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Decay estimates for evolutionary equations with fractional time-diffusion

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Abstract. We consider an evolution equation whose time-diffusion is of fractional type, and we provide decay estimates in time for the L^s-norm of the solutions in a bounded domain. The spatial operator that we take into account is very general and comprises classical local and nonlocal diffusion equations.

1. Introduction

1.1. General time-fractional diffusion equations

The goal of this paper is to consider evolutionary equations with nonlocal time-diffusion of fractional type, which is modeled by an integro-differential operator. The space-diffusion that we take into account can be both local and nonlocal, and in fact our approach aims at general energy estimates in an abstract framework which will in turn provide asymptotic decay estimates in a series of concrete cases, including nonlocal nonlinear operators, nonlocal porous medium equations and possibly nonlocal mean curvature operators.

More specifically, we consider equations of the form

$$\partial_t^\alpha u + \mathcal{N}[u] = 0, \tag{1.1}$$

with $\alpha \in (0, 1)$. In this setting, the solution u is a function $u = u(x, t)$, with x lying in a nice Euclidean domain, $t > 0$, and Dirichlet boundary data. The variable x will be referred to as "space," and in the examples that we take into account, the operator $\mathcal{N}$ possesses some kind of "elliptic" features, which make (1.1) a sort of "diffusive," or "parabolic," equation. In this spirit, the variable t, which will be referred to as "time," appears in (1.1) with a fractional derivative of order $\alpha \in (0, 1)$, and we

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thereby consider (1.1) as a fractional time-diffusion. In the examples that we take into account, the diffusion modeled on the operator $\mathcal{N}$ can be either "classical" (i.e., involving derivatives of integer order, up to order two) or "anomalous" (since it can involve fractional derivatives as well, in which case we refer to it as a fractional space-diffusion).

We also recall that integro-differential equations are a classical topic in mathematical analysis, see, e.g., [54,70]. Fractional calculus also appears under different forms in several real-world phenomena, see, e.g., [42,51,66]. In particular, time-fractional derivatives find applications in the magneto-thermoelastic heat conduction [30], wave equations [17,49], hydrodynamics [7], quantum physics [9], etc. See also [47,65] for existence and uniqueness results and [4,5,68] for related regularity results in the local and nonlocal spatial regime. The recent literature has also widely considered time-fractional diffusion coupled with p-Laplacian space-diffusion, see, e.g., [18,45,46,73] and the references therein.

In the framework of nonlocal equations, a deep and useful setting is that provided by the Volterra integral equations, which often offers a general context in which one develops existence, uniqueness, regularity and asymptotic theories, see, for instance, [6,21,22,33,36,37,56,63,71,72] and the references therein.

The setting in which we work in this paper is the following. We consider the so-called Caputo derivative of order $\alpha \in (0, 1)$, defined as

$$\partial_t^\alpha u(t) := \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{u(\tau) - u(0)}{(t-\tau)^\alpha}\, \mathrm{d}\tau, \tag{1.2}$$

up to a positive normalization constant that we neglect (see, e.g., [14]).

Goal of this paper is to study solutions

$$u = u(x, t) : \mathbb{R}^n \times [0, +\infty) \to [0, +\infty)$$

of the initial value problem

$$\begin{cases} \partial_t^\alpha u(x,t) + \mathcal{N}[u](x,t) = 0 & \text{for any } x \in \Omega \text{ and } t > 0, \\ u(x,t) = 0 & \text{for any } x \in \mathbb{R}^n \setminus \Omega \text{ and } t \geqslant 0, \\ u(x,0) = u_0(x) & \text{for any } x \in \Omega. \end{cases} \tag{1.3}$$

In our notation, Ω is a bounded subset of $\mathbb{R}^n$ with smooth boundary and $\mathcal{N}$ is a possibly nonlinear operator. For concreteness, we suppose that the initial datum u_0 does not vanish identically and lies in $L^q(\Omega)$ for any $q \in [1, +\infty)$ (as a matter of fact, weaker assumptions can be taken according to suitable choices of the parameters). In any case, from now on, the initial datum u_0 will be always implicitly supposed to be nonnegative, nontrivial and integrable at any power and the solution u to be nonnegative and smooth.

The main structural assumption that we take is that there exist $s \in (1, +\infty)$, $\gamma \in (0, +\infty)$ and $C \in (0, +\infty)$ such that if u is as in (1.3), then

$$\|u\|_{L^s(\Omega)}^{s-1+\gamma}(t) \leqslant C \int_\Omega u^{s-1}(x,t)\, \mathcal{N}[u](x,t)\, \mathrm{d}x, \tag{1.4}$$

where we used the notation

$$\|u\|_{L^s(\Omega)}(t) := \left(\int_\Omega u^s(x,t)\,\mathrm{d}x\right)^{1/s}.$$

For simplicity, we considered smooth solutions of (1.3) (in concrete cases, the notion of weak solutions may be treated similarly, see, e.g., the regularization methods discussed on page 235 of [69]).

After providing a general result on the decay of the solutions of (1.3), we will specify the operator $\mathcal{N}$ to the following concrete cases:

- the case of the Laplacian,
- the case of the p-Laplacian,
- the case of the porous medium equation,
- the case of the doubly nonlinear equation,
- the case of the mean curvature equation,
- the case of the fractional Laplacian,
- the case of the fractional p-Laplacian,
- the sum of different space-fractional operators,
- the case of the fractional porous medium equation,
- the case of the fractional mean curvature equation.

The general result will be obtained by energy methods for nonlinear operators (see, e.g., [27]). Our approach will largely exploit a very deep and detailed analysis of the time-fractional evolution problems recently performed in [39,69], and in a sense, our results can be seen as a generalization of those in [39,40,69] to comprise cases arising from space-fractional equations, nonlinear nonlocal operators, geometric operators and nonlocal porous medium equations.

Also, the general framework that we provide can be useful to give a unified setting in terms of energy inequalities.

Our "abstract" result is the following:

THEOREM 1.1. *Let u be as in* (1.3)*, under the structural condition in* (1.4)*. Then,*

$$\partial_t^\alpha \|u\|_{L^s(\Omega)}(t) \leqslant -\frac{\|u\|^\gamma_{L^s(\Omega)}(t)}{C}. \tag{1.5}$$

Furthermore,

$$\|u\|_{L^s(\Omega)}(t) \leqslant \frac{C_\star}{1+t^{\alpha/\gamma}}, \tag{1.6}$$

for some $C_\star > 0$, possibly depending on C, γ, α and $\|u_0\|_{L^s(\Omega)}$.

We point out that the result in (1.6) is quite different from the decay estimates for classical time-diffusion (compared, e.g., with [27]).

Indeed, in (1.6), a power-law decay is provided, while the classical uniformly elliptic time-diffusion case presents exponential decays. The power-law decay can be understood by looking at the solution of

$$\partial_t^\alpha e(t) = -e(t)$$

for $t \in (0, +\infty)$ with initial datum $e(0) = 1$ and at the first Dirichlet eigenfunction ϕ of a ball B normalized in such a way that the corresponding eigenvalue is equal to 1, namely,

$$\begin{cases} \Delta\phi = -\phi & \text{in } B, \\ \phi = 0 & \text{on } \partial B, \\ \|\phi\|_{L^2(B)} = 1. \end{cases}$$

Then, the function $u(x, t) := e(t)\,\phi(x)$ satisfies the fractional heat equation $\partial_t^\alpha u = \Delta u$ in B, with zero Dirichlet datum, and

$$\|u\|_{L^2(B)} = |e(t)|. \tag{1.7}$$

The function e is explicit in terms of the Mittag-Leffler function (see, e.g., [48,53] and the references therein), and it satisfies $e(t) \sim \frac{1}{t^\alpha}$ as $t \to +\infty$.

This fact and (1.7) imply a polynomial decay of the L^2-norm of the solution, in agreement with (1.6).

The decay presented in (1.6) is also different from the case of fast nonlinear diffusion, in which the solution gets extinct in finite time, see, e.g., Theorem 17 in [27].

We now specify Theorem 1.1 to several concrete cases, also recovering the main results in [40,69] and providing new applications. Several new applications will be also given in [3].

1.2. The cases of the Laplacian, of the p-Laplacian, of the porous medium equation and of the doubly nonlinear equation

The doubly nonlinear operator (see, e.g., [58]) is a general operator of the form

$$u \longmapsto \Delta_p u^m, \tag{1.8}$$

with $m \in (0, +\infty)$ and $p \in (1, +\infty)$.

When $m = 1$, this operator reduces to the p-Laplacian

$$\Delta_p u := \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big),$$

which in turn reduces to the classical Laplacian as $p = 2$.

When $p = 2$, the operator in (1.8) reduces to the porous medium operator (see, e.g., [67] and the references therein)

$$u \longmapsto \Delta u^m,$$

which again reduces to the Laplacian when $m = 1$.

In this setting, we have the following result:

THEOREM 1.2. *Suppose that u is a solution of*

$$\partial_t^\alpha u = \Delta_p u^m$$

in $\Omega \times (0, +\infty)$*, with* $u(x,t) = 0$ *for any* $x \in \partial\Omega$ *and any* $t \geqslant 0$*. Then, for any* $s \in (1, +\infty)$*,*

$$\|u\|_{L^s(\Omega)}(t) \leqslant \frac{C}{1 + t^{\frac{\alpha}{m(p-1)}}}, \tag{1.9}$$

for some $C > 0$*.*

As special cases of Theorem 1.2, we can take $m := 1$ and $p := 2$, which correspond to the p-Laplacian case and to the porous medium case, respectively. We state these results explicitly for the convenience of the reader.

