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**Abstract**

The Jordan-Moore-Gibson-Thompson equation is a prominent example of a Partial Differential Equation model which describes the acoustic velocity potential in ultrasound wave propagation, and where the paradox of infinite speed of propagation of thermal signals is eliminated; the use of the constitutive Cattaneo law for the heat flux, in place of the Fourier law, accounts for its being of third order in time. A great deal of attention has been recently devoted to its linearization – referred to in the literature as the Moore-Gibson-Thompson equation – whose analysis poses already several questions and mathematical challenges. In this work we consider and solve a quadratic control problem associated with the linear equation, formulated consistently with the goal of keeping the acoustic pressure close to a reference pressure during ultrasound excitation, as required in medical and industrial applications. While optimal control problems with *smooth* controls have been considered in the recent literature, we aim at relying on controls which are just $L^2$ in time; this leads to a singular control problem and to non-standard Riccati equations. In spite of the unfavorable combination of the semigroup describing the free dynamics that is *not* analytic [in contrast to models based on the Fourier law], with the pattern displayed by the dynamics subject to boundary control, a feedback synthesis of the optimal control, via the solution to an associated operator Riccati equation, is established.

**Key words**: ultrasound waves, optimal boundary control, absorbing boundary conditions, high intensity focused ultrasound, singular control, nonstandard Riccati equations, feedback synthesis

## 1 Introduction and motivation

Partial Differential Equation (PDE) models for the propagation of ultrasound waves – more specifically, high intensity ultrasound propagation – are relevant to a number of medical and industrial applications. To name but a few, lithotripsy, thermoterapy, (ultrasound) welding, sonochemistry; cf., e.g., [15]. The excitation of induced acoustic fields in order to attain a given task, such as destroying certain obstructions (such as, e.g., stones in kidneys or deposits resulting from chemical reactions), renders the presence of control functions within the model well-founded.

The subject of the present investigation is an optimal control problem for a third order in time PDE, referred to in the literature as the Moore-Gibson-Thompson equation, which is the linearization of the Jordan-Moore-Gibson-Thompson (JMGT) equation, arising in the modeling of ultrasound waves; see [16, 17], [19], [35]. In contrast with the renowned Westervelt ([37]) and Kuznetsov equations, the JMGT equation displays a *finite* speed of propagation of acoustic waves, thereby providing a solution to the infinite speed of propagation paradox. This is achieved by replacing the Fourier's law of heat conduction by the Cattaneo law ([8]); the distinct constitutive law brings about an additional time derivative of the acoustic velocity field (or acoustic pressure).

Restricting the analysis to the relevant spatial dimensions $n = 2, 3$, a Neumann boundary control will be acting as a force on a manifold $\Gamma_0$ of dimension $n-1$; $\Gamma_0$ will eventually represent a boundary portion of a bounded domain $\Omega \subset \mathbb{R}^n$. (It is an established procedure to reduce the analysis of wave processes on *unbounded* domains to boundary or initial/boundary value problems (IBVP) on *bounded* domains via the introduction of artificial boundaries.) Thus, *absorbing* boundary conditions (BC) will be taken on a complementary part of the boundary $\Gamma_1 = \partial\Omega \setminus \Gamma_0$; see section 1.2. We shall assume that the two parts of the boundary do not intersect. The optimal control problem arises from the minimization of the acoustic pressure in $\Omega$. This setup, which is motivated by significant applications and technologies, has been already adopted in the literature in connection with the said nonlinear PDEs; see [10], [9], [19], [30], [31] (and the references therein).

From the mathematical point of view, two main challenges appear. The first one is due to the presence of *boundary* controls, which naturally bring about unbounded input operators $B$ into the (linear) abstract state equation $y' = Ay + Bg$; see [5], [27]. It is well known that this issue can be dealt with by exploiting the additional regularity of the PDE dynamics: this occurs in the case of PDEs plainly governed by analytic semigroups $e^{At}$. The reader is referred to the classical texts [5] and [27, Vol. I] for a thorough study of the Linear-Quadratic (LQ) problem for parabolic-like PDEs, along with the related differential and algebraic Riccati equations.
(We note that the same is actually valid in the case of PDE problems whose corresponding abstract control systems satisfy appropriate *singular estimates* for $e^{At}B$, even if the semigroup $e^{At}$ is not analytic ([24]). And, further, appropriate regularity properties can be displayed by certain coupled systems of hyperbolic-parabolic PDEs subject to boundary control – including thermoelastic systems, acoustic-structure and fluid-structure interactions –, which ensure the solvability of the associated optimal control problems (with quadratic functionals), along with well-posed Riccati equations. The ultimate finite and infinite time horizon theories, as well as references to the motivating PDE systems, are found respectively in [1] and [2].)

Returning to the PDE under investigation, as we know from [29] and [21], the dynamics of the (uncontrolled) Moore-Gibson-Thompson equation, with classical Dirichlet or Neumann BC, is described by a *group* of operators, displaying an intrinsic hyperbolic character, and hence a lack of regularity of its dynamics. In addition, a major challenge is brought about by the presence – that cannot be eluded – of the time derivative of the control function $g(t,x)$ within the control system, which becomes

$$y' = Ay + B_0 g + B_1 g_t\,, \tag{1.1}$$

whereas on the other hand, penalization involves only the $L^2$ (in time) norm of the controls. This means that the cost functional is not coercive with respect to $g_t$. The resulting linear-quadratic problem becomes *singular*. It must be recalled that these features have been already encountered and dealt with in the study of optimal boundary control of (second-order in time) wave equations with structural damping; see the former study [6] and the subsequent analysis (and solutions) proposed in [36], [25], [26]. Because of the strong damping, in that case the free dynamics is described by an analytic semigroup, and displays an enhanced regularity of the control-to-state map; this feature has been exploited in the previous studies [6] and [36, 25, 26]. Instead, the present PDE problem is of hyperbolic type.

The goal of the present paper is to provide a framework for such class of singular control problems, in the case of a hyperbolic-like dynamics which intrinsically does not exhibit regularizing effects on its evolution. It is important to emphasize that while the singularity of the control is reflected in difficulties when treating *time* dependence, unbounded inputs affect the analysis of *space* dependence. So, the infinite-dimensional aspect of evolution is at the heart of the problem studied. To the authors' best knowledge this is a first investigation where a singular control problem associated with the control system (1.1) appear, in an infinite dimensional context and with a general semigroup governing the free dynamics.

### 1.1 The nonlinear model and its linearization

The Jordan-Moore-Gibson-Thompson (JMGT) equation is one of the fundamental equations in nonlinear acoustics which describes wave propagation in viscous thermally relaxing fluids. Its linearization is found in the literature as the Moore-Gibson-Thompson (MGT) equation. (In recognition of the original work on it by Stokes ([34]), it might rather be termed Stokes-Moore-Gibson-Thompson equation, as Pedro Jordan himself suggested; hence the acronym SMGT (in place of MGT) will be utilized throughout the paper.) The fully nonlinear PDE, that is the JMGT equation, is the following one:

$$\tau\psi_{ttt} + \alpha\psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = \frac{\partial}{\partial t}\Big(\frac{1}{c^2}\frac{B}{2A}\psi_t^2 + |\nabla\psi|^2\Big) \tag{1.2}$$

where $\tau > 0$ is a time relaxation parameter, the unknown $\psi = \psi(t,x)$ is the *acoustic velocity potential*, the space variable $x$ varies in a bounded domain $\Omega \subset \mathbb{R}^n$, $c$ is the speed of sound, the parameter $b$ stands for diffusivity, $\alpha > 0$ is a damping parameter and $A, B$ are suitable nonlinearity constants; then, $-\nabla\psi$ is the acoustic *particle velocity*.

When $\tau = 0$ the model becomes the Kuznetsov equation, that is

$$\alpha\psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = \frac{\partial}{\partial t}\Big(\frac{1}{c^2}\frac{B}{2A}\psi_t^2 + |\nabla\psi|^2\Big)\,, \tag{1.3}$$

a (second order in time) quasilinear PDE characterized by an infinite speed of propagation. The positive diffusivity coefficient $b$ provides a regularizing effect on its evolution; the corresponding linearized equation is of parabolic type, as its dynamics is governed by an analytic semigroup. Instead, as found out in the former works [21] and [29], in the case $\tau > 0$ the PDE turns to a hyperbolic character.

Optimal control problems with quadratic functional for both the Kuznetsov and Westervelt equations have been studied first in [10] and [9]; see also [19]. The latter reads as

$$\alpha u_{tt} - c^2\Delta u - b\Delta u_t = \beta \frac{\partial^2}{\partial t^2}(u^2)$$

in terms of the acoustic pressure $u$, where $\beta > 0$ is a suitable parameter of nonlinearity. (The relation $u = \rho\psi_t$ between the acoustic pressure and velocity potential – $\rho(x)$ being the mass density – allows another formulation of the Kuznetsov equation, with the pressure as the unknown variable.) Then, the ultrasound excitation on a certain manifold $\Gamma_0$ (of dimension $n-1$) can be represented by means of the Neumann boundary condition $\frac{\partial u}{\partial \nu} = g$ on $\Gamma_0$, where $g$ is the control function. A question which arises is to minimize appropriate cost functionals associated with the controlled PDE.

In the works [10] and [9] quadratic functionals of tracking type are taken into consideration, such as

$$J(g) = \frac{1}{2}\int_0^T\int_\Omega |u-u^d|^2\,dx\,dt + \frac{\alpha}{2}\int_0^T\int_{\Gamma_0} |g|^2\,d\sigma\,dt\,,$$

and

$$J(g) = \frac{1}{2}\int_\Omega |u(T,x)-u^d(x)|^2\,dx + \frac{\alpha}{2}\int_0^T\int_{\Gamma_0} |g|^2\,d\sigma\,dt$$

respectively, where $u^d$ is a given reference pressure; the class of admissible controls $G^{ad}$ is a suitably chosen space whose topology is induced by

$$H^1(0,T;H^{1/2}(\Gamma_0)) \cap H^2(0,T;H^{-1/2}(\Gamma_0))\,. \tag{1.4}$$

A critical role in these studies was played by (i) the assumption that $G^{ad}$ represents a space of smooth controls – more precisely, differentiable in time and subject to appropriate compatibility conditions (with respect to initial data) –, as well as (ii) the control constructed is an open-loop one, rather than a feedback one; (iii) the solutions considered are suitably small and the state equation is of parabolic type.
For such class of controls existence, uniqueness of solutions for *small data* (due to quasilinearity) has been derived; see [22], [19]. The optimal control is characterized via the Pontryagin Maximum Principle; see [10].

The present study, although focused on a simpler linear equation, departs from the avenues (i)–(iii), guided by two major goals. On one hand, we aim at minimizing a quadratic functional that penalizes controls functions in the $L^2$ (in time and space) norm, with (state) solutions under consideration not necessarily smooth (in space). A set of admissible controls that possess a low regularity is consistent with physical and engineering applications; see, e.g., [15]. In addition, feedback or closed-loop controls are of particular interest.

On the other hand, as already apparent in the case of the Westervelt equation – as well as in the case of its linearization, that is the strongly damped wave equation ([6]) –, the modeling of boundary control actions naturally brings about the time derivative of the control function, which is somehow 'hidden' within the PDE problem. This intrinsic analytical aspect will be made clear later, once we derive the input-to-state solution formula, after the third order abstract equation (2.5). (If one were to pursue such a study in the case of the JMGT equation, a natural choice would be to begin with the linear dynamics: it is already there where non-smoothness of controls will provide sufficient challenge. In fact, the minimization problem overall $L^2$ controls may not ensure an optimal solution even in the linear case, as already noted in [26]. We shall confirm this finding in the case of the problem under consideration.)

The above suggests that appropriate adjustments in the formulation of the problem and its modeling need to be made. We shall show that by enlarging slightly the class of controls resolves the issue of existence of optimal solution. Having established this, we shall proceed with the optimality analysis and the construction of a feedback control for the PDE which will still display 'rough' states (namely, displaying low space regularity). However, the feedback solution will be shown to generate sufficiently regular outputs (i.e., observed states) which can be used to control the system on-line – via the solution to a *non-standard* differential Riccati Equation (RE). The well-posedness of these corresponding non-standard Riccati equations provides a contribution of independent interest. In fact, the construction of solutions to the RE requires the extension of the dynamics to extrapolation spaces with very low regularity. This is needed in order to make the dynamics invariant.

To recapitulate, the novel contribution of the present work pertains to optimal feedback control of the acoustic SMGT equation; the closed-loop control will be generated by solving an appropriate non-standard Riccati equation. (The non-standard structure is due to the singular nature of the optimization problem.) Focus is placed on the linearized version of the model, which already provides significant challenges in terms of the underlying analysis and constitutes a necessary step for a further treatment of nonlinear problems. The expectation is that once a solution is given for the optimal feedback control of the linearized dynamics, such control may be used for the nonlinear problem, which then will have to be considered with small initial data. A similar approach has been pursued successfully in the case of the Navier-Stokes equations; cf. [3], [4].

### 1.2 Mathematical setting

We consider the problem of controlling the acoustic excitation on a certain closed region $\Gamma_0$ while maintaining the acoustic pressure below a certain threshold; $\Gamma_0$ will be identified as a part of the boundary of an introduced bounded domain $\Omega$. Then, an artificial boundary $\Gamma_1$ is introduced in order to limit the area of observation/computation. The *absorbing* boundary conditions (BC) on $\Gamma_1$ are then used to avoid reflections: roughly, no waves can ‘come back’. Accordingly, and consistently with the analysis carried out in [10] (on a classical nonlinear model for ultrasound wave propagation like the Westervelt equation), we will complement the SMGT equation with the BC which are the most pertinent: namely,

- Neumann *boundary control* acting on $\Gamma_0$ (the so called excited boundary); $g$ below represents a surface force;
- absorbing BC on the complement $\Gamma_1 = \partial\Omega\backslash\Gamma_0$ (the so called *absorbing boundary*).

(Higher order nonlinear absorbing BC have been introduced in [33] for the study of the Westervelt equation in an unbounded domain, that allow for efficient and robust numerical simulations.)

Thus, the boundary value problem (BVP) is as follows:

$$\begin{cases} \tau u_{ttt} + \alpha u_{tt} - c^2\Delta u - b\Delta u_t = 0 & \text{on } (0,T)\times\Omega \\ \frac{\partial u}{\partial\nu} = g & \text{on } (0,T)\times\Gamma_0 \\ c\frac{\partial u}{\partial\nu} + u_t = 0 & \text{on } (0,T)\times\Gamma_1 \end{cases} \tag{1.5}$$

to be supplemented with initial conditions.
Aiming at studying optimal control problems with quadratic functionals associated with the BVP (1.5), the following features need to be taken into account:
(i) *finite* time horizon problems, in the absence of penalization of the final time are the most pertinent ones (e.g., in lithotripsy);
(ii) with $u$ representing the acoustic pressure, the quantity to be minimized (under the action of the surface force $g$) is $\|u-u^d\|^2_{L^2(0,T;L^2(\Omega))}$, where $u^d$ is a reference pressure;
(iii) longer times (i.e. $T=+\infty$) might be taken into consideration (e.g., in connection with thermotherapy).

