Groups whose character degree graph has diameter three

GROUPS WHOSE CHARACTER DEGREE GRAPH HAS DIAMETER THREE

CARLO CASOLO, SILVIO DOLFI, EMANUELE PACIFICI, AND LUCIA SANUS

Dedicated to the memory of Laci Kovács

ABSTRACT. Let G be a finite group, and let $\Delta(G)$ denote the *prime graph* built on the set of degrees of the irreducible complex characters of G. It is well known that, whenever $\Delta(G)$ is connected, the diameter of $\Delta(G)$ is at most 3. In the present paper, we provide a description of the finite solvable groups for which the diameter of this graph attains the upper bound. This also enables us to confirm a couple of conjectures proposed by M.L. Lewis.

1. INTRODUCTION

Let G be a finite group; we denote by $\mathrm{Irr}(G)$ the set of all irreducible complex characters of G, and write

$$\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$$

for the set of the degrees of such characters. The character degree graph $\Delta(G)$ is thus defined as the graph with vertex set the set $\rho(G)$ of all the primes that divide some $\chi(1) \in \mathrm{cd}(G)$, and two distinct primes p and q are adjacent if and only if pq divides some degree in $\mathrm{cd}(G)$. The study of the graph $\Delta(G)$ and of the relationships between the properties of $\Delta(G)$ and the structural features of the group G, has by now a rich literature (we recommend the survey paper [8] for a general overview of the subject), and the purpose of this paper is to contribute to one particular aspect of this research.

A fundamental result of P.P. Palfy ([15]) ensures that if G is a solvable group, then given any three distinct primes in $\rho(G)$, at least two of them are adjacent in $\Delta(G)$. From this it immediately follows that, for a solvable group G, $\Delta(G)$ has at most two connected components (both inducing a complete subgraph of $\Delta(G)$), and that, when $\Delta(G)$ is connected, the diameter of $\Delta(G)$ is at most 3 (that this latter inequality holds in any finite group is proved in [11]). For some time it has been unknown whether there existed solvable groups whose character degree graph

2000 *Mathematics Subject Classification.* 20C15.

The first three authors are partially supported by the Italian INdAM-GNSAGA. The fourth author is partially supported by the Spanish MINECO proyecto MTM2013-40464-P, partly with FEDER funds and Prometeo2011/030-Generalitat Valenciana.

has diameter 3, until the question was settled by M.L. Lewis, who constructed in [9] a solvable group G such that $\Delta(G)$ has 6 vertices and diameter 3.

It was Lewis construction and his related comments (made particularly explicit in [8]) that prompted us to study in more detail solvable groups G such that the diameter of $\Delta(G)$ is 3. Through our analysis in the present paper, we pin their structure down enough to show that they all closely resemble Lewis' examples, and to allow to confirm a couple of conjectures appearing in [8].

In the following statement, which is the main result of this paper, $\mathbf{F}(G)$ denotes the Fitting subgroup of the group G and, for $i \geq 1$, $\gamma_i(P)$ is the i-th term of the lower central series of the p-group P.

Theorem A. *Let G be a finite solvable group such that $\Delta(G)$ is connected and $\mathrm{diam}(\Delta(G)) = 3$. Then the following conclusions hold.*

(a) *There exists a prime p such that $G = PH$, with P a normal non-abelian Sylow p-subgroup of G and H a p-complement.*
(b) *$\mathbf{F}(G) = P \times A$, where $A = \mathbf{C}_H(P) \leq \mathbf{Z}(G)$, H/A is not nilpotent and has cyclic Sylow subgroups.*
(c) *$\Delta(G/\gamma_3(P))$ is disconnected.*
(d) *If c is the nilpotency class of P, then all factors $M_1 = [P, G]/P'$ and $M_i = \gamma_i(P)/\gamma_{i+1}(P)$, for $2 \leq i \leq c$, are chief factors of G of the same order p^n, where n is divisible by at least two distinct odd primes; moreover, for all $1 \leq i \leq c$, $G/\mathbf{C}_G(M_i)$ embeds as an irreducible subgroup in the group of semi-linear transformations $\Gamma(p^n)$.*

In particular, we have that $\Delta(G/\gamma_3(P))$ is a disconnected subgraph of $\Delta(G)$ with the same set of vertices, thus confirming a suggestion of Lewis ([8]). In fact, it will not be hard to derive a proof of a related conjecture concerning the structure of the graph $\Delta(G)$, when G is solvable and $\mathrm{diam}(\Delta(G)) = 3$. Let r, s be two vertices of $\Delta(G)$ with $d(r, s) = 3$, and denote by π_1 and π_2 the sets consisting of r, respectively s, and all vertices adjacent to it; then $\pi_1 \cap \pi_2 = \emptyset$ and (by Palfy's three primes condition) $\rho(G) = \pi_1 \cup \pi_2$. Moreover, denoting by $\mathbf{F}_2(G)$ the second Fitting subgroup of G and supposing $p \in \pi_1$ (we will see that $r \neq p \neq s$), we will show that $\pi_1 = \pi(\mathbf{F}_2(G)/\mathbf{Z}(G))$, $\pi_2 = \pi(G/\mathbf{F}_2(G))$, $2 \notin \pi_2$, $|\pi_2| \geq 2$ and π_1, π_2 both induce complete subgraphs of $\Delta(G)$ (see Remark 4.4). Indeed, π_1 and π_2 are the set of vertices of the two connected components of $\Delta(G/\gamma_3(P))$; it follows (see Remark 4.4) that

$$|\pi_1| \geq 2^{|\pi_2|}.$$

Hence, Conjecture 4.8 of [8] is established; in particular, $|\rho(G)| \geq 6$ (indeed, it turns out that Lewis' example has the smallest possible order).

Still in the spirit of another suggestion by Lewis (see the paragraph following 5.8 in [8]), an immediate consequence of Theorem A is the following result.

Corollary B. *Let G be a finite solvable group, and assume that $\Delta(G)$ is connected with diameter 3. Then the Fitting height of G is precisely 3. In fact, G is a nilpotent-by-metacyclic group.*

As it is apparent from the above remarks, a central role in our treatment is played by solvable groups with disconnected degree graph, the main features of those we will need to have almost constantly in hand. For this, our main source is their description in [10] (although similar results also appear in [16] and [14]), and we in particular refer to the list of six subcases in section 2 (and 3) of that paper. From the same arguments that prove Theorem A, we derive a result which we believe adds to the understanding of case 2.6 in [10].

Theorem C. *Let G be a finite solvable group such that $\Delta(G)$ is disconnected and $\mathbf{F}(G)$ is not abelian. Then there is a unique prime p such that $P = \mathbf{O}_p(G)$ is not contained in $\mathbf{Z}(G)$ and*

(a) *either p is an isolated vertex of $\Delta(G)$, or*
(b) *$\Delta(G/P')$ is disconnected and, if c is the nilpotency class of P, all factors $M_1 = [P,G]/P'$ and $M_i = \gamma_i(P)/\gamma_{i+1}(P)$, for $2 \leq i \leq c$, are chief factors of G of the same order p^n, with $n \geq 3$; moreover, for all $1 \leq i \leq c$, $G/\mathbf{C}_G(M_i)$ embeds as an irreducible subgroup in the group of semi-linear transformations $\Gamma(p^n)$.*

We conjecture that, both in the disconnected and in the diameter-three case, the chief factors M_i are pairwise non-isomorphic as G-modules over $\mathrm{GF}(p)$; this is true for the first pair M_1 and M_2 (see point (a) in the proof of Proposition 4.2), but we were not able to prove it in general. Another question that we leave open is whether, in both Theorem A and Theorem C, one has $P = [P,G] \times \mathbf{Z}(G)$ (again, this is true modulo $\gamma_3(P)$). Finally, by looking at the known examples, one might ask if it is true that, in Theorem A, not only $\Delta(G/\gamma_3(P))$ but also $\Delta(G/\gamma_c(P))$ is disconnected.

2. Notation and preliminaries

Throughout this paper, every group is tacitly assumed to be a finite group. We write $\mathrm{V}(G)$ and $\mathrm{E}(G)$ for the sets of vertices and edges, respectively, of the prime graph $\Delta(G)$ on irreducible character degrees. We denote by $d_G(u,v)$ the distance in $\Delta(G)$ between the two (distinct) vertices u and v (i.e. the length of a shortest path joining u and v; set $d_G(u,v) = \infty$ if there is no such path), and by $\mathrm{diam}(\Delta(G))$ the maximum of $d_G(u,v)$ for $u,v \in \mathrm{V}(G)$ if $\Delta(G)$ is connected (whereas we set $\mathrm{diam}(\Delta(G)) = \infty$ if $\Delta(G)$ is not connected).

As customary, we denote by $\Gamma(p^n)$ the semi-linear group on the field $\mathrm{GF}(p^n)$, and by $\Gamma_0(p^n)$ the subgroup of $\Gamma(p^n)$ induced by the field multiplications. If V is an n-dimensional vector space over $\mathrm{GF}(p)$, then V can be identified with the additive

group of a field of order p^n, and in this sense we write $\Gamma(V)$ and $\Gamma_0(V)$ for $\Gamma(p^n)$ and $\Gamma_0(p^n)$ respectively.

Let $a > 1$ and n be positive integers. A prime t is called a *primitive prime divisor* for (a, n) if t divides $a^n - 1$ but t does not divide $a^j - 1$ for $1 \leq j < n$. Recall that, by a well-known result by Zsigmondy ([12, Theorem 6.2]), such a prime always exists except when $n = 6$ and $a = 2$, or $n = 2$ and $a + 1$ is a power of 2.

Let N be a normal subgroup of G and let $\lambda \in \mathrm{Irr}(N)$. We denote by $\mathrm{Irr}(G|\lambda)$ the set of irreducible characters χ of G such that λ is an irreducible constituent of χ_N. In this setting, χ and λ are said to be *fully ramified* with respect to G/N (but sometimes, when the context is clear enough, we also say that λ is fully ramified in G) if $\chi_N = e\lambda$ with $e^2 = |G : N|$. By [7, Problem 6.3], this is equivalent to the fact that χ vanishes on $G \setminus N$ with λ invariant in G, and also to the fact that χ is the unique irreducible constituent of λ^G still with λ invariant in G.

If A is an abelian group, we write $\widehat{A}$ to denote the dual group of A, that is, the set $\mathrm{Irr}(A)$ endowed with multiplication of characters.

Also, we freely use without references some basic facts of Character Theory such as Clifford Correspondence, Gallagher's Theorem, Ito-Michler's Theorem, results concerning character extension and coprime actions (see [7]).

We shall also take into account the following well-known result concerning character degrees.

Lemma 2.1 ([12, Proposition 17.3])**.** *Let G be a solvable group. Let $F = \mathbf{F}(G)$ and $K = \mathbf{F}_2(G)$. Then there exists $\chi \in \mathrm{Irr}(G)$ such that $\pi(K/F) \subseteq \pi(\chi(1))$.*

As mentioned in the Introduction, we will make an intensive use of the classification, provided in [10], of solvable groups whose character degree graph is disconnected. The next statement summarizes some aspects of that classification: the groups in (a), (b) and (c) are respectively those of types 2.1, 2.4 and 2.6 (described further in 3.1, 3.4 and 3.6, respectively) in [10].