COROLLARY 1.3. *Suppose that* u *is a solution of*

$$\partial_t^\alpha u = \Delta_p u$$

in $\Omega \times (0, +\infty)$*, with* $u(x,t) = 0$ *for any* $x \in \partial\Omega$ *and any* $t \geqslant 0$*. Then, for any* $s \in (1, +\infty)$*,*

$$\|u\|_{L^s(\Omega)}(t) \leqslant \frac{C}{1 + t^{\alpha/(p-1)}},$$

for some $C > 0$*.*

COROLLARY 1.4. *Suppose that* u *is a solution of*

$$\partial_t^\alpha u = \Delta u^m$$

in $\Omega \times (0, +\infty)$*, with* $u(x,t) = 0$ *for any* $x \in \partial\Omega$ *and any* $t \geqslant 0$*. Then, for any* $s \in (1, +\infty)$*,*

$$\|u\|_{L^s(\Omega)}(t) \leqslant \frac{C}{1 + t^{\alpha/m}},$$

for some $C > 0$*.*

When $p \in (2, +\infty)$ in Corollary 1.3, the case $\alpha \nearrow 1$ recovers the classical decay, see, e.g., Theorem 21 in [27].

For results related to Corollary 1.3 when $p = 2$, see [50,52]. Corollaries 1.3 and 1.4 can be compared with Theorems 8.1 and 9.1 in [69], respectively.

1.3. The case of the mean curvature equation

The setting in Theorem 1.1 is general enough to deal with nonlinear operators of mean curvature type and to consider equations of the type

$$\partial_t^\alpha u(x,t) = \operatorname{div}\left(\frac{\nabla u(x,t)}{\sqrt{1 + |\nabla u(x,t)|^2}}\right). \tag{1.10}$$

We recall that the right-hand side of (1.10) corresponds to the mean curvature of the hypersurface described by the graph of the function u, see, e.g., formula (13.1) in [35]. The result obtained in this setting goes as follows:

THEOREM 1.5. *Suppose that u is a solution of* (1.10) *in* $\Omega \times (0, +\infty)$*, with* $u(x, t) = 0$ *for any* $x \in \partial\Omega$ *and any* $t \geqslant 0$. *Assume that either*

$$n \in \{1, 2\} \quad \textit{and} \quad \sup_{t>0} \int_{\Omega} \sqrt{1 + |\nabla u(x, t)|^2}\, dx < +\infty \tag{1.11}$$

or

$$\sup_{x\in\Omega,\ t>0} |\nabla u(x, t)| < +\infty. \tag{1.12}$$

Then, for any $s \in (1, +\infty)$,

$$\|u\|_{L^s(\Omega)}(t) \leqslant \frac{C}{1 + t^{\alpha}}, \tag{1.13}$$

for some $C > 0$.

1.4. The case of the fractional Laplacian, of the fractional p-Laplacian and of the sum of different space-fractional operators

The setting in Theorem 1.1 is general enough to comprise also the case of operators modeling spatial nonlocal diffusion of fractional kind. The main example of such operators is given by the fractional Laplacian of order $\sigma \in (0, 1)$, which can be defined (up to a multiplicative constant that we neglect for simplicity) by

$$(-\Delta)^{\sigma} u(x) := \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2\sigma}}\, dy. \tag{1.14}$$

Here and in the following, we implicitly suppose that these types of singular integrals are taken in the principal value sense. The fractional Laplacian provides a natural framework for many problems in theoretical and applied mathematics, see, e.g., [10, 44, 64] and the references therein.

Several nonlinear variations in the fractional Laplacian can be taken into account, see, e.g., [23, 28, 38, 43, 57] and the references therein. In particular, for any $p \in (1, +\infty)$, one can consider the operator

$$(-\Delta)^{\sigma}_p u(x) := \int_{\mathbb{R}^n} \frac{\big|u(x) - u(y)\big|^{p-2}\big(u(x) - u(y)\big)}{|x - y|^{n+\sigma p}}\, dy. \tag{1.15}$$

Of course, when $p = 2$ the operator in (1.15) reduces to that in (1.14). In this setting, we have the following decay result for solutions of fractional time equations whose spatial diffusion is driven by the nonlinear fractional operator in (1.15).

THEOREM 1.6. *Suppose that u is a solution of*

$$\partial_t^{\alpha} u(x, t) + (-\Delta)^{\sigma}_p u(x, t) = 0$$

in $\Omega \times (0, +\infty)$*, with* $u(x, t) = 0$ *for any* $x \in \mathbb{R}^n \setminus \Omega$ *and for any* $t > 0$. *Then, for any* $s \in (1, +\infty)$,

$$\|u\|_{L^s(\Omega)}(t) \leqslant \frac{C}{1 + t^{\alpha/(p-1)}}, \tag{1.16}$$

for some $C > 0$.

When $p = 2$, decay estimates for nonlocal equations have been very recently obtained in [40].

An extension of Theorem 1.6 holds true also for sums of different, possibly nonlinear, space-fractional diffusion operators:

THEOREM 1.7. *Let $N \in \mathbb{N}$, $N \geqslant 1$. Let $\sigma_1, \dots, \sigma_N \in (0,1)$, $p_1, \dots, p_N \in (1,+\infty)$ and $\beta_1, \dots, \beta_N \in (0,+\infty)$. Suppose that u is a solution of*

$$\partial_t^\alpha u(x,t) + \sum_{j=1}^{N} \beta_j \, (-\Delta)_{p_j}^{\sigma_j} u(x,t) = 0$$

in $\Omega \times (0,+\infty)$, with $u(x,t) = 0$ for any $x \in \mathbb{R}^n \setminus \Omega$ and for any $t > 0$. Then, for any $s \in (1,+\infty)$,

$$\|u\|_{L^s(\Omega)}(t) \leqslant \frac{C}{1 + t^{\alpha/(p_{\max}-1)}}, \tag{1.17}$$

for some $C > 0$, where

$$p_{\max} := \max\{p_1, \dots, p_N\}.$$

More general settings for superpositions of fractional operators can be also considered in our setting, see, e.g., [12].

Another interesting case arises from the sum of fractional operators in different directions. Precisely, fixed $j \in \{1, \dots, n\}$ one can consider the unit vector e_j (i.e., the jth element of the Euclidean basis of $\mathbb{R}^n$), and define the fractional Laplacian in direction e_j, namely,

$$(-\partial_{x_j}^2)^{\sigma_j} u(x) := \int_{\mathbb{R}} \frac{u(x) - u(x + \rho e_j)}{\rho^{1+2\sigma_j}} \, \mathrm{d}\rho,$$

with $\sigma_j \in (0,1)$. Then, given $\beta_1, \dots, \beta_n > 0$, one can consider the superposition of such operators, that is,

$$(-\Delta_\beta)^\sigma u(x) := \sum_{j=1}^{n} \beta_j (-\partial_{x_j}^2)^{\sigma_j} u(x).$$

Here, we are using the formal notation $\sigma := (\sigma_1, \dots, \sigma_n)$ and $\beta := (\beta_1, \dots, \beta_n)$. Notice that $(-\Delta_\beta)^\sigma$ is similar to, but structurally very different from, the fractional Laplacian, since the nonlocal character of the fractional Laplacian also takes into account the interactions in directions different than $e_1, \dots, e_n$: for instance, even if $\sigma_1 = \dots = \sigma_n = 1/2$ and $\beta_1 = \dots = \beta_n = 1$, the operator $(-\Delta_\beta)^\sigma$ does not reduce to the square root of the Laplacian.

These types of anisotropic fractional operators (and even more general ones) have been considered in [31,59–61]. In our setting, we have the following decay result:

THEOREM 1.8. *Suppose that u is a solution of*

$$\partial_t^\alpha u(x,t) + (-\Delta_\beta)^\sigma u(x,t) = 0$$

in $\Omega \times (0,+\infty)$*, with* $u(x,t) = 0$ *for any* $x \in \mathbb{R}^n \setminus \Omega$ *and for any* $t > 0$*. Then, for any* $s \in (1,+\infty)$*,*

$$\|u\|_{L^s(\Omega)}(t) \leqslant \frac{C}{1+t^\alpha},$$

for some $C > 0$*.*

More general operators, such as the sum of fractional Laplacians along linear subspaces of $\mathbb{R}^n$, as well as operators in integral superposition, may also be taken into account in Theorem 1.8, but we focused on an explicit case for simplicity of notations.

1.5. The case of the fractional porous medium equation

We consider here a porous medium diffusion operator of fractional type, given by

$$u \longmapsto (-\Delta)^\sigma u^m,$$

with $\sigma \in (0,1)$ and $m \in (0,+\infty)$, where $(-\Delta)^\sigma$ is the fractional Laplace operator defined in (1.14).

In the classical time-diffusion case, such equation has been introduced and analyzed in [25,26] (remarkably, in this case, any nontrivial nonnegative solution becomes strictly positive instantaneously, and this is a different feature with respect to the classical porous medium equation).