Depending on the applications, different cost functionals may be considered. In what follows we shall focus on the minimization of the following (simple, and yet physically significant) cost functional:

$$J(g) = \int_0^T\int_\Omega |u-u^d|^2\,dx\,dt + \int_0^T\int_{\Gamma_0} |g|^2\,d\sigma\,dt\,. \tag{1.6}$$

**Remark 1.1.** The fact that the functional cost penalizes the control $g$ only in the $L^2$ norm renders the optimization problem a singular one. Indeed, if one penalizes also the velocity $g_t$ of the control, then we would obtain a standard boundary control problem with coercive cost functional (cf. [6]).

Control problems associated with acoustic equations (Westervelt, Kuznetsov, JMGT ones) have been recently studied in the literature; see the review paper [19]. However, the principal difference is that the present minimization involves control functions which belong to $L^2(\Sigma)$, $\Sigma := (0,T)\times\Gamma_0$, rather than more regular – time-space differentiable – controls (see (1.4), and the optimal control problems studied in [10] and [31]). In addition, control laws provided in the past literature were *open loop* controls. Our goal is to construct a *feedback* control, with controls of limited regularity and control gains involving a solution to a corresponding Riccati equation. This last aspect is the main trait of our contribution. A brief outline-guide to the paper follows below.

In order to state our results and to explain the ramifications of the low regularity of the control, it is necessary to derive an abstract input-to-state formula for the BVP (1.5) (supplemented with initial conditions, that is problem (2.1) below), within the realm of classical control theory. This means we will seek an explicit representation for the map

$$g \longrightarrow (u, u_t, u_{tt}) \tag{1.7}$$

This will be accomplished in the next Section 2 by using semigroup theory. Starting with the uncontrolled dynamics and its representation via the generator of a strongly continuous semigroup, we shall then proceed introducing the (boundary) controls into the "variation of parameters formula" which will provide an explicit map (1.7) – singular and defined on appropriately selected extrapolation spaces, though.

In the next step we shall formulate the control problem associated with the input-state dynamics and we shall discuss existence and non-existence of optimal solutions. The final result pertaining to well-posedness of Riccati equations and to the feedback synthesis of the optimal control is presented in Section 3. It is important to notice here that in spite of the singularity of input-state dynamics, the feedback synthesis and the resulting Riccati equations are defined and well-posed on the basic state and control spaces. This is due to the effects of the observation.

The proofs of the auxiliary and main results are deferred to Sections 4 and 5. The proofs will rely on techniques introduced in the study of the LQ problem for hyperbolic-like equations with unbounded inputs, where the dynamics does not provide beneficial regularizing effects. To handle this issue, we establish appropriate bounds by exploiting structural properties of the observation; see [27, Vol. II].

# 2 Input-to-state formulation of the PDE problem

A prerequisite step for the understanding of the control-theoretic properties of the IBVP

$$\begin{cases} \tau u_{ttt} + \alpha u_{tt} - c^2\Delta u - b\Delta u_t = 0 & \text{on } (0,T)\times\Omega \\ \frac{\partial u}{\partial \nu} = g & \text{on } (0,T)\times\Gamma_0 \\ \frac{\partial u}{\partial \nu} + \frac{1}{c}u_t = 0 & \text{on } (0,T)\times\Gamma_1 \\ u(0,x) = u_0(x)\,,\ u_t(0,x) = u_1(x)\,; u_{tt}(0,x) = u_2(x) & \text{on } \Omega \end{cases} \tag{2.1}$$

for the SMGT equation is to introduce a corresponding abstract system in an appropriate function space.

## 2.1 Abstract setup. Preliminary analysis

In order to incorporate into the equation the boundary control acting on $\Gamma_0$, along with the absorbing BC on $\Gamma_1$, we follow a well-established method.

Let $\mathcal{A}$ be the realization of $-\Delta$ in $L^2(\Omega)$ with Neumann BC: namely,

$$\mathcal{A} = -\Delta\,, \quad \mathcal{D}(\mathcal{A}) = \Big\{ f \in H^2(\Omega): \ \frac{\partial f}{\partial \nu}\Big|_{\partial\Omega} = 0 \Big\}\,.$$

It is well known that $\mathcal{A}$ is not boundedly invertible on $L^2(\Omega)$; it has bounded inverse on

$$L^2_0(\Omega) := L^2(\Omega)/\ker(\mathcal{A}) = \Big\{ f \in L^2(\Omega)\colon \ \int_\Omega f\, d\Omega = 0 \Big\}\,,$$

where $\ker(\mathcal{A})$ is the null space of $\mathcal{A}$ spanned by the normalized constant functions. Then, introduce the Green maps $N_i$, $i = 0, 1$, which define appropriate harmonic extensions into $\Omega$ of data defined on $\partial\Omega$. More precisely, for $\varphi \in L^2(\Gamma_i)$, $i = 0, 1$, $N_i$ will be defined as follows:

$$N_i\colon \varphi \longmapsto N_i\varphi =: v \iff \begin{cases} \Delta v - v = 0 & \text{on } \Omega \\ \frac{\partial v}{\partial \nu} = \varphi & \text{on } \Gamma_i \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \setminus \Gamma_i. \end{cases} \tag{2.2}$$

Either elliptic problem that defines the operator $N_i$ in (2.2) admits a unique solution $v_i \in H^{3/2}(\Omega)$, for (respective) boundary data $\varphi \in L^2(\Gamma_i)$, $i = 0, 1$. Then, by elliptic theory one has for each $i = 0, 1$ and any positive $\sigma < 3/4$

$$N_i \text{ continuous}\colon L^2(\Gamma_i) \longrightarrow H^{3/2}(\Omega) \subset H^{3/2-2\sigma}(\Omega) \equiv \mathcal{D}((I + \mathcal{A})^{3/4-\sigma})\,, \tag{2.3}$$

with identification of the Sobolev spaces $H^s(\Omega)$ with the fractional powers of the operator $(I + \mathcal{A})$, and equivalent norms, that will be especially useful in the sequel.

If now $N_i^*$ denote the respective adjoint operators of $N_i$, $i = 0, 1$ – defined by $(N_i\phi, w)_{L^2(\Omega)} = (\phi, N_i^* w)_{L^2(\Gamma_i)}$ –, it then follows for each $i = 0, 1$ and any $\sigma \in (0, 3/4)$,

$$(I + \mathcal{A})^{3/4-\sigma} N_i \in \mathcal{L}(L^2(\Gamma_i), L^2(\Omega))\,, \quad N_i^*(I + \mathcal{A})^{3/4-\sigma} \in \mathcal{L}(L^2(\Omega), L^2(\Gamma_i))\,.$$

A computation which utilizes the (second) Green Theorem yields, for $f \in \mathcal{D}(\mathcal{A})$, the following well known results:

$$N_i^*(\mathcal{A} + I) f = f|_{\Gamma_i} \qquad i = 0, 1. \tag{2.4}$$

see, e.g., [27, Chapter 3]. (For the reader's convenience: take $v \in \mathcal{D}(\mathcal{A})$, $\varphi \in L^2(\Gamma_0)$, and compute

$$\begin{aligned} -\big(N_0^*(\mathcal{A} + I)v, \varphi\big)_{\Gamma_0} &= \big(-(\mathcal{A} + I)v, N_0\varphi\big)_\Omega = (\Delta v, N_0\varphi)_\Omega - (v, N_0\varphi)_\Omega = \\ &= \big(v, \Delta(N_0\varphi)\big)_\Omega + \cancel{\Big(\frac{\partial v}{\partial \nu}, N_0\varphi\Big)_{\partial\Omega}} - \Big(v, \frac{\partial N_0\varphi}{\partial \nu}\Big)_{\partial\Omega} - (v, N_0\varphi)_\Omega = \\ &= \cancel{(v, N_0\varphi)_\Omega} - (v, \varphi)_{\Gamma_0} \cancel{- (v, N_0\varphi)_\Omega} = -(v, \varphi)_{\Gamma_0}\,. \end{aligned}$$

The above shows that (2.4) holds true when $i = 0$; the case $i = 1$ is proved in the same way. We note that it has been used that since $v$ belongs to $\mathcal{D}(\mathcal{A})$, then $\frac{\partial v}{\partial \nu} = 0$ on

$\partial\Omega$; in addition, the definition of $N_0\varphi$ – as the solution of the elliptic problem in (2.2) – gives in particular $\Delta(N_0\varphi) = N_0\varphi$.) By standard density argument the formula in (2.4) can be extended to all $f \in H^1(\Omega)$.

In view of the definition of the introduced operators $N_i$, $i = 0, 1$, we see that

$$\begin{cases} (\Delta - I)\big(u + \frac{1}{c}N_1 u_t|_{\Gamma_1} - N_0 g\big) = (\Delta - I)u & \text{on } \Omega \times (0,T) \\ \frac{\partial}{\partial\nu}(u + \frac{1}{c}N_1 u_t - N_0 g) = 0 & \text{on } \Gamma_0 \times (0,T) \\ \frac{\partial}{\partial\nu}(u + \frac{1}{c}N_1 u_t - N_0 g) = 0 & \text{on } \Gamma_1 \times (0,T) \end{cases} .$$

Proceeding formally we get

$$\begin{aligned} \Delta u &= (\Delta - I)\big(u + \frac{1}{c_1}N_1 u_t|_{\Gamma_1} - N_0 g\big) + u\,, \\ \Delta u_t &= (\Delta - I)\big(u_t + \frac{1}{c_1}N_1 u_{tt}|_{\Gamma_1} - N_0 g_t\big) + u_t\,, \end{aligned}$$

which enable us to rewrite the SMGT equation as

$$\begin{aligned} \tau u_{ttt} + \alpha u_{tt} - c^2(\Delta - I)\big(u + \frac{1}{c}N_1 u_t|_{\Gamma_1} - N_0 g\big) - c^2 u - \\ - b(\Delta - I)\big(u_t + \frac{1}{c}N_1 u_{tt}|_{\Gamma_1} - N_0 g_t\big) - b u_t = 0\,, \end{aligned}$$

where

$$\frac{\partial}{\partial\nu}\big(u + \frac{1}{c_1}N_1 u_t|_{\Gamma_1} - N_0 g\big)\big|_{\Gamma} = 0$$

(we set $\Gamma := \partial\Omega$). Thus, by using the abstract representation of the traces given by (2.4), the BVP (1.5) for the SMGT equation translates to the following abstract equation, where both the absorbing BC on $\Gamma_1$ and the boundary control action on $\Gamma_0$ are incorporated:

$$\begin{aligned} \tau u_{ttt} + \alpha u_{tt} + c^2(\mathcal{A} + I)\Big[u + \frac{1}{c}N_1 N_1^*(\mathcal{A} + I)u_t - N_0 g\Big] - c^2 u + \\ + b(\mathcal{A} + I)\Big[u_t + \frac{1}{c}N_1 N_1^*(\mathcal{A} + I)u_{tt} - N_0 g_t\Big] - b u_t = 0\,, \end{aligned}$$

that is

$$\begin{aligned} \tau u_{ttt} + \alpha u_{tt} + c^2\mathcal{A}u + c(\mathcal{A} + I)N_1 N_1^*(\mathcal{A} + I)u_t + b\mathcal{A}u_t + \\ + \frac{b}{c}(\mathcal{A} + I)N_1 N_1^*(\mathcal{A} + I)u_{tt} = c^2(\mathcal{A} + I)N_0 g + b(\mathcal{A} + I)N_0 g_t\,; \end{aligned} \tag{2.5}$$

the equality is understood with respect to the duality pairing, i.e. in $[\mathcal{D}(\mathcal{A})]'$.

The third order abstract equation (2.5) gives rise readily to a first order control system, initially defined on an extended space $L^2(\Omega) \times L^2(\Omega) \times [\mathcal{D}(\mathcal{A})]'$:

$$\frac{d}{dt}\begin{pmatrix} u \\ u_t \\ u_{tt} \end{pmatrix} = A \begin{pmatrix} u \\ u_t \\ u_{tt} \end{pmatrix} + B_0 g + B_1 g_t\,, \tag{2.6}$$

where the operator describing the *free* dynamics is

$$A = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -\tau^{-1}c^2\mathcal{A} & -\tau^{-1}\big[b\mathcal{A}+c(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)\big] & -\tau^{-1}\big[\alpha I+\frac{b}{c}(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)\big] \end{pmatrix} \tag{2.7}$$

while the input operators $B_i \in \mathcal{L}(L^2(\Gamma_i), [\mathcal{D}(\mathcal{A})]'), i = 0, 1$, are

$$B_0 = \begin{pmatrix} 0 \\ 0 \\ \tau^{-1}c^2(\mathcal{A}+I)N_0 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0 \\ 0 \\ \tau^{-1}b(\mathcal{A}+I)N_0 \end{pmatrix} = \frac{b}{c^2}B_0\,. \tag{2.8}$$

The (free dynamics) operator $A$ in (2.7) will be shown to generate a $C_0$-semigroup on the space $Y = H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$.

**Remark 2.1.** The first order equation (2.6) is a control system in (extended to) the dual space $[\mathcal{D}(A^*)]'$ as it will be shown later; and more precisely, it holds $(I-A)^{-1}B_i \in \mathcal{L}(U, Y)$, $i = 0, 1$ (see Appendix A). However, the given formulation involves the time derivative of the control, which does not enter the cost functional; as a consequence, the minimization problem lacks coercivity. To cope with this, we will follow [25]: integration by parts in the semigroup solution formula enables to eliminate the time derivative of the control function, however with the drawback that the states will become 'rougher'. The smoothing properties of the observation operator $R$ – here, *intrinsic* – will play a major role in the entire subsequent analysis, which will eventually bring about the solution of the optimization problem.

Before we proceed, let us consider the uncontrolled equation first. This step is necessary in order to formulate a correct notion of duality – which is always with respect to the generator of the semigroup underlying the dynamics.

