned}$$

We remark that $\mathfrak{m}_{1,\beta_0}^{\rho,\mathrm{can},b} = 0$ since we are working on a canonical model.

The first two terms on the right-hand side vanish by the induction hypothesis since $\deg \mathbf{f}_i < \deg \mathbf{f}$ and $\mu(\beta_i) \le \mu(\beta)$. The third term also vanishes since $\mu(\beta_1) < \mu(\beta)$. The proof of Lemma 13.1 is complete. □

Lemma 13.1 immediately implies Theorem 4.10. □

Proof of Proposition 5.4
Let us specialize to the case of two dimensions. In case $\dim L(u_0) = 2$, we can prove $\mathfrak{m}_{1,\beta}^{\rho,\mathrm{can},b} = 0$ for $\mu(\beta) \ge 4$ also by dimension counting. We can use that to prove Proposition 5.4 in the same way as above. □

Proof of Proposition 4.12
Let $\omega_i, u_i, \mathfrak{x}_{i,\mathcal{N}}$ be as in Definition 4.11. We assume that $\psi : X \to X$ does not satisfy (4.19) or (4.20), and we deduce a contradiction. We use the same (time dependent) Hamiltonian as ψ to obtain $\psi_i : (X, \omega_i) \to (X, \omega_i)$. Take an integer $\mathcal{N}$ such that $\|\psi_i\| < 2\pi\mathcal{N}$ for large i. Then for sufficiently large i, $L(u_0^i)$ and ψ_i does not satisfy (4.19) or (4.20). In fact, if $\psi(L(u_0)) \cap L(u_0) = \emptyset$, then for sufficiently large i, we have $\psi_i(L(u_0^i)) \cap L(u_0^i) = \emptyset$. If $\psi(L(u_0))$ is transversal to $L(u_0)$ and if (4.20) is not

satisfied, then

$$\#\big(\psi(L(u_0)) \cap L(u_0)\big) \geq \#\big(\psi_{\varepsilon_i}(L(u_0^i)) \cap L(u_0^i)\big).$$

On the other hand, by Theorem 4.10 we have

$$HF\big((L(u_i), \mathfrak{x}_{i,k}), (L(u_i), \mathfrak{x}_{i,k}); \Lambda_0^{\mathbb{C}}/(T^{\mathcal{N}})\big) \cong H\big(T^n; \Lambda_0^{\mathbb{C}}/(T^{\mathcal{N}})\big).$$

It follows from the universal coefficient theorem that

$$HF\big((L(u_i), \mathfrak{x}_{i,k}), (L(u_i), \mathfrak{x}_{i,k}); \Lambda_0^{\mathbb{C}}\big) \cong \Lambda_0^{\oplus a} \oplus \bigoplus_{i=1}^{b} \Lambda_0/(T^{c(i)}) \tag{13.4}$$

such that $c(i) \geq \mathcal{N}$ and $a + 2b \geq 2^n$. This contradicts [FOOO3, Theorem J]. (In fact, [FOOO3, Theorem J], which is equivalent to Theorems (3.12) and (13.4), imply that (4.19) and (4.20) hold for $L(u_i)$ and ψ_i with $\|\psi_i\| < 2\pi\mathcal{N}$.) Proposition 4.12 is proved. □

Proof of Theorem 5.11
The proof is the same as the proof of Proposition 4.12 above. □

Now we are ready to complete the proof of Theorem 1.5.

Proof of Theorem 1.5
In the case where the vertices of P are contained in $\mathbb{Q}^n$, Proposition 4.7 and Theorem 4.10 imply that $L(u_0)$ is balanced in the sense of Definition 4.11. Therefore, Proposition 4.12 implies Theorem 1.5 in this case. If the leading-term equation is strongly nondegenerate, Theorem 1.5 also follows from Theorem 4.10, Theorem 10.4, and Proposition 4.12.

We finally present an argument to remove the rationality assumption. In view of Lemma 4.12, it suffices to prove the following.

PROPOSITION 13.2
In the situation of Theorem 1.5, there exists u_0 such that $L(u_0)$ is a balanced Lagrangian fiber.

Proof
Let $\pi : X \to P$ be as in Theorem 1.5. Let us consider s_k, S_k, and P_k as in Section 9. We obtain $u_0 \in P$ such that $\{u_0\} = P_K$. We prove that $L(u_0)$ is balanced.

We perturb the Kähler form ω of X a bit so that we obtain ω'. Let P' be the corresponding moment polytope, and let $s_k^{\omega'}$, $S_k^{\omega'}$, and $P_k^{\omega'}$ be the corresponding piecewise affine function, number, and subset of $P_{\omega'}$, respectively, obtained for ω', $P_{\omega'}$ as in Section 9.

PROPOSITION 13.3
We can choose ω_h so that ω_h is rational and $\lim_{h\to\infty} s_k^{\omega_h} = s_k$, $\lim_{h\to\infty} S_k^{\omega_h} = S_k$, $\lim_{h\to\infty} P_k^{\omega_h} = P_k$, $\dim P_k^{\omega_h} = \dim P_k$ for all k.

Proof
We write $I_k^{\omega'}$ for the set I_k defined in (9.7) for ω', P'. We prove Lemma 13.4. We observe that the set $\mathfrak{K}$ of T^n-invariant Kähler structure ω' is regarded as an open set of an affine space defined on $\mathbb{Q}$ (i.e., the Kähler cone). We may regard $\mathfrak{K}$ as a moduli space of moment polytope as follows. We consider a polyhedron P' each of whose edges is parallel to a corresponding edge of P. Translation defines an $\mathbb{R}^n$-action on the set of such P'. The quotient space can be identified with $\mathfrak{K}$.

LEMMA 13.4
There exists a subset $\mathfrak{P}_k$ of $\mathfrak{K}$ which is a nonempty open subset of an affine subspace defined over $\mathbb{Q}$ such that any element $\omega' \in \mathfrak{P}_k$ has the following properties:
(1) $\dim P_l^{\omega'} = \dim P_l^{\omega}$ *for $l \le k$;*
(2) $I_l^{\omega'} = I_l^{\omega}$ *for $l \le k$.*

Remark 13.5
In the case of Example 8.1, the set $P_k^{\omega'}$ and so forth jumps at the point $\alpha = 1/3$ in the Kähler cone. Hence the set $\mathfrak{P}_k$ may have a strictly smaller dimension than $\mathfrak{K}$.

Proof of Lemma 13.4
Let $A_l^{\omega'}$ be the affine space defined in Section 9. (We put ω' to specify the symplectic form.) We write ℓ_i^{ω}, $\ell_i^{\omega'}$ in place of ℓ_i to specify symplectic form and moment polytope. We remark that the linear part of ℓ_i^{ω} is equal to the linear part of $\ell_i^{\omega'}$.

The proof of Lemma 13.4 is given by induction on k. Let us first consider the case $k = 1$. We put

$$\widehat{A}_1^{\omega'} = \bigl\{u \in M_{\mathbb{R}} \bigm| \ell_{1,1}^{\omega'}(u) = \cdots = \ell_{1,a_1}^{\omega'}(u)\bigr\}.$$

We remark that $\{\ell_{1,1}^{\omega}, \ldots, \ell_{1,a_1}^{\omega}\} = I_1^{\omega}$ and so $\widehat{A}_1^{\omega} = A_1^{\omega}$.

We put

$$\mathfrak{P}_1' = \{\omega' \mid \dim \widehat{A}_1^{\omega'} = \dim A_1^{\omega}\}.$$

It is easy to see that $\mathfrak{P}_1'$ is a nonempty affine subset of $\mathfrak{K}$ and that it is defined over $\mathbb{Q}$.

SUBLEMMA 13.6
If $\omega' \in \mathfrak{P}'_1$ and is sufficiently close to ω, then $P_1^{\omega'}$ is an equidimensional polyhedron in $\widehat{A}_1^{\omega'}$. In particular, $\widehat{A}_1^{\omega'} = A_1^{\omega'}$.

Proof
The tangent space of $\widehat{A}_1^{\omega'}$ is parallel to the tangent space of A_1^{ω}. Therefore, $\ell_{1,j}^{\omega'}$ is constant on $\widehat{A}_1^{\omega'}$. We put

$$\widehat{S}_1^{\omega'} = \ell_{1,1}^{\omega'}(u)$$

for some $u \in \widehat{A}_1^{\omega'}$.

On the other hand, if $\ell_i^{\omega} \notin I_1^{\omega}$, then $\ell_i^{\omega}(u) > S_1^{\omega}$ on Int P_1^{ω}. Therefore, if ω' is sufficiently close to ω, we have $\ell_i^{\omega'}(u) > \widehat{S}_1^{\omega'}$ on a neighborhood of a compact subset of Int P_1^{ω}, which we identify with a subset of P'. This implies the sublemma. □

Conditions (1) and (2) of Lemma 13.4 in the case $k = 1$ follow from Sublemma 13.6 easily.

Let us assume that Lemma 13.4 is proved up to $k - 1$. We note that $\{\ell_{k,1}^{\omega}, \dots, \ell_{k,a_k}^{\omega}\} = I_k^{\omega}$. We put

$$\widehat{A}_k^{\omega'} = \left\{u \in A_{k-1}^{\omega'} \,\middle|\, \ell_{k,1}^{\omega'}(u) = \cdots = \ell_{k,a_k}^{\omega'}(u)\right\}$$

and

$$\mathfrak{P}'_k = \{\omega' \in \mathfrak{P}'_{k-1} \mid \dim \widehat{A}_k^{\omega'} = \dim A_k^{\omega}\}.$$

Then $\mathfrak{P}'_k$ is a nonempty affine subset of $\mathfrak{K}$ and is defined over $\mathbb{Q}$. We can show that a sufficiently small open neighborhood $\mathfrak{P}_k$ of ω in $\mathfrak{P}'_k$ has the required properties in the same way as the first step of the induction. The proof of Lemma 13.4 is complete. □

Proposition 13.3 follows immediately from Lemma 13.4. In fact, the set of rational points are dense in $\mathfrak{P}_K$. □

Proposition 13.2 follows from Proposition 13.3, Proposition 4.7, and Theorem 4.10. □

The proof of Theorem 1.5 is now complete. □

Proof of Proposition 10.8
The proof is similar to the proof of Proposition 13.3. Let I_k be as in (10.2). We write it as $I_k(P, u_0)$, where P is the moment polytope of (X, ω). We define $I_k(P', u'_0)$ as follows.

Let P' be a polytope which is a small perturbation of P and such that each of the faces are parallel to the corresponding face of P. Let $u'_0 \in \mathrm{Int}\ P'$. Let us consider the set $\mathfrak{K}^+$ of all such pairs (P', u'_0). It is an open set of an affine space defined over $\mathbb{Q}$. Each such P' is a moment polytope of certain Kähler form on X. (We remark that Kähler form on X determines P' only up to translation.)