In our setting, we will consider the time-fractional version of the space-fractional porous medium equation and establish the following decay estimate:

THEOREM 1.9. *Suppose that u is a solution of*

$$\partial_t^\alpha u + (-\Delta)^\sigma u^m = 0$$

in $\Omega \times (0,+\infty)$*, with* $u(x,t) = 0$ *for any* $x \in \mathbb{R}^n \setminus \Omega$ *and any* $t \geqslant 0$*. Then, for any* $s \in (1,+\infty)$*,*

$$\|u\|_{L^s(\Omega)}(t) \leqslant \frac{C}{1+t^{\frac{\alpha}{m}}}, \tag{1.18}$$

for some $C > 0$*.*

Similar results may also be obtained in more general settings for doubly nonlinear and doubly fractional porous medium equations. We also observe that, for $m = 1$, Theorem 1.9 boils down to Theorem 1.6 with $p := 2$.

1.6. The case of the fractional mean curvature equation

The notion of nonlocal perimeter functional has been introduced and analyzed in [13]. While the first variation in the classical perimeter functional consists in the mean curvature operator, the first variation in the nonlocal perimeter produces an object, which can be seen as a nonlocal mean curvature and which corresponds to a weighted average of the characteristic function of a set with respect to a singular kernel. The study of such fractional mean curvature operator is a very interesting topic of research in itself, and the recent literature produced several contributions in this context, see, e.g., [1,8,11,20,24,32,34,55]. The nonlocal mean curvature also induces a geometric flow, as studied in [15,16,19,62]. See also [29] for a recent survey on the topic of nonlocal minimal surfaces and nonlocal mean curvature equations.

For smooth hypersurfaces with a structure of complete graphs, the notion of nonlocal mean curvature can be introduced as follows (see, e.g., formula (3.5) in [32]). For any $r \in \mathbb{R}$ and $\sigma \in (0, 1)$, we set

$$F(r) := \int_0^r \frac{d\tau}{(1+\tau^2)^{(n+1+\sigma)/2}} \tag{1.19}$$

and we consider the (minus) nonlocal mean curvature operator corresponding to the choice

$$\mathcal{N}[u](x,t) := \int_{\mathbb{R}^n} \frac{1}{|y|^{n+\sigma}}\, F\left(\frac{u(x,t)-u(x+y,t)}{|y|}\right) dy. \tag{1.20}$$

In this setting, we provide a decay estimate for graphical solutions of the fractional mean curvature equation, as stated in the following result:

THEOREM 1.10. *Suppose that u is a solution of*

$$\partial_t^\alpha u(x,t) = \int_{\mathbb{R}^n} \frac{1}{|y|^{n+\sigma}}\, F\left(\frac{u(x+y,t)-u(x,t)}{|y|}\right) dy$$

with $u(x,t) = 0$ for any $x \in \mathbb{R}^n \setminus \Omega$ and for any $t > 0$.

Assume that

$$\sup_{x\in\Omega,\ t>0} |\nabla u(x,t)| < +\infty. \tag{1.21}$$

Then, for any $s \in (1, +\infty)$,

$$\|u\|_{L^s(\Omega)}(t) \leqslant \frac{C}{1+t^\alpha},$$

for some $C > 0$.

The recent literature has considered the evolution of graphs under the fractional mean curvature flow, see Section 6 of [62]. In this respect, Theorem 1.10 here can be seen as the first study of evolution equations driven by the fractional mean curvature in which the flow possesses a memory effect.

The rest of the paper is devoted to the proofs of the above mentioned results. First we prove the general statement of Theorem 1.1, and then, we check that condition (1.4) is verified in all the concrete cases taken into consideration.

2. Proofs

We will exploit the following result, which follows from Corollary 3.1 in [69].

LEMMA 2.1. *Let $s > 1$, $u : \Omega \times [0, +\infty) \to \mathbb{R}$ and $u_0(x) := u(x, 0)$. Let $v(t) := \|u\|_{L^s(\Omega)}(t)$ and suppose that $u_0 \in L^s(\Omega)$, and for every $T > 0$, that $u \in L^s((0, T), L^s(\Omega))$. Then,*

$$v^{s-1}(t)\, \partial_t^\alpha v(t) \leqslant \int_\Omega u^{s-1}(x, t)\, \partial_t^\alpha u(x, t)\, \mathrm{d}x.$$

As a technical remark, we recall that in our setting u is supposed to be smooth in (space and) time, and this allows us to exploit Corollary 3.1 in [69]. Indeed, in Corollary 3.1 of [69], the interaction kernel is in principle assumed to be in $H_1^1((0, T))$, and in particular, no singularity is permitted at $t = 0$. Nevertheless, in [69] there is also a remark after Corollary 3.1, stating that the desired inequality remains true in the singular case provided that u is sufficiently smooth (which is the case under consideration in our setting), and hence, Lemma 2.1 here follows from Corollary 3.1 in [69], e.g., by means of a Yosida approximation argument for the fractional derivative.

2.1. Proof of Theorem 1.1

Without loss of generality, we can suppose that $\|u_0\|_{L^s(\Omega)} \neq 0$, and we set $v(t) := \|u\|_{L^s(\Omega)}(t)$. Hence, recalling (1.3) and Lemma 2.1,

$$v^{s-1}(t)\, \partial_t^\alpha v(t) \leqslant -\int_\Omega u^{s-1}(x, t)\, \mathcal{N}[u](x, t)\, \mathrm{d}x. \tag{2.1}$$

Using this and (1.4), we thus find that

$$v^{s-1}(t)\, \partial_t^\alpha v(t) \leqslant -\frac{1}{C}\, \|u\|_{L^s(\Omega)}^{s-1+\gamma}(t) = -\frac{v^{s-1+\gamma}(t)}{C}. \tag{2.2}$$

From (2.2), we plainly obtain (1.5) at all t for which $v(t) \neq 0$.

But at the points t at which $v(t) = 0$, we see that (1.5) is also automatically[1] satisfied in view of the following observation: using that u is smooth, combined with the Hölder's Inequality with exponents $s/(s-1)$ and s, we have that, a.e. $t > 0$,

$$\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}t} v(t)\right| &= \left|\frac{\partial}{\partial t}\left(\int_\Omega |u(x, t)|^s\, \mathrm{d}x\right)^{1/s}\right| \\
&\leqslant \left(\int_\Omega |u(x, t)|^s\, \mathrm{d}x\right)^{(1-s)/s} \int_\Omega |u(x, t)|^{s-1} \left|\frac{\partial u}{\partial t}(x, t)\right| \mathrm{d}x \\
&\leqslant \left(\int_\Omega |u(x, t)|^s\, \mathrm{d}x\right)^{(1-s)/s} \left(\int_\Omega |u(x, t)|^s\, \mathrm{d}x\right)^{(s-1)/s} \left(\int_\Omega \left|\frac{\partial u}{\partial t}(x, t)\right|^s \mathrm{d}x\right)^{1/s} \\
&= \left\|\frac{\partial u}{\partial t}\right\|_{L^s(\Omega)}(t).
\end{aligned}$$

[1] For another approach allowing for the division by the prefactor in (2.2), see Lemma 2.1 in the recent preprint [41].

Hence, v is Lipschitz continuous, and therefore, we see that

$$\lim_{\tau \nearrow t} \frac{v(t)-v(\tau)}{(t-\tau)^{\alpha}} = 0,$$

and as a consequence,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{v(t)-v(\tau)}{(t-\tau)^{\alpha}}\,\mathrm{d}\tau &= \int_0^t \frac{\partial_t v(t)}{(t-\tau)^{\alpha}}\,\mathrm{d}\tau - \alpha\int_0^t \frac{v(t)-v(\tau)}{(t-\tau)^{1+\alpha}}\,\mathrm{d}\tau \\ &= \frac{\partial_t v(t)\, t^{1-\alpha}}{1-\alpha} - \alpha\int_0^t \frac{v(t)-v(\tau)}{(t-\tau)^{1+\alpha}}\,\mathrm{d}\tau.\end{aligned}$$

Comparing this with (1.2), we find that

$$\begin{aligned}\partial_t^{\alpha} v(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{v(\tau)-v(t)}{(t-\tau)^{\alpha}}\,\mathrm{d}\tau + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{v(t)-v(0)}{(t-\tau)^{\alpha}}\,\mathrm{d}\tau \\ &= -\frac{\partial_t v(t)\, t^{1-\alpha}}{1-\alpha} + \alpha\int_0^t \frac{v(t)-v(\tau)}{(t-\tau)^{1+\alpha}}\,\mathrm{d}\tau + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\big(v(t)-v(0)\big)\, t^{1-\alpha}}{1-\alpha} \\ &= -\frac{\partial_t v(t)\, t^{1-\alpha}}{1-\alpha} + \alpha\int_0^t \frac{v(t)-v(\tau)}{(t-\tau)^{1+\alpha}}\,\mathrm{d}\tau + \frac{\partial_t v(t)\, t^{1-\alpha}}{1-\alpha} + \frac{v(t)-v(0)}{t^{\alpha}} \\ &= \alpha\int_0^t \frac{v(t)-v(\tau)}{(t-\tau)^{1+\alpha}}\,\mathrm{d}\tau + \frac{v(t)-v(0)}{t^{\alpha}}.\end{aligned}$$

Therefore, at points t where $v(t)=0$, using that $v \geqslant 0$, we see that

$$\partial_t^{\alpha} v(t) = -\alpha\int_0^t \frac{v(\tau)}{(t-\tau)^{1+\alpha}}\,\mathrm{d}\tau - \frac{v(0)}{t^{\alpha}} \leqslant 0,$$

which gives that (1.5) is satisfied in this case as well.