Theorem 2.2. *Let G be a solvable group, and set $F = \mathbf{F}(G)$, $K = \mathbf{F}_2(G)$. Assume that $\Delta(G)$ has two connected components. Then the following conclusions hold.*

(a) *Assume that G is metanilpotent. Then $G = PH$, where $P \trianglelefteq G$ is a non-abelian Sylow p-subgroup for a suitable prime p, and H is an abelian p-complement. Moreover, $P' \leq \mathbf{C}_P(H)$, and every non-linear irreducible character of P is fully ramified with respect to $P/\mathbf{C}_P(H)$. Finally, the sets of vertices of the two connected components of $\Delta(G)$ are respectively $\{p\}$ and $\pi(G/F)$.*

(b) *Assume that F is abelian, and that $|\mathrm{V}(G)| > 2$. Then $G = MH$, where $M \trianglelefteq G$ is an elementary abelian p-group for a suitable prime p, and H is a complement for M. Also, $F = M \times \mathbf{Z}(G)$, $\mathbf{Z}(G) = \mathbf{C}_H(M)$ and $G/F \leq \Gamma(M)$. The subgroup K acts irreducibly (by conjugation) on M, and both K/F and G/K are cyclic*

groups. Finally, the sets of vertices of the two connected components of $\Delta(G)$ *are respectively* $\pi(K/F)$ *and* $\pi(G/K)$.

(c) *Assume that* F *is non-abelian and that, whenever* $\mathbf{O}_r(G)$ *is non-abelian, the prime* r *is not an isolated vertex of* $\Delta(G)$. *Then* $G = PH$, *where* $P \trianglelefteq G$ *is a non-abelian Sylow* p*-subgroup for a suitable prime* p, *and* H *is a* p*-complement. Also,* $F = P \times U$ *where* $U \leq \mathbf{Z}(G)$. *The factor group* G/P' *is a group as in* (b), *so, in particular,* K/F *and* G/K *are cyclic groups. Finally, the sets of vertices of the two connected components of* $\Delta(G)$ *are respectively* $\{p\} \cup \pi(K/F)$ *and* $\pi(G/K)$.

We stress that a group G as in (b) or (c) of Theorem 2.2 is such that every Sylow subgroup of G/F is cyclic.

We also quote the following result, which is Theorem 5.5 of [10].

Lemma 2.3. *Let* G *be a solvable group such that* $\Delta(G)$ *is a disconnected graph. Then there exists a unique prime* p *such that* $\mathbf{O}_p(G)$ *is non-central in* G.

Next, another preliminary lemma.

Lemma 2.4. *Let* G *be a group such that* $\mathbf{F}(G) = M \times Z$, *with* $Z \leq \mathbf{Z}(G)$ *and* $M \trianglelefteq G$. *Assume also that every irreducible character of* $\mathbf{F}(G)$ *extends to its inertia subgroup. Then* $\Delta(G) = \Delta(G/Z)$.

Proof. Observe first that, by our assumptions, every irreducible character of Z has an extension to G: in fact, if θ is in $\widehat{Z}$, then $\theta \times 1_M \in \mathrm{Irr}(\mathbf{F}(G))$ extends to $G = I_G(\theta \times 1_M)$. Now, let d be a number in $\mathrm{cd}(G)$, χ an irreducible character of G of degree d, and θ an irreducible constituent of χ_Z; denoting by ξ an extension of θ to G, by Gallagher's Theorem there exists $\psi \in \mathrm{Irr}(G/Z)$ such that $\chi = \xi\psi$. As a consequence, $d = \chi(1) = \psi(1) \in \mathrm{cd}(G/Z)$, and the desired conclusion follows. ∎

Finally, the following result by C.P. Morresi Zuccari ([13, Corollary C]) will also be relevant for our purposes.

Theorem 2.5. *Let* G *be a solvable group such that* $\Delta(G)$ *is connected. If* $\mathbf{F}(G)$ *is abelian, then* $\mathrm{diam}(\Delta(G)) \leq 2$.

3. Some proofs

We start with two lemmas concerning modules over finite fields for cyclic groups.

Lemma 3.1. *Let* G *be a cyclic group,* $\mathbb{K}$ *a finite field of order* q, *and* M *a faithful irreducible* m*-dimensional* $\mathbb{K}[G]$*-module. Also, let* ϵ *be an element of order* $|G|$ *in the multiplicative group of* $\mathbb{F} = \mathrm{GF}(q^m)$. *Then the following hold.*

(a) *If* M *is a constituent of* $M \otimes_{\mathbb{K}} M$, *then there exist* σ_1 *and* σ_2 *in* $\mathrm{Gal}(\mathbb{F} \,|\, \mathbb{K})$ *such that* $\epsilon^{\sigma_1} \cdot \epsilon^{\sigma_2} = \epsilon$.

(b) *If* M *is self-contragredient, then there exists* σ *in* $\mathrm{Gal}(\mathbb{F} \,|\, \mathbb{K})$ *such that* $\epsilon^{\sigma} = \epsilon^{-1}$.

Proof. Observe that $\mathbb{F}$ is a splitting field for G over $\mathbb{K}$. By [5, II.3.10], we can identify M with the additive group of $\mathbb{F}$, and the action of a suitable generator x of G with the multiplication by ϵ. We denote by $M_{\mathbb{F}}$ the 1-dimensional $\mathbb{F}[G]$-module arising in this way. Setting $M^{\mathbb{F}} = M \otimes_{\mathbb{K}} \mathbb{F}$, by [6, VII, 1.16 a)] we get

$$M^{\mathbb{F}} = \bigoplus_{\sigma \in \mathrm{Gal}(\mathbb{F} \,|\, \mathbb{K})} (M_{\mathbb{F}})^{\sigma}.$$

Now, if M is a constituent of $M \otimes_{\mathbb{K}} M$, then $M^{\mathbb{F}}$ is a direct summand (as an $\mathbb{F}[G]$-module) of

$$(M \otimes_{\mathbb{K}} M)^{\mathbb{F}} \simeq M^{\mathbb{F}} \otimes_{\mathbb{F}} M^{\mathbb{F}} \simeq \bigoplus_{\sigma_1, \sigma_2 \in \mathrm{Gal}(\mathbb{F} \,|\, \mathbb{K})} (M_{\mathbb{F}})^{\sigma_1} \otimes_{\mathbb{F}} (M_{\mathbb{F}})^{\sigma_2}.$$

In particular, there exist σ_1 and σ_2 in $\mathrm{Gal}(\mathbb{F} \,|\, \mathbb{K})$ such that $M_{\mathbb{F}} \simeq (M_{\mathbb{F}})^{\sigma_1} \otimes_{\mathbb{F}} (M_{\mathbb{F}})^{\sigma_2}$. Considering now the action of x on these two isomorphic $\mathbb{F}[G]$-modules, claim (a) follows.

As for (b), if M is $\mathbb{K}[G]$-isomorphic to its contragredient module M^*, then we get

$$\bigoplus_{\sigma \in \mathrm{Gal}(\mathbb{F} \,|\, \mathbb{K})} (M_{\mathbb{F}})^{\sigma} \simeq M^{\mathbb{F}} \simeq (M^*)^{\mathbb{F}} \simeq \bigoplus_{\sigma \in \mathrm{Gal}(\mathbb{F} \,|\, \mathbb{K})} ((M^*)_{\mathbb{F}})^{\sigma}.$$

In particular, there exists σ in $\mathrm{Gal}(\mathbb{F} \,|\, \mathbb{K})$ such that $(M_{\mathbb{F}})^{\sigma}$ is $\mathbb{F}[G]$-isomorphic to $(M^*)_{\mathbb{F}}$, which is in turn $\mathbb{F}[G]$-isomorphic to $(M_{\mathbb{F}})^*$. Claim (b) is now achieved by comparing the action of x on the two relevant $\mathbb{F}[G]$-modules. ∎

Lemma 3.2. *Let G be a cyclic group, p a prime, and M a faithful irreducible $\mathrm{GF}(p)[G]$-module. Setting $|M| = p^m$, assume that there exists a divisor r of m, $1 \leq r < m$, such that $\frac{p^m-1}{p^r-1}$ divides $|G|$. Then the following hold.*

(a) *If M is a constituent of $M \wedge_{\mathrm{GF}(p)} M$, then $(p, m/r) = (2, 2)$.*
(b) *If M is self-contragredient, then $(|G|, m/r) = (p^r + 1, 2)$.*

Proof. Let us first prove Claim (a). If M is a constituent of $M \wedge_{\mathrm{GF}(p)} M$, then it is clearly a constituent of $M \otimes_{\mathrm{GF}(p)} M$ as well. Therefore, by Lemma 3.1(a), there exist $a, b \in \{0, ..., m-1\}$ (say $a \geq b$) such that $p^a + p^b \equiv 1 \pmod{|G|}$; in particular, setting $t = \frac{p^m-1}{p^r-1}$, we have that t divides $p^a + p^b - 1$.

Since we have $t > p^{m-r}$, we also have $a \geq m - r$; in fact, assuming the contrary, we would get $p^b \leq p^a \leq p^{m-r-1}$, thus $p^a + p^b - 1 \leq 2p^{m-r-1} - 1 < p^{m-r} < t$, contradicting the fact that t divides $p^a + p^b - 1$. Therefore we can write $a = m - n$ where n lies in $\{1, ..., r\}$. Now, t is a divisor of $(p^{m-n} + p^b - 1) \cdot p^n - (p^m - 1) = p^{b+n} - p^n + 1$, whence $m - r \leq b + n \leq a + n = m$. Again, defining $\ell = m - (b + n)$, we get $0 \leq \ell \leq r$ and t divides $(p^{b+n} - p^n + 1) \cdot p^{\ell} - (p^m - 1) = p^{\ell} - p^{n+\ell} + 1$. As a result, t is a divisor of $p^{\ell} \cdot (p^n - 1) - 1$.

On the other hand, if $m/r \geq 3$, then $m - r \geq 2r$, and so $n + \ell \leq 2r \leq m - r$. This implies $p^{n+\ell} - p^{\ell} < p^{m-r} < t$, and now the only possibility is

$$p^{\ell} \cdot (p^n - 1) = 1,$$

which yields $\ell = 0$, $p = 2$, $n = 1$. So, $p^a = p^b = 2^{m-1}$. Setting $\mathbb{F} = \mathrm{GF}(2^m)$ and denoting by σ the element of $\mathrm{Gal}(\mathbb{F} \mid \mathrm{GF}(2))$ which maps every $f \in \mathbb{F}$ to $f^{2^{m-1}}$, the conclusion so far is that $(M_{\mathbb{F}})^\sigma \otimes_{\mathbb{F}} (M_{\mathbb{F}})^\sigma$ is the unique constituent of $M^{\mathbb{F}} \otimes_{\mathbb{F}} M^{\mathbb{F}}$ which is isomorphic to $M_{\mathbb{F}}$. But $(M_{\mathbb{F}})^\sigma \otimes_{\mathbb{F}} (M_{\mathbb{F}})^\sigma$ is not a constituent of $M^{\mathbb{F}} \wedge_{\mathbb{F}} M^{\mathbb{F}} \simeq (M \wedge_{\mathrm{GF}(2)} M)^{\mathbb{F}}$, against the fact that M is a constituent of $M \wedge_{\mathrm{GF}(2)} M$.

It remains to treat the case $m/r = 2$, whence $t = p^r + 1$ divides $p^a + p^b - 1$ with $0 < a \leq 2r - 1$. Note that we must have $a \geq r$, so we can write $a = r + n$ with $0 \leq n < r$. Now, t divides $p^{n+r} + p^b - 1 - p^n \cdot (p^r + 1) = -p^n + p^b - 1$. In particular, $t \leq |-p^n + p^b - 1| < p^n + p^b - 1$. This in turn implies $b \geq r$, therefore we write $b = r + k$ where $0 \leq k < r$. Finally, $p^r + 1$ divides $-p^n + p^{r+k} - 1 - p^k \cdot (p^r + 1) = -p^n - p^k - 1$, whence $p^r + 1 \leq p^n + p^k + 1 \leq 2p^{r-1} + 1$. It follows that $p = 2$, and (a) is proved.