For each P' we take the corresponding Kähler form on X, and it determines a potential function. Therefore, $I_k(P', u'_0)$ is determined by (10.2). We define $A_l^{\perp}(P', u'_0)$ in the same way as Definition 10.1.

LEMMA 13.7
There exists a subset $\mathfrak{Q}_k$ of $\mathfrak{K}^+$, which is a nonempty open set of an affine subspace defined over $\mathbb{Q}$. All the elements (P', u'_0) of $\mathfrak{Q}_k$ have the following properties:
(1) $\dim A_l^{\perp}(P', u'_0) = \dim A_l^{\perp}(P, u_0)$ *for* $l \le k$;
(2) $I_l(P', u'_0) = I_l(P, u_0)$ *for* $l \le k$.

The proof of Lemma 13.7 is the same as the proof of Lemma 13.4 and is omitted.

Now we take a sequence of rational points $(P_h, u_0^h) \in \mathfrak{Q}_k$ converging to (P, u_0). Lemma 13.7(2) implies that the leading-term equation at u_0^h is the same as the leading-term equation at u_0. The proof of Proposition 10.8 is complete. □

Proof of Lemma 10.5
Let $[\omega] \in H^2(X; \mathbb{Q})$. We may take the moment polytope P such that its vertices are rational. (This time we do not change P throughout the proof.) Let $u_0 \in \mathrm{Int}\, P$, and assume that $\mathfrak{PO}_0^{u_0}$ has a nondegenerate critical point in $(\Lambda_0 \setminus \Lambda_+)^n$.

We define $I_k(P, u)$ as above. In the same way as in the proof of Lemma 13.7, we can prove the following.

SUBLEMMA 13.8
The set $\mathscr{P}_k$ of all $u \in \mathrm{Int}\, P$ satisfying conditions (1) *and* (2) *below contains an open neighborhood u_0 in certain affine subspace A of $\mathbb{R}^n$. The affine space A is defined on $\mathbb{Q}$:*
(1) $\dim A_l^{\perp}(P, u) = \dim A_l^{\perp}(P, u_0)$ *for all* $l < k$;
(2) $I_l(P, u) = I_l(P, u_0)$ *for all* $l < k$.

We omit the proof. We take K such that $\{d\ell_i \mid \ell_i \in I_1(P, u_0) \cup \cdots \cup I_K(P, u_0)\}$ generates $N_{\mathbb{R}}$. (Note that $P \subset N_{\mathbb{R}} = M_{\mathbb{R}}^*$.) By Sublemma 13.8 there exists a sequence $\{u_i\}_{i=1,2,\ldots}$ of rational points u_i in $\mathscr{P}_K$ which converges to u_0.

Items (1) and (2) in Sublemma 13.8 imply that $\mathfrak{PO}_0^{u_0}$ and $\mathfrak{PO}_0^{u_i}$ have the same leading-term equation. Therefore by assumption, $\mathfrak{PO}_0^{u_i}$ has a critical point on $(\Lambda_0 \setminus \Lambda_+)^n$. Since $\mathrm{Jac}(\mathfrak{PO}_0; \Lambda)$ is finite-dimensional, it follows that we may take a subsequence u_{k_i} such that $u_{k_1} = u_{k_2} = \cdots$. Hence $u_0 = u_{k_i}$ is rational as required. □

Remark 13.9
We can replace Definition 4.11(3) by

$$HF\big((L(u_i), \mathfrak{x}_{i,\mathcal{N}}), (L(u_i), \mathfrak{x}_{i,\mathcal{N}}); \Lambda^{\mathbb{C}}/(T^{\mathcal{N}})\big) \supseteq \Lambda^{\mathbb{C}}/(T^{\mathcal{N}}).$$

In fact, the following three conditions are equivalent to one another:
(1) $HF\big((L(u), \mathfrak{x}), (L(u), \mathfrak{x}); \Lambda^{\mathbb{C}}/(T^{\mathcal{N}})\big) \cong H(T^n; \mathbb{C}) \otimes \Lambda^{\mathbb{C}}/(T^{\mathcal{N}})$;
(2) $HF\big((L(u), \mathfrak{x}), (L(u), \mathfrak{x}); \Lambda^{\mathbb{C}}/(T^{\mathcal{N}})\big) \supseteq \Lambda^{\mathbb{C}}/(T^{\mathcal{N}})$;
(3) $\frac{\partial \mathfrak{PO}^u}{\partial y_k} \equiv 0, \quad \bmod T^{\mathcal{N}} \qquad k = 1, \ldots, n$, at $\mathfrak{x}$.
Here (1) $\Rightarrow$ (2) is obvious; (3) $\Rightarrow$ (1) is Theorem 4.10. Let us prove (2) $\Rightarrow$ (3). Suppose that (3) does not hold. We put $\frac{\partial \mathfrak{PO}^u}{\partial y_k} \equiv cT^{\lambda} \bmod T^{\lambda}\Lambda_+$, where $c \in \mathbb{C} \setminus \{0\}$ and $0 \le \lambda < N$. Then (13.2) implies that $T^{\mathcal{N}-\lambda} PD[L(u)] = 0$ in $HF\big((L(u), \mathfrak{x}), (L(u), \mathfrak{x}); \Lambda^{\mathbb{C}}/(T^{\mathcal{N}})\big)$. Since $PD[L(u)]$ is a unit, item (2) does not hold.

Remark 13.10
The proof of Theorem 4.10 (or of Lemma 13.1) does *not* imply that

$$\mathfrak{m}_{k,\beta}(\rho_1, \ldots, \rho_k) = 0 \tag{13.5}$$

for $\mu(\beta) \ge 4$. So it does not imply that the numbers c_β (Definition 11.6) determine the filtered A_∞-algebra $\big(H(L(u); \Lambda_0), \mathfrak{m}\big)$ up to homotopy equivalence. However, we believe that this is indeed the case. In fact, the homology group $H(\mathscr{L}(T^n); \mathbb{Q})$ of the free loop space $\mathscr{L}(T^n)$ is trivial for degree $> n$. On the other hand, $\dim \mathscr{M}_1^{\mathrm{main}}(L(u_0); \beta) = n + \mu(\beta) - 2$. Hence if $\mu(\beta) \ge 4$, there is no nonzero homology class on the corresponding degree in the free loop space. Using the argument of [F4], it may imply that the contribution of those classes to the homotopy type of filtered A_∞-structure is automatically determined from the contribution of the classes of Maslov index 2.

On the other hand, the pseudoholomorphic disc with Maslov index ≥ 4 certainly contributes to the operator $\mathfrak{q}_{\ell,k,\beta}$ introduced in [FOOO3, Section 3.8] and [FOOO2, Section 13]: namely, $\mathfrak{q}_{\ell,k,\beta}$ is the operator that involves a cohomology class of the ambient symplectic manifold X (see Remark 6.15). It seems that tropical geometry plays a role in this calculation.

Appendices

A. Algebraic closedness of Novikov fields

LEMMA A.1

If F is an algebraically closed field of characteristic zero, then Λ^F is algebraically closed.

Proof

Let $\sum_{k=0}^{n} a_k x^k = 0$ be a polynomial equation with Λ^F-coefficients. We prove that it has a solution in Λ^F by induction on n. We may assume that $a_n = 1$. Replacing x by $x - \frac{a_{n-1}}{n}$, we may assume that $a_{n-1} = 0$. (Here we use the fact that the characteristic of F is zero.) We may assume furthermore that $a_0 \neq 0$, since otherwise zero is a solution. We put

$$c = \inf_{k=0,\dots,n-2} \frac{\mathfrak{v}_T(a_k)}{n-k}.$$

We put $x = T^c y$, $b_k = T^{c(k-n)} a_k$. Then our equation is equivalent to $P(y) = \sum_{k=0}^{n} b_k y^k = 0$. We observe that $b_n = 1$, $b_{n-1} = 0$, $b_0 \neq 0$. Moreover,

$$\mathfrak{v}_T(b_k) = c(k-n) + \mathfrak{v}_T(a_k) \geq 0.$$

Specifically, $b_k \in \Lambda_0$. We define $\overline{b}_k \in F$ to be the zero-order term of b_k (i.e., to satisfy $b_k \equiv \overline{b}_k \mod \Lambda_+$). We consider the equation $\overline{P}(\overline{y}) = \sum_{k=0}^{n} \overline{b}_k \overline{y}^k = 0$. By our choice of c there exists $k < n-1$ such that $\overline{b}_k \neq 0$ and $\overline{b}_n = 1$, $\overline{b}_{n-1} = 0$. Therefore, $\overline{P}$ has at least two distinct roots. (We use the fact that the characteristic of F is zero here.) Since F is algebraically closed, we can decompose $\overline{P} = \overline{QR}$, where $\overline{Q}$ and $\overline{R}$ are monic, of nonzero degree, and coprime. Therefore, by Hensel's lemma, there exists $Q, R \in \Lambda_0[y]$ such that $P = QR$ and $\deg Q = \deg \overline{Q}$, $Q \equiv \overline{Q} \mod \Lambda_+$, $R \equiv \overline{R} \mod \Lambda_+$.*

Since the degree of Q is smaller than the degree of P, it follows from induction hypothesis that Q has a root in Λ^F. The proof of Lemma A.1 is now complete. □

By a similar argument we can show that if F has characteristic zero, then a finite algebraic extension of Λ^F is contained in Λ^K for some finite extension K of F (we used this fact in Section 7).

*A proof of Hensel's lemma, in the case when valuation is not necessarily discrete, is given, for example, in [BGR, page 144]. See also the proof of Lemma 8.5 given here.

Proof of Lemma 8.5
In view of the proof of Lemma A.1, it suffices to show that $\Lambda_0^{\mathrm{conv}}$ is Henselian (namely, Hensel's lemma holds for it). Let

$$P(X) = \sum_{i=0}^{n-1} a_i X^i + X^n \in \Lambda_0^{\mathrm{conv}}[X].$$

We assume that its $\mathbb{C}$-reduction $\overline{P} \in \mathbb{C}[X]$ is decomposed as $\overline{P} = \overline{QR}$, where $\overline{Q}$ and $\overline{R}$ are monic and coprime. We put

$$a_i = a_{i,0} + a_{i,+}, \quad a_{i,0} \in \mathbb{C}, \quad a_{i,+} \in \Lambda_+^{\mathrm{conv}},$$

and

$$\widetilde{P}(X) = \sum_{i=0}^{n-1} (a_{i,0} + Z_i) X^i + X^n \in \mathbb{C}[Z_0, \ldots, Z_{n-1}][X],$$

where Z_i are formal variables.