Now, we prove (1.6). To this aim, we consider the solution $w(t)$ of the nonlinear fractional differential equation

$$\begin{cases} \partial_t^{\alpha} w(t) = -\dfrac{w^{\gamma}(t)}{C} & \text{for any } t>0, \\ w(0) = v(0). \end{cases} \tag{2.3}$$

When $\gamma = 1$, the function w is explicitly known in terms of the Mittag-Leffler function, see [48,53]. The general case $\gamma > 0$ has been dealt with in detail in Section 7 of [69]. In particular (see Theorem 7.1 in [69]), it holds that

$$w(t) \leqslant \frac{C_{\star}}{1+t^{\alpha/\gamma}}, \tag{2.4}$$

for some $C_{\star} > 0$, possibly depending on C, γ, α and $v(0)$. Moreover, by (1.5), (2.3) and the comparison principle (see, e.g., Lemma 2.6 in [69]), we have that $v(t) \leqslant w(t)$ for all $t \geqslant 0$. Using this and (2.4), we obtain (1.6). $\square$

REMARK 2.2. We observe that the constant C in (1.5) is exactly the one coming from (1.4). If needed, the long-time behavior in (1.6) can be also made more precise in terms of $\|u_0\|_{L^s(\Omega)}$. Indeed, recalling formula (41) in [69], in the notation used for the proof of Theorem 1.1 one can define

$$t_0 := \bar{C}\, w^{(1-\gamma)/\alpha}(0),$$

with $\bar{C} > 0$ depending only on α, γ and C and

$$\bar{w}(t) := \begin{cases} w(0) & \text{if } t \in [0, t_0], \\ w(0)\,(t_0/t)^{\alpha/\gamma} & \text{if } t \in (t_0, +\infty), \end{cases}$$

and conclude that $w(t) \leqslant \bar{w}(t)$. In this way, for large t, we have that

$$\begin{aligned} \|u\|_{L^s(\Omega)}(t) = v(t) \leqslant \bar{w}(t) = \frac{w(0)\, t_0^{\alpha/\gamma}}{t^{\alpha/\gamma}} = \frac{\|u_0\|_{L^s(\Omega)} \left(\bar{C}\, \|u_0\|_{L^s(\Omega)}^{(1-\gamma)/\alpha}\right)^{\alpha/\gamma}}{t^{\alpha/\gamma}} \\ = \frac{\tilde{C}\, \|u_0\|_{L^s(\Omega)}^{1/\gamma}}{t^{\alpha/\gamma}}, \end{aligned}$$

with $\tilde{C} > 0$ depending only on α, γ and C.

2.2. Proof of Theorem 1.2

We set

$$v := u^{\frac{s-2+p+(m-1)(p-1)}{p}} \tag{2.5}$$

and we point out that

$$\begin{aligned} |\nabla v|^p &= \left(\frac{s-2+p+(m-1)(p-1)}{p}\right)^p u^{s-2+(m-1)(p-1)}\, |\nabla u|^p \\ &= \left(\frac{s-2+p+(m-1)(p-1)}{p}\right)^p u^{s-2} \nabla u \cdot \left(u^{(m-1)(p-1)} |\nabla u|^{p-2} \nabla u\right) \\ &= \left(\frac{s-2+p+(m-1)(p-1)}{p}\right)^p \frac{1}{(s-1)\, m^{p-1}}\, \nabla u^{s-1} \cdot \left(|\nabla u^m|^{p-2} \nabla u^m\right). \end{aligned} \tag{2.6}$$

Now, when $p \in (1, n)$, we recall the Sobolev exponent

$$p_\star := \frac{np}{n-p}$$

and we claim that if $p \in (1, n)$ and $q \in [1, p_\star]$, as well as if $p \in [n, +\infty)$ and $q \in [1, +\infty)$, it holds that

$$\|v\|_{L^q(\Omega)}^p(t) \leqslant C_0 \int_\Omega |\nabla v(x,t)|^p \, dx, \tag{2.7}$$

for some $C_0 > 0$. Indeed, when $p \in (1, n]$, the inequality in (2.7) follows from the Sobolev Embedding Theorem. When instead $p \in (n, +\infty)$, we can use the Sobolev Embedding Theorem with exponent n to write

$$\|v\|^n_{L^q(\Omega)}(t) \leqslant C_0 \int_\Omega |\nabla v(x,t)|^n \,\mathrm{d}x.$$

Combining this with the Hölder's Inequality for the norm of the gradient, we obtain (2.7) (up to renaming constants).

We also observe that when $p \in (1, n)$ and

$$s \geqslant \max\left\{ m - \frac{1}{p-1},\ \frac{n\big(1 - m(p-1)\big)}{p} \right\}, \tag{2.8}$$

it holds that

$$\frac{sp}{s - 2 + p + (m-1)(p-1)} \in [1, p_\star]. \tag{2.9}$$

Indeed, we have that

$$s - 2 + p + (m-1)(p-1) > 1 - 2 + p + (m-1)(p-1) = m(p-1) \geqslant 0.$$

Moreover,

$$s - 2 + p + (m-1)(p-1) - sp = m(p-1) - 1 - s(p-1) \leqslant 0$$

thanks to (2.8). This gives that $\frac{sp}{s-2+p+(m-1)(p-1)} \geqslant 1$.

In addition,

$$s(n-p) - n\big(s - 2 + p + (m-1)(p-1)\big) = -sp - n\big(p - 2 + (m-1)(p-1)\big) \leqslant 0,$$

due to (2.8), which gives that $\frac{sp}{s-2+p+(m-1)(p-1)} \leqslant p_\star$. These considerations prove (2.9).

By means of (2.9), when either $p \in [n, +\infty)$ or (2.8) holds true, we can choose $q := \frac{sp}{s-2+p+(m-1)(p-1)}$ in (2.7). Hence, recalling (2.5), we find that

$$\begin{aligned}
\|u\|^{s-2+p+(m-1)(p-1)}_{L^s(\Omega)}(t) &= \left(\int_\Omega u^s(x,t)\,\mathrm{d}x \right)^{\frac{s-2+p+(m-1)(p-1)}{s}} \\
&= \left(\int_\Omega u^{\frac{s-2+p+(m-1)(p-1)}{p} \cdot \frac{sp}{s-2+p+(m-1)(p-1)}}(x,t)\,\mathrm{d}x \right)^{\frac{s-2+p+(m-1)(p-1)}{s}} \\
&= \left(\int_\Omega v^{\frac{sp}{s-2+p+(m-1)(p-1)}}(x,t)\,\mathrm{d}x \right)^{\frac{s-2+p+(m-1)(p-1)}{s}} \\
&= \|v\|^p_{L^{\frac{sp}{s-2+p+(m-1)(p-1)}}(\Omega)}(t) \\
&\leqslant C_0 \int_\Omega |\nabla v(x,t)|^p \,\mathrm{d}x.
\end{aligned}$$

As a consequence, making use of (2.6), we conclude that

$$\|u\|^{s-2+p+(m-1)(p-1)}_{L^s(\Omega)}(t) \leqslant C_1 \int_\Omega \nabla u^{s-1} \cdot \big(|\nabla u^m|^{p-2} \nabla u^m\big)\,\mathrm{d}x,$$

provided that either $p \in [n, +\infty)$ or (2.8) holds true.

This gives that condition (1.4) is satisfied in this case with $\gamma := m(p-1)$. This and (1.6) imply that if either $p \in [n, +\infty)$ or (2.8) holds true, then

$$\|u\|_{L^s(\Omega)}(t) \leqslant \frac{C_\star}{1 + t^{\frac{\alpha}{m(p-1)}}}, \tag{2.10}$$

for some $C_\star > 0$. Also, when (2.8) is not satisfied, we have that

$$s < \max\left\{m - \frac{1}{p-1},\ \frac{n\big(1 - m(p-1)\big)}{p}\right\} =: \bar{s}$$

and in this case the Hölder's Inequality implies that $\|u\|_{L^s(\Omega)}(t) \leqslant C\,\|u\|_{L^{\bar{s}}(\Omega)}(t)$, for some $C > 0$, and $\bar{s}$ lies in the range satisfying (2.10).