We move now to Claim (b). If M is self-contragredient, then Lemma 3.1(b) yields that there exists $k \in \{0, ..., m-1\}$ such that $p^k \equiv -1 \pmod{|G|}$. Therefore, $\frac{p^m - 1}{p^r - 1}$ divides $p^k + 1$. Let us first exclude the possibility $k = 0$; in that case, in fact, we would have $|G| = 2$ and $m = 1$, contradicting the existence of a proper positive divisor of m. Observe also that, as by [5, II.3.10] m is the smallest positive integer such that $p^m \equiv 1 \pmod{|G|}$, for every integer $z \neq 0$ such that $|G|$ divides $p^z - 1$ we get $m \mid z$, so m divides $2k$; on the other hand, since $0 < k \leq m - 1$, we have in fact $m = 2k$.

Our conclusion so far is that $\frac{p^{2k} - 1}{p^r - 1}$ is a divisor of $p^k + 1$; this yields $p^k - 1 \mid p^r - 1$, which in turn implies $k \mid r$. But since r properly divides $2k$, the only possibility is $r = k$, i.e., $m/r = 2$. Moreover, $|G|$ is divisible by $\frac{p^m - 1}{p^r - 1} = p^k + 1$ and $|G|$ divides $p^k + 1$, so we have in fact $|G| = p^k + 1$, as desired. ∎

Remark 3.3. Let P be a p-group, and N a subgroup of P such that $P' \leq N \leq \mathbf{Z}(P)$. For $\lambda \in \widehat{N}$, set $Z_\lambda / \ker \lambda = \mathbf{Z}(P / \ker \lambda)$, and observe that $Z_\lambda = \mathbf{Z}(\theta)$ for every $\theta \in \mathrm{Irr}(P|\lambda)$. In fact, $[\mathbf{Z}(\theta), P] \leq N \cap \ker \theta = \ker \lambda$ (recall that, N being central in P, θ_N is a multiple of λ), hence $\mathbf{Z}(\theta) \leq Z_\lambda$; but indeed equality holds, as clearly $[Z_\lambda, P] \leq \ker \lambda \leq \ker \theta$.

Note also that, if μ lies in $\mathrm{Irr}(Z_\lambda|\lambda)$, then μ is a character of $\mathbf{Z}(P/\ker \lambda)$; therefore, for $\theta \in \mathrm{Irr}(P|\mu)$, we have that θ_{Z_λ} is a multiple of μ (and in fact μ is fully ramified in P, because $N \leq \mathbf{Z}(\theta)$, thus $P/\mathbf{Z}(\theta)$ is abelian and [7, Theorem 2.31] yields $\theta(1)^2 = |P : \mathbf{Z}(\theta)| = |P : Z_\lambda|$). Now, taking into account that $\theta \in \mathrm{Irr}(P|\lambda)$ certainly does not vanish on any element of $\mathbf{Z}(\theta) = Z_\lambda$, it is easy to see that $\lambda \in \widehat{N}$ is fully ramified in P if and only if $Z_\lambda = N$.

The proof of next Lemma uses ideas from the proofs of [1, Satz 1] and [4, Satz 1].

Lemma 3.4. *Let P be a p-group, and N an elementary abelian subgroup of P such that $\mathbf{\Phi}(P) \leq N \leq \mathbf{Z}(P)$. Write $|P/N| = p^n$ and $|N| = p^m$. Assume $m > n/2$. Then there are at least $p^{m - \lfloor n/2 \rfloor}$ characters $\lambda \in \widehat{N}$ such that λ is not fully ramified with respect to P/N.*

Proof. We can assume that n is even, as otherwise no character $\lambda \in \widehat{N}$ could be fully ramified with respect to P/N. Let $\epsilon \in \mathbb{C}$ be a fixed primitive pth-root of unity and $\mathbb{K} = \mathbb{Z}/p\mathbb{Z}$. Then we can associate to every $\lambda \in \widehat{N}$ an alternating bilinear form $\langle\ ,\ \rangle_\lambda$ on P/N (as a $\mathbb{K}$-space) by setting, for $aN, bN \in P/N$, $\epsilon^{\langle aN,bN\rangle_\lambda} = \lambda([a,b])$. From the above remark, it is clear that λ is fully ramified with respect to P/N if and only if the form $\langle\ ,\ \rangle_\lambda$ is non-degenerate.

Choosing a basis $\lambda_1, \lambda_2, \ldots, \lambda_m$ of the dual group $\widehat{N}$ of N, there is a bijection between $\mathbb{K}^m$ and $\widehat{N}$, by associating $\lambda = \lambda_1^{x_1}\lambda_2^{x_2}\cdots\lambda_m^{x_m} \in \widehat{N}$ to $(x_1, x_2, \ldots, x_m) \in \mathbb{K}^m$; moreover,

$$\langle\ ,\ \rangle_\lambda = \sum_{i=1}^{m} x_i \langle\ ,\ \rangle_{\lambda_i}\ .$$

If A_i are the matrices associated to the forms $\langle\ ,\ \rangle_{\lambda_i}$ (with respect to a suitable basis of P/N), $\langle\ ,\ \rangle_\lambda$ is degenerate if and only if

$$d(x_1, x_2, \ldots, x_m) = \det \sum_{i=1}^{m} x_i A_i = 0.$$

Now, $d(x_1, x_2, \ldots, x_m) = f^2(x_1, x_2, \ldots, x_m)$ where f is a homogeneous polynomial of degree $n/2$ (see [3, (IV), page 46]. By [17, Satz 3], f has at least $p^{m-n/2}$ roots in $\mathbb{K}^m$ and the result follows. ∎

We next proceed through a series of results concerning semi-linear actions.

Lemma 3.5. *Let p be a prime, V a vector space of order p^n, and H a subgroup of $\Gamma(V)$. Also, setting $X_0 = H \cap \Gamma_0(V)$, let δ be a set of primes in $\pi(H) \setminus \pi(X_0)$, let D be a Hall δ-subgroup of H. Then $|D|$ divides n and, defining $k = \dfrac{p^n - 1}{p^{n/|D|} - 1}$, the following facts are equivalent.*

(a) *For every $v \in V$, $\mathbf{C}_H(v)$ contains a suitable conjugate of D.*
(b) *$|\{D^h : h \in H\}| = k$.*
(c) *k divides $|X_0|$ and $|D|$ is coprime to $p^n - 1$.*

Proof. As $D \cap \Gamma_0(V) = (D \cap H) \cap \Gamma_0(V) = D \cap X_0 = 1$, then $|D|$ divides n. Moreover, by Lemma 3(ii) in [2] we get $|\mathbf{C}_V(D)| = p^{n/|D|}$. Since X_0 acts fixed-point freely on V, for $v \in V \setminus \{0\}$ we have that $\mathbf{C}_H(v) \simeq \mathbf{C}_H(v)X_0/X_0$ is cyclic, and therefore $\mathbf{C}_H(v)$ contains at most one Hall δ-subgroup of H; now the equivalence between (a) and (b) follows by counting in $V \setminus \{0\}$. Assume now (b). Observing that $X_0 D$ is normal in H (because H/X_0 is abelian), by the Frattini argument we have $X_0\mathbf{N}_H(D) = H$, and (c) follows at once. Conversely, assume (c). If r is a prime divisor of $p^{n/|D|} - 1$, then $p^{n/|D|} \equiv 1 \pmod r$ and so $k \equiv |D| \pmod r$; since $|D|$ is coprime with $|X_0|$ (hence with k), it follows that $|D|$ is coprime with $p^{n/|D|} - 1$, and therefore with $|\Gamma_0(V)| = p^n - 1$. As a consequence, $[\Gamma_0(V), D] \simeq \Gamma_0(V)/\mathbf{C}_{\Gamma_0(V)}(D)$ has order k, so $[\Gamma_0(V), D]$ is contained in X_0. Finally, $[\Gamma_0(V), D] = [\Gamma_0(V), D, D] \leq$

$[X_0, D]$, hence $k = |[X_0, D]|$, and therefore $k = |X_0/\mathbf{C}_{X_0}(D)| = |H : \mathbf{N}_H(D)| = |\{D^h : h \in H\}|$, as desired. ∎

Lemma 3.6. *Let H act on a group A and let s be a prime divisor of $|H/\mathbf{C}_H(A)|$. Assume that, for every $a \in A \setminus \{1\}$, $\mathbf{C}_H(a)$ contains a Sylow s-subgroup of H as a normal subgroup. Then A is an elementary abelian p-group for some prime p, and either $p = s = 3$ (and $|A| = 3^2$), or $H/\mathbf{C}_H(A) \leq \Gamma(A)$, $(H/\mathbf{C}_H(A)) \cap \Gamma_0(A)$ acts irreducibly on A, and $s \nmid |(H/\mathbf{C}_H(A)) \cap \Gamma_0(A)|$.*

Proof. Set $\overline{H} = H/\mathbf{C}_H(A)$. By [10, Lemma 4.4], then either $|A| = 9$ and $s = 3$, or there exists a normal abelian subgroup $\overline{X_0}$ of $\overline{H}$ such that A is an irreducible $\overline{X_0}$-module. By [12, Theorem 2.1], this implies that $\overline{H} \leq \Gamma(A)$ and that $\overline{X_0} \leq \overline{H} \cap \Gamma_0(A)$. The last claim is obvious, as $s \mid |\varnothing H|$ and every Sylow s-subgroup of $\overline{H}$ centralizes some non-trivial element of A, whereas every non-trivial element of $\overline{H} \cap \Gamma_0(A)$ acts fixed-point freely on A. ∎

The following fact is folklore, but we include a proof for convenience.

Lemma 3.7. *Let $H \leq \Gamma(V)$ be a group of semilinear maps on V. Let $|V| = p^n$, p prime. Let T_0 be a subgroup of H such that $|T_0|$ is a primitive prime divisor of $p^n - 1$. Then $\mathbf{C}_H(T_0) = H \cap \Gamma_0(V) = \mathbf{F}(H)$.*

Proof. Let $\mathbb{K} = \mathrm{GF}(p)$. By [5, II.3.10], V is an irreducible $\mathbb{K}[T_0]$-module. Hence by Schur's Lemma $\mathbb{L} = \mathrm{End}_{\mathbb{K}[T_0]}(V)$ is division ring. Since $\mathbb{L}$ is finite, it is a field by Wedderburn's Theorem. So, as $\mathbf{C}_H(T_0)$ is a subgroup of the multiplicative group of $\mathbb{L}$, $\mathbf{C}_H(T_0)$ is cyclic. But $\mathbf{C}_H(T_0)$ acts irreducibly on V (as it contains T_0), and hence by [5, II.3.10] we get that $\mathbf{C}_H(T_0) \leq H \cap \Gamma_0(V)$. So, as $H \cap \Gamma_0(V)$ is cyclic, we conclude that $\mathbf{C}_H(T_0) = H \cap \Gamma_0(V) \leq \mathbf{F}(H)$. Note that $t > n$ because t is a primitive prime divisor of $p^n - 1$, so a Sylow t-subgroup of H is contained in $\mathbf{C}_H(T_0)$, hence $\mathbf{F}(H)$ centralizes T_0. ∎

Lemma 3.8. *Let H be a solvable group, p a prime, and V_1, V_2 two H-modules over $\mathrm{GF}(p)$. Assume that there exists a prime $s \in \pi(H/\mathbf{F}(H)) \setminus \{p\}$ such that, for $i \in \{1, 2\}$ and for every $v \in V_i \setminus \{0\}$, $\mathbf{C}_H(v)$ contains a Sylow s-subgroup of H as a normal subgroup. Then both V_1 and V_2 are irreducible H-modules, $|V_1| = |V_2|$, and $H/\mathbf{C}_H(V_i) \leq \Gamma(V_i)$ for $i \in \{1, 2\}$.*

Proof. For $i \in \{1, 2\}$, set $|V_i| = p^{n_i}$ and $C_i = \mathbf{C}_H(V_i)$; also, let S be a Sylow s-subgroup of H. Note that s divides $|H/C_i|$, as otherwise S would be a characteristic subgroup of $C_i \trianglelefteq H$ yielding $S \leq \mathbf{F}(H)$, against our assumption.