The *convergent* power series ring $\mathbb{C}\{Z_0, \ldots, Z_{n-1}\}$ is Henselian (see [N, Section 45]). Therefore, there exist monic polynomials $\widetilde{Q}, \widetilde{R} \in \mathbb{C}\{Z_0, \ldots, Z_{n-1}\}[X]$ such that

$$\widetilde{Q}\widetilde{R} = \widetilde{P}$$

and the $\mathbb{C}$-reduction of $\widetilde{Q}, \widetilde{R}$ are $\overline{Q}, \overline{R}$, respectively. On the other hand, it is easy to see that $Z_i \mapsto a_{i,+}$ induces a continuous ring homomorphism $\mathbb{C}\{Z_0, \ldots, Z_{n-1}\} \to \Lambda_0^{\mathrm{conv}}$. Thus $\widetilde{Q}, \widetilde{R}$ induce $Q, R \in \Lambda_0^{\mathrm{conv}}[X]$ such that $QR = P$. Hence $\Lambda_0^{\mathrm{conv}}$ is Henselian, as required. □

B. T^n-equivariant Kuranishi structure

In this section, we define the notion of T^n-equivariant Kuranishi structure and prove that our moduli space $\mathcal{M}_{k+1}^{\mathrm{main}}(\beta)$ has one. We also show the existence of T^n-equivariant perturbation of the Kuranishi map.

Let $\mathcal{M}$ be a compact space with Kuranishi structure. The space $\mathcal{M}$ is covered by a finite number of Kuranishi charts $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$, $\alpha \in \mathfrak{A}$ which satisfy the following.

Condition B.1

(1) The space V_α is a smooth manifold (with boundaries or corners) and Γ_α is a finite group acting effectively on V_α.
(2) The map $\mathrm{pr}_\alpha : E_\alpha \to V_\alpha$ is a finite-dimensional vector bundle on which Γ_α acts so that pr_α is Γ_α- equivariant.
(3) The map s_α is a Γ_α-equivariant section of E_α.

(4) The map $\psi_\alpha : s_\alpha^{-1}(0)/\Gamma_\alpha \to \mathscr{M}$ is a homeomorphism to its image.
(5) The union of $\psi_\alpha(s_\alpha^{-1}(0)/\Gamma_\alpha)$ for various α is $\mathscr{M}$.

We assume that $\{(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha) \mid \alpha \in \mathfrak{A}\}$ is a good coordinate system, in the sense of [FO, Definition 6.1] or [FOOO3, Lemma A1.11] and [FOOO2, Lemma A1.11]. This means the following. The set $\mathfrak{A}$ has a partial order $<$, where either $\alpha_1 \le \alpha_2$ or $\alpha_2 \le \alpha_1$ holds for $\alpha_1, \alpha_2 \in \mathfrak{A}$ if

$$\psi_{\alpha_1}\big(s_{\alpha_1}^{-1}(0)/\Gamma_{\alpha_1}\big) \cap \psi_{\alpha_2}\big(s_{\alpha_2}^{-1}(0)/\Gamma_{\alpha_2}\big) \neq \emptyset.$$

Let $\alpha_1, \alpha_2 \in \mathfrak{A}$, and let $\alpha_1 \le \alpha_2$. Then there exists a Γ_{α_1}-invariant open subset $V_{\alpha_2,\alpha_1} \subset V_{\alpha_1}$, a smooth embedding $\varphi_{\alpha_2,\alpha_1} : V_{\alpha_2,\alpha_1} \to V_{\alpha_2}$, and a bundle map $\widehat{\varphi}_{\alpha_2,\alpha_1} : E_{\alpha_1}|_{V_{\alpha_2,\alpha_1}} \to E_{\alpha_2}$, which covers $\varphi_{\alpha_2,\alpha_1}$. Moreover, there exists an injective homomorphism $\widehat{\widehat{\varphi}}_{\alpha_2,\alpha_1} : \Gamma_{\alpha_1} \to \Gamma_{\alpha_2}$. We require that they satisfy the following.

Condition B.2

(1) The maps $\varphi_{\alpha_2,\alpha_1}, \widehat{\varphi}_{\alpha_2,\alpha_1}$ are $\widehat{\widehat{\varphi}}_{\alpha_2,\alpha_1}$-equivariant.
(2) The maps $\varphi_{\alpha_2,\alpha_1}$ and $\widehat{\widehat{\varphi}}_{\alpha_2,\alpha_1}$ induce an embedding of orbifold

$$\overline{\varphi}_{\alpha_2,\alpha_1} : \frac{V_{\alpha_2,\alpha_1}}{\Gamma_{\alpha_1}} \to \frac{V_{\alpha_2}}{\Gamma_{\alpha_2}}. \tag{B.1}$$

(3) We have $s_{\alpha_2} \circ \varphi_{\alpha_2,\alpha_1} = \widehat{\varphi}_{\alpha_2,\alpha_1} \circ s_{\alpha_1}$.
(4) We have $\psi_{\alpha_2} \circ \overline{\varphi}_{\alpha_2,\alpha_1} = \psi_{\alpha_1}$ on $\frac{V_{\alpha_2,\alpha_1} \cap s_{\alpha_1}^{-1}(0)}{\Gamma_{\alpha_1}}$.
(5) If $\alpha_1 < \alpha_2 < \alpha_3$, then $\varphi_{\alpha_3,\alpha_2} \circ \varphi_{\alpha_2,\alpha_1} = \varphi_{\alpha_3,\alpha_1}$, on $\varphi_{\alpha_2,\alpha_1}^{-1}(V_{\alpha_3,\alpha_2})$. Also $\widehat{\varphi}_{\alpha_3,\alpha_2} \circ \widehat{\varphi}_{\alpha_2,\alpha_1} = \widehat{\varphi}_{\alpha_3,\alpha_1}$ and

$$\widehat{\widehat{\varphi}}_{\alpha_3,\alpha_2} \circ \widehat{\widehat{\varphi}}_{\alpha_2,\alpha_1} = \widehat{\widehat{\varphi}}_{\alpha_3,\alpha_1},$$

hold in the similar sense.
(6) Also, $V_{\alpha_2,\alpha_1}/\Gamma_{\alpha_1}$ contains $\psi_{\alpha_1}^{-1}\big(\psi_{\alpha_1}(s_{\alpha_1}^{-1}(0)/\Gamma_{\alpha_1}) \cap \psi_{\alpha_2}(s_{\alpha_2}^{-1}(0)/\Gamma_{\alpha_2})\big)$.

Condition B.3

The space with Kuranishi structure $\mathscr{M}$ has a tangent bundle, that is, the differential of s_{α_2} in the direction of the normal bundle induces a bundle isomorphism

$$ds_{\alpha_2} : \frac{\varphi_{\alpha_2,\alpha_1}^{*} T V_{\alpha_2}}{T V_{\alpha_2,\alpha_1}} \to \frac{\widehat{\varphi}_{\alpha_2,\alpha_1}^{*} E_{\alpha_2}}{E_{\alpha_1}}.$$

We say that $\mathscr{M}$ is *oriented* if V_α, E_α is oriented, the Γ_α-action is orientation-preserving, and ds_α is orientation-preserving.

Definition B.4
Suppose that $\mathscr{M}$ has a T^n-action. We say our Kuranishi structure on $\mathscr{M}$ is T^n-*equivariant in the strong sense* if the following conditions hold:
(1) V_α has a T^n-action which commutes with the given Γ_α-action;
(2) E_α is a T^n-equivariant bundle;
(3) the Kuranishi map s_α is T^n-equivariant and ψ_α is a T^n-equivariant map;
(4) the coordinate changes $\varphi_{\alpha_2,\alpha_1}$ and $\varphi_{\alpha_2,\alpha_1}$ are T^n-equivariant.

Remark B.5
We note that Condition (1) above is more restrictive than the condition that the orbifold V_α / Γ_α has a T^n-action. This is the reason why we use the phrase *in the strong sense* in the above definition; hereafter, we say T^n-equivariant instead for simplicity.

Let L be a smooth manifold. A strongly continuous smooth map $\mathrm{ev} : \mathscr{M} \to L$ is a family of Γ_α-invariant smooth maps

$$\mathrm{ev}_\alpha : V_\alpha \to L \tag{B.2}$$

which induce $\overline{\mathrm{ev}}_\alpha : V_\alpha / \Gamma_\alpha \to L$ such that $\overline{\mathrm{ev}}_{\alpha_2} \circ \overline{\varphi}_{\alpha_2,\alpha_1} = \overline{\mathrm{ev}}_{\alpha_1}$ on $V_{\alpha_2,\alpha_1} / \Gamma_\alpha$. (Note that Γ_α-action on L is trivial.)

We say that ev is *weakly submersive* if each of ev_α in (B.2) is a submersion.

Definition B.6
Assume that there exist T^n-actions on L and on $\mathscr{M}$. We say that $\mathrm{ev} : \mathscr{M} \to L$ is T^n-equivariant if ev_α in (B.2) is T^n-equivariant.

Now we show the following.

PROPOSITION B.7
The moduli space $\mathscr{M}_{k+1}(\beta)$ has a T^n-equivariant Kuranishi structure such that ev_0 : $\mathscr{M}_{k+1}(\beta) \to L$ *is a T^n-equivariant strongly continuous weakly submersive map.*

Proof
Except for the T^n-equivariance, this is proved in [FOOO3, Section 7.1] and [FOOO2, Section 29]. Below we explain how we choose our Kuranishi structure so that it is T^n-equivariant. We also include the case of the moduli space $\mathscr{M}_{k+1,\ell}(\beta)$ with interior marked points.

We first review the construction of the Kuranishi neighborhood from [FOOO3, Section 7.1] and [FOOO2, Section 29].

Let $\mathbf{x} = ((\Sigma, \vec{z}), w) \in \mathscr{M}_{k+1,\ell}(\beta)$. Let Σ_a be an irreducible component of Σ. We consider the operator

$$D_{w,a}\overline{\partial} : W^{1,p}\big(\Sigma_a; w^*(TX); L, \vec{z}_a\big) \to W^{0,p}\big(\Sigma_a; w^*(TX) \otimes \Lambda^{0,1}\big). \tag{B.3}$$

Here $W^{1,p}(\Sigma_a; w^*(TX); L, \vec{z}_a)$ is the space of section v of $w^*(TX)$ of $W^{1,p}$ class with the following properties:

(1) The restriction of v to $\partial\Sigma_a$ is in $w^*(TL)$.

(2) Also, $\vec{z}_a$ is the set of Σ which is either singular or marked. We assume that v is zero at those points.

Here $\Lambda^{0,1}$ is the bundle of (0, 1)-forms on Σ_a, and $W^{0,p}(\Sigma_a; w^*(TX) \otimes \Lambda^{0,1})$ is the set of sections of the $W^{0,p}$ class of $w^*(TX) \otimes \Lambda^{0,1}$. Also, $D_{w,a}\overline{\partial}$ is the linearization of (nonlinear) Cauchy-Riemann equations (see [Fo]). The operator (B.3) is Fredholm by the ellipticity thereof.