This observation and (2.10) imply (1.9), as desired. □

2.3. Proof of Theorem 1.5

We set $v := u^{s/2}$. Notice that

$$|\nabla v|^2 = \frac{s^2}{4}\, u^{s-2}|\nabla u|^2 = \frac{s^2}{4(s-1)}\, \nabla u \cdot \nabla u^{s-1}. \tag{2.11}$$

We distinguish two cases, according to (1.11) and (1.12). We first consider the case in which (1.11) holds true. Then, by Cauchy–Schwarz Inequality,

$$\begin{aligned}\int_\Omega |\nabla v(x,t)|\,\mathrm{d}x &= \int_\Omega \frac{|\nabla v(x,t)|}{\big(1 + |\nabla u(x,t)|^2\big)^{1/4}}\, \big(1 + |\nabla u(x,t)|^2\big)^{1/4}\,\mathrm{d}x \\ &\leqslant \sqrt{\int_\Omega \frac{|\nabla v(x,t)|^2}{\sqrt{1 + |\nabla u(x,t)|^2}}\,\mathrm{d}x}\ \sqrt{\int_\Omega \sqrt{1 + |\nabla u(x,t)|^2}\,\mathrm{d}x}.\end{aligned} \tag{2.12}$$

Moreover, when $n > 1$, from the Gagliardo–Nirenberg–Sobolev Inequality, we know that

$$\left(\int_\Omega u^{\frac{sn}{2(n-1)}}(x,t)\,\mathrm{d}x\right)^{\frac{n-1}{n}} = \|v\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)}(t) \leqslant C_0 \int_{\mathbb{R}^n} |\nabla v(x,t)|\,\mathrm{d}x$$

for some $C_0 > 0$. Also, when $n = 1$, one can use the Fundamental Theorem of Calculus and check that, for any $q \in [1, +\infty)$,

$$\left(\int_\Omega u^{\frac{sq}{2}}(x,t)\,\mathrm{d}x\right)^{\frac{1}{q}} = \|v\|_{L^q(\mathbb{R}^n)}(t) \leqslant C_1 \int_{\mathbb{R}^n} |\nabla v(x,t)|\,\mathrm{d}x$$

for some $C_1 > 0$.

Using this, (2.11) and (2.12), we obtain that

$$\left(\int_\Omega u^{\frac{sq}{2}}(x,t)\,\mathrm{d}x\right)^{\frac{1}{q}} \leqslant C_2 \sqrt{\int_\Omega \frac{\nabla u(x,t) \cdot \nabla u^{s-1}(x,t)}{\sqrt{1 + |\nabla u(x,t)|^2}}\,\mathrm{d}x}\ \sqrt{\int_\Omega \sqrt{1 + |\nabla u(x,t)|^2}\,\mathrm{d}x},$$

where $q = \frac{n}{n-1}$ when $n = 2$, and any $q \in [1, +\infty)$ when $n = 1$. From this and assumption (1.11), we find that

$$\left(\int_\Omega u^{\frac{sq}{2}}(x,t)\,\mathrm{d}x \right)^{\frac{2}{q}} \leqslant C_3 \int_\Omega \frac{\nabla u(x,t) \cdot \nabla u^{s-1}(x,t)}{\sqrt{1+|\nabla u(x,t)|^2}}\,\mathrm{d}x,$$

where $q = \frac{n}{n-1}$ when $n = 2$, and any $q \in [1, +\infty)$ when $n = 1$. In any case, when $n \in \{1, 2\}$, we have that we can take $q = 2$ and write

$$\int_\Omega u^s(x,t)\,\mathrm{d}x \leqslant C_3 \int_\Omega \frac{\nabla u(x,t) \cdot \nabla u^{s-1}(x,t)}{\sqrt{1+|\nabla u(x,t)|^2}}\,\mathrm{d}x.$$

Therefore, we have that (1.4) is satisfied for any $s \in (1, +\infty)$ and $\gamma := 1$. This and (1.6) imply (1.13), as desired.

Now we deal with the case in which (1.12) is satisfied. We can also assume that $n \geqslant 3$ (since the cases $n \in \{1, 2\}$ are covered by (1.11)). Then, exploiting the Gagliardo–Nirenberg–Sobolev Inequality in this situation and recalling (2.11), we see that

$$\begin{aligned} \frac{s^2}{4(s-1)} \int_\Omega \nabla u(x,t) \cdot \nabla u^{s-1}(x,t)\,\mathrm{d}x &= \int_\Omega |\nabla v(x,t)|^2\,\mathrm{d}x \\ &\geqslant C_0 \, \|v\|^2_{L^{\frac{2n}{n-2}}(\Omega)} \\ &= C_0 \left(\int_\Omega u^{\frac{sn}{n-2}} \right)^{\frac{n-2}{n}}, \end{aligned}$$

for some $C_0 > 0$. Hence, by Hölder's Inequality,

$$\int_\Omega \nabla u(x,t) \cdot \nabla u^{s-1}(x,t)\,\mathrm{d}x \geqslant C_1 \, \|u\|^s_{L^s(\Omega)},$$

for some $C_1 > 0$. Thus, in light of (1.12),

$$\int_\Omega \frac{\nabla u(x,t) \cdot \nabla u^{s-1}(x,t)}{\sqrt{1+|\nabla u(x,t)|^2}}\,\mathrm{d}x \geqslant C_2 \, \|u\|^s_{L^s(\Omega)}.$$

This gives that (1.4) holds true in this case with $\gamma := 1$. Therefore, by means of (1.6) we obtain (1.13), as desired. □

2.4. Proof of Theorem 1.6

We define $v := u^{(s-2+p)/p}$, and we claim that

$$|v(x,t) - v(y,t)|^p \leqslant C_0 |u(x,t) - u(y,t)|^{p-2}(u(x,t) - u(y,t))(u^{s-1}(x,t) - u^{s-1}(y,t)), \tag{2.13}$$

for some $C_0 > 0$. To this aim, we consider the auxiliary function

$$(1, +\infty) \ni \lambda \longmapsto g(\lambda) := \frac{(\lambda^{(s-2+p)/p} - 1)^p}{(\lambda - 1)^{p-1}(\lambda^{s-1} - 1)}.$$

We recall that $s-2+p>-1+p>0$ and observe that

$$\lim_{\lambda\to+\infty} g(\lambda)=\lim_{\lambda\to+\infty}\frac{\left(1-\frac{1}{\lambda^{(s-2+p)/p}}\right)^p}{\left(1-\frac{1}{\lambda}\right)^{p-1}\left(1-\frac{1}{\lambda^{s-1}}\right)}=\frac{(1-0)^p}{(1-0)^{p-1}(1-0)}=1$$

and that

$$\begin{aligned}\lim_{\lambda\searrow 1} g(\lambda)&=\lim_{\varepsilon\searrow 0}\frac{\left((1+\varepsilon)^{(s-2+p)/p}-1\right)^p}{\left((1+\varepsilon)-1\right)^{p-1}\left((1+\varepsilon)^{s-1}-1\right)}\\&=\lim_{\varepsilon\searrow 0}\frac{\left(1+((s-2+p)/p)\varepsilon+o(\varepsilon)-1\right)^p}{\varepsilon^{p-1}\left(1+(s-1)\varepsilon+o(\varepsilon)-1\right)}=\frac{((s-2+p)/p)^p}{s-1}.\end{aligned}$$

Consequently,

$$C_0:=\sup_{\lambda\in(1,+\infty)} g(\lambda)<+\infty.$$

Now, to prove (2.13), we may suppose, up to exchanging x and y, that $u(x,t)\geqslant u(y,t)$. Also, when either $u(y,t)=0$ or $u(x,t)=u(y,t)$, then (2.13) is obvious. Therefore, we can assume that $u(x,t)>u(y,t)>0$ and set

$$\lambda(x,y,t):=\frac{u(x,t)}{u(y,t)}\in(1,+\infty)$$

and conclude that

$$\begin{aligned}C_0&\geqslant g(\lambda(x,y,t))\\&=\frac{\left(\frac{u^{(s-2+p)/p}(x,t)}{u^{(s-2+p)/p}(y,t)}-1\right)^p}{\left(\frac{u(x,t)}{u(y,t)}-1\right)^{p-1}\left(\frac{u^{s-1}(x,t)}{u^{s-1}(y,t)}-1\right)}\\&=\frac{\left(u^{(s-2+p)/p}(x,t)-u^{(s-2+p)/p}(y,t)\right)^p}{(u(x,t)-u(y,t))^{p-1}\left(u^{s-1}(x,t)-u^{s-1}(y,t)\right)},\end{aligned}\tag{2.14}$$

and this proves (2.13).

Now, when $p\in\left(1,\frac{n}{\sigma}\right)$, we consider the fractional critical exponent

$$p_\sigma:=\frac{np}{n-\sigma p}.\tag{2.15}$$

We claim that

$$\|v\|^p_{L^q(\Omega)}(t)\leqslant C_1\iint_{\mathbb{R}^{2n}}\frac{|v(x,t)-v(y,t)|^p}{|x-y|^{n+\sigma p}}\,dx\,dy,\tag{2.16}$$

for some $C_1>0$, for every $q\in[1,p_\sigma]$ when $p\in\left(1,\frac{n}{\sigma}\right)$, and for every $q\in[1,+\infty)$ when $p\in\left[\frac{n}{\sigma},+\infty\right)$. Indeed, when $p\in\left(1,\frac{n}{\sigma}\right)$, then (2.16) follows by the Gagliardo–Sobolev Embedding (see, e.g., Theorem 3.2.1 in [10]). If instead $p\in\left[\frac{n}{\sigma},+\infty\right)$ and $q\in[1,+\infty)$, we set $\tilde q:=1+\max\{p,q\}$. Notice that

$$0<\frac{n}{p}-\frac{n}{\tilde q}<\frac{n}{p}\leqslant\sigma<1.$$

Hence, we can take

$$\tilde{\sigma} \in \left(\frac{n}{p} - \frac{n}{\tilde{q}},\ \frac{n}{p}\right),$$