## 2.2 The uncontrolled equation. Semigroup well-posedness.

In order to pinpoint the control-theoretic properties of the abstract system (2.6) – an ineludible preliminary step for the analysis of the optimal control problem –, we consider first the uncontrolled equation, that is equation (2.5) in the absence of the boundary action $g$. With $g \equiv 0$, the equation (2.5) reads as

$$\begin{aligned} \tau u_{ttt} + \alpha u_{tt} + c^2\mathcal{A}u + c(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)u_t + b\mathcal{A}u_t + \\ + \frac{b}{c}(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)u_{tt} = 0\,. \end{aligned} \tag{2.9}$$

We follow an idea introduced and utilized in [21] and [29]. Calculations below might appear formal: however, they are fully justified with respect to the duality in $[\mathcal{D}(A^*)]'$. After having set $\tau = 1$ for the sake of simplicity, the rewriting of equation (2.9) as

$$(u_t + \alpha u)_{tt} + b\mathcal{A}\Big(u_t + \frac{c^2}{b}u\Big) + \frac{b}{c}(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)\Big(u_{tt} + \frac{c^2}{b}u_t\Big) = 0\,, \tag{2.10}$$

suggests the introduction of the auxiliary variable

$$z := u_t + \frac{c^2}{b} u \,. \tag{2.11}$$

The new variable $z$ plays a major role in deriving well-posedness results for the third order equation (2.9) in the unknown $u$; this is because it allows to connect the (free) equation under investigation with the following system in the unknowns $(u, z)$:

$$\begin{cases} u_t = -\frac{c^2}{b} u + z \\ z_{tt} = -b\mathcal{A}z - \frac{b}{c}(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)z_t - \gamma z_t + \gamma \frac{c^2}{b} z - \gamma \left(\frac{c^2}{b}\right)^2 u \end{cases} \tag{2.12}$$

where $\gamma := \alpha - \frac{c^2}{b}$ will be assumed to be positive. The explicit statement and proof of this claim, that is an immediate generalization of what done in [21], is given below for the reader's convenience and the sake of completeness.

**Lemma 2.2.** *The uncontrolled third order (in time) equation* (2.9) *is equivalent to the coupled ODE-PDE system* (2.12)*, with* $\gamma = \alpha - \frac{c^2}{b}$*.*

*Proof.* The starting point is equation (2.10) that is nothing but a rewriting of (2.9). With the new variable $z = u_t + \frac{c^2}{b} u$, the term $u_t + \alpha u$ in (2.10) is rewritten in terms of $z$ and $u$ as follows:

$$u_t + \alpha u = z + \gamma u \,, \quad \gamma := \alpha - \frac{c^2}{b} \,,$$

so that (2.10) becomes

$$z_{tt} + b\mathcal{A}z + \frac{b}{c}(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)z_t + \gamma u_{tt} = 0 \,. \tag{2.13}$$

On the other hand, using once again the definition of $z$ we see that $u_t = z - \frac{c^2}{b} u$, which gives

$$u_{tt} = z_t - \frac{c^2}{b} u_t = z_t - \frac{c^2}{b} z + \left(\frac{c^2}{b}\right)^2 u \,; \tag{2.14}$$

the above, inserted in (2.13) yields the following equation

$$z_{tt} + b\mathcal{A}z + \frac{b}{c}(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)z_t + \gamma z_t - \gamma \frac{c^2}{b} z + \left(\frac{c^2}{b}\right)^2 u = 0$$

on the space $[\mathcal{D}(A^*)]'$ (just like the previous (2.13)).

The latter second order in time equation for $z$, combined with (2.14) leads to the following coupled system of (second-order in time) equations in the unknowns $(u, z)$

$$\begin{cases} u_{tt} = z_t - \frac{c^2}{b} u_t \\ z_{tt} + b\mathcal{A}z + \frac{b}{c}(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)z_t + \gamma z_t - \gamma \frac{c^2}{b} u_t = 0 \,; \end{cases}$$

or, equivalently, to the coupled ODE-PDE system (2.12). $\square$

We establish (semigroup) well-posedness of the Cauchy problems associated with (2.12), in three different function spaces.

**Theorem 2.3** (Equivalent system. Well-posedness, I)**.** *The (first order in time) system in the unknown* $(u, z, z_t)$ *corresponding to system* (2.12) *is well-posed in the space*

$$Y = \underbrace{H^1(\Omega)}_{u} \times \underbrace{H^1(\Omega) \times L^2(\Omega)}_{(z,z_t)} .$$

*Its dynamics is described by a closed operator* $\tilde{A} : \mathcal{D}(\tilde{A}) \subset Y \to Y$ *which is the generator of a* $C_0$*-semigroup* $e^{\tilde{A}t}$ *on* $Y$*,* $t \geq 0$.

*Proof.* The second-order system (2.12) is rewritten as a first-order system

$$\begin{pmatrix} u \\ z \\ z_t \end{pmatrix}_t = \tilde{A} \begin{pmatrix} u \\ z \\ z_t \end{pmatrix},$$

with dynamics operator

$$\tilde{A} = \begin{pmatrix} -\frac{c^2}{b}\, I & I & 0 \\ 0 & 0 & I \\ -\gamma\big(\frac{c^2}{b}\big)^2 I & -b\mathcal{A} + \gamma\frac{c^2}{b}\, I & -\gamma I - \frac{b}{c}(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I) \end{pmatrix}.$$

It is then natural to observe that the decomposition

$$\tilde{A} = \tilde{A}_1 + C_1 + K_1$$

holds true, where we set

$$\tilde{A}_1 = \begin{pmatrix} -\frac{c^2}{b}\, I & 0 & 0 \\ 0 & 0 & I \\ 0 & -b(\mathcal{A}+I) & -\gamma I - \frac{b}{c}(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I) \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ -\gamma\big(\frac{c^2}{b}\big)^2 I & 0 & 0 \end{pmatrix}, \qquad K_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & (\gamma\frac{c^2}{b} + b)\, I & 0 \end{pmatrix}.$$

It is enough to single out the following respective features:
(i) the operator $\tilde{A}_1 : \mathcal{D}(\tilde{A}_1) \subset Y \longrightarrow Y$ is a (maximally) dissipative operator on

$$\underbrace{H^1(\Omega)}_{u} \times \underbrace{\mathcal{D}((\mathcal{A}+I)^{1/2}) \times L^2(\Omega)}_{(z,z_t)}$$

and hence it is the generator of a $C_0$-semigroup of *contractions* $e^{\tilde{A}_1 t}$ on $Y$ (which, however, is *not* analytic);

(ii) $C_1$ is a *bounded* operator from $Y$ into itself;
(iii) $K_1$ is a *compact* operator: in fact, with $f \in \mathcal{D}((\mathcal{A}+I)^{1/2}) = \mathcal{D}(\mathcal{A}^{1/2})$ one has

$$\gamma\Big(\frac{c^2}{b}+b\Big)\, f = \gamma\Big(\frac{c^2}{b}+b\Big)(\mathcal{A}+I)^{-1/2}[(\mathcal{A}+I)^{1/2} f]\,.$$

The generation of a $C_0$-semigroup $e^{\tilde{A}t}$ on $Y$ follows by semigroup theory. $\square$

**Remark 2.4.** The space $Y$ will provide an appropriate functional setting where the original uncontrolled system is well-posed, and a state space for the optimal control problem under investigation. It is however important to add that well-posedness remains valid in distinct functional spaces; the corresponding results are stated below for the sake of completeness, while the relative proofs are omitted.

**Corollary 2.5** (Equivalent system. Well-posedness, II)**.** *The uncontrolled problem is well-posed in*

$$Y_2 = \underbrace{H^2(\Omega)}_{u} \times \underbrace{H^1(\Omega)\times L^2(\Omega)}_{(z,z_t)}\,.$$

Thus, in view of the definition of the domain of the generator $\tilde{A}$, that is

$$\begin{aligned}\mathcal{D}(\tilde{A}) &= \big\{(u,z,z_t)\in [H^1(\Omega)]^3\colon\, z+\frac{1}{c}N_1\,N_1^*(\mathcal{A}+I)z_t\in\mathcal{D}(\mathcal{A})\big\} = \\ &= \Big\{(u,z,z_t)\in H^1(\Omega)\times H^2(\Omega)\times H^1(\Omega):\ \frac{\partial z}{\partial\nu}\Big|_{\Gamma_0}=0\,,\ \Big[c\frac{\partial z}{\partial \nu}+z_t\Big]_{\Gamma_1}=0\Big\}\,,\end{aligned}$$

taking the dual $[\mathcal{D}(\tilde{A})]'$ (duality with respect to $Y_2$), we are able to infer the following result.

**Corollary 2.6** (Equivalent system. Well-posedness, III)**.** *The uncontrolled problem is well-posed in*

$$Y_0 \sim \underbrace{H^1(\Omega)}_{u} \times \underbrace{L^2(\Omega)\times [H^1(\Omega)]'}_{(z,z_t)}, \tag{2.15}$$

*where $\sim$ indicates topological equivalence.*

The next Theorem 2.3 summarizes relevant well-posedness results which will be used throughout.

**Theorem 2.7** (The uncontrolled equation. Well-posedness and stability)**.** *With reference to the third order abstract equation* (2.9) *describing the free dynamics, the following statements hold true.*

*i) The boundary value problem* (1.5) *with $g\equiv 0$ admits the abstract formulation* (2.9) *as a third order equation; equivalently, it is rewritten as a first order abstract system $y' = Ay$, where $y$ denotes the state variable $(u,u_t,u_{tt})$.*

*ii) The operator $A$ which governs the free dynamics, detailed in* (2.7)*, is the generator of a $C_0$-semigroup $\{e^{At}\}_{t\ge 0}$ on the function space $Y = H^1(\Omega)\times H^1(\Omega)\times L^2(\Omega)$.*

*iii) The semigroup $e^{At}$ is exponentially stable when $\gamma = \alpha - \frac{c^2}{b} > 0$.*

**Remark 2.8.** In the critical case, when $\gamma = 0$, it is expected that with $\Gamma_1$ subject to the "star-shaped" Geometric Condition (see [23]), the resulting semigroup is exponentially stable.

**Remarks 2.9.** The first assertion in Theorem 2.7 establishes the existence of a linear semigroup defined on $Y$ which describes the original uncontrolled dynamics. It is worth noting that if the SMGT equation is complemented with either Dirichlet or Neumann BC the same result holds true, as it was first proved in [29] and [21]; in that case the semigroup is actually a *group* on $Y$. Instead, the group property is not valid any more in the presence of absorbing BC on $\Gamma_1$.

The studies [21] and [29] – the latter, providing a clarifying spectral analysis – obtain that (still in the case of Dirichlet or Neumann BC) the semigroup $e^{tA}$ is exponentially stable on the factor space $Y/\ker(A)$, provided $\gamma > 0$; it is marginally stable when $\gamma = 0$ and unstable when $\gamma < 0$. In the present case, assuming appropriate geometric conditions on $\Gamma_0$, the absorbing boundary conditions turn marginal stability ($\gamma = 0$) to stability. This issue has not been fully investigated so far, yet it is expected that the multipliers' method combined with a background on wave equations would provide the tools.

**Remark 2.10.** (*A distinct perspective)* The connection between the SMGT equation with wave equations with memory has been pointed out in the recent independent works [14] and [7]. The critical role of $\gamma$ as a threshold for uniform stability is revisited and recovered in [14] via the analysis of a corresponding viscoelastic equation. It is apparent that appropriate compatibility conditions on initial data must be assumed, in order to study the third order (in time) equation by using theories pertaining to wave equations with a non-local term.

And yet, the perspective of equations with memory opens a distinct avenue of investigation of the (interior and trace) regularity properties of the corresponding solutions, fruitfully explored in [7] – as well as, possibly, of other control-theoretic properties. In this connection, we mention the paper [32], which provides an analysis of the LQ problem and Riccati equations for finite dimensional systems with memory.

## 2.3 Domain of the generator

We give an explicit description of the natural domain of the (free) dynamics generator $A$ introduced in (2.7): given the state space $Y = H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$, one has

$$y \in \mathcal{D}(A) \Longleftrightarrow y \in \Big\{ y = (y_1, y_2, y_3) \in Y : y_3 \in H^1(\Omega)\,, \\ c^2 y_1 + b y_2 + N_1 N_1^*(\mathcal{A} + I)\Big(c y_2 + \frac{b}{c} y_3\Big) \in \mathcal{D}(\mathcal{A}) \Big\}\,,$$

whose PDE interpretation is as follows:

$$y \in \mathcal{D}(A) \Longleftrightarrow y \in \Big\{ y \in [H^1(\Omega)]^3 \colon \Delta(c^2 y_1 + b y_2) \in L^2(\Omega)\,,\ \frac{\partial}{\partial\nu}(c^2 y_1 + b y_2) = 0 \text{ on } \Gamma_0\,,$$
$$c\frac{\partial}{\partial\nu}(c^2 y_1 + b y_2)\Big|_{\Gamma_1} = -\Big[c^2 y_2 + b y_3\Big]\Big|_{\Gamma_1} \text{ on } \Gamma_1 \Big\}\,.$$

Notice that by a standard variational argument the normal derivatives are first well defined on $H^{-1/2}(\Gamma)$. Then, the $H^{1/2}(\Gamma)$-regularity of $y_i$, $i = 1, 2, 3$, along with elliptic theory gives

$$\begin{split}\mathcal{D}(A) = \Big\{ y \in [H^1(\Omega)]^3 \colon (c^2 y_1 + b y_2) \in H^2(\Omega)\,,\ \frac{\partial}{\partial\nu}(c^2 y_1 + b y_2) = 0 \text{ on } \Gamma_0\,,\\ c\frac{\partial}{\partial\nu}(c^2 y_1 + b y_2)\Big|_{\Gamma_1} = -\Big[c^2 y_2 + b y_3\Big]\Big|_{\Gamma_1} \text{ on } \Gamma_1 \Big\}\,.\end{split} \tag{2.16}$$

We also note that the resolvent of $A$ is not compact, which is important to be pointed out.

## 2.4 The SMGT equation subject to smooth controls

Now let us turn our attention to the controlled (abstract) equation (2.5) corresponding to the BVP (1.5) and to its reformulation as the first-order control system (2.6). This system produces readily a solution formula, assuming that $g \in H^1(0, T; L^2(\Gamma_0))$: the following Proposition provides a rigorous justification.