We choose open subsets W_a of Σ_a whose closure is disjoint from the boundary of each of the irreducible component Σ_a of Σ and from the singular points and marked points. By the unique continuation theorem, we can choose a finite-dimensional subset $E_{0,a}$ of $C_0^\infty(W_a; w^*TX)$ (the set of smooth sections of compact support in W_a) such that

$$\mathrm{Im} D_{w,a}\overline{\partial} + E_{0,a} = W^{0,p}\big(\Sigma_a; w^*(TX) \otimes \Lambda^{0,1}\big).$$

When $\mathbf{x}$ has nontrivial automorphisms, we choose $\bigoplus_a E_{0,a}$ to be invariant under the automorphisms.

We next associate a finite-dimensional subspace $E_{0,a}((\Sigma, \vec{z}), w')$ for w' which is "C^0 close to w." We need some care to handle the case where some component $(\Sigma_a, \vec{z}_a)$ is not stable, that is, the case for which the automorphism group G_a of $(\Sigma_a, \vec{z}_a)$ is of positive dimension. (Note that G_a is different from the automorphism group of $((\Sigma_a, \vec{z}_a), w|_{\Sigma_a})$. The latter group is finite.) We explain this choice of $E_{0,1}$ below following [FO, Appendix].

For each unstable component Σ_a, pick points $z_{a,i} \in \Sigma$ $(i = 1, \ldots, d_a)$ with the following properties.

Condition B.8

We have the following:

(1) w is immersed at $z_{a,i}$;

(2) $z_{a,i}$ in the interior of Σ_a, $z_{a,i} \neq z_{a,j}$ for $i \neq j$ and $z_{a,i} \notin \vec{z}$;

(3) $(\Sigma_a; (\vec{z} \cap \Sigma_a) \cup (z_{a,1}, \ldots, z_{a,d_a}))$ is stable;

(4) let $\Gamma = \Gamma(\mathbf{x})$ be the group of automorphisms of $\mathbf{x} = ((\Sigma, \vec{z}), w)$, so if $\gamma \in \Gamma$, $\gamma(\Sigma_a) = \Sigma_{a'}$, then $d_a = d_{a'}$ and $\{\gamma(z_{a,i}) \mid i = 1, \ldots, d_a\} = \{z_{a',i'} \mid i' = 1, \ldots, d_a\}$.

For each a, i, we choose a submanifold $N_{a,i}$ of codimension 2 in X that transversely intersects with (Σ_a, w) at $w_a(z_{a,i})$. In relation to Condition B.8(4), we choose $N_{a,i} = N_{a',i'}$ if $\gamma(z_{a,i}) = z_{a',i'}$.

We add extra interior marked points $\{z_{a,i} \mid a, i\}$ in addition to $\vec{z}$ on $(\Sigma, \vec{z})$, and obtain a stable curve $(\Sigma, \vec{z}^+)$ (namely, $\vec{z}^+ = \vec{z} \sqcup \{z_{a,i} \mid a, i\}$). (We choose an order of the added marked points and fix it.)

We consider a neighborhood $\mathfrak{U}_0$ of $[\Sigma, \vec{z}^+]$ in $\mathscr{M}^{\text{main}}_{k+1,\ell'}$ (i.e., the moduli space of bordered stable curve of genus zero with $k+1$ boundary and ℓ' interior marked points). Let Γ_0 be the group of automorphisms of $((\Sigma, \vec{z}^+), w)$. Now both Γ and Γ_0 are finite groups and $\Gamma \supseteq \Gamma_0$.

An element $\gamma \in \Gamma$ induces an automorphism $\gamma : \Sigma \to \Sigma$, which fixes marked points in $\vec{z}$ and permutes $\ell - \ell'$ marked points $\{z_{a,i} \mid a, i\}$ by Condition B.8(4). Therefore, we obtain an element of $[\gamma_*(\Sigma, \vec{z}^+)]$ that is different from $[\Sigma, \vec{z}^+]$ only by the reordering of marked points. We take the union of neighborhoods of $[\gamma_*(\Sigma, \vec{z}^+)]$ for various $\gamma \in \Gamma$ in $\mathscr{M}^{\text{main}}_{k+1,\ell'}$ and denote it by $\mathfrak{U}$.

The space $\mathfrak{U}_0$ is written as $\mathfrak{V}_0/\Gamma_0$, where $\mathfrak{V}_0$ is a manifold. Moreover, there exists a manifold $\mathfrak{V}$ on which Γ acts such that $\mathfrak{V}/\Gamma_0 = \mathfrak{U}$, $\mathfrak{V}/\Gamma \cong \mathfrak{U}_0$.

Furthermore, there is a "universal family" $\mathfrak{M} \to \mathfrak{V}$, where the fiber $\Sigma(\mathbf{v})$ of $\mathbf{v} \in \mathfrak{V}$ is identified with the bordered marked stable curve that represents the element $[\mathbf{v}] \in \mathfrak{V}/\Gamma \subset \mathscr{M}^{\text{main}}_{k+1,\ell'}$. There is a Γ-action on $\mathfrak{M}$ such that $\mathfrak{M} \to \mathfrak{V}$ is Γ-equivariant.

By construction, each member $\Sigma(\mathbf{v})$ of our universal family is diffeomorphic to Σ away from singularity. More precisely, we have the following.

Let $\Sigma_0 = \Sigma \setminus S$, where S is a small neighborhood of the union of the singular point sets and the marked point sets. Then for each $\mathbf{v}$ there exists a smooth embedding $i_{\mathbf{v}} : \Sigma_0 \to \Sigma(\mathbf{v})$. The error of this embedding for becoming a biholomorphic map goes to zero as $\mathbf{v}$ goes to zero. We may assume that $\mathbf{v} \mapsto i_{\mathbf{v}}$ is Γ-invariant in an obvious sense and that $i_{\mathbf{v}}$ depends smoothly on $\mathbf{v}$.

We may choose W_a so that $W_a \subset \Sigma_0$ for each a. Moreover, we assume that $\bigoplus_a E_{0,a}$ is invariant under the Γ-action in the following sense : if $\gamma \in \Gamma$ and $\Sigma_{a'} = \gamma(\Sigma_a)$, then the isomorphism induced by γ sends $E_{0,a}$ to $E_{0,a'}$.

Now we consider a pair $(w', \mathbf{v})$ where

$$w' : \big(\Sigma(\mathbf{v}), \partial\Sigma(\mathbf{v})\big) \to (X, L).$$

We assume the following.

Condition B.9

There exists $\epsilon > 0$ depending only on $\mathbf{x}$ such that the following items hold:

(1) $\sup_{x\in\Sigma_0} \operatorname{dist}\big(w'(i_{\mathbf{v}}(x)), w(x)\big) \le \epsilon$;

(2) for any connected component D_c of $\Sigma(\mathbf{v}) \setminus \operatorname{Im}(i_{\mathbf{v}})$, the diameter of $w'(D_c)$ in X (with a fixed Riemannian metric on it) is smaller than ϵ.

For each point $x \in W_a$, we use the parallel transport to make the identification

$$T_{w(x)}X \otimes \Lambda^{0,1}_x(\Sigma) \equiv T_{w'(i_{\mathbf{v}}(x))}X \otimes \Lambda^{0,1}_{i_{\mathbf{v}}(x)}\big(\Sigma(\mathbf{v})\big).$$

Using this identification, we obtain an embedding

$$I_{(\mathbf{v},w')} : \bigoplus_a E_{0,a} \longrightarrow w'^*(TX) \otimes \Lambda^{0,1}\big(\Sigma(\mathbf{v})\big).$$

Via this embedding $I_{(\mathbf{v},w')}$, we consider the equation

$$\overline{\partial} w' \equiv 0 \mod \bigoplus_a I_{(\mathbf{v},w')}(E_{0,a}) \tag{B.4}$$

together with the additional conditions

$$w'(z_{a,i}) \in N_{a,i}. \tag{B.5}$$

Let $V(\mathbf{x})$ be the set of solutions of (B.4) and (B.5). This is a smooth manifold (with boundary and corners) by the implicit function theorem and a gluing argument (see [FOOO3, Section A1.4] and [FOOO2, Section A1.4] for the smoothness at boundary and corner). Since we can make all the construction above invariant under the $\Gamma(\mathbf{x})$-action, the space $V(\mathbf{x})$ has a $\Gamma(\mathbf{x})$-action. (Note that we may choose $N_{a,i}$ so that $\{N_{a,i} \mid a, i\}$ is invariant under the action of $\Gamma(\mathbf{x})$.)

The obstruction bundle E is the space $\bigoplus_a E_{0,a}$ at $\mathbf{x}$ and $\bigoplus_a I_{(\mathbf{v},w')}(E_{0,a})$ at $(\mathbf{v}, w')$.

We omit the construction of coordinate changes (see [FOOO3, Section 7.1] and [FOOO2, Section 29], which in turn is similar to [FO, Section 15]).

The Kuranishi map is given by

$$\big((\Sigma', \vec{z}'), w'\big) \mapsto \overline{\partial} w' \in \bigoplus_a E_{0,a}. \tag{B.6}$$

We have thus finished our review of the construction of Kuranishi charts.

Now we assume that X has a T^n-action, which preserves both the complex and the symplectic structures on X. We also assume that L is a T^n-orbit.

We want to construct a family of vector spaces $(\mathbf{v}, w') \mapsto \bigoplus_a I_{(\mathbf{v},w')}(E_{0,a})$ so that it is invariant under the T^n-action. We need to slightly modify the above construction for this purpose. In fact, it is not totally obvious to make Condition (B.5) T^n-invariant.

For this purpose, we proceed in the following way. We fix a point $p_0 \in L$ and consider an element

$$\mathbf{x} \in \mathrm{ev}_0^{-1}(p_0) \cap \mathscr{M}_{k+1,\ell}(\beta).$$

We are going to construct a T^n-equivariant Kuranishi neighborhood of the T^n-orbit of $\mathbf{x}$. Let $\mathbf{v} \in \mathfrak{V}$ and $w' : (\Sigma(\mathbf{v}), \partial\Sigma(\mathbf{v})) \to (X, L)$ be as before. We replace Condition B.9 by the following.

Condition B.10
Let z_0 be the zeroth boundary marked point, and let $g \in T^n$ be the unique element satisfying $w'(z_0) = g(p_0)$. There exists $\epsilon > 0$ depending only on $\mathbf{x}$ such that
(1) $\sup_{x\in\Sigma_0} \operatorname{dist}\big(w'(i_{\mathbf{v}}(x)), g(w(x))\big) \le \epsilon$;
(2) for any connected component D_c of $\Sigma(\mathbf{v}) \setminus \operatorname{Im}(i_{\mathbf{v}})$, the diameter of $w'(D_c)$ in X (with a fixed Riemannian metric on it) is smaller than ϵ.