and since

$$p \in \left(1, \frac{n}{\tilde{\sigma}}\right) \quad \text{and} \quad \tilde{q} \leqslant \frac{np}{n - \tilde{\sigma} p} = p_{\tilde{\sigma}},$$

we can make use of Gagliardo–Sobolev Embedding (see, e.g., Theorem 3.2.1 in [10]) with exponents $\tilde{\sigma}$ and $\tilde{q}$. In this way, we find that

$$\|v\|^p_{L^{\tilde{q}}(\Omega)}(t) \leqslant C_\sharp \iint_{\mathbb{R}^{2n}} \frac{|v(x,t) - v(y,t)|^p}{|x-y|^{n+\tilde{\sigma} p}}\, dx\, dy, \tag{2.17}$$

for some $C_\sharp > 0$. Now, we fix $M > 0$, to be taken appropriately large, and we observe that

$$\begin{aligned}&\iint_{\mathbb{R}^{2n} \cap \{|x-y| \leqslant M\}} \frac{|v(x,t) - v(y,t)|^p}{|x-y|^{n+\tilde{\sigma} p}}\, dx\, dy \\ &\quad \leqslant M^{(\sigma - \tilde{\sigma})p} \iint_{\mathbb{R}^{2n} \cap \{|x-y| \leqslant M\}} \frac{|v(x,t) - v(y,t)|^p}{|x-y|^{n+\sigma p}}\, dx\, dy\end{aligned}$$

and

$$\begin{aligned}&C_\sharp \iint_{\mathbb{R}^{2n} \cap \{|x-y| > M\}} \frac{|v(x,t) - v(y,t)|^p}{|x-y|^{n+\tilde{\sigma} p}}\, dx\, dy \\ &\leqslant C' \iint_{\mathbb{R}^{2n} \cap \{|x-y| > M\}} \frac{|v(x,t)|^p}{|x-y|^{n+\tilde{\sigma} p}}\, dx\, dy \\ &= \frac{C''}{M^{\tilde{\sigma} p}} \int_\Omega |v(x,t)|^p\, dx \leqslant \frac{C'''}{M^{\tilde{\sigma} p}} \|v\|^p_{L^{\tilde{q}}(\Omega)}(t) \leqslant \frac{1}{2} \|v\|^p_{L^{\tilde{q}}(\Omega)}(t),\end{aligned}$$

as long as M is large enough. Here above, we have denoted by C', C'' and C''' suitable positive constants and used that $\tilde{q} > p$ in order to use the Hölder's Inequality. These inequalities and (2.17) imply that

$$\frac{1}{2}\|v\|^p_{L^{\tilde{q}}(\Omega)}(t) \leqslant C_\sharp \left(1 + M^{(\sigma - \tilde{\sigma})p}\right) \iint_{\mathbb{R}^{2n}} \frac{|v(x,t) - v(y,t)|^p}{|x-y|^{n+\tilde{\sigma} p}}\, dx\, dy. \tag{2.18}$$

We also have that

$$\|v\|_{L^q(\Omega)}(t) \leqslant C \|v\|_{L^{\tilde{q}}(\Omega)}(t), \tag{2.19}$$

for some $C > 0$, in view of the Hölder's Inequality and the fact that $\tilde{q} > q$. Thanks to (2.18) and (2.19), we have completed the proof of (2.16).

Using (2.13), (2.16) and the fact that u and v vanish outside Ω, we see that, for every $q \in [1, p_\sigma]$ when $p \in \left(1, \frac{n}{\sigma}\right)$, and for every $q \in [1, +\infty)$ when $p \in \left[\frac{n}{\sigma}, +\infty\right)$,

$$\begin{aligned}
&\left(\int_\Omega u^{(s-2+p)q/p}(x,t)\,\mathrm{d}x\right)^{p/q} \\
&= \left(\int_{\mathbb{R}^n} v^q(x,t)\,\mathrm{d}x\right)^{p/q} \\
&= \|v\|^p_{L^q(\mathbb{R}^n)}(t) \\
&\leqslant C_1 \iint_{\mathbb{R}^{2n}} \frac{|v(x,t)-v(y,t)|^p}{|x-y|^{n+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant C_2 \iint_{\mathbb{R}^{2n}} \frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))(u^{s-1}(x,t)-u^{s-1}(y,t))}{|x-y|^{n+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y \\
&= 2C_2 \iint_{\mathbb{R}^{2n}} \frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))\,u^{s-1}(x,t)}{|x-y|^{n+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y \\
&= 2C_2 \int_{\mathbb{R}^n} (-\Delta)^\sigma_p u(x,t)\,u^{s-1}(x,t)\,\mathrm{d}x \\
&= 2C_2 \int_\Omega (-\Delta)^\sigma_p u(x,t)\,u^{s-1}(x,t)\,\mathrm{d}x,
\end{aligned} \tag{2.20}$$

for some $C_2 > 0$.

We also claim that when $p \in \left(1, \frac{n}{\sigma}\right)$ and

$$s \geqslant \frac{n(2-p)}{\sigma p}, \tag{2.21}$$

it holds that

$$\frac{sp}{s-2+p} \in [1, p_\sigma]. \tag{2.22}$$

Indeed, $s-2+p > 1-2+1 = 0$ and $sp-s+2-p = s(p-1)+2-p > (p-1)+2-p = 1$, which gives that $\frac{sp}{s-2+p} \geqslant 1$. In addition,

$$s(n-\sigma p) - n(s-2+p) = -s\sigma p + n(2-p) \leqslant 0,$$

thanks to (2.21), which, recalling (2.15), says that $\frac{sp}{s-2+p} \leqslant p_\sigma$. These considerations prove (2.22).

From (2.22) it follows that if either $s \geqslant \frac{n(2-p)}{\sigma p}$ or $p \geqslant \frac{n}{\sigma}$, then we can choose $q := \frac{sp}{s-2+p}$ in (2.20). Consequently, we have that

$$\left(\int_\Omega u^s(x,t)\,\mathrm{d}x\right)^{\frac{s-2+p}{s}} \leqslant 2C_2 \int_\Omega (-\Delta)^\sigma_p u(x,t)\,u^{s-1}(x,t)\,\mathrm{d}x. \tag{2.23}$$

This says that (1.4) is satisfied with $\gamma := p-1$. Hence, we are in position of exploiting (1.6), obtaining that if either $s \geqslant \frac{n(2-p)}{\sigma p}$ or $p \geqslant \frac{n}{\sigma}$,

$$\|u\|_{L^s(\Omega)}(t) \leqslant \frac{C_\star}{1+t^{\alpha/(p-1)}}. \tag{2.24}$$

We also observe that when $s \in \left(1, \frac{n(2-p)}{\sigma p}\right)$, we have that

$$\|u\|_{L^s(\Omega)} \leqslant \hat{C}\,\|u\|_{L^{\frac{n(2-p)}{\sigma p}}(\Omega)},$$

thanks to the Hölder's Inequality. This and (2.24) imply (1.16) for all $s > 1$ and $p > 1$. □

2.5. Proof of Theorem 1.7

The main idea is to use (2.23) for each index $j \in \{1, \dots, N\}$. That is, we fix

$$\tilde{s} := \max\left\{s, \frac{n(2-p_1)}{\sigma_1 p_1}, \dots, \frac{n(2-p_N)}{\sigma_N p_N}\right\}$$

and we exploit (2.23) to write that

$$\|u\|^{\tilde{s}-2+p_j}_{L^{\tilde{s}}(\Omega)}(t) = \left(\int_\Omega u^{\tilde{s}}(x,t)\,\mathrm{d}x\right)^{\frac{\tilde{s}-2+p_j}{\tilde{s}}} \leqslant C \int_\Omega (-\Delta)^{\sigma_j}_{p_j} u(x,t)\, u^{\tilde{s}-1}(x,t)\,\mathrm{d}x, \tag{2.25}$$

for some $C > 0$.