**Proposition 2.11.** *Assume that $g \in H^1(0, T; L^2(\Gamma_0))$. The boundary value problem* (1.5) *for the SMGT equation can be recast as the (third order in time) abstract equation* (2.5)*; equivalently, it is rewritten as a first order abstract system of the form* (2.6)*, that is*

$$y' = Ay + B_0 g + B_1 g_t\,; \tag{2.17}$$

*$y$ denotes the state variable $(u, u_t, u_{tt})$ and $g$ is the control variable, while the linear operators $A$ and $B_i$, $i = 0, 1$, satisfy the following analytical properties.*

- *i) The operator $A \colon Y \supset \mathcal{D}(A) \longrightarrow Y$ which describes the free dynamics, detailed in* (2.7)*, is the generator of a $C_0$-semigroup $\{e^{At}\}_{t \ge 0}$ on the function space $Y = H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$, with domain $\mathcal{D}(A)$ as in* (2.16)*;*
- *ii) the control operators $B_i$, $i = 0, 1$ defined in* (2.8) *satisfy $B_i \in \mathcal{L}(U, [\mathcal{D}(A^*)]')$.*

*Then, the third order equation* (2.5) *is understood on the extrapolation space $[\mathcal{D}(\mathcal{A})]'$.*

*Proof.* Since the Neumann maps $N_i$ defined in (2.2) enjoy the regularity in (2.3), that is $N_i \in \mathcal{L}(L^2(\Gamma_i), \mathcal{D}(\mathcal{A}^{3/4-\sigma}))$, we accordingly have that the distributional range of the control maps $B_i$ is such that

$$\mathcal{R}(B_i) \subset \{0\} \times \{0\} \times [\mathcal{D}(\mathcal{A}^{1/4+\sigma})]'\,.$$

To see this, just recall the explicit form of the input operators $B_0$ in (2.8), which gives

$$\begin{aligned}|(B_0 g, y)|_Y &= |(c^2(\mathcal{A}+I)N_0 g, y)|_Y = c^2|(g, y_3|_{\Gamma_0})_{L^2(\Gamma_0)} = c^2\,|y_3|_{H^{1/2+2\sigma}(\Omega)}|g|_{L^2(\Gamma_0)} \\ &= c^2\,|\mathcal{A}^{1/4+\sigma}y_3|_{L^2(\Omega)}|g|_{L^2(\Gamma_0)}\end{aligned}$$

which proves that there exists a positive constant $C$ such that

$$|(B_i g, y)|_Y| \le C\,|\mathcal{A}^{1/4+\sigma}y_3|_{L^2(\Omega)}|g|_{L^2(\Gamma_0)} \le C\,|Ay|_Y\,|g|_{L^2(\Gamma_0)}\,, \qquad i=0,1\,,$$

since $B_1 = \frac{b}{c^2}B_0$.

By using interpolation trace results, a stronger inequality is obtained: for any $\epsilon > 0$ one has

$$(B_i g, y)_Y \le C\,|Ay|_Y^{1/2}|y|_Y^{1/2}|g|_{L^2(\Gamma_i)} \le \big(\epsilon|Ay|_Y + C_\epsilon|y|_Y\big)\,|g|_{L^2(\Gamma_i)}$$

which gives

$$|B_i^* y|_{L^2(\Gamma_i)} \le \epsilon|Ay|_Y + C_\epsilon|y|_Y \qquad \forall\epsilon > 0\,.$$

□

In view of Proposition 2.11 – hence, still under the assumption $g \in H^1(0,T,U)$) – semigroup theory yields a first input-to-state formula in the extrapolation space $[\mathcal{D}(A^*)]'$.

**Corollary 2.12.** *For any initial state $y_0 \in [\mathcal{D}(A^*)]'$ and any control $g \in H^1(0,T,U)$, the control system* (2.17) *has a unique mild solution $y \in C([0,T];[\mathcal{D}(A^*)]')$ given by*

$$\begin{aligned}y(t) &= e^{At}y(0) + \int_0^t e^{A(t-s)}\big(B_0 g(s) + B_1 g_t(s)\big)\,ds = \\ &= e^{At}y(0) + \int_0^t e^{A(t-s)}B_0\Big(g(s) + \frac{b}{c^2}g_t(s)\Big)\,ds\,.\end{aligned} \tag{2.18}$$

# 3 The control problem. Main results

If the cost functional (1.6) penalized (quadratically) the time derivative of the control function, we might choose as space of admissible controls $\mathcal{U} = H^1(0,T;L^2(\Gamma_0))$, and the obtained semigroup solution formula (2.18) as the state equation. Remember however that we seek to minimize the functional (1.6) over all controls $g$ which belong to $L^2(0,T;L^2(\Gamma_0))$, where the acoustic pressure $u$ satisfies the IBVP (2.1). Hence, in this Section we first derive from (2.18) a solution formula which requires controls which just belong to $L^2(0,T;L^2(\Gamma_0))$ and are continuous at time $t=0$; this is done by an elementary integration (in time) by parts. Then, following an idea proposed in [25] and [26], we introduce an (auxiliary) optimal control problem associated to an equation depending on a parameter $g_0 \in L^2(\Gamma_0) =: U$. The main result pertaining to the auxiliary problem and the connection with the original one are stated collectively in the section. The respective proofs are the subject of the subsequent two sections.

## 3.1 Control problem with the observation

Our next step is to provide a representation formula for the solutions to the controlled dynamics by assuming that controls belong to $L^2(0,T;U)$. This is done, as usual, integrating by parts (in a dual space) and exploiting the structure of the domain of the generator.

**Remark 3.1** (**Notation).** From now on the dual of the operator $A^2$, that is $(A^2)^*$, will often occurr in the paper. To simplify the notation, the parentheses will be omitted and the notation $A^{*2}$, or $A^{*2}$ – in place of $(A^*)^2$, which is the same – will be utilized throughout.

**Lemma 3.2.** *Given an initial state $y_0 \in [\mathcal{D}(A^{*2})]'$ and any control function $g \in C([0,T];U)$, the solution to the original control system* (2.6)*, represented via the input-to-state formula* (2.18)*, is equivalently given by*

$$y(t) = e^{At}[y_0 - B_1 g(0)] + Lg(t)\,, \tag{3.1}$$

*with*

$$\begin{aligned}
&(Lg)(t) = B_1 g(t) + (L_0 g)(t)\,,\\
&(L_0 g)(t) = \int_0^t e^{A(t-s)} B_0 g(s)ds + A\int_0^t e^{A(t-s)} B_1 g(s)ds\,.
\end{aligned} \tag{3.2}$$

*The map $(y_0, g) \to y(\cdot)$ is bounded from $[\mathcal{D}(A^{*2})]' \times C([0,T];U) \to C([0,T];[\mathcal{D}(A^{*2})]')$.*

*Proof.* The novel representation formula (3.1) is easily established integrating by parts in (2.18); what we need to justify rigorously is the claimed regularity. We know already that $e^{At}$ generates a $C_0$-semigroup on $[\mathcal{D}(A^{*2})]'$, and that $(I-A)^{-1}B_i \in \mathcal{L}(U,Y)$. Then, it suffices to analyze the regularity of the operator $L_0$ in (3.2), which depends on the one of the operator $AB_1$. Recalling the definitions of $A$ and $B_1$, it is easily seen that

$$AB_1 = b\begin{pmatrix} 0 \\ (\mathcal{A}+I)N_0 \\ -\alpha(\mathcal{A}+I)N_0 \end{pmatrix}\,,$$

where it has been used that the distributions on $\Gamma_0$ and $\Gamma_1$ have disjoint support; this property, combined with the contribution of the operator $N_1$ in the definition of $A$, brings about $(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)\,(\mathcal{A}+I)N_0 \equiv 0$. As a consequence, we obtain that

$$\mathcal{R}(AB_1) \subset \{0\} \times [\mathcal{D}(\mathcal{A}^{1/4+\epsilon})]' \times [\mathcal{D}(\mathcal{A}^{1/4+\epsilon})]' \subset [\mathcal{D}(A^{*2})]'\,,$$

which gives the desired conclusion. □

Observe that – just as in the works [25] and [26] – the drawback of the chosen approach is that the space regularity of the state function gets worse. Moreover, in contrast with the dynamics under investigation therein, whose underlying semigroup is

analytic, we are dealing with a purely hyperbolic problem.
On the other hand, recall that the goal is to minimize the $L^2(\Omega)$-norm of the acoustic pressure, described by the state variable $u$, that is the first component of the state variabile $y$. By setting $u^d = 0$ in (1.6) just for the sake of simplicity, the cost functional is abstractly rewritten as

$$J(g) = \int_0^T \|Ry\|_Y^2\, dt + \int_0^T \|g\|_U^2\, dt\,, \tag{3.3}$$

where $U$ denotes the control space, i.e. $U = L^2(\Gamma_0)$, and the observation operator $R$ is acting as follows: for any $y = [y_1, y_2, y_3]^T$, it holds

$$Ry = \begin{pmatrix} (\mathcal{A}+I)^{-1/2} y_1 \\ 0 \\ 0 \end{pmatrix}. \tag{3.4}$$

In fact, after identifying $H^1(\Omega)$ with $\mathcal{D}(\mathcal{A}^{1/2})$, we see that

$$\|Ry\|_Y = \|(\mathcal{A}+I)^{1/2}(\mathcal{A}+I)^{-1/2} y_1\|_{L^2(\Omega)} = |y_1|_{L^2(\Omega)}\,.$$

Thus, the simple – and yet natural – quadratic functional taken into consideration, attributes to the observation operator $R$ a very special structure and an intrinsic (rather strong) smoothing effect. The improved regularity of the observed states enables us to pursue an adaptation of the theory developed in [27, Vol. II] in the study of hyperbolic-like PDE's with boundary or point control actions and "smoothing" observations.

## 3.2 Main Results

In this subsection we shall formulate the main results, while the proofs are all postponed to the next sections. We shall begin with a negative result.

Consider the following minimization problem.

**Problem 3.3.** *For any $y_0 \in Y$, minimize the cost functional* (3.3) *over all controls $L^2((0,T)\times\Gamma_0)$, where $y(\cdot)$ satisfies the controlled equation* (3.1).

**Theorem 3.4.** *If the initial state $y_0$ belongs to $\mathcal{R}(B_1)$, $y_0 \neq 0$, then Problem 3.3 does not have a solution.*

Given this negative result, one might wonder what are the additional constraints which render the problem solvable. The proof of the negative result (cf. [26]) reveals that the issue is in singularity of control, as the 'candidate' to be the optimal control is no longer in the space $L^2(0,T;U)$. (This depends upon the appearance of a (time-)trace operator – intrisincally uncloseable – in the definition of the state.)

In view of the above, we shall consider an input-to-state formula depending on a given (and yet arbitrary) parameter $g_0 \in U$, that is

$$y_{g_0}(t) = e^{At}(y_0 - B_1 g_0) + Lg(t)\,, \tag{3.5}$$

with $L$ defined in (3.2). This idea has been developed in [25, 26]. When $g(0) = g_0$ the above controlled dynamics coincides with the one given by (3.1). With (3.5) we associate the same cost functional (3.3). A new (extended) optimal control problem is formulated as follows.

**Problem 3.5.** *For any $y_0 \in [\mathcal{D}(A^{*2})]'$, $g_0 \in U$, minimize the cost functional* (3.3) *overall controls $g \in L^2((0,T) \times \Gamma_0)$, with $y$ subject to* (3.5).

For this problem the following results holds true.

**Theorem 3.6.** *The optimization Problem 3.5 has a unique solution $\hat{g}_{g_0} \in L^2(0,T;U)$. The corresponding optimal trajectory satisfies*

$$\hat{y}_{g_0} \in C([0,T];[\mathcal{D}(A^{*2}]'), \quad R\hat{y}_{g_0} \in C([0,T];Y). \tag{3.6}$$

The first main result of the paper establishes the feedback synthesis of the optimal control referred to in Theorem 3.6. For clarity of the exposition, we shall take $u^d = 0$.

**Theorem 3.7.** *With reference to the minimization Problem 3.5, the following statements are valid.*

*i) (Partial regularity) For any $y_0 \in [\mathcal{D}(A^{*2})]'$, and any $g_0 \in U$, the unique optimal control $\hat{g}_{g_0}$ belongs to $C([0,T];U]$, and produces the output $R\hat{y}_{g_0} \in C([0,T];Y)$.*

*ii) (Riccati Equation) For every $t \in [0,T]$, there exists a self-adjoint positive operator $P(t)$ on $\mathcal{L}(Y)$, whose regularity is as follows,*

$$A^*P(t) \in \mathcal{L}(Y), \quad B_1^*A^*P(t) \in \mathcal{L}(Y,U) \textit{ continuously in time,}$$

*and which satisfies the following (non-standard) Riccati equation:*

$$\begin{aligned} &\frac{d}{dt}(P(t)y,w)_Y + (Ay,P(t)w)_Y + (P(t)y,Aw)_Y + (Ry,Rw)_Y = \\ &\quad = ((B_0^* + B_1^*A^*)P(t)y,([B_0^* + B_1A^*)P(t)w)_U \quad \textit{for all } y,w \in \mathcal{D}(A), \end{aligned} \tag{3.7}$$

*with terminal condition $P(T) = 0$. The equation* (3.7) *actually extends to all $y, w \in Y$.*

*iii) (Feedback synthesis) The optimal control $\hat{g}_{g_0}(\cdot)$ has the following feedback representation:*

$$\hat{g}_{g_0}(t) = -G(t)^{-1}[B_0^* + B_1^*A^*]P(t)\,\hat{y}_{g_0}(t),$$

*where the operator $G(t) := I - [B_0^* + B_1^*A^*]P(t)B_1$ is boundedly invertible on $U$ for each $t \in [0,T]$.*

From the structure of the Riccati equation (3.7), along with the space regularity of the operator $P(t)$ asserted in Theorem 3.7, some additional regularity of the operator $P(t)$ follows.

**Corollary 3.8.** *The Riccati operator $P(t)$ is time differentiable from $Y$ into itself. More precisely, the operator $\frac{d}{dt}P(t)\colon Y \to C([0,T];Y)$ is bounded.*

**Remark 3.9.** We note that the Riccati equation (3.7) is termed *non-standard* (already in [26]) because of the special structure of its quadratic term. This feature results from the lack of coercivity in the functional cost, a cause for singularity of the minimization problem. Then, the feedback formula which allows the synthesis of the optimal control of Problem 3.5 involves the inverse of certain operator defined on the control space $U$. Invertibility of the said operator is an issue already encountered in [25] and [26]: however, differently from those studies, in the present case we cannot appeal to the analyticity of the semigroup underlying the controlled dynamics.

Theorem 3.7 provides the optimal control and the optimal synthesis for the input-state dynamics (3.5), given $y_0$ and the parameter $g_0$. One aims then at exploring the relation between the parameter $g_0$ with the optimal control $\hat{g}$, which is known from Theorem 3.7 to be continuous on $[0,T]$. Thus, a question of major concern is whether the parameter $g_0 \in U$ can be selected in order that $\hat{g}(0) = g_0$. The validity of this property will prove the equivalence of the state description in (3.1) with the one in (3.5), thereby ensuring that the latter system corresponds to the original PDE model. The answer to this question is positive, as asserted by the Theorem below.