Now we define an embedding

$$I_{(\mathbf{v},w')} : \bigoplus_a E_{0,a} \longrightarrow w'^*(TX) \otimes \Lambda^{0,1}\big(\Sigma(\mathbf{v})\big)$$

as follows. We first use the g-action to define an isomorphism

$$g_* : T_{w(x)}X \otimes \Lambda^{0,1}_x(\Sigma) \cong T_{g(w(x))}X \otimes \Lambda^{0,1}_x(\Sigma).$$

Then we use the parallel transport in the same way as before to define

$$\bigoplus_a g_*(E_{0,a}) \longrightarrow w'^*(TX) \otimes \Lambda^{0,1}\big(\Sigma(\mathbf{v})\big).$$

By composing the two we obtain the embedding $I_{(\mathbf{v},w')}$. Now we consider the equation

$$\overline{\partial} w' \equiv 0 \mod \bigoplus_a I_{(\mathbf{v},w')}(E_{0,a}), \tag{B.7}$$

together with the additional conditions

$$w'(z_{a,i}) \in g(N_{a,i}) \tag{B.8}$$

as before. Clearly, these equations are T^n-invariant. It is also $\Gamma(\mathbf{x})$-invariant.

Note that the automorphism group $\Gamma(\mathbf{x})$ of $\mathbf{x}$ of our Kuranishi structure, which is a finite group, acts on the source while T^n acts on the target. Therefore, it is obvious that two actions commute.

By definition the obstruction bundle has a T^n-action. Moreover, (B.6) is T^n-equivariant. It is fairly obvious from the construction that coordinate changes of the constructed Kuranishi structure is also T^n-equivariant.

The proof of Proposition B.7 is now complete. □

Remark B.11

(1) From the above construction, it is easy to see that our T^n-equivariant Kuranishi structure is compatible with the gluing at the boundary marked points. Under the embedding

$$\mathcal{M}_{k_1+1}(\beta_1)_{\mathrm{ev}_0} \times_{\mathrm{ev}_i} \mathcal{M}_{k_2+1}(\beta_2) \subset \mathcal{M}_{k_1+k_2}(\beta_1 + \beta_2)$$

the restriction of the Kuranishi structure of the right-hand side coincides with the fiber product of the Kuranishi structure of the left-hand side. More precisely, we can construct the system of our Kuranishi structures so that this statement holds: this Kuranishi structure is constructed inductively over the number of disc components and the energy of β.

(2) On the other hand, when we construct the T^n-invariant Kuranishi structure in the way we described above, it may not be compatible with the gluing at the interior marked point. By the embedding

$$\mathcal{M}_1(\alpha) \times_X \mathcal{M}_{k+1,1}(\beta) \subset \mathcal{M}_{k+1}(\beta + \alpha) \tag{B.9}$$

the restriction of the Kuranishi structure of the right-hand side may not coincide with the fiber product of the Kuranishi structure of the left-hand side. This compatibility is not used in this article and hence is not required (see Remark 11.4, where a similar point is discussed for the choice of multisections).

However, contrary to the choice of multisections mentioned in Remark 11.4, we note that it is possible to construct T^n-equivariant Kuranishi structure compatible with (B.9). Since we do not use this point in the article, we do not elaborate it here.

We next review the multisections (see [FO, Section 3]). Let $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$ be a Kuranishi chart of $\mathcal{M}$. For $x \in V_\alpha$, we consider the fiber $E_{\alpha,x}$ of the bundle E_α at x. We take its l copies and consider the direct product $E^l_{\alpha,x}$. We take the quotient thereof by the action of symmetric group of order $l!$, and let $\mathcal{S}^l(E_{\alpha,x})$ be the quotient space. There exists a map $tm_m : \mathcal{S}^l(E_{\alpha,x}) \to \mathcal{S}^{lm}(E_{\alpha,x})$, which sends $[a_1, \dots, a_l]$ to $[\underbrace{a_1, \dots, a_1}_{m \text{ copies}}, \dots, \underbrace{a_l, \dots, a_l}_{m \text{ copies}}]$. A smooth *multisection* s of the bundle $E_\alpha \to V_\alpha$ consists of an open covering $\bigcup_i U_i = V_\alpha$ and s_i which maps $x \in U_i$ to $s_i(x) \in \mathcal{S}^{l_i}(E_{\alpha,x})$. They are required to have the following properties.

Condition B.12

(1) The open set U_i is Γ_α-invariant, and the map s_i is Γ_α-equivariant. (We note that there exists an obvious map $\gamma : \mathcal{S}^{l_i}(E_{\alpha,x}) \to \mathcal{S}^{l_i}(E_{\alpha,\gamma x})$ for each $\gamma \in \Gamma_\alpha$.)

(2) If $x \in U_i \cap U_j$, then we have

$$tm_{l_j}\big(s_i(x)\big) = tm_{l_i}\big(s_j(x)\big) \in \mathscr{S}^{l_i l_j}(E_{\alpha,\gamma x}).$$

(3) Also, s_i is *liftable and smooth* in the following sense. For each x there exists a smooth section $\tilde{s}_i$ of $\underbrace{E_\alpha \oplus \cdots \oplus E_\alpha}_{l_i \text{ times}}$ in a neighborhood of x such that

$$\tilde{s}_i(y) = \big(s_{i,1}(y), \dots, s_{i,l_i}(y)\big), \quad s_i(y) = [s_{i,1}(y), \dots, s_{i,l_i}(y)]. \tag{B.10}$$

We identify two multisections $(\{U_i\}, \{s_i\}, \{l_i\})$, $(\{U_i'\}, \{s_i'\}, \{l_i'\})$ if

$$tm_{l_i}\big(s_i(x)\big) = tm_{l_j'}\big(s_j'(x)\big) \in \mathscr{S}^{l_i l_j'}(E_{\alpha,\gamma x})$$

on $U_i \cap U_j'$. We say that $s_{i,j}$ is a *branch* of s_i in the situation of (B.10).

We next prove the following lemma, which we use in Section 11.

LEMMA B.13

Let $\mathscr{M}$ have a T^n-action and a T^n-equivariant Kuranishi structure. Suppose that the T^n-action on each of the Kuranishi neighborhood is free. Then we can descend the Kuranishi structure to $\mathscr{M}/T^n$ in a canonical way.

Proof

Let $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$ be a Kuranishi chart. Since the T^n-action on V_α is free, V_α/T^n is a smooth manifold and $E_\alpha/T^n \to V_\alpha/T^n$ is a vector bundle. Since the Γ_α-action commutes with the T^n-action, it follows that it acts on this vector bundle. The T^n-equivariance of s_α implies that we have a section $\overline{s}_\alpha$ of $E_\alpha/T^n \to V_\alpha/T^n$. Moreover, since $\psi_\alpha : s_\alpha^{-1}(0) \to \mathscr{M}$ is T^n-equivariant, it induces a map $\overline{s}_\alpha^{-1}(0) \to \mathscr{M}/T^n$. Thus we obtain a Kuranishi chart. It is easy to define the coordinate changes. □

Definition B.14

In the situation of Lemma B.13, we say that a system of multisections of the Kuranishi structure of $\mathscr{M}$ *is T^n-equivariant* if it is induced from the multisection of the Kuranishi structure on $\mathscr{M}/T^n$ in an obvious way.

COROLLARY B.15

In the situation of Lemma B.13, we assume that we have a T^n-equivariant multisection at the boundary of $\mathscr{M}$, which is transversal to zero. Then it extends to a T^n-equivariant multisection of $\mathscr{M}$.

This is an immediate consequence of [FO, Lemma 3.14], that is, the nonequivariant version.

C. Smooth correspondence via the zero set of multisection

In this section, we explain the way we use the zero set of multisection to define smooth correspondence, when appropriate submersive properties are satisfied.

Such a construction is a special case of the techniques of using a continuous family of multisections and integration along the fiber on their zero sets so that a smooth correspondence by the space with Kuranishi structure induces a map between the de Rham complex.

This (more general) technique is not new and is known to some experts. In fact, [R] and [F3, Section 16] use a similar technique and [FOOO3, Section 7.5], [FOOO2, Section 33], [F4], and [F5] contain the details of this more general technique.

We explain the special case (namely, the case in which we use a single multisection) in our situation of toric manifolds for the sake of completeness and the reader's convenience.

Let $\mathscr{M}$ be a space with Kuranishi structure and $\mathrm{ev}_s : \mathscr{M} \to L_s$, $\mathrm{ev}_t : \mathscr{M} \to L_t$ be strongly continuous smooth maps (here s and t stand for source and target, respectively). We assume that our smooth manifolds L_s, L_t are compact and oriented without boundary. We also assume that $\mathscr{M}$ has a tangent bundle and is oriented in the sense of Kuranishi structure.

Suppose that L_t has a free and transitive T^n-action and that L_s and $\mathscr{M}$ have T^n-action. We assume that the Kuranishi structure on $\mathscr{M}$ is T^n-equivariant and that the maps ev_s, ev_t are T^n-equivariant.

Let $\{(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)\}$ be a T^n-equivariant Kuranishi coordinate system (good coordinate system) of $\mathscr{M}$. We use Corollary B.15 to find a T^n-equivariant (system) of multisection $\mathfrak{s}_\alpha : V_\alpha \to E_\alpha$ that is transversal to zero.

Let θ_α be a smooth differential form of compact support on V_α. We assume that θ_α is Γ_α-invariant. Let $f_\alpha : V_\alpha \to L_s$ be a Γ_α-equivariant submersion. (The Γ_α-action on L_s is trivial.) We next define *integration along the fiber*:

$$\big((V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha), \mathfrak{s}_\alpha, f_\alpha\big)_*(\theta_\alpha) \in \Omega^{\deg\theta_\alpha + \dim L_t - \dim \mathscr{M}}(L_t).$$

We first fix α. Let $(U_i, \mathfrak{s}_{\alpha,i})$ be a representative of $\mathfrak{s}_\alpha$ (namely, $\{U_i \mid i \in I\}$ is an open covering of V_α, and $\mathfrak{s}_\alpha$ is represented by $\mathfrak{s}_{\alpha,i}$ on U_i). By the definition of the multisection, U_i is Γ_α-invariant. We may shrink U_i, if necessary, so that there exists a lifting $\tilde{\mathfrak{s}}_{\alpha,i} = (\tilde{\mathfrak{s}}_{\alpha,i,1}, \dots, \tilde{\mathfrak{s}}_{\alpha,i,l_i})$ as in (B.10).

Let $\{\chi_i \mid i \in I\}$ be a partition of unity subordinate to the covering $\{U_i \mid i \in I\}$. By replacing χ_i with its average over Γ_α, we may assume that χ_i is Γ_α-invariant.