We also observe that

$$\|u\|_{L^{\tilde{s}}(\Omega)}(t) \leqslant \|u\|_{L^{\tilde{s}}(\Omega)}(0). \tag{2.26}$$

Indeed, we have that

$$\big(u(x,t)-u(y,t)\big)\big(u^{\tilde{s}-1}(x,t)-u^{\tilde{s}-1}(y,t)\big) \geqslant 0,$$

and therefore,

$$\begin{aligned}
&2\int_\Omega u^{\tilde{s}-1}(x,t)\,\mathcal{N}[u](x,t)\,\mathrm{d}x\\
&\quad= 2\sum_{j=1}^N \beta_j \int_\Omega u^{\tilde{s}-1}(x,t)\,(-\Delta)^{\sigma_j}_{p_j} u(x,t)\,\mathrm{d}x\\
&\quad= 2\sum_{j=1}^N \beta_j \iint_{\mathbb{R}^{2n}} \frac{|u(x,t)-u(y,t)|^{p-2}\big(u(x,t)-u(y,t)\big)}{|x-y|^{n+\sigma_j p_j}}\, u^{\tilde{s}-1}(x,t)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad= \sum_{j=1}^N \beta_j \iint_{\mathbb{R}^{2n}} \frac{|u(x,t)-u(y,t)|^{p-2}\big(u(x,t)-u(y,t)\big)\big(u^{\tilde{s}-1}(x,t)-u^{\tilde{s}-1}(y,t)\big)}{|x-y|^{n+\sigma_j p_j}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\geqslant 0.
\end{aligned} \tag{2.27}$$

Furthermore, from (2.1), we know that

$$\|u\|^{\tilde{s}-1}_{L^{\tilde{s}}(\Omega)}(t)\;\partial^\alpha_t \|u\|_{L^{\tilde{s}}(\Omega)}(t) \leqslant -\int_\Omega u^{\tilde{s}-1}(x,t)\,\mathcal{N}[u](x,t)\,\mathrm{d}x.$$

This and (2.27) give that

$$\|u\|^{\tilde{s}-1}_{L^{\tilde{s}}(\Omega)}(t)\;\partial^\alpha_t \|u\|_{L^{\tilde{s}}(\Omega)}(t) \leqslant 0. \tag{2.28}$$

We now observe that if $\mu > 0$, $f \geqslant 0$, and

$$f^{\mu}(t)\, \partial_t^{\alpha} f(t) \leqslant 0 \text{ with } f(0) > 0, \text{ then } \partial_t^{\alpha} f(t) \leqslant 0. \tag{2.29}$$

We prove[2] this by contradiction, supposing that $\partial_t^{\alpha} f(t_\star) > 0$ for some $t_\star > 0$. Hence, we find an open interval $(a_\star, b_\star)$, with $0 < a_\star < t_\star$ such that $\partial_t^{\alpha} f(t) > 0$ for all $t \in (a_\star, b_\star)$, $f(t) = 0$ for all $t \in (a_\star, b_\star)$, and $f(t) > 0$ for all $t \in [0, a_\star)$. This gives that $f(a_\star) = 0$. Now, from (1.2), integrating by parts twice we obtain that

$$\begin{aligned}
\partial_t^{\alpha} f(t) &= \frac{1}{\alpha - 1} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \big(f(\tau) - f(0)\big) \frac{\mathrm{d}}{\mathrm{d}\tau} (t-\tau)^{1-\alpha}\, \mathrm{d}\tau \\
&= \frac{1}{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \left[\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau} \big(f(\tau) - f(0)\big)(t-\tau)^{1-\alpha}\, \mathrm{d}\tau \right] \\
&= \frac{1}{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \left[\int_0^t \dot{f}(\tau)(t-\tau)^{1-\alpha}\, \mathrm{d}\tau \right] \\
&= \int_0^t \dot{f}(\tau)(t-\tau)^{-\alpha}\, \mathrm{d}\tau \\
&= \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau} \big(f(\tau) - f(t)\big)(t-\tau)^{-\alpha}\, \mathrm{d}\tau \\
&= \frac{f(t) - f(0)}{t^{\alpha}} + \alpha \int_0^t \frac{f(t) - f(\tau)}{(t-\tau)^{1+\alpha}}\, \mathrm{d}\tau
\end{aligned}$$

Consequently,

$$\begin{aligned}
0 &\leqslant \lim_{t \searrow a_\star} \partial_t^{\alpha} f(t) \\
&= \partial_t^{\alpha} f(a_\star) \\
&= \frac{f(a_\star) - f(0)}{a_\star^{\alpha}} + \alpha \int_0^{a_\star} \frac{f(a_\star) - f(\tau)}{(a_\star - \tau)^{1+\alpha}}\, \mathrm{d}\tau \\
&= -\frac{f(0)}{a_\star^{\alpha}} - \alpha \int_0^{a_\star} \frac{f(\tau)}{(a_\star - \tau)^{1+\alpha}}\, \mathrm{d}\tau \\
&< 0.
\end{aligned}$$

This contradiction establishes (2.29).

By (2.28) and (2.29), we find that $\partial_t^{\alpha} \|u\|_{L^{\tilde{s}}(\Omega)}(t) \leqslant 0$. From this, we obtain (2.26) by inverting the Caputo derivative by a Volterra integral kernel (see, e.g., formula (2.61) in [2]; alternatively, one could also use the comparison principle, e.g., Lemma 2.6 in [69]).

Then, using (2.25) and (2.26), we conclude that

$$\|u\|_{L^{\tilde{s}}(\Omega)}^{\tilde{s}-2+p_{\max}}(t) \leqslant C' \, \|u\|_{L^{\tilde{s}}(\Omega)}^{\tilde{s}-2+p_j} \leqslant C'' \int_{\Omega} (-\Delta)_{p_j}^{\sigma_j} u(x,t)\, u^{s-1}(x,t)\, \mathrm{d}x,$$

[2] After this work was completed, archived and submitted, the very interesting preprint [41] became available: with respect to this, we mention that formula (2.29) here also follows from Lemma 2.1 in [41].

for some $C', C'' > 0$, and so, multiplying by $\beta_j > 0$ and summing up over $j \in \{1, \dots, N\}$,

$$\|u\|_{L^{\tilde{s}}(\Omega)}^{\tilde{s}-2+p_{\max}}(t) \leqslant C''' \int_\Omega \sum_{j=1}^N \beta_j \, (-\Delta)_{p_j}^{\sigma_j} u(x,t)\, u^{\tilde{s}-1}(x,t)\, \mathrm{d}x.$$

This says that (1.4) is satisfied with s replaced by $\tilde{s}$ and $\gamma := p_{\max} - 1$. Accordingly, we can exploit (1.6) and find that

$$\|u\|_{L^{\tilde{s}}(\Omega)}(t) \leqslant \frac{C_\star}{1 + t^{\frac{\alpha}{p_{\max}-1}}}. \tag{2.30}$$

Since, by Hölder's Inequality and the fact that $s \leqslant \tilde{s}$, we have that $\|u\|_{L^s(\Omega)} \leqslant C\|u\|_{L^{\tilde{s}}(\Omega)}$, for some $C > 0$, we deduce from (2.30) that (1.17) holds true, as desired. □

2.6. Proof of Theorem 1.8

We fix $j \in \{1, \dots, n\}$ and $(\rho_1, \dots, \rho_{j-1}, \rho_{j+1}, \dots, \rho_n) \in \mathbb{R}^{n-1}$ and denote $\Omega_j(\rho_1, \dots, \rho_{j-1}, \rho_{j+1}, \dots, \rho_n) := \Omega \cap \mathbb{R}_j(\rho_1, \dots, \rho_{j-1}, \rho_{j+1}, \dots, \rho_n)$, where

$$\mathbb{R}_j(\rho_1, \dots, \rho_{j-1}, \rho_{j+1}, \dots, \rho_n) := \big\{(\rho_1, \dots, \rho_{j-1}, 0, \rho_{j+1}, \dots, \rho_n) + r e_j,\ \ r \in \mathbb{R}\big\}.$$

The function $\mathbb{R} \ni \rho_j \mapsto u(\rho_1 e_1 + \dots + \rho_n e_n, t)$ is supported inside the closure of the bounded set $\Omega_j(\rho_1, \dots, \rho_{j-1}, \rho_{j+1}, \dots, \rho_n)$, and using (2.23) with $p := 2$, we get that

$$\begin{aligned}
&\int_{\mathbb{R}} u^s(\rho_1 e_1 + \dots + \rho_n e_n, t)\, \mathrm{d}\rho_j \\
&= \int_{\Omega_j(\rho_1, \dots, \rho_{j-1}, \rho_{j+1}, \dots, \rho_n)} u^s(\rho_1 e_1 + \dots + \rho_n e_n, t)\, \mathrm{d}\rho_j \\
&\leqslant C \int_{\mathbb{R}} (-\partial_{x_j}^2)^{\sigma_j} u(\rho_1 e_1 + \dots + \rho_n e_n, t)\, u^{s-1}(\rho_1 e_1 + \dots + \rho_n e_n, t)\, \mathrm{d}\rho_j,
\end{aligned}$$

for some $C > 0$.

We now integrate such inequality over the other coordinates $(\rho_1, \dots, \rho_{j-1}, \rho_{j+1}, \dots, \rho_n)$, and we thus obtain that

$$\begin{aligned}
\int_\Omega u^s(x,t)\, \mathrm{d}x &= \int_{\mathbb{R}^n} u^s(x,t)\, \mathrm{d}x \\
&= \int_{\mathbb{R}^n} u^s(\rho_1 e_1 + \dots + \rho_n e_n, t)\, \mathrm{d}\rho \\
&\leqslant C \int_{\mathbb{R}^n} (-\partial_{x_j}^2)^{\sigma_j} u(\rho_1 e_1 + \dots + \rho_n e_n, t)\, u^{s-1}(\rho_1 e_1 + \dots + \rho_n e_n, t)\, \mathrm{d}\rho \\
&= C \int_\Omega (-\partial_{x_j}^2)^{\sigma_j} u(x,t)\, u^{s-1}(x,t)\, \mathrm{d}x.
\end{aligned}$$

We multiply this inequality by $\beta_j > 0$ and we sum over j, and we find that

$$\int_\Omega u^s(x,t)\,dx \leqslant C' \int_\Omega (-\Delta_\beta)^\sigma u(x,t)\,u^{s-1}(x,t)\,dx,$$

for some $C' > 0$. Hence, (1.4) holds true with $\gamma := 1$, and then, the desired result follows from (1.6). $\square$