**Theorem 3.10.** *Let $G(t)$ be the operator defined in Theorem 3.7. Then, the operator $[I + G(0)B_1]$ is bounded invertible on $U$; in particular, $[I + G(0)B_1]^{-1} \in \mathcal{L}(U)$. By choosing $g_0 = [I + G(0)B_1]^{-1}G(0)y_0$, one obtains that*

$$\hat{y}(t) = e^{At}[y_0 - B_1\hat{g}(0)] + (L\hat{g})(t)\,,$$

*so that the original dynamics* (3.1) *coincides with the one in* (3.5). *Moreover, the obtained $\hat{g}$ is continuous in time, i.e. $\hat{g} \in C([0,T];U)$.*

Forcing the original model with continuity of the control at the origin may compromise the optimality. Instead, the additional ‘player’ $g_0 \in U$ is advantageous from the optimality point of view. While we know that in general there is no optimal control in the class of $L^2(0,T;U)$ functions (cf. Theorem 3.4), reformulating the solution formula as in (3.5), with an additional parameter, gives additional possibilities for optimization with respect to the parameter.

**Theorem 3.11.** *Let $U_0 \subset U$ be a bounded and weakly closed set in $U$. Then, there exists a $g^* \in U_0$ such that the resulting control $\hat{g}_{g^*}$ attains the infimum of the functional $J(g)$ with respect to $g_0 \in U_0$, $g \in L^2(0,T;U)$ and $y$ satisfying (3.5). Moreover, the following characterization holds true: either $g^*$ is such that $y_0 - B_1 g^* \in \ker(B_1^* P(0))$, or $g^* \in \partial U_0$.*

**Remark 3.12.** Note that the optimal control of Theorem 3.11 provides a control which is in a larger space than just $L^2(0,T;U)$. This is a singular control. The corresponding state is described by (3.5) and it satisfies $R\hat{y}(\hat{g}_{g^*}) \in C([0,T];Y)$.

It is important to note that from both the point of view of applications as well as of mathematical developments, it is significant to have two versions of optimal solutions corresponding to two different formulations of the input-state map. If one is to develop nonlinear versions of the problem, where regularity of controls and of the states is of paramount importance, the first version in Theorem 3.10 is the most relevant. However, from the point of view of automatic control – where discontinuous inputs are feasible and lead to 'better' optimization solutions –, Theorem 3.11 becomes more relevant. In particular, the result stated in Theorem 3.11 leads to the following algorithm for an "almost" on line optimal feedback control:

- Step 1: Solve Riccati Equation $\to P(t)$.
- Step 2 : Find the optimal value of the parameter $g^*$ from $(ker B_1^* P(0)) \cup \partial U_0$.
- Step 3: Find $G(t)^{-1}$ where $G(t) \equiv I - [B_)^* + B_1^* A^*] P(t) B_1$.
- Step 4: Resolve optimal feedback synthesis with a parameter $g^*$: obtaining $\hat{g}_{g^*}, \hat{y}_{g^*}$.

$$\hat{g}_{g^*}(t) = -G(t)^{-1}[B_0^* + B_1^* A^*] P(t)[\hat{y}_{g^*}(t)]$$
$$\hat{y}_{g^*} = \hat{y}(\hat{g}_{g^*}) - e^{At} B_1 [\hat{g}_{g^*}(0) - g^*]$$

**Remark 3.13.** There are several open problems sparked off by the present work. We name but a few.

i) Extension of the theory to more general observation operators $R$. However, it is clear that $R$ should display some kind of smoothing effect. Moreover, the structure of the problem – namely, an appropriate interplay between control and observation operators – will need to be carefully chosen, in order that the optimal ($L^2$) solution does exist.

ii) The infinite time horizon LQ-problem in both the stable and the critical case. It is expected that under suitable geometric conditions imposed on $\Gamma_1$ one could guarantee solvability of the optimization problem, along with a feedback synthesis of the optimal control.

iii) Application of the previous result to the feedback control of the nonlinear equation. A local theory for small initial data should emerge, while the feedback control should provide a stabilizing effect on the nonlinear dynamics.

The remaining parts of the paper are devoted to proofs of four Theorems.

# 4 Proofs of Theorems 3.4, 3.6

We point out at the outset that the main challenge in proving the stated results is to be able to 'run' the dynamics on much larger dual spaces, still preserving the invariance of the said dynamics. The following Proposition singles out some basic regularity and structural properties pertaining to the observation operator $R$.

**Proposition 4.1.** *The observation $R$ satisfies the following properties.*

- $R\colon Y \to \mathcal{D}(\mathcal{A}) \times \{0\} \times \{0\}$ *is bounded;*
- $R \in \mathcal{L}(Y, \mathcal{D}(A))$*;*
- $R = R^*$ *on* $Y$*, hence* $R \in \mathcal{L}([\mathcal{D}(A^*)]', Y)$*.*

*Proof.* For the first statement, take $y \in Y$: then $y_1 \in \mathcal{D}(\mathcal{A}^{1/2})$, and since $Ry = ((\mathcal{A}+I)^{-1/2}y_1, 0, 0)^T$ we obtain $(\mathcal{A}+I)^{-1/2}y_1 \in \mathcal{D}(\mathcal{A})$.

The second statement follows from the calculation with $y \in Y$

$$ARy = [0, 0, -\tau^{-1}c^2\mathcal{A}(\mathcal{A}+I)^{-1/2}y_1]^T \in Y$$

We also note that $(I-A)^{-1} \in \mathcal{L}(Y, [\mathcal{D}(\mathcal{A}^{1/2})]^3)$. The third statement follows from direct calculations using the inner product in $Y$.

The fourth statement follows combining the third with the second one. □

## 4.1 Properties of the input-to-output map

The following Lemma captures a set of functional-analytic properties pertaining to appropriate combination of the involved abstract operators – namely, the dynamics, control and observation operators –, which will play a major role in the proof of well-posedness for the (generalized) differential/integral Riccati equations, eventually leading to solvability of the optimal control problem.

**Lemma 4.2.** *Let $A$, $B_i$ and $R$ the dynamics, control, observation operators defined by* (2.7), (2.8), (3.4)*, respectively. Then,*

*i)* $RA^2$ *can be extended to a* bounded *operator on the state space* $Y$*;*

*ii)* $RB_1 = 0$*;*

*iii)* $(I-A)^{-1}B_i$ *are bounded and compact operators from* $L^2(\Gamma_i)$ *into* $Y$*,* $i = 0, 1$*.*

*Proof.* i) We take an element $y = (y_1, y_2, y_3)$ initially assumed in $\mathcal{D}(A^2)$, and compute

$$A^2y = A(Ay) =$$

$$= A\begin{pmatrix} y_2 \\ y_3 \\ -c^2\mathcal{A}y_1 - [b\mathcal{A} + c(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)]y_2 - \left[\alpha I + \frac{b}{c}(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)\right]y_3 \end{pmatrix}$$

$$= \begin{pmatrix} y_3 \\ \dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \end{pmatrix}$$

where the second and third component of $A^2 y$ are neglected, owing to the structure of the observation operator $R$ to be applied. Then,

$$R\,A^2 y = \begin{pmatrix} (I+\mathcal{A})^{-1/2} y_3 \\ 0 \\ 0 \end{pmatrix}$$

which gives

$$\|R\,A^2 y\|_Y = \left\| \begin{pmatrix} (I+\mathcal{A})^{-1/2} y_3 \\ 0 \\ 0 \end{pmatrix} \right\|_Y = \left\|(I+\mathcal{A})^{1/2}(I+\mathcal{A})^{-1/2} y_3\right\|_{L^2(\Omega)} = \|y_3\|_{L^2(\Omega)}$$

ii) It is immediately verified that for any $h \in L^2(0,T;L^2(\Gamma_1))$

$$R\,B_1 h = R \begin{pmatrix} 0 \\ 0 \\ b(\mathcal{A}+I)N_1\,h \end{pmatrix} = (I+\mathcal{A})^{-1/2}\,0 = 0\,.$$

iii) It is clear that the resolvent $(I-A)^{-1}$ is not compact. However, a careful computation gives

$$(I-A)^{-1}B_0 = \begin{pmatrix} c^2 S^{-1}(\mathcal{A}+I)N_0 \\ c^2 S^{-1}(\mathcal{A}+I)N_0 \\ c^2 S^{-1}(\mathcal{A}+I)N_0 \end{pmatrix}\,, \qquad (I-A)^{-1}B_1 = \frac{b}{c^2}(I-A)^{-1}B_0\,,$$

that is (A.5) in Appendix A and where the operator $S$ is defined by (A.3). Because $\mathcal{R}(N_0) \subset H^{3/2}(\Omega) \subset \mathcal{D}(\mathcal{A}^{1/2})$ (the latter being a compact embedding), then the operators $(I-A)^{-1}B_i$ are not only bounded from $L^2(\Gamma_0)$ into $Y$, but also compact. $\square$

The following Lemma pertains to the regularity of the map $RL_0$.

**Lemma 4.3.** *Let $L_0$ be the operator defined in* (3.2)*. Then*

- *$RL_0$ is a compact operator from $L^2(0,T;L^2(\Gamma_0))$ into $C([0,T];Y)$.*
- *$Re^{A\cdot}B_i\colon L^2(0,T;L^2(\Gamma_0)) \to C([0,T];Y)$, $i=0,1$, are compact.*

*Proof.* The first statement follows computing

$$\begin{aligned}(RL_0 g)(t) &= R\int_0^t e^{A(t-s)}B_0 g(s)ds - RA\int_0^t e^{A(t-s)}B_1 g(s)ds \\ &= R(I-A)^2\int_0^t e^{A(t-s)}(I-A)^{-2}B_0 g(s) - \\ &\quad - RA(I-A)\int_0^t e^{A(t-s)}(I-A)^{-1}B_1 g(s)ds\end{aligned} \tag{4.1}$$

in view of Lemma 4.2, combined with Aubin-Simon compactness criterion.

The second statement follows rewriting $Re^{At}B_0$ as follows,

$$Re^{At}B_0 = R(I-A)e^{At}(I-A)^{-1}B_0\,,$$

where $RA \in \mathcal{L}(Y)$ and $(I-A)^{-1}B_0 \colon U \to Y$ compactly. The strong additional regularity $RA^2 \in \mathcal{L}(Y)$ allows to handle the time derivative

$$\frac{d}{dt}R(I-A)e^{At}(I-A)^{-1}B_0 = RA(I-A)e^{At}(I-A)^{-1}B_0 \in \mathcal{L}(U,Y)\,,$$

as needed for the applicability of the Aubin-Simon compactness criterion. □

### 4.2 Proof of Theorem 3.4

We will denote by $J(g)$ the cost functional $J(g,y)$, where $y(\cdot) = y(\cdot;g)$ corresponds to the state variable given by (3.1). Take $y_0 \in \mathcal{R}(B_1) \in [\mathcal{D}(A^*)]'$ and select a sequence of controls $g_n \in H^1(0,T;U)$ such that

i) $B_1 g_n(0) = y_0$,

ii) $g_n \to 0$ in $L^2(0,Y;U)$.

Then, with $y_n(t) = y_n(t,g_n) = e^{At}\big(y_0 - B_1 g_n(0)\big) + (L_0 g_n)(t) + B_1 g_n(t)$ we have

$$Ry_n = RL_0 g_n \longrightarrow 0 \quad \text{in } L^2(0,T;Y),$$

on the strength of Lemma 4.3. Consequently, $J(g_n) \to 0$.

Since $g_n \to 0$ in $L^2(0,T;U)$, we turn to $J(0) = \int_0^T |Re^{At}y_0|_Y^2 dt > 0$, which combined with $g_n \to 0$ contradicts the existence of a minimizer.

### 4.3 Proof of Theorem 3.6

The argument is in principle standard, as it is based on proving weak lower semicontinuity of the cost functional. Thus, the challenge is to establish appropriate regularity of the input-to-state map, which is not obvious in view of the high unboundedness of the control input operators. However, this is possible exploiting the smoothing effect of the observation operator as well as the properties specifically established for the input-to-output map (cf. Lemma 4.3). To wit: for a given $g_0 \in U$ consider a minimizing sequence $g_n \in L^2(0,T;U)$, so that $J(g_n) \to d = \inf_{g \in L^2(0,T;U)} J(g)$. Then, coercivity of the cost in $L^2(0,T;U)$ gives the bound $\|g_n\|_{L_2(0,T;U)} \le M$ which implies that

$$g_n \to g \quad \text{weakly in } L^2(0,T;U). \tag{4.2}$$

We also have

$$Ry_n(t) = Re^{At}(y_0 - B_1 g_0) + (RL_0 g_n)(t)\,.$$

On the strength of Lemma 4.2 and Lemma 4.3, for a subsequence – denoted by the same symbol – it follows $RL_0 g_n \longrightarrow RL_0 g$ in $L^2(0,T;Y)$. In addition, $Re^{At}B_1 =$

$R(I-A)e^{At}(I-A)^{-1}B_1$ is bounded from $L^2(0,T;U)$ into $L^2(0,T;Y)$. This implies the weak lower semicontinuity of $J(g)$, along with $J(g) \le d$, which proves optimality.

The regularity in (3.6) pertaining to the observed optimal state, follows in view of the obtained regularity of the three summands in

$$Ry(t) = Re^{At}(y_0 - B_1 g_0) + (RL_0 g)(t)\,,$$

where in particular $Re^{At}y_0 = R(I-A)^2 e^{At}(I-A)^{-2}y_0 \in C([0,T];Y)$ for any $y_0 \in [\mathcal{D}(A^{*2})]'$, thanks to the property i) of Lemma 4.2.

# 5 Proof of Theorem 3.7

Given the solution formula (3.5), with the input-to-state map $L$ defined in (3.2), let us consider the dynamics

$$y_\alpha(t) = e^{At}\alpha + (Lg)(t) \tag{5.1}$$

depending on the parameter $\alpha \in [\mathcal{D}(A^*)]'$. This choice is justified by $B_1 g_0 \in [\mathcal{D}(A^*)]'$ for $g_0 \in U$. (We note that $y_{g_0}(\cdot)$ has been used to denote the function in (3.5), with emphasis on the dependence of $y$ on $g_0 \in U$, beside to $y_0$. In the present section, although with a certain abuse of notation, with $y_\alpha(\cdot)$ we shall be always referring to the 'full' parameter $\alpha$, rather than to its component $g_0$.)