We put

$$\tilde{\mathfrak{s}}_{\alpha,i,j}^{-1}(0) = \left\{x \in U_i \,\middle|\, \tilde{\mathfrak{s}}_{\alpha,i,j}(x) = 0\right\}. \tag{C.1}$$

By assumption, $\tilde{\mathfrak{s}}_{\alpha,i,j}^{-1}(0)$ is a smooth manifold. Since the T^n-action on L_t is free and transitive it follows that

$$\mathrm{ev}_{t,\alpha}|_{\tilde{\mathfrak{s}}_{\alpha,i,j}^{-1}(0)} : \tilde{\mathfrak{s}}_{\alpha,i,j}^{-1}(0) \to L_t \tag{C.2}$$

is a submersion.

Definition C.1
We define a differential form on L_t by

$$\begin{aligned} &\big((V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha), \mathfrak{s}_\alpha, \mathrm{ev}_{t,\alpha}\big)_*(\theta_\alpha) \\ &= \frac{1}{\#\Gamma_\alpha} \sum_{i=1}^{I} \sum_{j=1}^{l_i} \frac{1}{l_i} (\mathrm{ev}_{t,\alpha})_!\big(\chi_i \theta_\alpha|_{\tilde{\mathfrak{s}}_{\alpha,i,j}^{-1}(0)}\big). \end{aligned} \tag{C.3}$$

Here $(\mathrm{ev}_{t,\alpha})_!$ is the integration along the fiber of the smooth submersion (C.2).

LEMMA C.2
The right-hand side of (C.3) depends only on $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$, $\mathfrak{s}_\alpha$, $\mathrm{ev}_{t,\alpha}$, *and* θ_α *but is independent of*

(1) *the choice of representatives* $(\{U_i\}, \mathfrak{s}_{\alpha,i})$ *of* $\mathfrak{s}_\alpha$*;*

(2) *the partition of unity* χ_i.

Proof
The proof is straightforward generalization of the proof of well-definedness of integration on the manifold, which can be found in any textbook on manifold theory and is left to the reader. □

So far we have been working on one Kuranishi chart $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$. We next describe the compatibility conditions of multisections for various α. During the construction, we need to shrink V_α slightly several times; we do not explicitly mention this point henceforth.

Let $\alpha_1 < \alpha_2$. For $\alpha_1 < \alpha_2$, we take the normal bundle $N_{V_{\alpha_2,\alpha_1}} V_{\alpha_2}$ of $\varphi_{\alpha_2,\alpha_1}(V_{\alpha_2,\alpha_1})$ in V_{α_2}. We use an appropriate Γ_{α_2}-invariant Riemannian metric on V_{α_2} to define the exponential map

$$\mathrm{Exp}_{\alpha_2,\alpha_1} : B_\varepsilon N_{V_{\alpha_2,\alpha_1}} V_{\alpha_2} \to V_2. \tag{C.4}$$

(Here $B_\varepsilon N_{V_{\alpha_2,\alpha_1}} V_{\alpha_2}$ is the ϵ-neighborhood of the zero section of $N_{V_{\alpha_2,\alpha_1}} V_{\alpha_2}$.)

Using $\mathrm{Exp}_{\alpha_2,\alpha_1}$, we identify $B_\varepsilon N_{V_{\alpha_2,\alpha_1}} V_{\alpha_2}/\Gamma_{\alpha_1}$ to an open subset of $V_{\alpha_1}/\Gamma_{\alpha_1}$ and denote it by $U_\epsilon(V_{\alpha_2,\alpha_1}/\Gamma_{\alpha_1})$.

Using the projection

$$\mathrm{Pr}_{V_{\alpha_2,\alpha_1}} : U_\epsilon(V_{\alpha_2,\alpha_1}/\Gamma_{\alpha_1}) \to V_{\alpha_2,\alpha_1}/\Gamma_{\alpha_1}$$

we extend the orbibundle E_{α_1} to $U_\epsilon(V_{\alpha_2,\alpha_1}/\Gamma_{\alpha_1})$. Also we extend the embedding $E_{\alpha_1} \to \widehat{\varphi}^*_{\alpha_2,\alpha_1} E_{\alpha_2}$, (which is induced by $\widehat{\varphi}_{\alpha_2,\alpha_1}$) to $U_\epsilon(V_{\alpha_2,\alpha_1}/\Gamma_{\alpha_1})$.

We fix a Γ_α-invariant inner product of the bundles E_α. We then have a bundle isomorphism

$$E_{\alpha_2} \cong E_{\alpha_1} \oplus \frac{\widehat{\varphi}^*_{\alpha_2,\alpha_1} E_{\alpha_2}}{E_{\alpha_1}} \tag{C.5}$$

on $U_\varepsilon(V_{\alpha_2,\alpha_1}/\,\Gamma_{\alpha_1})$. We can use Condition B.3 to modify $\mathrm{Exp}_{\alpha_2,\alpha_1}$ in (C.4) so that the following is satisfied.

Condition C.3
If $y = \mathrm{Exp}_{\alpha_2,\alpha_1}(\tilde{y}) \in U_\epsilon(V_{\alpha_2,\alpha_1}/\Gamma_{\alpha_1})$, then

$$ds_{\alpha_2}(\tilde{y} \mod TV_{\alpha_1}) \equiv s_{\alpha_2}(y) \mod E_{\alpha_1}. \tag{C.6}$$

Let us explain the notation of (C.6). We note that $\tilde{y} \in T_{\varphi_{\alpha_2,\alpha_1}(x)} V_{\alpha_2}$ for $x = \mathrm{Pr}(\tilde{y}) \in V_{\alpha_2,\alpha_1}$. Hence

$$\tilde{y} \mod TV_{\alpha_1} \in \frac{T_{\varphi_{\alpha_2,\alpha_1}(x)} V_{\alpha_2}}{T_x V_{\alpha_1}}.$$

Therefore,

$$ds_{\alpha_2}(\tilde{y} \mod TV_{\alpha_1}) \in \frac{(E_{\alpha_2})_{\varphi_{\alpha_2,\alpha_1}(x)}}{(E_{\alpha_1})_x}.$$

Note that (C.6) claims that it coincides with s_{α_2} modulo $(E_{\alpha_1})_x$.

We note that Condition B.3 implies that

$$[\tilde{y}] \mapsto ds_{\alpha_2}(\tilde{y} \mod TV_{\alpha_1}) : \frac{T_{\varphi_{\alpha_2,\alpha_1}(x)} V_{\alpha_2}}{T_x V_{\alpha_1}} \longrightarrow \frac{(E_{\alpha_2})_{\varphi_{\alpha_2,\alpha_1}(x)}}{(E_{\alpha_1})_x}$$

is an isomorphism. Therefore, we can use the implicit function theorem to modify $\mathrm{Exp}_{\alpha_2,\alpha_1}$ so that Condition C.3 holds.

Definition C.4
A multisection $\mathfrak{s}_{\alpha_2}$ of V_{α_2} is said to be *compatible* with $\mathfrak{s}_{\alpha_1}$ if the following holds for each $y = \mathrm{Exp}_{\alpha_2,\alpha_1}(\tilde{y}) \in U_\epsilon(V_{\alpha_2,\alpha_1}/\Gamma_{\alpha_1})$:

$$\mathfrak{s}_{\alpha_2}(y) = \mathfrak{s}_{\alpha_1}\big(\mathrm{Pr}(\tilde{y})\big) \oplus ds_{\alpha_2}(\tilde{y} \mod TV_{\alpha_1}). \tag{C.7}$$

We note that $\mathfrak{s}_{\alpha_1}(w, \mathrm{Pr}(\tilde{y}))$ is a multisection of $\pi^*_{\alpha_1} E_{\alpha_1}$ and $ds_{\alpha_2}(\tilde{y} \mod TV_{\alpha_1})$ is a (single-valued) section. Therefore, via the isomorphism (C.5), the right-hand side of (C.7) defines an element of $\mathscr{S}^{l_i}(E_{\alpha_2})_x$ ($x = \mathrm{Pr}(\tilde{y})$), and hence it is regarded as a multisection of $\pi^*_{\alpha_2} E_{\alpha_2}$. In other words, we omit $\widehat{\varphi}_{\alpha_2,\alpha_1}$ in (C.7).

We next choose a partition of unity χ_α subordinate to our Kuranishi charts. To define the notion of partition of unity, we need some notation. Let $\mathrm{Pr}_{\alpha_2,\alpha_1} : N_{V_{\alpha_2,\alpha_1}} V_{\alpha_2} \to V_{\alpha_2,\alpha_1}$ be the projection. We fix a Γ_{α_1}-invariant positive definite metric of $N_{V_{\alpha_2,\alpha_1}} V_{\alpha_2}$, and we let $r_{\alpha_2,\alpha_1} : N_{V_{\alpha_2,\alpha_1}} V_{\alpha_2} \to [0, \infty)$ be the norm with respect to this metric. We fix a sufficiently small δ, and we let $\chi^\delta : \mathbb{R} \to [0, 1]$ be a smooth function such that

$$\chi^\delta(t) = \begin{cases} 0 & t \geq \delta, \\ 1 & t \leq \delta/2. \end{cases}$$

Let $U_\delta(V_{\alpha_2,\alpha_1}/\Gamma_{\alpha_1})$ be the image of the exponential map. In other words,

$$U_\delta(V_{\alpha_2,\alpha_1}/\Gamma_{\alpha_1}) = \big\{\mathrm{Exp}(v) \,\big|\, v \in N_{V_{\alpha_2,\alpha_1}} V_{\alpha_2}/\Gamma_{\alpha_1} \,\big|\, r_{\alpha_2,\alpha_1}(v) < \delta\big\}.$$

We push out our function r_{α_2,α_1} to $U_\delta(V_{\alpha_2,\alpha_1}/\Gamma_{\alpha_1})$ and denote it by the same symbol. We call r_{α_2,α_1} a *tubular distance function*. We require r_{α_2,α_1} to satisfy the compatibility conditions for various tubular neighborhoods and tubular distance functions, which are formulated in [Ma, Sections 5, 6].

Let $x \in V_\alpha$. We put

$$\mathfrak{A}_{x,+} = \{\alpha_+ \mid x \in V_{\alpha_+,\alpha},\ \alpha_+ > \alpha\},$$
$$\mathfrak{A}_{x,-} = \big\{\alpha_- \,\big|\, [x \mod \Gamma_\alpha] \in U_\delta(V_{\alpha,\alpha_-}/\Gamma_{\alpha_-}),\ \alpha_- < \alpha\big\}.$$

For $\alpha_- \in \mathfrak{A}_{x,-}$, we take x_{α_-} such that $\mathrm{Exp}(x_{\alpha_-}) = x$.