2.7. Proof of Theorem 1.9

It is convenient to define $\tilde{u} := u^m$, $\tilde{s} := 1+\frac{s-1}{m}$ and $\tilde{v} := \tilde{u}^{\tilde{s}/2}$. Let also $v := u^{\frac{m+s-1}{2}}$. We remark that

$$\tilde{u}^{\tilde{s}-1} = u^{s-1}, \quad \text{and} \quad \tilde{v} = u^{m\tilde{s}/2} = u^{(m+s-1)/2} = v. \tag{2.31}$$

We also exploit (2.13) with $p := 2$ to the functions $\tilde{u}$ and $\tilde{v}$, with exponent $\tilde{s}$. In this way, we have that

$$|\tilde{v}(x,t) - \tilde{v}(y,t)|^2 \leqslant C_0(\tilde{u}(x,t) - \tilde{u}(y,t))(\tilde{u}^{\tilde{s}-1}(x,t) - \tilde{u}^{\tilde{s}-1}(y,t)),$$

for some $C_0 > 0$. This estimate and (2.31) give that

$$|v(x,t) - v(y,t)|^2 \leqslant C_0(u^m(x,t) - u^m(y,t))(u^{s-1}(x,t) - u^{s-1}(y,t)). \tag{2.32}$$

Also, exploiting formula (2.16) with $p := 2$, we have that

$$\left(\int_\Omega u^{\frac{q(m+s-1)}{2}}(x,t)\,dx\right)^{2/q} = \|v\|^2_{L^q(\Omega)}(t) \leqslant C_1 \iint_{\mathbb{R}^{2n}} \frac{|v(x,t) - v(y,t)|^2}{|x-y|^{n+2\sigma}}\,dx\,dy, \tag{2.33}$$

for some $C_1 > 0$, for every $q \in \left[1, \frac{2n}{n-2\sigma}\right]$ when $2 \in \left(1, \frac{n}{\sigma}\right)$, and for every $q \in [1,+\infty)$ when $2 \in \left[\frac{n}{\sigma}, +\infty\right)$.

Now we observe that when

$$s \geqslant m-1, \tag{2.34}$$

it holds that

$$\frac{2s}{m+s-1} \in \left[1, \frac{2n}{n-2\sigma}\right]. \tag{2.35}$$

Indeed, we have that

$$s(n-2\sigma) - n(m+s-1) = -2\sigma s - n(m-1) \leqslant 0,$$

which says that $\frac{2s}{m+s-1} \leqslant \frac{2n}{n-2\sigma}$. On the other hand, from (2.34), we see that

$$2s - (m+s-1) = s - m + 1 \geqslant 0,$$

giving that $\frac{2s}{m+s-1} \geqslant 1$. This proves (2.35).

Now, by (2.35), when either $2 \in \left[\frac{n}{\sigma}, +\infty\right)$ or $s \geqslant m-1$, we are allowed to choose $q := \frac{2s}{m+s-1}$ in (2.33) and conclude that

$$\left(\int_\Omega u^s(x,t)\,\mathrm{d}x\right)^{(m+s-1)/s} \leqslant C_1 \iint_{\mathbb{R}^{2n}} \frac{|v(x,t)-v(y,t)|^2}{|x-y|^{n+2\sigma}}\,\mathrm{d}x\,\mathrm{d}y.$$

Combining this with (2.32), we find that

$$\begin{aligned}
&\left(\int_\Omega u^s(x,t)\,\mathrm{d}x\right)^{(m+s-1)/s} \\
&\leqslant C_2 \iint_{\mathbb{R}^{2n}} \frac{(u^m(x,t)-u^m(y,t))(u^{s-1}(x,t)-u^{s-1}(y,t))}{|x-y|^{n+2\sigma}}\,\mathrm{d}x\,\mathrm{d}y \\
&= 2C_2 \iint_{\mathbb{R}^{2n}} \frac{(u^m(x,t)-u^m(y,t))\,u^{s-1}(x,t)}{|x-y|^{n+2\sigma}}\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

provided that either $2 \in \left[\frac{n}{\sigma}, +\infty\right)$ or $s \geqslant m-1$.

This says that, under these circumstances, condition (1.4) is fulfilled with $\gamma := m$. Therefore, we can exploit (1.6) and obtain (1.18), provided that either $2 \in \left[\frac{n}{\sigma}, +\infty\right)$ or $s \geqslant m-1$.

Then, when $2 < \frac{n}{\sigma}$, we first establish (1.18) for a large exponent of the Lebesgue norm, and then, we reduce it by using the Hölder's Inequality.

This completes the proof of (1.18) in all the cases, as desired. □

2.8. Proof of Theorem 1.10

We claim that

$$\begin{aligned}
&F\left(\frac{u(x,t)-u(y,t)}{|x-y|}\right)\left(u^{s-1}(x,t)-u^{s-1}(y,t)\right) \\
&\qquad \geqslant c_0\,\frac{\big(u(x,t)-u(y,t)\big)\big(u^{s-1}(x,t)-u^{s-1}(y,t)\big)}{|x-y|}
\end{aligned} \tag{2.36}$$

for some $c_0 > 0$. To check this, we observe that, by (1.19), F is odd; hence, we can reduce to the case in which

$$u(x,t) \geqslant u(y,t). \tag{2.37}$$

Also, by (1.19), we see that when $|r|$ is bounded, then $F(r) \simeq r$, and therefore, by (1.21) and (2.37),

$$F\left(\frac{u(x,t)-u(y,t)}{|x-y|}\right) \geqslant c_0\,\frac{u(x,t)-u(y,t)}{|x-y|},$$

and this implies (2.36).

Now, we let $v := u^{s/2}$. We use (2.13) (with $p := 2$) and (2.36) to deduce that

$$
\begin{aligned}
&2 \iint_{\mathbb{R}^{2n}} \frac{1}{|y|^{n+\sigma}} F\left(\frac{u(x,t)-u(x+y,t)}{|y|}\right) u^{s-1}(x,t)\,\mathrm{d}x\,\mathrm{d}y \\
&= 2 \iint_{\mathbb{R}^{2n}} \frac{1}{|x-y|^{n+\sigma}} F\left(\frac{u(x,t)-u(y,t)}{|x-y|}\right) u^{s-1}(x,t)\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_{\mathbb{R}^{2n}} \frac{1}{|x-y|^{n+\sigma}} F\left(\frac{u(x,t)-u(y,t)}{|x-y|}\right) u^{s-1}(x,t)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \iint_{\mathbb{R}^{2n}} \frac{1}{|x-y|^{n+\sigma}} F\left(\frac{u(y,t)-u(x,t)}{|x-y|}\right) u^{s-1}(y,t)\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_{\mathbb{R}^{2n}} \frac{1}{|x-y|^{n+\sigma}} F\left(\frac{u(x,t)-u(y,t)}{|x-y|}\right) \big(u^{s-1}(x,t)-u^{s-1}(y,t)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&\geqslant c_0 \iint_{\mathbb{R}^{2n}} \frac{\big(u(x,t)-u(y,t)\big)\big(u^{s-1}(x,t)-u^{s-1}(y,t)\big)}{|x-y|^{n+\sigma+1}}\,\mathrm{d}x\,\mathrm{d}y \\
&\geqslant \frac{c_0}{C_0} \iint_{\mathbb{R}^{2n}} \frac{\big|v(x,t)-v(y,t)\big|^2}{|x-y|^{n+2\sigma'}}\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}
\tag{2.38}
$$

where $\sigma' := \frac{\sigma+1}{2} \in (0,1)$.

Furthermore, we recall (2.16), used here with fractional exponent σ' and with $p := 2$, and we see that

$$
\left(\int_\Omega u^{sq/2}(x,t)\,\mathrm{d}x\right)^{2/q} = \|v\|^2_{L^q(\Omega)}(t) \leqslant C_1 \iint_{\mathbb{R}^{2n}} \frac{\big|v(x,t)-v(y,t)\big|^2}{|x-y|^{n+2\sigma'}}\,\mathrm{d}x\,\mathrm{d}y,
\tag{2.39}
$$

for some $C_1 > 0$, for every $q \in \left[1, \frac{2n}{n-2\sigma'}\right]$ when $2 \in \left(1, \frac{n}{\sigma'}\right)$, and for every $q \in [1,+\infty)$ when $2 \in \left[\frac{n}{\sigma'}, +\infty\right)$.

In any case, since $\frac{2n}{n-2\sigma'} > 2$, we can always choose $q := 2$ in (2.39) and conclude that

$$
\int_\Omega u^s(x,t)\,\mathrm{d}x \leqslant C_1 \iint_{\mathbb{R}^{2n}} \frac{\big|v(x,t)-v(y,t)\big|^2}{|x-y|^{n+2\sigma'}}\,\mathrm{d}x\,\mathrm{d}y.
$$

Combining this with (2.38), we infer that

$$
\int_\Omega u^s(x,t)\,\mathrm{d}x \leqslant C_2 \iint_{\mathbb{R}^{2n}} \frac{1}{|y|^{n+\sigma}} F\left(\frac{u(x,t)-u(x+y,t)}{|y|}\right) u^{s-1}(x,t)\,\mathrm{d}x\,\mathrm{d}y
$$

for some $C_2 > 0$, which, together with (1.20), establishes (1.4) with $\gamma := 1$. This and (1.6) yield the thesis of Theorem 1.10. □

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