Recall that $g \in L^2(0,T;U)$ gives $(I-A)^{-1}B_i g \in L^2(0,T;Y)$, $i=0,1$; then, since $AB_1 = B_1 - (I-A)B_1 = B_1 - (I-A)^2\,(I-A)^{-1}B_1$, we obtain

$$L \in \mathcal{L}(L^2(0,T;U), C([0,T];[\mathcal{D}(A^{*2})]'\,. \tag{5.2}$$

The following auxiliary control problem is naturally associated to (5.1).

**Problem 5.1** (**Problem $\mathcal{P}_\alpha$**). *For any $\alpha \in [\mathcal{D}(A^{*2})]'$, minimize the functional*

$$J(g, y_\alpha) = \int_0^T \|Ry_\alpha\|_Y^2\,dt + \int_0^T \|g\|_U^2\,dt\,, \tag{5.3}$$

*overall controls $g \in L^2(0,T;U)$, with $y_\alpha(\cdot)$ solution to* (5.1).

Of course, our goal is to obtain the results in the topology of the original spaces $Y$ and $U$. While this is not possible for the entire control system, it turns out that the optimal solution displays an additional regularity that will make it possible the return to the original state space. The corresponding result is formulated below. For simplicity of notation we shall set $C(Y) = C([0,T];Y)$ and $L^2(Y) = L^2 2(0,T;Y)$; a similar notation will be adopted with $Y$ replaced by $U$.

**Proposition 5.2.** *With reference to the parametrized control Problem 5.1, the following statements are valid.*

*i) For any $\alpha \in [\mathcal{D}(A^{*2})]'$, there exists a unique optimal control $g^0(\cdot) \in L^2(0,T;U)$, which additionally satisfies $g^0 \in C([0,T];U)$. Moreover, $Ry^0_\alpha \in C([0,T];Y)$.*

*ii) There exists a selfadjoint, positive operator $P(t)$ on $\mathcal{L}(Y)$ with the following regularity,*

$$A^*P(t)A \in \mathcal{L}(Y, C(Y))\,, \quad B_1^*A^*P(t) \in \mathcal{L}(Y, C(U))\,, \quad \frac{d}{dt}P \in \mathcal{L}(Y, C(Y))\,;$$

*$P(t)$ satisfies the following (non-standard) Riccati equation, valid for any $y, w \in \mathcal{D}(A)$:*

$$\begin{aligned}&\frac{d}{dt}(P(t)y, w)_Y + (Ay, P(t)w)_Y + (P(t)y, Aw)_Y + (Ry, R\hat{y})_Y = \\ &\big((B_0^* + B_1^*A^*)Py, [I + B_1^*R^*RB_1]^{-1}[(B_0^* + B_1A^*)P(t)w]\big)_U\,,\end{aligned} \tag{5.4}$$

*with terminal condition $P(T) = 0$.*

*iii) For every $\alpha \in \mathcal{D}(A^{*2})]'$, the optimal cost $J(g^0) = \min_{g\in L_2(0,T;U)} J(g, y_\alpha)$ is given by $J(g^0) = (P(0)\alpha, \alpha)_Y$.*

*iv) The optimal control has the following feedback representation:*

$$g^0(t) = -\big[I - (B_0^* + B_1^*A^*)P(t)B_1\big]^{-1}\big[(B_0^* + B_1^*A^*)P(t)\big]y_\alpha^0(t)\,,$$

*where the operator $I - (B_0^* + B_1^*A^*)P(t)B_1$ is boundedly invertible on $U$ for each $t \geq 0$.*

## 5.1 Proof of Proposition 5.2

### 5.1.1 The parametrized LQ-problem

The starting point is the semigroup solution $y(t) = e^{At}\alpha + Lg(t)$. In order to derive the synthesis for the 'enlarged' problem by introducing a parameter $\alpha \in Y$ and later considering the family of control problems depending on a parameter $\alpha \in Y \oplus \mathcal{R}(B_1)$, one needs to develop a dynamics that is invariant on the space compatible with initial data.

It is then essential to extend the action of the semigroup $e^{At}$, originally defined on $Y$, to a larger space which contains $Y \oplus \mathcal{R}(B_1)$. This can be done on the strength of the extended regularity of the operator $B_1$ as acting into the dual space of $\mathcal{D}(A^*)$ (this will be seen below). The low regularity of the input-to-state mapping $L$ in

$$y(t) = e^{At}\alpha + Lg(t) = e^{At}\alpha + B_1g(t) + \big(L_0g\big)(t) \tag{5.5}$$

suggest that we take $\alpha$ in $[\mathcal{D}(A^{*2})]'$. It is important to emphasize that $y(0) \neq \alpha$, whereas $y(0) = \alpha + B_1g(0)$.

Via the same arguments as the ones used for the proof of Theorem 3.6 we obtain the following result.

**Lemma 5.3** (Auxiliary optimal control problem)**.** *Given $\alpha \in [\mathcal{D}(A^{*2})]'$, there exists a control function $g^0 \in L^2(0,T;U)$ which minimizes the cost functional* (5.3), *where $y(\cdot)$ is the solution to* (5.5) *corresponding to the control $g(\cdot)$.*

Our main goal is to provide a feedback synthesis of the optimal control $g^0$.

While the existence of the optimal solution for the parametrized problem follows from Lemma 5.3, in order to provide a (pointwise in time) feedback representation of the optimal control – via the optimal cost operator $P(t)$ – one needs to introduce, for any $s \in [0, T)$, the dynamics described by the equation

$$y(t, s; \alpha) = e^{A(t-s)}\alpha + L_s g(t)\,, \qquad s \le t \le T\,, \tag{5.6}$$

as well as the cost functional

$$J_{s,T}(g) \equiv \int_s^T \left(\|Ry(t)\|_Y^2 + \|g(t)\|_U^2\right) dt\,, \tag{5.7}$$

where as before $y = (u, u_t, u_t)$ and $L_{s,T}$ – $L_s$, in short – is the operator defined by

$$\{L_s g\}(t) = \int_s^t e^{A(t-\tau)} B_0 g(\tau)\, d\tau + A\int_s^t e^{A(t-\tau)} B_1 g(\tau)\, d\tau + B_1 g(t) \qquad \forall t \in [s, T]\,. \tag{5.8}$$

(Note that the subscript $s$ refers to initial time: in order to avoid confusion, the former operator $L_0 = L - B_1$ is written $L^0$.)

**Lemma 5.4.** *One has the following basic regularity of the input-to-state map:*

$$L_s^0 \textit{ is continuous}\colon L^2(s, T; U) \longrightarrow C([s, T]; [\mathcal{D}(A^{*2})]')\,,$$

$$L_s \textit{ is continuous}\colon L^2(s, T; U) \longrightarrow L^2(s, T; [\mathcal{D}(A^*)]') \oplus C([s, T]; [\mathcal{D}(A^{*2})]')\,,$$

*The above regularity improves when the input-to-state map is combined with the observation operator R; indeed, for the operator RL and its adjoint it holds*

$$RL_s \textit{ continuous}\colon L^1(s, T; U) \longrightarrow C([s, T]; Y)\,;$$

$$L_s^* R^* \textit{ continuous}\colon L^1(s, T; Y) \longrightarrow C([s, T]; U)\,.$$

*In addition, the operator $L_s^* R^*$ satisfies*

$$L_s^* R^* \textit{ continuous}\colon L^2(s, T; Y) \longrightarrow C([s, T]; U)$$

*uniformly with respect to $s \in [0, T)$.*

*Proof.* The regularity of the input-to-state map $L_s$ follows from the quantified regularity of the operator $B_1$, which takes boundedly $U$ into $[\mathcal{D}(A^*)]'$. Then the first statements in the Lemma follow from the very structure of $L_s$.

The key for the assertions about the regularity of the mappings $RL_s$ and $L_s^* R^*$ lies in the three properties $(I - A)^{-2} B_i \in \mathcal{L}(U, Y)$, $i = 1, 2$, $RA^2 \in \mathcal{L}(Y)$, $RB_1 = 0$. □

**Lemma 5.5.** *With reference to the optimal control problem* (5.6)-(5.7)*, the following statements are valid:*

*i)* **(Optimal pair).** *Given $\alpha \in [\mathcal{D}(A^{*2})]'$, there exists a unique optimal pair*

$$(\hat{y}(t,s;\alpha), \hat{g}(t,s;\alpha))$$

*for Problem 5.3, with*

$$\hat{g}(t,s;\alpha) = [I + L_s^* R^* R L_s]^{-1} L_s^* R^* R e^{A(\cdot - s)} \alpha \in C([s,T];U)\,, \tag{5.9a}$$

$$\hat{y}(t,s;\alpha) = e^{A(t-s)}\alpha + \{L_s \hat{g}(\cdot, s;\alpha)\}(t) \in C([s,T];[\mathcal{D}(A^{*2})]')\,, \tag{5.9b}$$

$$R\hat{y}(t,s;\alpha) = [I + R L_s L_s^* R^*]^{-1} R e^{A(\cdot - s)}\alpha \in C([s,T];Y)\,. \tag{5.9c}$$

*ii)* **(Riccati operator).** *The operator $P(t) \in \mathcal{L}(Y)$, $t \in [s,T]$, is given by*

$$P(t)\alpha = \int_t^T e^{A^*(\tau - t)} R^* R \hat{y}(\tau, t;\alpha)\, d\tau\,, \tag{5.10}$$

*The operator $P(t)$ is positive selfadjoint on $Y$, and represents the optimal cost (or Riccati) operator; its regularity properties are detailed separately (cf. Proposition 5.8 below).*

*iii)* **(Implicit feedback formula).** *The optimal control satisfies*

$$\hat{g}(t,s;\alpha) = -[B_0^* P(t) + B_1^* A^* P(t)]\Phi(t,s)\alpha\,,$$

*that is the following implicit equation*

$$\hat{g}(t,s;\alpha) = -[B_0^* P(t) + B_1^* A^* P(t)]\hat{y}(t,s;\alpha) + [B_0^* P(t) + B_1^* A^* P(t)]B_1 \hat{g}(t,s;\alpha)\,,$$

*where the operator $\Phi(t,s)$ is defined in* (5.14)*.*

*iv)* **(Optimal cost).** *The optimal cost for Problem 5.3 is given by*

$$J_s(\hat{g}) = \int_s^T \left(\|R\hat{y}\|_Y^2 + |\hat{g}(t)|_U^2\right) dt = \|[I + R L_s L_s^* R^*]^{-1/2}\, R e^{A(\cdot - s)}\alpha\|^2_{L^2(s,T;Y)}$$

*which is rewritten in terms of the optimal cost (or Riccati) operator as follows*

$$\begin{aligned} J_s(\hat{g}) &= (P(s)\alpha, \alpha) = \\ &= \left([I + R L_s L_s^* R^*]^{-1}\, R e^{A(\cdot - s)}\alpha, R e^{A(\cdot - s)}\alpha\right)_{L^2(s,T;Y)}, \end{aligned} \tag{5.11}$$

*thereby providing*

$$P(s)\alpha = e^{A^*(\cdot - s)} R^* \,[I + R L_s L_s^* R^*]^{-1}\, R e^{A(\cdot - s)}\alpha \quad \forall \alpha \in [\mathcal{D}(A^{*2}]'\,. \tag{5.12}$$

*Proof.* 1. The first statement follows by standard variational arguments applied to the LQ-problem (cf. [27]), after taking into consideration the regularity of input-output map stated in the preceding Lemma. The formulas for the optimal control, optimal state, observed state are derived as usual from the optimality conditions. The regularity of the optimal quantities follows from the regularity of the map $L$. In fact $(I-A)^{-2}\alpha \in Y$ gives $Re^{At}\alpha = R(I-A)^2 e^{At}(I-A)^{-2}\alpha \in C([0,T];Y)$ and by Lemma 5.4

$$L_s^* R^* Re^{A\cdot}\alpha \in C([0,T];U)\,.$$

We note that the invertibility of the operator $I + L_s^* R^* RL_s$ on $C([s,T];U)$ follows combining the self-adjointness and positivity of $L_s^* R^* RL_s$ – which guarantees the invertibility on $L^2(U)$ – with boundedness of the latter operator on $C([s,T];U)$. A classical bootstrap argument yields the claimed regularity: one starts from

$$v = [I + L_s^* R^* RL_s]^{-1} g\,,$$

with $g \in C(U)$, obtaining first $v \in L^2(U)$; then, since $v = -L_s^* R^* RL_s v + g$, the regularity improves to $v \in C(U)$.

The regularity of $R\hat{y}(t,s;\alpha)$ is a consequence of the regularity of the operator $RL$ in Lemma 5.4. Then, by the optimality condition

$$\hat{g}(t,s;\alpha) = -\{L_s^* R^*[R\hat{y}(\cdot,s;\alpha)]\}(t)\,, \tag{5.13}$$

which combined with the regularity of the operator $L_s^* R^*$ yields continuity (in time) of the optimal control.

2. All the statements ii)-iv) follow by variational arguments, by using the structure of the optimal quantities, once several properties that specifically pertain the operators $\Phi(\cdot,\cdot)$ and $P(\cdot)$ are proved. These technical results are given in the Propositions which follow next. □

**Remark 5.6.** A peculiarity of the parametrized minimization problem is that the optimal trajectory does not satisfy the evolution property. (For this reason the Riccati operator and the resulting synthesis cannnot be standard, as certain cancellations do not occur.) In the next section we study the evolution operator, defined only on a dual (extrapolation) space. This is a consequence of the low regularity of the control-to-state map.

### 5.1.2 The operator $\Phi(t,s)$

One of the most critical ingredients of Riccati theory is the evolution operator which describes the controlled dynamics. While in the standard theory the evolution operator is constructed directly from the optimal trajectory, this is not the case in singular theory. The reason is that such operator will not display the evolution property, that is the most fundamental feature, as it appears immediately from its definition below.

For any couple $(t,s)$ such that $0 \le s \le t \le T$, let $\Phi(t,s)\colon [\mathcal{D}(A^{*2})]' \to [\mathcal{D}(A^{*2})]'$ defined by

$$\Phi(t,s)\alpha := \hat{y}(t,s;\alpha) - B_1\hat{g}(t,s;\alpha) = e^{A(t-s)}\alpha + \{L_s^0\hat{g}(\cdot,s;\alpha)\}(t)\,. \tag{5.14}$$

The regularity properties of the operator $\Phi(\cdot,\cdot)$, which a priori belongs to $\mathcal{L}([\mathcal{D}(A^{*2})]')$ (for $(t,s)$ given), are collected in the following Proposition.