Definition C.5
A system $\{\chi_\alpha \mid \alpha \in \mathfrak{A}\}$ of Γ_α-equivariant smooth functions $\chi_\alpha : V_\alpha \to [0, 1]$ of compact support is said to be a partition of unity subordinate to our Kuranishi chart if

$$\chi_\alpha(x) + \sum_{\alpha_- \in \mathfrak{A}_{x,-}} \chi^\delta\big(r_{\alpha,\alpha_-}(x_{\alpha_-})\big)\chi_{\alpha_-}\big(\mathrm{Pr}_{\alpha,\alpha_-}(x_{\alpha_-})\big) + \sum_{\alpha_+ \in \mathfrak{A}_{x,+}} \chi_{\alpha_+}\big(\varphi_{\alpha_+,\alpha}(x)\big) = 1.$$

LEMMA C.6
There exists a partition of unity subordinate to our Kuranishi chart. We may choose them so that they are T^n-equivariant.

Proof
We may assume that $\mathfrak{A}$ is a finite set since $\mathscr{M}$ is compact. By shrinking V_α if necessary, we may assume that there exists V_α^- such that V_α^- is a relatively compact subset of V_α and that E_α, $\varphi_{\alpha_2,\alpha_1}$, s_α, and so forth restricted to V_α^- still defines a good coordinate system. We take a Γ_α-invariant smooth function χ'_α on V_α, which has compact support and satisfies $\chi'_\alpha = 1$ on V_α^-. We define

$$h_\alpha(x) = \chi'_\alpha(x) + \sum_{\alpha_- \in \mathfrak{A}_{x,-}} \chi^\delta\big(r_{\alpha,\alpha_-}(x_{\alpha_-})\big)\chi'_{\alpha_-}\big(\mathrm{Pr}_{\alpha,\alpha_-}(x_{\alpha_-})\big) + \sum_{\alpha_+ \in \mathfrak{A}_{x,+}} \chi'_{\alpha_+}\big(\varphi_{\alpha_+,\alpha}(x)\big).$$

Using compatibility of tubular neighborhoods and tubular distance functions, we can show that h_α is Γ_α-invariant and that

$$h_{\alpha_2}\big(\varphi_{\alpha_2,\alpha_1}(x)\big) = h_{\alpha_1}(x)$$

if $x \in V_{\alpha_2,\alpha_1}$. Therefore,

$$\chi_\alpha(x) = \chi'_\alpha(x)/h_\alpha(x)$$

has the required properties. □

Now we consider the situation we start with; namely, we have two strongly continuous T^n-equivariant smooth maps

$$\mathrm{ev}_s : \mathscr{M} \to L_s, \qquad \mathrm{ev}_t : \mathscr{M} \to L_t$$

and ev_t is weakly submersive. (In fact, T^n-action on L_t is transitive and free.)

Let a differential form h on L_s be given. We choose a T^n-equivariant good coordinate system $\{(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)\}$ of $\mathscr{M}$ and a T^n-equivariant multisection represented by $\{\mathfrak{s}_\alpha\}$ in this Kuranishi chart. Assume that the multisection is transversal to zero.

We also choose a partition of unity $\{\chi_\alpha\}$ subordinate to our Kuranishi chart. Then we put

$$\theta_\alpha = \chi_\alpha(\mathrm{ev}_{s,\alpha})^* h, \tag{C.8}$$

which is a differential form on V_α.

Definition C.7
Define

$$(\mathscr{M}; \mathrm{ev}_s, \mathrm{ev}_t)_*(h) = \sum_\alpha \big((V_\alpha, \Gamma_\alpha, E_\alpha, \psi_\alpha, s_\alpha), \mathfrak{s}_\alpha, \mathrm{ev}_{t,\alpha}\big)_*(\theta_\alpha). \tag{C.9}$$

This is a smooth differential form on L_t. It is T^n-equivariant if h is T^n-equivariant.

Remark C.8
(1) Actually, the right-hand side of (C.9) *depends* on the choice of $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$, $\mathfrak{s}_\alpha$. We write $\mathfrak{s}$ to demonstrate this choice and write $(\mathscr{M}; \mathrm{ev}_s, \mathrm{ev}_t, \mathfrak{s})_*(h)$.
(2) The right-hand side of (C.9) is independent of the choice of partition of unity. The proof is similar to the well-definedness of integration on manifolds.

In case $\mathscr{M}$ has a boundary $\partial\mathscr{M}$, the choices $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$, $\mathfrak{s}_\alpha$ on $\mathscr{M}$ induce one for $\partial\mathscr{M}$. We then have the following.

LEMMA C.9 (Stokes's theorem)
We have

$$d\big((\mathscr{M}; \mathrm{ev}_s, \mathrm{ev}_t, \mathfrak{s})_*(h)\big) = (\mathscr{M}; \mathrm{ev}_s, \mathrm{ev}_t, \mathfrak{s})_*(dh) + (\partial\mathscr{M}; \mathrm{ev}_s, \mathrm{ev}_t, \mathfrak{s})_*(h). \tag{C.10}$$

We discuss the sign at the end of this section.

Proof
Using the partition of unity χ_α it suffices to consider the case when $\mathscr{M}$ has only one Kuranishi chart V_α. We use the open covering U_i of V_α and the partition of unity again to see that we need only to study on one U_i. In that case, (C.10) is immediate from the usual Stokes's formula. □

We next discuss composition of smooth correspondences. We consider the following situation. Let

$$\mathrm{ev}_{s;st} : \mathscr{M}_{st} \to L_s, \qquad \mathrm{ev}_{t;st} : \mathscr{M}_{st} \to L_t$$

be as before such that T^n-action on L_t is free and transitive. Let

$$\mathrm{ev}_{r;rs} : \mathscr{M}_{rs} \to L_r, \qquad \mathrm{ev}_{s;rs} : \mathscr{M}_{rs} \to L_s$$

be a similar diagram such that T^n on L_s is free and transitive. Using the fact that $\mathrm{ev}_{s;rs}$ is weakly submersive, we define the fiber product

$$\mathscr{M}_{rs\ \mathrm{ev}_{s;rs}} \times_{\mathrm{ev}_{s;st}} \mathscr{M}_{st}$$

as a space with Kuranishi structure. We write it as $\mathscr{M}_{rt}$. We have a diagram of strongly continuous smooth maps

$$\mathrm{ev}_{r;rt} : \mathscr{M}_{rt} \to L_r, \qquad \mathrm{ev}_{t;rt} : \mathscr{M}_{rt} \to L_t.$$

We next make choices $\mathfrak{s}^{st}$, $\mathfrak{s}^{rs}$ for $\mathscr{M}_{st}$ and $\mathscr{M}_{rs}$. It is easy to see that it determines a choice $\mathfrak{s}^{rt}$ for $\mathscr{M}_{rt}$.

Now we have the following.

LEMMA C.10 (Composition formula)
We have the following formula for each differential form h on L_r.

$$\begin{aligned} &(\mathscr{M}_{rt}; \mathrm{ev}_{r;rt}, \mathrm{ev}_{t;rt}, \mathfrak{s}^{rt})_*(h) \\ &= \big((\mathscr{M}_{st}; \mathrm{ev}_{s;st}, \mathrm{ev}_{t;st}, \mathfrak{s}^{st})_* \circ (\mathscr{M}_{rs}; \mathrm{ev}_{r;rs}, \mathrm{ev}_{s;rs}, \mathfrak{s}^{rs})_*\big)(h). \end{aligned} \tag{C.11}$$

Proof
Using a partition of unity it suffices to study locally on $\mathscr{M}_{rs}$, $\mathscr{M}_{st}$. In that case it suffices to consider the case of usual manifold, which is well known. □

Finally, we discuss the signs in Lemmas C.9 and C.10. It is rather cumbersome to fix appropriate sign conventions and show those lemmas with signs. So, instead, we use the trick of [FOOO3, Section 8.10.3] and [FOOO2, Section 53.3] (see also [F5, Section 13]) to reduce the orientation problem to the case already discussed in [FOOO3, Chapter 8] and [FOOO2, Chapter 9], as follows.

The correspondence $h \mapsto (\mathscr{M}; \mathrm{ev}_s, \mathrm{ev}_t, \mathfrak{s})_*(h)$ extends to the currents h that satisfy appropriate transversality properties about its wave-front set (see [Hm]). We can also represent the smooth form h by an appropriate average (with respect to certain smooth measure) of a family of currents realized by smooth singular chains. So, as far as sign concerns, it suffices to consider a current realized by a smooth singular chain. Then the right-hand side of (C.3) turns out to be a current realized by a smooth singular chain which is obtained from a smooth singular chain on L_s by a transversal smooth correspondence. In fact, we may assume that all the fiber products appearing here are transversal, since it suffices to discuss the sign in the generic case. Thus the problem reduces to finding a sign convention (and orientation) for the correspondence of the singular chains by a smooth manifold. In the situation of our application, such sign convention (singular homology version) was determined and analyzed in detail in [FOOO3, Chapter 8] and [FOOO2, Chapter 9]. Especially, existence of an appropriate orientation that is consistent with the sign appearing in A_∞ formulas and so forth was proved there. Therefore, we can prove that there is a sign (orientation) convention which induces all the formulas we need with sign, in our de Rham version, as well (see [FOOO3, Section 8.10.3] and [FOOO2, Section 53.3] or [F5, Section 13] for the details).