So, by Lemma 3.6 we get that $H/C_i \leq \Gamma(V_i)$, that V_i is an irreducible H-module and that s does not divide the order of $X_i/C_i = (H/C_i) \cap \Gamma_0(V_i)$. Moreover, by Lemma 3.5, s divides n_i and $(p^{n_i} - 1)/(p^{n_i/s} - 1)$ divides $|X_i/C_i|$. Observe that there exists a primitive prime divisor t_i for (p, n_i): otherwise either $n_i = 2$ or

$p^{n_i} = 2^6$, and in both cases (as $s \neq p$) s divides $(p^{n_i}-1)/(p^{n_i/s}-1)$, contradicting the fact that $s \nmid |X_i/C_i|$. Note also that $t_i \neq s$, as $t_i > n_i$. Let T_i/C_i be a subgroup of prime order t_i of X_i/C_i; so $T_i/C_i \trianglelefteq H/C_i$.

We claim that T_1 is not contained in C_1C_2. Otherwise, writing $\emptyset H = H/C_1$, we have $\emptyset T_1 \leq \emptyset C_2$ and, choosing a non-trivial $v \in V_2$ such that $S \leq \mathbf{C}_H(v)$, both $\emptyset S$ and $\emptyset T_1$ are normal subgroups of $\emptyset \mathbf{C}_H(v)$, so $[\emptyset S, \emptyset T_1] = 1$. But this, by Lemma 3.7, implies that s divides $|\emptyset X_1|$, a contradiction.

As T_1C_2/C_2 centralizes T_2/C_2 (also if $t_1 = t_2$), then again Lemma 3.7 yields that t_1 divides $|X_2/C_2|$ and hence t_1 divides $p^{n_2}-1$. We conclude that $n_2 \geq n_1$.

Similarly, one shows that $n_1 \geq n_2$, completing the proof. ∎

With the following lemmas, we will gather some relevant information on the character degree graph of solvable groups.

Lemma 3.9. *Let G be a solvable group, and E an abelian normal subgroup of G. Assume that E has a complement H in G and that $\mathbf{F}(G) = E \times Z$ with $Z \leq H \cap \mathbf{Z}(G)$. Setting $X = \mathbf{F}(H)$, let $q \in \pi(X/Z) \setminus \pi(E)$, and let $s \in \pi(H/Z) \setminus \pi(E)$ be such that q and s are not adjacent in $\Delta(G)$. Let $Q \in \mathrm{Syl}_q(X)$, $S \in \mathrm{Syl}_s(H)$ and $L = (QS)^H$ (the normal closure of QS in H). Then the following conclusions hold.*

(a) *We have $[E,Q] = [E,L]$ and $\mathbf{C}_E(Q) = \mathbf{C}_E(L)$; moreover, $E = [E,L] \times \mathbf{C}_E(L)$.*
(b) *Set $A = [E,Q]$. Then A is an elementary abelian p-group, say of order p^n, where p is a suitable prime. Also, $Z = \mathbf{C}_{LZ}(A)$ and, for every non-trivial $a \in A$, $\mathbf{C}_{LZ}(a)$ contains a conjugate of S as a normal subgroup. Moreover, $LZ/Z \leq \Gamma(A)$ and $L_0/Z = (LZ/Z) \cap \Gamma_0(A)$ acts irreducibly on A. We also have that $d = |SZ/Z|$ divides n, and $(p^n-1)/(p^{n/d}-1)$ divides $|L_0/Z|$.*
(c) *There exists a primitive prime divisor t of $p^n - 1$.*
(d) *$L_0/Z = \mathbf{F}(LZ/Z)$. Moreover, $LZ = L_0S$ and p does not divide LZ/Z.*

Proof. Set $A = [E,Q]$ and $B = \mathbf{C}_E(Q)$. As q does not divide $|E|$ and E is abelian, we have $E = A \times B$. Consider now the action of G on the dual group $\widehat{E} = \widehat{A} \times \widehat{B}$. For $\alpha \in \widehat{A} \setminus \{1\}$, q divides $|G : \mathbf{C}_G(\alpha)| = |H : \mathbf{C}_H(\alpha)|$. Also, the linear character α extends to $\mathbf{C}_G(\alpha)$, because A has a complement (namely $B\mathbf{C}_H(\alpha)$) in $\mathbf{C}_G(\alpha)$. Thus, by Gallagher's Theorem and Clifford Correspondence, this forces $\mathbf{C}_H(\alpha) \simeq \mathbf{C}_G(\alpha)/E$ to contain an H-conjugate of S as a normal subgroup (and also, S is abelian). Let $\alpha \in \widehat{A} \setminus \{1\}$ be such that $S \leq \mathbf{C}_H(\alpha)$ and let $\beta \in \widehat{B}$; then $\mathbf{C}_H(\alpha \times \beta) = \mathbf{C}_H(\alpha) \cap \mathbf{C}_H(\beta)$ and q divides $|H : \mathbf{C}_H(\alpha \times \beta)|$. As $\alpha \times \beta$ extends to its inertia subgroup in G, using as above Clifford Theory and that no irreducible character of G has degree divisible by qs, we get that the unique Sylow s-subgroup S of $\mathbf{C}_H(\alpha)$ must also be contained in $\mathbf{C}_H(\beta)$. We conclude that S acts trivially on $\widehat{B}$ and hence that $S \leq \mathbf{C}_H(B) \trianglelefteq H$. Thus $L = QS^H \leq \mathbf{C}_H(B)$, so that $B = \mathbf{C}_E(L)$. Moreover, we get $[E,L] = [A \times B, L] = [A,L] \leq A$, hence $A = [E,L]$ and (a) is proved.

Next, observe that $\mathbf{C}_H(E) = Z$: in fact, if $x \in \mathbf{C}_H(E)$, then x centralizes $EZ = \mathbf{F}(G)$, so $x \in \mathbf{F}(G) \cap H = Z$. Thus, it follows that $Z = \mathbf{C}_{LZ}(A)$. Since all Sylow s-subgroups of H are contained in L and $Z \leq \mathbf{Z}(G)$, we have that $\mathbf{C}_{LZ}(\alpha)$ contains an L-conjugate of S as a normal subgroup, for every $\alpha \in \widehat{A} \setminus \{1\}$. Hence, as s is coprime to $|\widehat{A}|$, an application of Lemma 3.6 together with Lemma 3.5 yields that $\widehat{A}$ (thus A) is an elementary abelian p-group of order p^n, where p is a suitable prime and n a suitable integer; moreover, setting $\overline{H} = H/Z$, we get $\overline{L} \leq \Gamma(\widehat{A})$ and $\overline{L_0} = \overline{L} \cap \Gamma_0(\widehat{A})$ acts irreducibly on $\widehat{A}$. We also have that s does not divide $|\overline{L_0}|$, whereas $d = |\overline{S}|$ divides n, and $(p^n - 1)/(p^{n/d} - 1)$ divides $|\overline{L_0}|$.

As in the proof of Lemma 3.8, there exists a primitive prime divisor t of $p^n - 1$. Otherwise, either $n = 2$ or $p^n = 2^6$. In both cases, as s and p are distinct primes, s divides $(p^n - 1)/(p^{n/d} - 1)$, so s divides $|\overline{L_0}|$, a contradiction. This proves (c), and Lemma 3.7 yields $L_0/Z = \mathbf{F}(LZ/Z)$. In order to conclude the proof of (d), it remains to show that $LZ = L_0S$.

Clearly t divides $(p^n - 1)/(p^{n/d} - 1)$, hence it divides $|\overline{L_0}|$. Denoting by $\o T_0$ the subgroup of $\overline{L_0}$ with $|\o T_0| = t$, by Lemma 3.7 it follows that $\mathbf{C}_{\overline{L}}(\overline{T_0}) = \overline{L_0}$. Note that t is larger than n and hence, as $|\overline{L}/\overline{L_0}|$ divides n, we get that a Sylow t-subgroup of $\overline{L}$ is contained in $\overline{L_0}$. This implies that $\overline{Q} \leq \overline{L_0}$, since $\overline{Q} \trianglelefteq \overline{H}$ centralizes $\overline{T_0}$. But now both $\overline{Q}$ and $\overline{S}$ lie in $\overline{L_0S}$; moreover, as $\overline{L}/\overline{L_0}$ is cyclic, $\overline{L_0S}$ is normal in $\overline{H}$. So $\overline{L} = \overline{L_0S}$ and hence $LZ = L_0S$. Recalling that $s \neq p$ and $|\o L_0|$ divides $p^n - 1$, we have also that p does not divide $|LZ/Z|$, and the proof of (d) is complete.

In particular, the actions of $\overline{L}$ on A and on $\widehat{A}$ are isomorphic and also (b) is proved. ∎

Lemma 3.10. *Let G be a solvable group, and assume that $\mathbf{F}(G) = M \times U$, where $M \trianglelefteq G$ is a non-abelian p-group, $U \trianglelefteq G$ is abelian, and p does not divide $|G : \mathbf{F}(G)|$. If every irreducible character of U has an extension to its inertia subgroup in G, then either $U \leq \mathbf{Z}(G)$ or $d_G(p, v) \leq 2$ for every $v \in \mathrm{V}(G)$.*

Proof. Our first claim is that, if $t \in \mathrm{V}(G) \setminus \{p\}$ is not adjacent to p, then every Sylow t-subgroup of G centralizes U. In fact, let θ be in $\mathrm{Irr}(M)$. Setting $P = M \times \mathbf{O}_p(U)$, we have that $\theta \times 1_{\mathbf{O}_p(U)} \in \mathrm{Irr}(P)$ extends to $I_G(\theta \times 1_{\mathbf{O}_p(U)}) = I_G(\theta)$ because $p \nmid |G : P|$, and therefore θ extends to $I_G(\theta)$. Now, denoting by K a complement for M in G containing U, the degrees of the characters in $\mathrm{Irr}(G|\theta)$ are of the kind $|G : I_G(\theta)| \cdot \theta(1) \cdot \lambda(1)$, where $\lambda \in \mathrm{Irr}(I_K(\theta))$. As a consequence, if θ is chosen to be non-linear, $I_K(\theta)$ contains a Sylow t-subgroup T of G, and T is abelian and normal in $I_K(\theta)$. Now, U is a nilpotent normal subgroup of $I_K(\theta)$, thus $[U, T] = 1$ as wanted.

Now, let us assume $\mathbf{C}_G(U) \neq G$ and let w be a prime divisor of $\mathbf{F}(G/\mathbf{C}_G(U))$; observe that, by the previous paragraph, w is a vertex of $\Delta(G)$ which is adjacent to p. Also, let $s \in \mathrm{V}(G)$ be non-adjacent to w. In this setting, we claim that

s is adjacent to p in $\Delta(G)$. In fact, we can certainly find $\phi \in \widehat{U}$ such that w divides $|G : \mathbf{C}_G(\phi)|$. Thus $\mathbf{C}_G(\phi)$ contains a Sylow s-subgroup S of G; by our assumptions, ϕ extends to $\mathbf{C}_G(\phi)$, and so $\mathbf{C}_G(\phi)/U$ has an abelian normal Sylow s-subgroup. It follows that $SU \trianglelefteq \mathbf{C}_G(\phi)$ and $\mathrm{Syl}_s(\mathbf{C}_G(\phi)) = \{S^u \mid u \in U\}$. Take now any $\theta \in \mathrm{Irr}(M)$; we get that $I_G(\theta \times \phi)$ (thus $I_G(\theta)$) contains a Sylow s-subgroup S^u of G for some $u \in U$, but then $I_G(\theta)$ contains S because $U \leq I_G(\theta)$. We conclude that S centralizes every irreducible character of M, thus it centralizes M by coprimality. But this forces $S \not\leq \mathbf{C}_G(U)$, which yields that s is adjacent to p in view of the previous paragraph.