**Proposition 5.7.** *For the operator $\Phi(\cdot,\cdot)$ defined in* (5.14) *the following properties are valid:*

*i)* $\Phi(t,t)\alpha = \alpha$ *for all* $\alpha \in [\mathcal{D}(A^{*2})]'$*;*

*ii) for any* $s, \tau$ *with* $0 \le s \le \tau \le T$*, it holds*

$$Re^{A(\cdot-\tau)}\Phi(\tau,s)\alpha \in C([\tau,T];Y) \qquad \forall \alpha \in [\mathcal{D}(A^{*2})]' \tag{5.15}$$

*continuously with respect to* $\alpha$ *and uniformly in* $s$ *and* $\tau$*;*

*iii) for any* $s, \tau$ *with* $0 \le s \le \tau \le T$*, it holds*

$$R\Phi(\cdot,\tau)\,\Phi(\tau,s)\alpha \in C([\tau,T];Y) \qquad \forall \alpha \in [\mathcal{D}(A^{*2})]'$$

*continuously with respect to* $\alpha$ *and uniformly in* $s$ *and* $\tau$*;*

*iv) for any* $s, \tau, t$ *with* $0 \le s \le \tau \le t \le T$*, it holds in* $Y$

$$R\Phi(t,\tau)\,\Phi(\tau,s)\alpha = R\Phi(t,s)\alpha \qquad \forall \alpha \in [\mathcal{D}(A^{*2})]'$$

*Proof.* Since the operator $\Phi(t,s)$ – as defined above – has the same algebraic structure as in the classical LQ-theory, we can treat this operator as an evolution on the dual space to $\mathcal{D}(A^{*2})$. The needed regularity is established by referring to preceding Lemmas: in particular, to Lemma 5.4. The proof of the above properties can be produced along the lines of Lemma 8.3.2.3 and Lemma 8.3.2.4 in [27], on the basis of the powerful facts $RA^2 \in \mathcal{L}(Y)$, $RB_1 = 0$, beside $(I-A)^{-1}B_i \in \mathcal{L}(U,Y)$, $i = 1,2$ established in Lemma4.2. □

### 5.1.3 The optimal cost operator

We note that the Riccati Operator defined via optimal trajectory coincides with

$$P(t)\alpha = \int_t^T e^{A^*(\tau-t)}R^*R\Phi(\tau,t)\alpha\,d\tau\,, \qquad 0 \le t \le T\,,\ \alpha \in [\mathcal{D}(A^{*2})]'\,, \tag{5.16}$$

where $\Phi(\tau,t)$ is defined in (5.14). It is readily seen that, combining $\Phi(\tau,t)\alpha = \hat{y}(\tau,t;\alpha) - B_1\hat{g}(\tau,t;\alpha)$ with $RB_1 = 0$, (5.16) is actually equivalently rewritten as follows

$$P(t)\alpha = \int_t^T e^{A^*(\tau-t)}R^*R\hat{y}(\tau,t;\alpha)\,d\tau\,, \qquad 0 \le t \le T\,,\ \alpha \in [\mathcal{D}(A^{*2})]'$$

which confirms the equivalent relation (5.10).

**Proposition 5.8.** *The optimal cost operator* $P(t)$ *defined by* (5.16) *(equivalently, by* (5.10)*) satisfies the following (enhanced) regularity properties:*

1. (**Space regularity**) *For any given $t \in [0,T]$, one has*

$$A^{*2}P(t)A^2 \in \mathcal{L}(Y)\,; \tag{5.17}$$

*equivalently,*

$$P(t) \in \mathcal{L}([\mathcal{D}(A^{*\gamma_1})]', \mathcal{D}(A^{*\gamma_2})) \qquad \forall \gamma_1, \gamma_2 \le 2\,. \tag{5.18}$$

*As a consequence, $B_i^* P(\cdot)A^2 \in \mathcal{L}(Y,U)$, $i = 1,2$ and the gain operator $B^*P(t) \equiv B_0^*P(t) + B_1^*A^*P(t)$ satisfies $B^*P(t)A^2 \in \mathcal{L}(Y,U)$; namely,*

$$B_i^*P(t) \in \mathcal{L}([\mathcal{D}(A^{*2})]', U))\,; \qquad i = 0,1\,. \tag{5.19}$$

2. (**Time regularity**) *As for the regularity in time of the optimal cost operator – then, of the value function – one has*

$$P(\cdot) \text{ continuous}\colon [\mathcal{D}(A^{*2})]' \longrightarrow C(0,T;\mathcal{D}(A^{*2}))\,. \tag{5.20}$$

*Proof.* 1. Let $\alpha \in [\mathcal{D}(A^{*2})]'$ be given. We write down and compute, with $0 \le t \le T$,

$$\begin{aligned}(A^2)^*P(t)\alpha &= (A^2)^* \int_t^T e^{A^*(\tau-t)} R^*R\Phi(\tau,t)\alpha\, d\tau \\ &= \int_t^T e^{A^*(\tau-t)} [(A^2)^*R^*]\, R\Phi(\tau,t)\alpha\, d\tau\,,\end{aligned}$$

where the application of the operator $(A^2)^*$ commutes with the integration in time on the extrapolation space.
Then, the conclusion in (5.17) follows recalling that the function $R\Phi(\cdot,t)\alpha$ takes values in $Y$ (cf. (5.15)), whilst $(A^2)^*R^*$ is a bounded operator on $Y$.

As for gain operator, on the basis of (5.17), we next obtain

$$B_i^*P(\cdot)A^2 = B_i^*(I-A^*)^{-1}\,(I-A^*)P(\cdot)A^2 \in \mathcal{L}(Y,U)\,, \qquad i = 1,2\,,$$

owing to $B_i^*(I-A^*)^{-1} \in \mathcal{L}(Y,U)$, thereby confirming the exceptional boundedness and smoothing effect of the gain operator in (5.19).
2. Finally, the regularity in time of (5.20) follows combining the continuity in time of the function $R\Phi(\cdot,t)\alpha$ (see, once again, (5.15)) with more standard semigroup properties; see the proof in [27, p. 697]. □

### 5.1.4 The Riccati equation

In this section we shall provide several key relations which lead to a characterization of the Riccati operator via Differential Riccati equation. One of the fundamental properties is the evolution (of the evolution operator) with respect to the initial time, that is the second argument. Whilein the case of semigroups both evolutions are the same, in the case of time dependent evolutions – as in the present case – proving differentiability with respect to the initial time is challenging. The challenge is due to the compromised regularity and the intrinsic lack of invariance.

**Lemma 5.9** (Differentiability of the evolution with respect to initial time)**.** *The evolution operator* $\Phi(\tau, t)$ *defined in* (5.14) *satisfies*

$$\frac{d}{dt}\big(R\Phi(\tau,t)\alpha\big) = -R\Phi(\tau,t)\big[A - BB^*P(t)\big]\alpha \qquad \forall \alpha \in [\mathcal{D}(A^*)]', \quad a.e.\ in\ t,$$

*where* $B$ *denotes* $B_0 + AB_1$*.*

*Proof.* We will sketch the major steps of the proof.
1. We have seen that $R\Phi(t,s)$ may be defined on the extrapolation space $[\mathcal{D}(A^{*2})]'$. In particular, $R\Phi(t,s)Bu$ does make sense and it holds

$$\sup_{0\le t\le T} \|R\Phi(\cdot,t)Bu\|_{L^1(t,T;Y)} \le c_T\|u\|_U\,.$$

To justify the above assertion: we recall that

$$R\Phi(\cdot,t)\alpha = Re^{A(\tau-t)}\alpha + R\{L_t\hat{g}(\cdot,t;\alpha)\}(\tau)$$

which combined with (5.13) gives

$$R\Phi(\tau,t)\alpha = \big\{\big[I + RL_tL_t^*R^*\big]^{-1} Re^{A(\tau-t)}\alpha\big\}(\tau)\,, \quad \alpha \in [\mathcal{D}(A^{*2})]' \tag{5.21}$$

Insertion of $Bu \in [D(A^{*2})]'$ in place of $\alpha$ yields the estimate

$$\sup_{0\le t\le T} \|R\Phi(\cdot,t)Bu\|_{L^1(t,T;Y)} \le \cdots \le \|Re^{A(\tau-t)}\alpha\|_{L^?(t,T;Y)} \le c_T\|u\|_U\,.$$

2. A major step is to show existence (as well as to pinpoint the regularity) of the derivative of $R\Phi(\tau,t)\alpha$ with respect to $t$, with $\alpha$ belonging to the largest possible space. The arguments here owe to [27, Vol. II, Lemma 8.3.4.2]. Rewrite

$$R\Phi(\tau,t)\alpha + \big\{RL_tL_t^*R^*\,R\Phi(\cdot,t)\alpha\big\}(\tau) = Re^{A(\tau-t)}\alpha \tag{5.22}$$

and notice that if $\alpha \in [\mathcal{D}(A^*)]'$ (please note that here it is *not* $\alpha \in [\mathcal{D}(A^{*2})]'$), then

$$Re^{A(\tau-t)}x = RA^2\,A^{-1}e^{A(\tau-t)}A^{-1}x\,,$$

which gives

$$\frac{d}{dt}Re^{A(\tau-t)}x = -[RA^2]e^{A(\tau-t)}\underbrace{A^{-1}x}_{\in Y}\,.$$

Rewrite next (5.22) explicitly:

$$R\Phi(\tau,t)\alpha + R\int_t^\tau e^{A(\tau-\sigma)}B\int_\sigma^T B^*e^{A^*(r-\sigma)}R^*R\Phi(r,t)\alpha\,dr\,d\sigma = \\ = Re^{A(\tau-t)}\alpha$$

which implies

$$
\begin{aligned}
&\frac{d}{dt}\big(R\Phi(\tau,t)\alpha\big) - Re^{A(\tau-t)}B\int_t^T B^* e^{A^*(r-t)}R^*R\Phi(r,t)\alpha\,dr + \\
&\quad + R\int_t^\tau e^{A(\tau-\sigma)}B\int_\sigma^T B^* e^{A^*(r-\sigma)}R^*R\frac{d}{dt}\big(R\Phi(\tau,t)\alpha\big)\,dr\,d\sigma \\
&\qquad = -Re^{A(\tau-t)}A\alpha\,.
\end{aligned}
$$

The above implicit equation is rewritten as

$$
\big[I + RL_tL_t^*R^*\big]\frac{d}{dt}\big(R\Phi(\cdot,t)\alpha\big) = -\underbrace{Re^{A(\tau-t)}A\alpha}_{T_1(\tau,t)} + \underbrace{Re^{A(\tau-t)}BB^*P(t)\alpha}_{T_2(\tau,t)}
$$

which makes sense at least in $H^{-1}(0,T;Y)$.

Then, noting that

$$
T_1(\cdot,t)\in C([t,T];Y)\,,\qquad T_2(\cdot,t)\in L^\infty(t,T;Y)
$$

we get

$$
\frac{d}{dt}\big(R\Phi(\tau,t)\alpha\big) = \big[I+RL_tL_t^*R^*\big]^{-1}\Big\{-Re^{A(\tau-t)}A\alpha+Re^{A(\tau-t)}BB^*P(t)\alpha\Big\}\in L^2(t,T;Y)\,.
$$

Recalling (5.21) we finally obtain

$$
\frac{d}{dt}\big(R\Phi(\tau,t)\alpha\big) = -R\Phi(\tau,t)A\alpha + R\Phi(\tau,t)BB^*P(t)\alpha
$$

(cf. [27, Vol. II, § 8.3.4, p. 701]), thereby providing with

$$
\begin{aligned}
&\frac{d}{dt}(\big(R\Phi(\tau,t)x\big),y)_Y = \\
&\quad = -(R\Phi(\tau,t)\,[A-(B_0+AB_1)\,(B_0^*+B_1^*A^*)P(t)]x,y)_Y\,,\quad x\in[\mathcal{D}(A^*)]',y\in Y\,.
\end{aligned}
$$

□

**Lemma 5.10 (First Feedback Synthesis).** *The optimal control $\hat g$ admits the representation*

$$
\hat g(\tau,t;\alpha) = -[B_0^*+B_1^*A^*]P(\tau)\Phi(\tau,t)\alpha\qquad \forall\alpha\in[\mathcal{D}(A^{*2})]'\,.
$$

*Proof.* From the optimality conditions we know that

$$
\hat g(\tau,t;\alpha) = -\{L_t^*R^*R\hat y(\cdot,t;\alpha)\}(\tau)\,.
$$

Because $RB_1=0$, and exploiting the evolution property enjoyed by $\Phi$, it follows

$$
\hat g(\tau,t;\alpha) = -L_t^*R^*R\Phi(\cdot,t)\alpha\,.
$$

Observing that for any $\alpha\in[\mathcal{D}(A^*)]'$ one has $R\Phi(t,s)\alpha\in Y$ and $L_t^*R^*\colon L^1(Y)\to C(U)$, makes the above composition of operators meaningful – as acting on appropriate domains. This concludes the optimal synthesis as stated in the Lemma. □

**Lemma 5.11 (Riccati Equation).** *For all $x, y \in \mathcal{D}(A)$ the Riccati operator $P(\cdot)$ satisfies*

$$\Big(\frac{d}{dt}P(t)x, y\Big)_Y = -(R^*Rx, y)_H - (A^*P(t)x, y)_Y -$$
$$- (P(t)Ax, y)_Y - ([B_0^* + B_1^*A^*]P(t)x, [B_0^* + B_1^*A^*]P(t)y)_Y \,,$$

*with*

$$\begin{cases} A^*P_t(t)A \in \mathcal{L}(Y)\,, \\ A^*P_t(t)A \text{ continuous} : Y \longrightarrow L^\infty(0,T;Y). \end{cases}$$

*Proof.* In order to derive the Riccati equation, we follow the so called direct approach (cf. [27]). Differentiation (in a weak sense) of the Riccati operator requires the characterization of the left derivative (with respect to the initial time) of the evolution. However, in the present case, Proposition 5.7 provides the needed regularity for the evolution when acted upon by the observation. This allows to obtain the critical representation for the right evolutionary derivative which is given by Lemma 5.9. The said representation, when combined with the "first feedback synthesis" in Lemma 5.10 gives the final conclusion.

Calculations are justified by the already proved regularity of the quantities involved. In particular, the compromised regularity of the derivative of the evolution (which requires $\alpha \in [\mathcal{D}(A^*)]'$, is sufficient to obtain the final conclusion. □

We note that the feedback synthesis given in Lemma 5.10 is in terms of the evolution operator $\Phi(t,s)$. What is needed, instead, is the feedback synthesis in terms of the actual trajectory $\hat{y}$. This is achieved below.