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References

[Ab] M. ABOUZAID, *Homogeneous coordinate rings and mirror symmetry for toric varieties*, Geom. Topol. **10** (2006), 1097 – 1157. MR 2240909

[Au] M. AUDIN, *Torus Actions on Symplectic Manifolds*, 2nd ed., Progr. Math. **93**, Birkhäuser, Basel, 2004. MR 2091310

[Ar] D. AUROUX, *Mirror symmetry and T-duality in the complement of an anticanonical divisor*, J. Gökova Geom. Topol. GGT **1** (2007), 51 – 91. MR 2386535

[AKO] D. AUROUX, L. KATZARKOV, and D. ORLOV, *Mirror symmetry for weighted projective planes and their noncommutative deformations*, Ann. of Math. (2) **167** (2008), 867 – 943. MR 2415388

[B1] V. V. BATYREV, "Quantum cohomology rings of toric manifolds" in *Journées de géométrie algébrique d'Orsay (Orsay, France, 1992),* Astérisque **218**, Soc. Math. France, Montrouge, 1993, 9 – 34. MR 1265306

[B2] ———, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Algebraic Geom. **3** (1994), 493 – 535. MR 1269718

[BGR] S. BOSCH, U. GÜNTZER, and R. REMMERT, *Non-Archimedean Analysis*, Grundlehren Math. Wiss. **261**, Springer, Berlin, 1984. MR 0746961

[CL] K. CHAN and N. C. LEUNG, *Mirror symmetry for toric Fano manifolds via SYZ transformations*, preprint, arXiv:0801.2830v3 [math.SG]

[Che] YU. V. CHEKANOV, *Lagrangian intersections, symplectic energy, and areas of holomorphic curves*, Duke Math. J. **95** (1998), 213 – 226. MR 1646550

[Cho] C.-H. CHO, *Non-displaceable Lagrangian submanifolds and Floer cohomology with non-unitary line bundle*, J. Geom. Phys. **58** (2008), 1465 – 1476. MR 2463805

[CO] C.-H. CHO and Y.-G. OH, *Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds*, Asian J. Math. **10** (2006), 773 – 814. MR 2282365

[CS] K. CIELIEBAK and D. SALAMON, *Wall crossing for symplectic vortices and quantum cohomology*, Math. Ann. **335** (2006), 133 – 192. MR 2217687

[CK] D. A. COX and S. KATZ, *Mirror Symmetry and Algebraic Geometry*, Math. Surveys Monogr. **68**, Amer. Math. Soc., Providence, 1999. MR 1677117

[D] T. DELZANT, *Hamiltoniens périodiques et image convexes de l'application moment*, Bull. Soc. Math. France **116** (1988), 315 – 339. MR 0984900

[Ei] D. EISENBUD, Commutative Algebra, Grad. Texts in Math. **150**, Springer, Berlin, 1994. MR 1322960

[EP1] M. ENTOV and L. POLTEROVICH, *Quasi-states and symplectic intersections*, Comment. Math. Helv. **81** (2006), 75 – 99. MR 2208798

[EP2] ———, "Symplectic quasi-states and semi-simplicity of quantum homology" in *Toric Topology (Osaka, Japan, 2006)*, Contemp. Math. **460**, Amer. Math. Soc., Providence, 2008, 47 – 70. MR 2428348

[EP3] ———, *Rigid subsets of symplectic manifolds*, Compos. Math. **145** (2009), 773 – 826. MR 2507748

[Fo] A. FLOER, *Morse theory for Lagrangian intersections*, J. Differential Geom. **28** (1988), 513 – 547. MR 0965228

[F1] K. FUKAYA, "Morse homotopy, A^{∞}-category, and Floer homologies" in *Proceedings of GARC Workshop on Geometry and Topology '93 (Seoul, 1993)*, Lecture Notes Ser. **18**, Seoul National University, Seoul, 1993. MR 1270931

[F2] ———, "Floer homology for families—A progress report" in *Integrable Systems, Topology, and Physics (Tokyo, 2000)*, Contemp. Math. **309**, Amer. Math. Soc., Providence, 2002, 33 – 68. MR 1953352

[F3] ———, *Mirror symmetry of abelian varieties and multi-theta functions*, J. Algebraic Geom. **11** (2002), 393 – 512. MR 1894935

[F4] ———, "Application of Floer homology of Lagrangian submanifolds to symplectic topology" in *Morse Theoretic Methods in Nonlinear Analysis and in Symplectic Topology (Montreal, 2004)*, NATO Sci. Ser. II Math. Phys. Chem. **217**, Springer, Dordrecht, 2006, 231 – 276. MR 2276953

[F5] ———, "Differentiable operand, Kuranishi correspondence, and foundation of topological field theories based on pseudo-holomorphic curve," to appear in *Arithmetic and Geometry around Quantization*, Prog. Math. **279**, Birkhäuser, Boston.

[FOOO1] K. FUKAYA, Y.-G. OH, H. OHTA, and K. ONO, *Lagrangian intersection Floer theory-anomaly and obstruction*, preprint, 2000.

[FOOO2] ———, *Lagrangian intersection Floer theory-anomaly and obstruction*, preprint, 2006, 2007.

[FOOO3] ———, *Lagrangian Intersection Floer Theory-Anomaly and Obstruction*, AMS/IP Stud. Adv. Math. **46**, Amer. Math. Soc., Providence, 2009.

[FOOO4] ———, "Canonical models of filtered A_{∞}-algebras and Morse complexes" in *New Perspectives and Challenges in Symplectic Field Theory*, CRM Proc. Lecture Notes **49**, Amer. Math. Soc., Providence, 2009, 201 – 228.

[FOOO5] ———, *Lagrangian Floer theory on compact toric manifolds, II: Bulk deformations*, preprint, arXiv:0810.5654v1 [math.SG]

[FO] K. FUKAYA and K. ONO, *Arnold conjecture and Gromov-Witten invariant*, Topology **38** (1999), 933 – 1048. MR 1688434

[Fu] W. FULTON, *Introduction to Toric Varieties*, Ann. of Math. Stud. **131**, Princeton Univ. Press, Princeton, 1993. MR 1234037

[G1] A. B. GIVENTAL, "Homological geometry and mirror symmetry" in *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, Birkhäuser, Basel, 1995, 472 – 480. MR 1403947

[G2] ———, *Equivariant Gromov-Witten invariants*, Internat. Math. Res. Notices **1996**, no. 13, 613 – 663. MR 1408320

[GP] T. GRABER and R. PANDHARIPANDE, *Localization of virtual classes*, Invent. Math. **135** (1999), 487 – 518. MR 1666787

[Gu] V. GUILLEMIN, *Kaehler structures on toric varieties*, J. Differential Geom. **40** (1994), 285 – 309. MR 1293656

[Ho] H. HOFER, *On the topological properties of symplectic maps*, Proc. Roy. Soc. Edinburgh Sect. A **115** (1990), 25 – 38. MR 1059642

[H] K. HORI, "Linear models of supersymmetric D-branes" in *Symplectic Geometry and Mirror Symmetry (Seoul, 2000)*, World Sci. Publishing, River Edge, New Jersey, 2001, 111 – 186. MR 1882329

[HKKP] K. HORI, S. KATZ, A. KLEMM, R. PANDHARIPANDE, R. THOMAS, C. VAFA, R. VAKIL, and E. ZASLOW, *Mirror Symmetry*, Clay Math. Monogr. **1**, Amer. Math. Soc., Providence, 2003. MR 2003030

[HV] K. HORI and C. VAFA, *Mirror symmetry*, preprint, 2000.

[Hm] L. HÖRMANDER, *Fourier integral operators, I*, Acta Math. **127** (1971), 79 – 183. MR 0388463

[I1] H. IRITANI, *Quantum D-modules and equivariant Floer theory for free loop spaces*, Math. Z. **252** (2006), 577 – 622. MR 2207760

[I2] ———, *Convergence of quantum cohomology by quantum Lefschetz*, J. Reine Angew. Math. **610** (2007), 29 – 69. MR 2359850

[K] M. KONTSEVICH, "Homological algebra of mirror symmetry" in *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, Birkhäuser, Basel, 1995. MR 1403918

[Ko] A. G. KOUCHNIRENKO [Kušnirenko], *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32** (1976), 1 – 31. MR 0419433

[LR] A.-M. LI and Y. RUAN, *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds*, Invent. Math. **145** (2001), 151 – 218. MR 1839289

[M] YU. I. MANIN, *Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces*, Amer. Math. Soc. Colloq. Publ. **47**, Amer. Math. Soc., Providence, 1999. MR 1702284

[MW] J. MARSDEN and A. WEINSTEIN, *Reduction of Symplectic Manifolds with Symmetry*, Rep. Mathematical Phys. **5** (1974), 121 – 130. MR 0402819

[Ma] J. N. MATHER, "Stratifications and mappings" in *Dynamical Systems (Salvador, Brazil, 1971)* Academic Press, New York, 1973, 195 – 232. MR 0368064

[Mt] H. MATSUMURA, *Commutative Algebra*, Benjamin, New York, 1970. MR 0266911

[Mc] D. MCDUFF, *Displacing Lagrangian toric fibers via probes*, preprint, arXiv:0904.1686v1 [math.SG]

[McT] D. MCDUFF and S. TOLMAN, *Topological properties of Hamiltonian circle actions*, IMRP Int. Math. Res. Pap. **2006**, no. 72826, 1 – 77. MR 2210662

[Mi] J. MILNOR, *Singular Points of Complex Hypersurfaces*, Ann. of Math. Stud. **61**, Princeton Univ. Press, Princeton, 1968. MR 0239612

[N] M. NAGATA, *Local rings*, Interscience Tracts in Pure and Applied Mathematics **13**, Interscience, New York, 1962. MR 0155856

[O1] Y.-G. OH, *Floer cohomology of Lagrangian intersections and pseudo-holomorphic discs, I*, Comm. Pure Appl. Math. **46** (1993), 949 – 994; *Addenda*, Comm. Pure Appl. Math. **48** (1995), 1299 – 1302. MR 1223659; MR 1367384

[O2] ———, *Symplectic topology as the geometry of action functional, II*, Comm. Anal. Geom. **7** (1999), 1 – 55. MR 1674121

[O3] ———, "Construction of spectral invariants of Hamiltonian paths on closed symplectic manifolds" in *The Breadth of Symplectic and Poisson Geometry*, Prog. Math. **232**, Birkhäuser, Boston, 2005, 525 – 570. MR 2103018

[O4] ———, *Floer mini-max theory, the Cerf diagram, and the spectral invariants*, J. Korean Math. Soc. **46** (2009), 363 – 447. MR 2494501

[Oh] H. OHTA, "Obstruction to and deformation of Lagrangian intersection Floer cohomology" in *Symplectic Geometry and Mirror Symmetry (Seoul, 2000)*, World Sci., River Edge, N.J., 2001, 281 – 309. MR 1882333

[R] Y. RUAN, *Virtual neighborhoods and pseudo-holomorphic curves*, Turkish J. Math. **23** (1999), 161 – 231. MR 1701645

[S] K. SAITO, *Period mapping associated to a primitive form*, Publ. Res. Inst. Math. Sci. **19** (1983), 1231 – 1264. MR 0723468

[Sc] M. SCHWARZ, *On the action spectrum for closed symplectically aspherical manifolds*, Pacific J. Math. **193** (2000), 419 – 461. MR 1755825

[Se1] P. SEIDEL, π_1 *of symplectic automorphism groups and invertibles in quantum homology rings*, Geom. Funct. Anal. **7** (1997), 1046 – 1095. MR 1487754

[Se2] ———, *Fukaya Categories and Picard-Lefschetz Theory*, Zur. Lect. Adv. Math., European Mathematical Society, Zürich, 2008. MR 2441780

[U] K. UEDA, *Homological mirror symmetry for toric del Pezzo surfaces*, Comm. Math. Phys. **264** (2006), 71 – 85. MR 2212216

[Us] M. USHER, *Spectral numbers in Floer theories*, Compos. Math. **144** (2008), 1581 – 1592. MR 2474322

[V] C. VITERBO, *Symplectic topology as the geometry of generating functions*, Math. Ann. **292** (1992), 685 – 710. MR 1157321

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