To sum up, under the assumption $U \not\leq \mathbf{Z}(G)$, we proved the existence of $w \in \mathrm{V}(G)$ which is adjacent in $\Delta(G)$ to p and to every vertex of $\Delta(G)$ not adjacent to p. It easily follows that every vertex of $\Delta(G)$ can be reached from p through a path of length at most 2, and the proof is complete. ∎

Lemma 3.11. *Let G be a group, p a prime, and P a normal p-subgroup of G. If $G/\mathbf{C}_G(P)$ is a p-group, then P is a hypercentral subgroup of G.*

Proof. Since $G/\mathbf{C}_G(P)$ is a p-group, the number of elements in P that are fixed under the action of $G/\mathbf{C}_G(P)$ is divisible by p (unless P is trivial, in which case there is nothing to prove), and hence $\mathbf{Z}(G) \cap P \neq 1$; in particular, $\mathbf{Z}(G)$ is non-trivial. Consider now the factor group $\overline{G} = G/\mathbf{Z}(G)$ and adopt the bar convention throughout; clearly $\overline{P}$ is a normal p-subgroup of $\overline{G}$ and $\overline{G}/\mathbf{C}_{\overline{G}}(\overline{P})$ is a p-group (because it is isomorphic to a quotient of $G/\mathbf{C}_G(P)$), therefore we can use induction on the order of the group and conclude that $\overline{P} \leq \mathbf{Z}_\infty(\overline{G})$. The claim now follows, as $\mathbf{Z}_\infty(\overline{G}) = \overline{\mathbf{Z}_\infty(G)}$. ∎

Lemma 3.12. *Let G be a solvable group, and p a prime. Setting $P = \mathbf{O}_p(G)$ and $N = P'$, assume that P is non-abelian, $P/N \leq \mathbf{Z}(G/N)$, and that every irreducible character of P has an extension to its inertia subgroup in G. Then p is a complete vertex in $\Delta(G)$.*

Proof. Note that, if H is a p-complement of G, then H centralizes P/N and hence it centralizes P by coprimality; this yields that $G/\mathbf{C}_G(P)$ is a p-group, thus P is hypercentral in G by Lemma 3.11.

Working by contradiction, we assume that p is not a complete vertex of $\Delta(G)$ and we consider a vertex $s \in \mathrm{V}(G)$, $s \neq p$, such that ps does not divide any irreducible character degree of G, and let θ be in $\mathrm{Irr}(P)$. Since θ extends to $I = I_G(\theta)$, the degrees of the characters in $\mathrm{Irr}(G|\theta)$ are of the kind $|G : I| \cdot \theta(1) \cdot \lambda(1)$, where $\lambda \in \mathrm{Irr}(I/P)$. As a consequence, if θ is chosen to be non-linear, I/P contains SP/P where S is a suitable Sylow s-subgroup of G (recall that $|G : I|$ is a p-power), and SP/P is abelian and normal in I/P; here S is in fact abelian, as $S \simeq SP/P$. Now, as $P \leq \mathbf{Z}_\infty(G)$, the nilpotency of SP/P yields the nilpotency of SP; moreover, SP is normal in I, which is in turn subnormal in G because $I/\mathbf{C}_G(P)$ is a subgroup of

the p-group $G/\mathbf{C}_G(P)$. We conclude that SP is a nilpotent subnormal subgroup of G, whence it lies in $F = \mathbf{F}(G)$. To sum up, S is a Sylow s-subgroup of G which is abelian and normal in G (as $S \leq F$), hence s is not a vertex of $\Delta(G)$, a contradiction; in other words, every prime in $\mathrm{V}(G) \setminus \{p\}$ is adjacent to p in $\Delta(G)$, as wanted. ∎

Proposition 3.13. *Let G be a solvable group such that $\Delta(G)$ is connected of diameter 3, and let p be a prime. Setting $P = \mathbf{O}_p(G)$, assume that P is non-abelian, P' is a minimal normal subgroup of G and that $\Delta(G/P')$ is a disconnected graph. Then P is the Sylow p-subgroup of G and P/P' is not contained in the center of G/P'.*

Proof. Write $N = P'$ and $F = \mathbf{F}(G)$. Observe first that N lies in $\mathbf{\Phi}(P)$, thus $N \leq \mathbf{\Phi}(G)$ and the ascending Fitting series of G/N is just the image of the ascending Fitting series of G under the natural homomorphism onto G/N.

First, we will show that P is a Sylow p-subgroup of G. Assume, working by contradiction, that this is not the case.

Note that, since $\Delta(G)$ has diameter three, the graph $\Delta(G/N)$ (whose vertex set is $\mathrm{V}(G)$ in this situation) has no isolated vertices, and therefore G/N is of type (b) or (c) of Theorem 2.2. Hence, the Sylow subgroups of G/F are all cyclic.

Let θ be any character in $\mathrm{Irr}(P)$. Setting $R = \mathbf{O}_{p'}(F)$, clearly we have that $\theta \times 1_R$ is an extension of θ to F, such that $I_G(\theta \times 1_R) = I_G(\theta)$. Moreover, since all Sylow subgroups of G/F are cyclic, $\theta \times 1_R$ (and therefore θ) extends to $I = I_G(\theta)$. If $P/N \leq \mathbf{Z}(G/N)$ then, by Lemma 3.12, p would be a complete vertex of $\Delta(G)$, against the assumption $\mathrm{diam}(\Delta(G)) = 3$. Hence, $P/N \not\leq \mathbf{Z}(G/N)$. By Lemma 2.3, this yields that F/N is abelian and hence that G/N is necessarily of type (b) of Theorem 2.2.

So $F/N = M/N \times Z/N$, where M/N is an elementary abelian p-group having a complement H/N in G/N and $Z/N = \mathbf{Z}(G/N) = \mathbf{C}_{H/N}(M/N)$. Set $K = \mathbf{F}_2(G)$ (so $K/N = \mathbf{F}_2(G/N)$). Also, M/N is an irreducible K/N-module, and K/F and G/K are cyclic groups of coprime order (in fact, $\pi(K/F)$ and $\pi(G/K)$ are the connected components of $\Delta(G/N)$). Note that $p \notin \pi(K/P)$ (as $P = \mathbf{O}_p(G)$).

Let π_0 be the set of vertices not adjacent to p in $\Delta(G)$; so, $\pi_0 \neq \emptyset$. We remark that $\pi_0 \subseteq \pi(K/P)$, as $p \in \pi(G/K)$ and $\pi(G/K)$ induces a complete subgraph of $\Delta(G)$.

Write $K = PX$, where $X \leq H$ is a p-complement of K. Note that X is abelian, because $X/(Z \cap X) \cong K/F$ is cyclic and $Z \cap X$ is central in X. Let Y be the Hall π_0-subgroup of X. Then $YN/N \trianglelefteq H/N$ (as $XN/N = \mathbf{F}(H/N)$). So $PY \trianglelefteq G$ and hence $\Delta(PY)$ is a subgraph of $\Delta(G)$. We deduce that $\Delta(PY)$ is disconnected with components $\{p\}$ and π_0 (observe that every $r \in \pi_0$ is a vertex of $\Delta(PY)$; in fact, a Sylow r-subgroup R of PY is also a Sylow r-subgroup of G and, if R is abelian and normal in PY, then the same is true in G (as $PY \trianglelefteq G$), against the fact that r is

in $\mathrm{V}(G)$), hence PY is of type (a) in Theorem 2.2. So, setting $C = \mathbf{C}_P(Y)$, we have that $N \leq C$ and that every non-linear irreducible character of P is fully ramified in P/C. Note also that $M \not\leq C$, since otherwise (as above) YN/N centralizes F/N, so $Y \leq F$ and no prime divisor of $|Y|$ would be a vertex of $\Delta(G)$. Let Z_p/N be the Sylow p-subgroup of Z/N. Since Y acts trivially on both N and Z_p/N, then $Z_p \leq C$. Moreover, $C \cap M = N$, as $(C \cap M)/N$ is a proper submodule of the K/N-irreducible module M/N. Now, since $P/N = M/N \times Z_p/N$, it follows that $C = Z_p$.

We next observe that M is non-abelian. In fact, as p is an isolated vertex of $\Delta(PY)$, every non-linear irreducible character of P is centralized by Y; therefore, an application of [12, Theorem 19.3] yields $N = [P, Y]'$, but $[P, Y] = [MC, Y] = [M, Y] \leq M$, whence $M' = N$. In particular, $\Delta(MY)$ is also disconnected with components $\{p\}$ and π_0. Now, working with MY instead of PY (as in the previous paragraph), we get that every non-linear irreducible character of M is fully ramified with respect to $M/\mathbf{C}_M(Y) = M/N$. By Lemma 3.4 this implies $|M/N| \geq |N|^2$.

Consider now $U = \mathbf{C}_X(N)$. Assume first $U = X$. Let $\chi \in \mathrm{Irr}(G)$ such that $N \not\leq \ker(\chi)$ and let ψ be an irreducible constituent of χ_P. Then ψ is fully ramified in P/C and $I_G(\psi) = I_G(\theta)$, where θ is the irreducible constituent of ψ_C. As X acts trivially on both $C/N = Z_p/N$ and N, then X centralizes C and hence ψ is K-invariant. So, recalling that $(|P|, |K/P|) = 1$, ψ extends to K. Since $K/P \cong X$ is abelian, Gallagher's Theorem implies that every irreducible character of K lying over ψ has degree coprime to $|K/P|$. It follows that $\pi(\chi(1)) \subseteq \pi(G/K)$ (recall that $p \in \pi(G/K)$). We conclude that $\Delta(G) = \Delta(G/N)$ is disconnected, a contradiction.

Hence, $U < X$. Choose $q \in \pi(X/U)$ and let $Q \in \mathrm{Syl}_q(X)$. So, $QN/N \trianglelefteq H/N$ and then, in particular, $PQ \trianglelefteq G$.

We remark that p and q are adjacent in $\Delta(G)$. If not, they are not adjacent in the subgraph $\Delta(PQ)$ as well, and hence $\Delta(PQ)$ is of type (a) of Theorem 2.2, giving $N \leq \mathbf{C}_P(Q)$, so $Q \leq U$, against the choice of q. So, since $(p, q) \in \mathrm{E}(G)$ and q is not a complete vertex of $\Delta(G)$, there exists a vertex $s \neq p$ of $\Delta(G)$ such that $(q, s) \notin \mathrm{E}(G)$. Note also that $s \in \pi(H/Z)$: otherwise, as the p-complement of Z is abelian, G would have an abelian normal Sylow s-subgroup and s would not be a vertex of $\Delta(G)$.