**Lemma 5.12 (Feedback Synthesis).** *For any $\alpha \in [\mathcal{D}(A^{*2})]'$, the following feedback representation of the optimal control $\hat{g}(t;\alpha)$ holds true:*

$$\hat{g}(t;\alpha) = -\big[I - [B_0^* + B_1^*A^*]P(t)B_1\big]^{-1}[B_0^* + B_1^*A^*]P(t)\hat{y}(t,\alpha)\,;$$

*the formula provides an "on line" optimal control $\hat{g}(\cdot,\alpha) \in L_2(U)$ for the $\alpha$-parametrized problem.*

*Proof.* For the feedback synthesis of the optimal control it remains to discuss the invertibility of the operator

$$I - [{B_0}^* + {B_1}^*A^*]P(t)B_1\,.$$

**Proposition 5.13.** *The operator $I - [{B_0}^* + {B_1}^*A^*]P(t)B_1$ is boundedly invertible on $U$ for each $t \in [0,T]$.*

*Proof. Step 1.* We shall first prove the injectivity of the operator $I-[{B_0}^*+{B_1}^*A^*]P(t)B_1$ for $t = 0$. Then, the dynamic programming argument will extend the conclusion to all $t \in [0,T]$.

By contradiction, let $v \in U$ be such that $v \neq 0$, and

$$v = [{B_0}^* + {B_1}^* A^*] P(t) B_1 v \,. \tag{5.23}$$

Consider then the optimal control problem with $y_0 = 0$, and $\alpha = -B_1 v$. The (implicit) optimal synthesis gives

$$\hat{g}_\alpha(0) = -[B_0^* + B_1^* A^*] P(0) \big(\hat{y}_\alpha(0) - B_1 \hat{g}_\alpha(0)\big) \,. \tag{5.24}$$

But from the continuity of optimal control, we also have $\hat{y}_\alpha(0) = \alpha + B_1 \hat{g}_\alpha(0)$. This, combined with (5.24) give

$$\hat{g}_\alpha(0) = -[B_0^* + B_1^* A^*] P(0) [\hat{y}_\alpha(0) - B_1 \hat{g}_\alpha(0)] = [B_0^* + B_1^* A^*] P(0) B_1 v \,. \tag{5.25}$$

From the contradiction argument (5.23) it follows that $g_\alpha^0(0) = v$. On the other hand, the optimal control problem with $y_0 = 0$ produces only one solution which is equal identically to zero. Therefore, the optimal control $g^0$ should be zero as well. This contradicts the fact that $v \neq 0$.

The same argument applied to the dynamics originating at the time $t$ yields injectivity of $I - [B_0^* + B_1^* P(t)] B_1$ on $U$, for any $t \in [0,T]$.

*Step 2.* Compactness of the operator $[B_0^* + B_1^* P(t)] B_1$. This follows from the regularity properties of $P(t)$ which asserts that $P(t) \colon \mathcal{D}(A^{*2})]' \to \mathcal{D}(A^{*2})$ is bounded. However, the injection $B_1 \colon U \to \mathcal{D}(A^{*2})$ is compact. Thus, the final conclusion follows from the theory of compact operators. □

Now, the conclusion in Lemma 5.12 follows from the Proposition 5.13 and the representation in Lemma 5.10 supported by definition of evolution operator $\Phi$. □

*Completion of the proof of Proposition 5.2*: combine the results stated in Proposition 5.8, Lemma 5.11 and Lemma 5.12.

*Completion of the proof of Theorem 3.7*: setting $\alpha = y_0 - B_1 g_0$ provides the conclusions stated in Theorem 3.7.

## 5.2 Proof of Theorem 3.10

It remains to be shown that $\hat{g}(0)$ coincides with the parameter $g_0$. This is done below.

Let $y_0 \in [\mathcal{D}(A^{*2})]'$ and $g_0 \in U$ be given. With $\alpha = y_0 - B_1 g_0$, we know from from Part 1 of Theorem 3.7 that the optimal control $g^0$ belongs to $C([0,T];U)$. Therefore, in order to comply with the original model one is asking for the following selection of the parameter $g_0$: $g_0 = g^0(0)$. This amounts to

$$g_\alpha^0(t=0) = g_0 \,, \qquad \alpha = y_0 - B_1 g_0 \,.$$

The above implicit relation is always uniquely solvable for some $g_0 \in U$. In fact, the matching condition amounts to solving $g_0 = F\alpha = F(y_0 - B_1 g_0)$, that is $(I - FB_1) g_0 = F y_0$, where we set $F := [B_0^* + B_1^* A^*] P(0)$. (We note the key property $F \in \mathcal{L}([\mathcal{D}(A^*)]', U)$.)

Thus, it is sufficient to recognize that $I - FB_1$ coincides with the operator $G(0)$, whose requisite boundeness and invertibility have been shown in Proposition 5.13. This gives $g_0 = (I - FB_1)^{-1}Fy_0$, with $(I - FB_1)^{-1} \in \mathcal{L}(U)$.

The previous Proposition provides the crucial result for the concluson of our analysis.

**Corollary 5.14.** *Let $y_0 \in \mathcal{D}(A^{*2})]'$ be given. Cosnider Problem $\mathcal{P}_\alpha$, with $\alpha = y_0 - B_1 g_0$, and $g_0 \in U$ given by*

$$g_0 = (I - FB_1)^{-1}Fy_0 \,, \tag{5.26}$$

*where $F := F := [B_0^* + B_1^* A^*]P(0)$.*

*Then, there exists a unique optimal control $g^0 \in C([0,T];U)$ and a corresponding trajectory* (3.1)*, with $y^0(0) = y_0$, such that the results of Proposition 5.2 hold with $\alpha = y_0 - B_1 g_0$ and $g_0$ given by* (5.26)*.*

In other words, by solving the parametrized optimal control problem with a given $\alpha = y_0 - B_1 g_0$ and a parameter $g_0 \in U$ we solve a family of LQ problems, which always have a unique solution. The original dynamics is included in this family. By selecting $g_0 \in U$ according to the matching condition, we make a selection of a problem whose dynamics coincides with the original one. However, the above does not imply that the constructed optimal control for the parametrized control problem is also optimal for the original problem – when considered within the $L^2(U)$ framework for optimal controls. In fact, the latter may not have an optimal solution at all when $y_0 \in \mathcal{R}(B_1)$, as shown in Theorem 3.4; see also [26]. Thus, the constructed control is suboptimal, yet it corresponds to the original dynamics. However, if the original problem does have an $L_2(U)$ optimal control, then such control coincides with a parametrized control where $g_0$ is selected according to the matching condition.

### 5.3 Proof of Theorem 3.11

Theorem 3.11 follows from Theorem 3.10 by using a rather standard argument in calculus of variations. To wit: we recall from Proposition 5.2 that the optimal value for the parametrized problem equals

$$J(\hat{g}, \hat{y}_{g_0}) = (P(0)\alpha, \alpha)_Y = (P(0)(y_0 - B_1 g_0), y_0 - B_1 g_0)_Y \,.$$

On the strength of positivity and selfadjointness of $P(0)$ we can write the above as

$$J(\hat{g}, \hat{y}_{g_0}) = ||P^{1/2}(0)(y_0 - B_1 g_0)||_Y^2 \,.$$

Appealing to the regularity properies of $P(0)$ listed in Theorem 3.10 we obtain that $J(g_0) \equiv J(\hat{g}, \hat{y}_{g_0})$ is weakly lower semicontinuous on $U$. Indeed, the latter follows from

$$\begin{aligned} J(\hat{g}, \hat{y}_{g_0}) = (P(0)(y_0 - B_1 g_0), y_0 - B_1 g_0)_Y &= (P(0)y_0, y_0)_Y \\ -2(P(0)y_0, B_1 g_0)_Y &+ (P(0)B_1 g_0, B_1 g_0)_Y \,, \end{aligned} \tag{5.27}$$

where $(I-A)^{-1}B_1 : U \to Y$ is compact and $A^* P(0) A : Y \to Y$ is bounded. This gives compactness of the map $g \to P^{1/2}(0) B_1 g$ from $U$ to $Y$, adressing the convergence of the last quadratic term in (5.27).

As for the first term, we simply recall Proposition 5.8 which states $A^{*2} P(0) A^2 : Y \to Y$ is also bounded. Strong continuity of the second term (linear in $g_0$ ) follows now from $A^{-1}B_1 \in L(Y)$ and $A^* P(0) A^2 \in L(Y)$. Thus the regularity of the Riccati operator $P(0)$ along with $(I-A)^{-1}B_1 \in \mathcal{L}(Y)$ implies weak lower-semicontinuity of the functional. Since $U_0$ is weakly compact, we obtain a minimizing sequence $g_n \in U_0$ such that $J(g_n) \to d = \inf_{g_0 \in U_0} J(g_0)$ and $g_n \to g^* \in U_0$ weakly in $U$. Weak lower semicontinuity of $J(g_0)$ gives an existence of a minimizer. The characterization of the minimizer follows now from a standard argument in calculus of variations, after taking into consideration the representation of the functional via Riccati operator. This leads to the final conclusion stated in Theorem 3.11.

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## A On the inversion of the free dynamics operator

This Appendix contains (pretty elementary and yet) explicit calculations showing that although $A$ does not admit a bounded inverse on $L^2(\Omega)$, the operator $I-A$ does. The obtained expression of the inverse $(I-A)^{-1}$ then enables us to compute $(I-A)^{-1}B_i$, disclosing its boundedness and compactness.

Let us return to the definition (2.7) of operator $A : Y \supset \mathcal{D}(A) \longrightarrow Y$ that governs the free dynamics, whose domain $\mathcal{D}(A)$ is described in both abstract and PDE terms at the beginning of Section 2.3. First we verify that $A$ does not adimit a bounded inverse on $Y$. Given $w = (w_1, w_2, w_3)^T \in Y$, seek an element $y = (y_1, y_2, y_3)^T \in \mathcal{D}(A)$, such that $Ay = w$. In view of (2.7), one has

$$Ay = \begin{pmatrix} y_2 \\ y_3 \\ -c^2 \mathcal{A} y_1 - \big[b\mathcal{A} + c(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)\big]y_2 - \big[\alpha I + \frac{b}{c}(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)\big]y_3 \end{pmatrix}$$

which gives

$$\begin{cases} y_2 = w_1 \\ y_3 = w_2 \\ -c^2\mathcal{A}y_1 = \big[b\mathcal{A} + c(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)\big]w_1 + \big[\alpha I + \frac{b}{c}(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)\big]w_2 + w_3\,. \end{cases}$$

Since $\mathcal{A}$ does not admit a bounded inverse on $L^2(\Omega)$, the third equation does not yield $y_1$ in terms of $w$.

We thus restart from $(I - A)y = w$, where $w \in Y$ is given. This is now

$$\begin{cases} y_1 - y_2 = w_1 \\ y_2 - y_3 = w_2 \\ y_3 + c^2\mathcal{A}y_1 + \big[b\mathcal{A} + c(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)\big]y_2 + \big[\alpha I + \frac{b}{c}(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)\big]y_3 = w_3\,, \end{cases}$$

which gives

$$\begin{cases} y_1 = w_1 + w_2 + y_3 \\ y_2 = w_2 + y_3 \end{cases} \tag{A.1}$$

along with

$$\begin{aligned} &\Big[(1+\alpha)I + (c^2+b)\mathcal{A} + \Big(c + \frac{b}{c}\Big)(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)\Big]\,y_3 = \\ &\qquad = w_3 - c^2\mathcal{A}(w_1+w_2) - \big[b\mathcal{A} + c(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)\big]w_2 = \\ &\qquad = -c^2\mathcal{A}w_1 - \big[(c^2+b)\mathcal{A} + c(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)\big]w_2 + w_3\,. \end{aligned} \tag{A.2}$$

Since the operator $(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)$ is non-negative and selfadjoint on $\mathcal{D}(\mathcal{A})$, then the operator

$$S := (1+\alpha)I + (c^2+b)\mathcal{A} + \Big(c + \frac{b}{c}\Big)(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I) \tag{A.3}$$

admits a bounded inverse $S^{-1}\colon L^2(\Omega) \longrightarrow \mathcal{D}(\mathcal{A})$; consequently, we obtain from (A.2)

$$y_3 = S^{-1}\big[-c^2\mathcal{A}w_1 - \big((c^2+b)\mathcal{A} + c(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)\big)\,w_2 + w_3\big]\,.$$

The above, combined with (A.1) establishes

$$\begin{aligned} &(I - A)^{-1} = \\ &= \begin{pmatrix} I - c^2S^{-1}\mathcal{A} & I - S^{-1}\big[(c^2+b)\mathcal{A} + c(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)\big] & S^{-1} \\ -c^2S^{-1}\mathcal{A} & I - S^{-1}\big[(c^2+b)\mathcal{A} + c(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)\big] & S^{-1} \\ -c^2S^{-1}\mathcal{A} & -S^{-1}\big[(c^2+b)\mathcal{A} + c(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)\big] & S^{-1} \end{pmatrix} \\ &= \begin{pmatrix} = I - c^2S^{-1}\mathcal{A} & (1+\alpha)S^{-1} + \frac{b}{c}S^{-1}(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I) & S^{-1} \\ -c^2S^{-1}\mathcal{A} & (1+\alpha)S^{-1} + \frac{b}{c}S^{-1}(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I) & S^{-1} \\ -c^2S^{-1}\mathcal{A} & -I + (1+\alpha)S^{-1} + \frac{b}{c}S^{-1}(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I) & S^{-1} \end{pmatrix}, \end{aligned} \tag{A.4}$$

where the latter equivalent representation of the matrix elements (of the operator) follows from the identity

$$(c^2+b)\mathcal{A}+c(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)\equiv S-(1+\alpha)I-\frac{b}{c}(\mathcal{A}+I)N_1N_1^*(\mathcal{A}+I)\,.$$

The obtained explicit expression of the inverse of $I-A$ in (A.4) combined with the actual structure of the operators $B_i$ in (2.8), $i=0,1$, gives in particular,

$$\begin{aligned}&(I-A)^{-1}B_0=(I-A)^{-1}\begin{pmatrix}0\\0\\c^2(\mathcal{A}+I)N_0\end{pmatrix}=\begin{pmatrix}c^2S^{-1}(\mathcal{A}+I)N_0\\c^2S^{-1}(\mathcal{A}+I)N_0\\c^2S^{-1}(\mathcal{A}+I)N_0\end{pmatrix}\,,\\&(I-A)^{-1}B_1=\frac{b}{c^2}(I-A)^{-1}B_0\,.\end{aligned}\tag{A.5}$$

The conclusion that both $(I-A)^{-1}B_i$ belong to $\mathcal{L}(U,Y)$, $i=0,1$, as well as their compactness, now follows from the structure of the operator $S$, along with the definitions (and properties) of the operators $\mathcal{A}$, $N_0$, $N_1$.

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