Let S be a Sylow s-subgroup of H and let $L = (QS)^H$. By applying Lemma 3.9 to G/N with M/N playing the role of E, we get that LZ/Z acts as a faithful, irreducible semi-linear group on $M/N = [M/N, Q] = [M/N, L]$ (here we are taking into account that $\mathbf{C}_{M/N}(Q)$ is normal in G/N, hence it is trivial because M/N is an irreducible K/N-module). So, recalling that C/N is central in G/N (thus it lies in $\mathbf{C}_{P/N}(L)$), we see that $\mathbf{C}_{P/N}(L) = C/N \times \mathbf{C}_{M/N}(L) = C/N$. Also, for every non-trivial element $a \in M/N$, $\mathbf{C}_{LZ/Z}(a)$ contains a Sylow s-subgroup of LZ/Z as

a normal subgroup. Finally, note that by part (d) of Lemma 3.9, $\mathbf{F}(LZ/Z)$ acts fixed-point freely on M/N and hence, in particular, s does not divide $|\mathbf{F}(LZ/Z)|$.

Next, observe that $[N,Q]=N$, as N is minimal normal in G and $1<[N,Q]=[N,PQ]\trianglelefteq G$. Hence, setting $B=\mathbf{C}_P(L)$, we have that $B\cap N=\mathbf{C}_N(L)\leq \mathbf{C}_N(Q)=1$. Since $C/N=\mathbf{C}_{P/N}(L)=\mathbf{C}_{P/N}(LZ/Z)$ and, recalling 3.13(d), the action of LZ/Z on P/N is a coprime action, we get $C=NB$ and hence $C=N\times B$, because N is central in P.

Let now $\gamma\in\widehat{C}$ such that $N\not\leq\ker(\gamma)$. So γ is fully ramified in P/C; let $\psi\in\mathrm{Irr}(P)$ the unique constituent of γ^P. Then $I_G(\psi)=\mathbf{C}_G(\gamma)$. As $\gamma=\alpha\times\beta$ with $\alpha\in\widehat{N}\setminus\{1\}$ and $\beta\in\widehat{B}$, then q divides $|G:\mathbf{C}_G(\gamma)|$. Since ψ extends to $\mathbf{C}_G(\gamma)$ (in fact, the Sylow p-subgroups of G/P are cyclic because $p\notin\pi(K/P)$) and $(q,s)\notin \mathrm{E}(G)$, as usual we get that $\mathbf{C}_G(\gamma)/P$ contains a Sylow s-subgroup of G/P as a normal subgroup; in particular, the same is true also for $\mathbf{C}_{LZ/Z}(\gamma)$. But $\mathbf{C}_{LZ/Z}(\gamma)=\mathbf{C}_{LZ/Z}(\alpha)$, as LZ acts trivially on B. So, for every $\alpha\in\widehat{N}\setminus\{1\}$, $\mathbf{C}_{LZ/Z}(\alpha)$ contains a Sylow s-subgroup of LZ/Z as a normal subgroup. Hence an application of Lemma 3.8 yields that $|M/N|=|\widehat{N}|=|N|$, a contradiction.

So far, we have shown that P is a Sylow p-subgroup of G. Thus, every irreducible character of P has an extension to its inertia subgroup in G and hence Lemma 3.12 yields that P/N is not central in G/N. This finishes the proof. ∎

Lemma 3.14. *Let P be a non-abelian normal Sylow p-subgroup of a solvable group G and let H be a p-complement of G. Assume that there is a prime divisor s of $|H/\mathbf{F}(H)|$ such that s is not adjacent to p in $\Delta(G)$. Then for all $2\leq i\leq c$, where c is the nilpotency class of P, the factor groups $M_i=\gamma_i(P)/\gamma_{i+1}(P)$ are chief factors of G of the same order p^n, with $n\geq 3$, and $G/\mathbf{C}_G(M_i)$ embeds in $\Gamma(p^n)$.*

Proof. For $2\leq i\leq c$, take any non-trivial μ in $\widehat{M_i}$: by [7, Theorem 13.28], and by the fact that M_i is central in $P/\gamma_{i+1}(P)$, there exists ϕ in $\mathrm{Irr}(P/\gamma_{i+1}(P)\mid\mu)$ such that $I_H(\phi)=\mathbf{C}_H(\mu)$. As $i\geq 2$ and μ is non-trivial, clearly $\phi(1)$ is a multiple of p. Now, viewing ϕ as a character of P by inflation, we have that ϕ extends to its inertia subgroup in G (by coprimality); as a consequence of our non-adjacency assumption, Clifford Correpondence together with Gallagher's Theorem yield that $I_H(\phi)$ (thus $\mathbf{C}_H(\mu)$) contains a Sylow s-subgroup of H as a normal subgroup. Observe also that $\mathbf{C}_H(M_i)$ does not contain any Sylow s-subgroup of H: in fact, if $S\in\mathrm{Syl}_s(H)$ lies in $\mathbf{C}_H(M_i)$ (which in turn lies in $\mathbf{C}_H(\mu)$), then S would be a characteristic subgroup of $\mathbf{C}_H(M_i)$, and therefore a normal subgroup of H, against the fact that s is a divisor of $|H/\mathbf{F}(H)|$. We are then in a position to apply Lemma 3.8, which yields that all groups $\widehat{M_i}$ are in fact irreducible H-modules of the same order; thus all M_i are irreducible H-modules as well (i.e., M_i is a chief factor of G), all of them have the same order p^n, and all the groups $G/\mathbf{C}_G(M_i)$ embed in $\Gamma(p^n)$. Moreover, Lemma 3.5 yields that s divides n and that s is coprime to p^n-1. As $s\neq p$, this implies that $n\geq 3$. ∎

4. The main results

In what follows, we write $\mathrm{diam}(\Delta(G)) \geq 3$ to indicate that either $\Delta(G)$ is connected and $\mathrm{diam}(\Delta(G)) = 3$, or that $\Delta(G)$ is not connected (i.e., $\mathrm{diam}(\Delta(G)) = \infty$).

Proposition 4.1. *Let G be solvable group such that $\mathbf{F}(G) = P$ is a non-abelian Sylow p-subgroup of G, and assume that P' is a minimal normal subgroup of G. If $\mathrm{diam}(\Delta(G)) \geq 3$ and p is not an isolated vertex of $\Delta(G)$, then $\mathbf{F}_2(G)$ does not centralize P'.*

Proof. We denote by H a p-complement of G, and we set $X = \mathbf{F}(H)$ and $N = P'$. Since $\mathbf{F}_2(G) = PX$, we have to prove that X does not centralize N. Working by contradiction we assume that X centralizes N, and go through a series of steps.

(a) *Every prime in $\pi(H/X)$ is adjacent to p in $\Delta(G)$.*

Proof. Let $q \in \mathrm{V}(G)$ be such that $(q,p) \notin \mathrm{E}(G)$ and consider a non-trivial character $\phi \in \widehat{N}$. As ϕ is X-invariant and p does not divide $|X|$, by [7, Theorem 13.28] there is a $\psi \in \mathrm{Irr}(P|\phi)$ such that $X \leq I = I_G(\psi)$. By coprimality, ψ extends to I; hence Gallagher's Theorem and Clifford Correspondence imply that $I \cap H \simeq I/P$ contains a Sylow q-subgroup Q of G, and that Q is abelian and normal in $I \cap H$. As $X \trianglelefteq I \cap H$, then Q centralizes the q-complement of X. But Q centralizes also the Sylow q-subgroup $Q \cap X$ of X, as Q is abelian. Thus, Q centralizes $X = \mathbf{F}(H)$ and hence $Q \leq X$, so $q \notin \pi(H/X)$. ∎

Note that a group G as in our hypotheses, in the disconnected case, is as in (c) of Theorem 2.2; in particular, the connected components of $\Delta(G)$ are $\{p\} \cup \pi(X)$ and $\pi(H/X)$, clearly against what obtained in the paragraph above. This contradiction settles the disconnected case, so we may henceforth assume that $\Delta(G)$ is connected of diameter 3.

(b) *There exist $q \in \pi(X)$ and $s \in \pi(H) \setminus \pi(X)$ such that q is non-adjacent to both p and s in $\Delta(G)$.*

Proof. By Lemma 2.1 the subgraph of $\Delta(G)$ induced by $\pi(X)$ is complete and by (a) the set of vertices of $\Delta(G)$ that are not adjacent to p is contained in $\pi(X)$. We may consider a path $q - r - s_1 - s_2$ in $\Delta(G)$ connecting two vertices with distance 3, where $q \neq p$, q is not adjacent to p, and one among the s_i is not p: naming it s we have the claim. ∎

(c) *Let $Q \in \mathrm{Syl}_q(X)$, $S \in \mathrm{Syl}_s(H)$. Set $M = [P,Q]$, $L = (QS)^H$ and $X_0 = \mathbf{F}(L)$. Then $M' = N$ and M/N is an irreducible X_0-module. Moreover, setting $|M/N| = p^n$, we have that X_0 is a cyclic group whose order is divisible by $(p^n - 1)/(p^{n/|S|} - 1)$.*

Proof. Note that Q stabilizes every non-linear irreducible character of P, as $Q \trianglelefteq H$ and q is not adjacent to p. So $M' = P' = N$ by [12, Theorem 19.3],

and it also follows that $M/N = [P/N, Q]$. Now we are in a position to apply Lemma 3.9 to the group G/N, with respect to the non-adjacent vertices q, s of $\Delta(G/N)$, and all the desired conclusions follow. ∎

(d) *$V = M/N$ is a self-contragredient X_0-module.*

Proof. Let ψ be a non-linear irreducible character of M, and λ an irreducible constituent of ψ_N. Then λ is X_0-invariant (as $X_0 \leq X$), and clearly it does not extend to M. Since M/N is irreducible as an X_0-module, then λ is fully ramified with respect to M/N (see [7, Exercise 6.12]). So by Remark 3.3 the bilinear form defined by $\epsilon^{\langle aN, bN\rangle} = \lambda([a, b])$ on M/N (where ϵ is a given primitive p-th root of unity) is non-degenerate, and it is also X_0-invariant, as X_0 acts trivially on N. Hence, it induces an isomorphism of X_0-modules between M/N and its contragredient module. ∎

Note that $q \in \pi(X_0) \subseteq \pi(X)$, so Lemma 2.1 implies that s does not divide $|X_0|$. We conclude by applying Lemma 3.2, which gives $p = 2 = s$, a contradiction. ∎

The next result, which is the core of this work, shows that actually no group G such that $\Delta(G)$ is connected can satisfy the assumptions of Proposition 4.1.

Proposition 4.2. *Let G be a solvable group such that $\mathbf{F}(G) = P$ is a non-abelian Sylow p-subgroup of G, and let $N = P'$ be a minimal normal subgroup of G. Assume also that* $\operatorname{diam}(\Delta(G)) \geq 3$ *and p is not an isolated vertex of $\Delta(G)$. Then $\Delta(G)$ is not connected, $[P, G]/N$ is a chief factor of G, and $|[P, G]/N| = |N|$; moreover, if H is a p-complement of G, then $H \leq \Gamma([P, G]/N)$ and $H/\mathbf{C}_H(N) \leq \Gamma(N)$.*

Proof. Let H be a p-complement of G, and set $X = \mathbf{F}(H)$ (observe that $\mathbf{C}_H(P) = 1 = \mathbf{C}_H(P/N)$). By the previous proposition, there exists a prime divisor q of $|X|$ such that $Q = \mathbf{O}_q(X)$ does not centralize N. As N is minimal normal in G, we have $[N, Q] = N$ and $\mathbf{C}_N(Q) = 1$, whence $Q \not\leq \mathbf{C}_H(\lambda)$ for every non-trivial $\lambda \in \widehat{N}$. So Q does not lie in the inertia subgroup of any non-linear irreducible character of P (as restrictions to $N \leq \mathbf{Z}(P)$ are homogeneous), and hence q is adjacent to p in $\Delta(G)$.

Since $\mathrm{V}(G/N) = \mathrm{V}(G) \setminus \{p\}$ and q is adjacent to p in $\Delta(G)$, there certainly exists $s \in \mathrm{V}(G/N)$ that is not adjacent to q in $\Delta(G/N)$. Observe also that s does not divide $|X| = |\mathbf{F}(HN/N)|$, as otherwise, by Lemma 2.1, q would be adjacent to s in $\Delta(G/N)$; therefore a Sylow s-subgroup S of H is not normal in H. Let $L = (QS)^H$, $V = [P/N, Q]$ and $C/N = \mathbf{C}_{P/N}(Q)$; then, as $N \leq [P, Q] = R$, $V = R/N$, $P/N = R/N \times C/N$. Setting $|V| = p^n$, by applying Lemma 3.9 we obtain that $C/N = \mathbf{C}_{P/N}(L)$, $R = [P, L]$, $L \leq \Gamma(V)$ and V is an irreducible X_0-module, where $X_0 = L \cap \Gamma_0(V) = \mathbf{F}(L)$; moreover, $|S|$ divides n, the order of X_0 is divisible by $(p^n - 1)/(p^{n/|S|} - 1)$, and there exists a primitive prime divisor t_0 of $p^n - 1$. As in the previous proposition, we shall proceed through a number of steps.

(a) *R is not abelian and, as an X_0-module, N has no irreducible constituent isomorphic to $V = R/N$.*

Proof. Suppose, by contradiction, that R is abelian, and consider the action of L on the dual group $\widehat{R}$: by coprimality, no non-trivial element of $\widehat{R}$ is centralized by Q, and therefore every irreducible character of RL whose kernel does not contain R has a degree divisible by q. Since $RL \trianglelefteq G$ (therefore $\Delta(RL)$ is a subgraph of $\Delta(G)$) and every irreducible character of R extends to its inertia subgroup in RL, we have that $\mathbf{C}_L(\lambda)$ contains a unique Sylow s-subgroup of L for every $\lambda \in \widehat{R} \setminus \{1_R\}$. An application of Lemma 3.6 yields that $\widehat{R}$ (thus R) is an irreducible L-module, contradicting the fact that N is a proper non-trivial L-invariant subgroup of R. Therefore R is not abelian.

Observe next that $\langle\ ,\ \rangle : R/N \times R/N \to N$, defined by $\langle aN, bN\rangle = [a, b]$, for $a, b \in R$, is a $\mathrm{GF}(p)$-bilinear map. This induces a homomorphism (of $\mathrm{GF}(p)$-spaces) $\delta : R/N \otimes_{\mathrm{GF}(p)} R/N \to N$, which is easily checked to be an X_0-homomorphism. Since R is not abelian and N is minimal normal in G, we have $R' = N$, whence δ is surjective. Thus, δ induces a surjective X_0-homomorphism from $R/N \wedge_{\mathrm{GF}(p)} R/N$ to N, because the symmetric tensors are in $\ker\delta$. If N, as an X_0-module, has an irreducible constituent isomorphic to R/N then, by Lemma 3.2, $p = 2 = s$, which is not our case. ∎

(b) *$\mathbf{C}_P(Q)$ is an abelian direct factor of G, and H acts faithfully on V.*

Proof. Note that $\mathbf{C}_P(L) = \mathbf{C}_P(Q)$ and, by coprimality, $C = \mathbf{C}_P(L) \times N$; moreover, we have $R \cap \mathbf{C}_P(L) = 1$ because $N = [N, Q]$. As $P = R\mathbf{C}_P(L)$, we get that $\mathbf{C}_P(L) \simeq P/R$ is abelian.

If C is not contained in $\mathbf{Z}(P)$ then, as $N \leq \mathbf{Z}(P)$, we can choose $y \in \mathbf{C}_P(L) \setminus \mathbf{Z}(P)$. The map $\varphi_y : R/N \to N$, defined by $\varphi_y(aN) = [a, y]$, is then a homomorphism of X_0-modules, because X_0 centralizes y. Since R/N is an irreducible X_0-module, φ_y is either injective or the trivial homomorphism. But in the latter case y would centralize both R and C, whence $y \in \mathbf{Z}(P)$. We conclude that φ_y is injective, which means that the X_0-module N has a constituent isomorphic to R/N, against step (a).

Hence $C \leq \mathbf{Z}(P)$. Let $M = \mathbf{C}_P(L)$; so $P = R \times M$ and HR is a complement for M in G. We next show that $M \leq \mathbf{Z}(G)$.

Write $\o G = G/N$. If M is not central in G, then $\o K = \mathbf{C}_{\o HR}(\o M) < \o HR$ (note that $\o L \leq \o K$) and so there exists a prime divisor t of $|\mathbf{F}(\o HR/\o K)|$. Then, clearly, t divides $|\o G : I_{\o G}(\lambda)|$ for some non-trivial irreducible character λ of $\o M$, whence t divides $\chi(1)$ for all $\chi \in \mathrm{Irr}(\o G|\lambda)$. But $\o M\o K = \o M \times \o K \trianglelefteq \o G$, so t is adjacent in $\Delta(\o G)$ to every vertex of $\Delta(\o K)$. As s is a vertex of $\Delta(\o K)$ (note that $\o S$ is not normal in $\o H$, so it is not normal in $\o L$, and the same holds in $\o K$), we conclude that s is adjacent to t. This

is true for every vertex s that is not adjacent to q in $\Delta(G/N)$ and for q as well; in fact, $\o Q \leq \o K$ and $\o K$ does not have a normal Sylow q-subgroup, as otherwise $\o Q$ would centralize $\o R = V$. It thus follows that $\Delta(G/N)$ is connected; in particular, $\Delta(G)$ is connected as well (and it has diameter 3, by our assumptions), otherwise p would be an isolated vertex of $\Delta(G)$.

Now, by Theorem 2.5, $\mathrm{diam}(\Delta(G/N)) \leq 2$, because $\mathbf{F}(G/N) = P/N$ is abelian. We then have that there exists $s' \in \mathrm{V}(G)$ such that $d_G(s', p) = 3$. Note that then s' is not adjacent to q in $\Delta(G)$ (as p is adjacent to q); therefore, s' is not adjacent to q in $\Delta(\o G)$ as well, and so, as observed above, it is adjacent to t in $\Delta(\o G)$ and hence in $\Delta(G)$. But $M \times K \trianglelefteq G$ implies that p is adjacent to t in $\Delta(G)$, yielding $d_G(s, p) \leq 2$. This contradiction shows that M lies in $\mathbf{Z}(G)$. Hence, $G = HR \times M$ and $M = \mathbf{C}_P(Q)$ is a central direct factor of G.

As a consequence, we get $\mathbf{C}_H(V) = \mathbf{C}_H(P/N) = 1$, so H acts faithfully on V and the proof of (b) is complete. ∎

In the following, we set $Y = \mathbf{C}_H(N)$, write $\o H = H/Y$, and adopt the bar convention. The next step settles, in particular, the first claim of the statement.

(c) *$[P, G]/N$ and N are chief factors of G of the same order p^n, $H \leq \Gamma([P, G]/N)$ and $\o H \leq \Gamma(N)$.*

Proof. By the previous step:

$$[P, G] = [[P, Q]\mathbf{C}_P(Q), G] = [[P, Q], G] \leq [P, Q] = R.$$

Therefore, $[P, G] = [P, Q] = R$. It follows that if μ lies either in $\widehat{V} \setminus \{1\}$ or in $\widehat{N} \setminus \{1\}$, then $\mathbf{C}_H(\mu)$ does not contain Q and, by [7, Theorem 13.28], there exists $\theta \in \mathrm{Irr}(P|\mu)$ such that $\mathbf{C}_H(\mu) \leq I_H(\theta)$. But the restriction of θ to R (respectively, to N) is a multiple of μ, so in fact $\mathbf{C}_H(\mu) = I_H(\theta)$; now, since θ extends to $I_G(\theta)$ (and recalling that s is a vertex of $\Delta(G)$ not adjacent to q), we get that $\mathbf{C}_H(\mu)$ contains a Sylow s-subgroup of H as a normal subgroup. Hence, by Lemma 3.8 we conclude that $V = R/N$ is a chief factor of G of order $|N|$, and that $H \leq \Gamma(V)$ and $H/\mathbf{C}_H(N) \leq \Gamma(N)$. ∎

In view of the above paragraph, our aim for the rest of the proof will be to show that $\Delta(G)$ cannot be connected under our hypotheses. To this end, we assume that G is a counterexample of minimal order; thus $\Delta(G)$ is connected, $\mathrm{diam}(\Delta(G)) = 3$ and, by minimality, G has no non-trivial abelian direct factors. In particular, step (b) yields $\mathbf{C}_P(Q) = 1$, $R = P$ and $V = P/N$.

Setting $m = |H/X|$, as $H \leq \Gamma(V)$ we have that m divides n; in particular, a primitive prime divisor t_0 of $p^n - 1$ (which exists, as observed before), being larger than n, does not divide m. Therefore, denoting by T_0 a Sylow t_0-subgroup of H, we have that T_0 lies in X_0 and it is in fact central in X. Now, Lemma 3.7 yields $\mathbf{C}_H(T_0) \leq \Gamma_0(V)$, whence $X \leq \Gamma_0(V)$ and X acts fixed-point freely on

V; also, as observed in the paragraph preceding (a), s does not divide the order of $X = \mathbf{C}_H(T_0)$. Now (with the notation introcuced before point (c)),

(d) *t_0 does not divide $|Y|$, $Y \leq X$, and $\o X = \o H \cap \Gamma_0(N)$.*

Proof. As observed above, $|T_0| \mid |\o H \cap \Gamma_0(N)|$. This in turn implies $t_0 \nmid |Y|$; thus (as Y and T_0 are both normal in H) we get $[Y, T_0] \leq Y \cap T_0 = 1$, whence $Y \leq \mathbf{C}_H(T_0) = X$. Thus, set $U/Y = \o H \cap \Gamma_0(N)$. We clearly have that $\o X$ centralizes $\o T_0$, so Lemma 3.7 yields $\o X \leq \o H \cap \Gamma_0(N)$. On the other hand we get $[U, T_0] \leq Y \cap T_0$ because, again by Lemma 3.7, $\o T_0$ is contained in the cyclic group $\o U$; thus $U \leq \mathbf{C}_H(T_0) = X$, and so $\o X = \o H \cap \Gamma_0(N)$. ∎

(e) *The subgraphs of $\Delta(G)$ induced on $\pi_1 = \{p\} \cup \pi(X)$ and $\pi_2 = \pi(m)$ are complete graphs. Hence, in particular, $\pi(X) \cap \pi(m) = \emptyset$ and X has a complement D in H.*

Proof. As $H \leq \Gamma(V)$ and $X = \mathbf{F}(H)$, then H/X is nilpotent (in fact, cyclic); thus, Lemma 2.1 implies that both $\pi(X)$ and $\pi(H/X) = \pi(m)$ induce complete subgraphs of $\Delta(G)$.

It remains to show that $(p, t) \in \mathrm{E}(G)$ for all $t \in \pi(X)$. Let $T \in \mathrm{Syl}_t(X)$; as $[V, T] \leq V$ is X-invariant and non-trivial, we see that $[V, T] = V$. If $(p, t) \notin \mathrm{E}(G)$ then, since $PT \trianglelefteq G$, the graph $\Delta(PT)$ is disconnected, and PT is as in case (a) of Theorem 2.2. In particular, $N \leq \mathbf{C}_P(T)$ and every non-linear irreducible character of P is fully ramified with respect to $P/\mathbf{C}_P(T)$. On the other hand, by coprimality, $\mathbf{C}_P(T)/N = \mathbf{C}_V(T)$ is trivial, and in fact every non-linear irreducible character of P is fully ramified with respect to P/N. In this setting, an application of Lemma 3.4 yields the contradiction $|P/N| \geq |N|^2$. ∎

(f) *The graph $\Delta(G/N)$ is disconnected with connected components $\pi(X)$ and $\pi(D) = \pi(m)$, where D is a complement for X in H. Moreover, $(p^n - 1)/(p^{n/|D|} - 1)$ divides $|X|$.*

Proof. Arguing by contradiction, assume that $\Delta(G/N)$ is connected. Then, by Theorem 2.5, $\Delta(G/N)$ has diameter (at most) two, and therefore a pair of vertices at distance 3 in $\Delta(G)$ must include the prime p. Let (p, v) be such a pair and $p - t - r - v$ a shortest path connecting them; then step (e) yields $r, v \in \pi(D) = \pi(m)$ and $t \in \pi(X)$. Take now any $\lambda \in \widehat{N} \setminus \{1\}$; then p divides $\theta(1)$ whenever θ lies in $\mathrm{Irr}(P|\lambda)$, and if r divides $|\o H : \mathbf{C}_{\o H}(\lambda)|$, then r divides $|\o H : \mathbf{C}_{\o H}(\theta)|$, and so pr divides $\chi(1)$ for every $\chi \in \mathrm{Irr}(G|\theta)$, a contradiction. Therefore, Lemma 3.5 yields that $k_r = (p^n - 1)/(p^{n/|R|} - 1)$ divides $|\o X|$, where R is a Sylow r-subgroup of H.

On the other hand, consider $\chi \in \mathrm{Irr}(G)$ such that tr divides $\chi(1)$. Then p does not divide $\chi(1)$, whence $\chi \in \mathrm{Irr}(G/N)$. Let $\mu \in \mathrm{Irr}(P/N)$ be an irreducible

constituent of χ_P; note that $\mu \neq 1_{P/N}$, as $t \mid \chi(1)$. Since $r \mid \chi(1)$, then r divides $|H : I_H(\mu)|$, and again Lemma 3.5 implies that k_r does not divide $|X|$, against what observed in the previous paragraph.

Thus, $\Delta(G/N)$ is disconnected and, by Lemma 2.1, it is clear that its connected components are $\pi(X)$ and $\pi(D)$. Consider now $\mu \in \mathrm{Irr}(P/N) \setminus \{1_{P/N}\}$; as X acts fixed-point freely on P/N, we have that $I_H(\mu) \cap X = 1$, hence, for every $\chi \in \mathrm{Irr}(G|\mu)$, the degree of χ is divisible by all the primes in $\pi(X)$. As a consequence, $I_H(\mu)$ must contain a conjugate of D and the last claim follows by Lemma 3.5. ∎

(g) *Let $\lambda \in \widehat{N} \setminus \{1\}$. Then there exists $\theta_0 \in \mathrm{Irr}(P|\lambda)$ such that $I_H(\theta_0) = \mathbf{C}_H(\lambda)$. For every $\theta \in \mathrm{Irr}(P|\lambda) \setminus \{\theta_0\}$, we have $I_H(\theta) \cap X = 1$, and $I_H(\theta)$ contains a complement D for X in H.*

Proof. The first claim follows from [7, Theorem 13.28], and from the fact that $I_H(\theta) \leq \mathbf{C}_H(\lambda)$ for every $\theta \in \mathrm{Irr}(P|\lambda)$, as $N \leq \mathbf{Z}(P)$.

Consider now a character $\theta \in \mathrm{Irr}(P|\lambda) \setminus \{\theta_0\}$. Recalling that X/Y acts fixed-point freely on $\widehat{N}$, in order to prove that $I_H(\theta) \cap X = 1$ it is enough to show that $Y_0 = I_H(\theta) \cap Y = 1$. If $Y_0 > 1$, then $\mathbf{C}_V(Y_0)$ cannot be the whole V; since V is irreducible X-module and $\mathbf{C}_V(Y_0)$ is X-invariant, $\mathbf{C}_V(Y_0)$ is trivial and, by [7, Exercise 13.10], θ_0 is the only character in $\mathrm{Irr}(P|\lambda)$ which is Y_0-invariant, a clear contradiction. We conclude that $Y_0 = 1$.

Now, if $I_H(\theta)$ does not contain any complement for X in H (i.e., any Hall $\pi(m)$-subgroup of H), then there exists a prime $r \in \pi(m)$ which does not divide $|H : I_H(\theta)|$; as a consequence, any $\chi \in \mathrm{Irr}(G|\theta)$ would be such that $pr|X|$ divides $\chi(1)$. This yields a contradiction, as r would be a complete vertex of $\Delta(G)$, and also the last claim is proved. ∎

For the next two steps of the proof, it will be convenient to introduce some specific notation. We define $\widehat{N}^*$ as the set of all $\lambda \in \widehat{N} \setminus \{1\}$ that are not fully ramified in P; since $|P/N| = |N|$, Lemma 3.4 ensures that $\widehat{N}^*$ is not empty. We shall also take into account Remark 3.3 and the notation introduced therein; in particular recall that, for $\lambda \in \widehat{N}$, the subgroup Z_λ is defined by $Z_\lambda / \ker \lambda = \mathbf{Z}(P/\ker \lambda)$.

(h) *Let λ be in $\widehat{N}^*$, and set $M = Z_\lambda$. Then the following conclusions hold.*
   (i) *There exists a complement D of X in H such that $\mathbf{C}_H(\lambda) = YD$.*
   (ii) *YD normalizes M and, for every $a \in M \setminus N$, we have $\mathbf{C}_{YD}(aN) = D^y$ for some $y \in Y$.*
   (iii) *If D_1 is a complement for X in H such that $\mathbf{C}_{M/N}(D_1)$ is non-trivial, then $D_1 \leq YD$.*
   (iv) *If $Y \not\leq \mathbf{Z}(H)$, then $YD/\mathbf{C}_{YD}(M/N) \leq \Gamma(M/N)$ and M/N is an irreducible Y-module.*

Proof. Since λ is not fully ramified in P, then $|\mathrm{Irr}(P|\lambda)| > 1$, and by (f) we find $\theta \in \mathrm{Irr}(P|\lambda)$ such that $I_H(\theta)$ contains a complement D for X in H. Now, $I_H(\theta)$ is contained in $\mathbf{C}_H(\lambda)$ because θ_N is a multiple of λ, and therefore $D \leq \mathbf{C}_H(\lambda)$. Clearly Y lies in $\mathbf{C}_H(\lambda)$ as well, and since X/Y acts fixed point freely on $\widehat{N}$ (and $\lambda \neq 1$), we deduce that $\mathbf{C}_H(\lambda)$ is in fact YD.

As for (ii), observe that $YD = \mathbf{C}_H(\lambda)$ normalizes $\ker\lambda$, so it acts on $P/\ker\lambda$ and on $\mathbf{Z}(P/\ker\lambda) = M/\ker\lambda$ as well; as a consequence, $\mathbf{C}_H(\lambda)$ normalizes M. Since $\lambda \in \mathrm{Irr}(N/\ker\lambda)$ and $N/\ker\lambda$ is a subgroup of the abelian group $M/\ker\lambda$, we have that λ extends to M, and in fact $\mathrm{Irr}(M|\lambda)$ consists of extensions of λ. By [7, Theorem 13.28], among those extensions we can choose μ_0 that is $\mathbf{C}_H(\lambda)$-invariant, and we can write $\mathrm{Irr}(M|\lambda) = \{\mu_0\rho \mid \rho \in \widehat{M/N}\}$. Consider now $\rho \in \widehat{M/N}$, and take $\theta \in \mathrm{Irr}(P|\mu_0\rho)$. Since θ lies in $\mathrm{Irr}(P|\lambda)$, step (f) together with the previous paragraph yield that $I_H(\theta)$ contains D^x for some $x \in X$; but $I_H(\theta) \leq \mathbf{C}_H(\lambda) = YD$, therefore there exists $y \in Y$ such that $D^x = D^y$. Moreover, recalling that θ_M is a multiple of $\mu_0\rho$, we get $D^y \leq I_{YD}(\theta) \leq I_{YD}(\mu_0\rho)$ and, as μ_0 is D^y-invariant, we easily deduce that D^y lies in $\mathbf{C}_{YD}(\rho)$ as well. If $\rho \neq 1$, taking into account that X acts fixed-point freely on P/N, we also have $\mathbf{C}_{YD}(\rho) \cap X = 1$. Thus $\mathbf{C}_{YD}(\rho) = D^y$. Now, by coprimality, (ii) follows.

Assume now that D_1 is a complement for X in H such that $aN \in \mathbf{C}_{M/N}(D_1)$, where a lies in $M \setminus N$. By (ii), aN is also centralized by D^y for a suitable $y \in Y$; but, as $X \cap \mathbf{C}_H(aN) = 1$, $\mathbf{C}_H(aN) \simeq \mathbf{C}_H(aN)X/X$ is cyclic, and therefore $D_1 = D^y \leq YD$.

Finally, if $Y \not\leq \mathbf{Z}(H)$, then $[Y, D] \neq 1$ and therefore D does not lie in $\mathbf{C}_{YD}(M/N)$ (otherwise, as Y acts fixed-point freely on M/N, we would have $D = \mathbf{C}_{YD}(M/N) \trianglelefteq YD$), so (iv) follows by Lemma 3.6. ∎

(i) $Y \leq \mathbf{Z}(H)$.

Proof. Aiming at a contradiction, let us assume $[Y, H] \neq 1$. Thus, as $Y \leq X$ and X is abelian, $[Y, D] \neq 1$ for any complement D for X in H.

Observe first that every $x \in P$ is contained in a subgroup Z_λ for some $\lambda \in \widehat{N}^*$. In fact, as this clearly holds for $x \in N$, let us focus on an element $x \notin N$. Since $\mathbf{C}_P(x) \supseteq N\langle x\rangle \supset N$, we get $|[P, x]| = |P|/|\mathbf{C}_P(x)| < |P|/|N| = |N|$, and it is enough to choose a non-trivial $\lambda \in \mathrm{Irr}(N/[P, x])$ in order to have $x \in Z_\lambda$ (with $\lambda \in \widehat{N}^*$, as $x \in Z_\lambda$ and so $Z_\lambda > N$). Observe also that, if $Z_{\lambda_1} \neq Z_{\lambda_2}$ for $\lambda_1, \lambda_2 \in \widehat{N}^*$, then $Z_{\lambda_1} \cap Z_{\lambda_2} = N$, since both Z_{λ_1}/N and Z_{λ_2}/N are irreducible Y-modules by (g).

Given $\lambda \in \widehat{N}^*$, we set $M = Z_\lambda$ and we denote by $\Xi(M)$ the dual group of $N/[P, M]$ (as a subgroup of the dual group $\widehat{N}$ of N). Note that if $\mu \neq 1_N$ lies in $\Xi(M)$, then one has $M \leq Z_\mu$, thus μ lies in $\widehat{N}^*$ and so in fact, by irreducibility

of the Y-module Z_μ/N, $M = Z_\mu$. In other words, we have $\Xi(M) \setminus \{1_N\} = \{\mu \in \widehat{N}^* : Z_\mu = M\}$. It is then clear that $\Xi(M) \cap \Xi(Z_\nu) = \{1_N\}$, if $Z_\nu \neq M$ for some $\nu \in \widehat{N}^*$.

Also, given a complement D_1 for X in H, if $\mathbf{C}_{M/N}(D_1) > 1$ then, for every $\lambda \in \Xi(M)$, by (g) $D_1 \leq \mathbf{C}_H(\lambda)$ and hence $\Xi(M) \leq \mathbf{C}_{\widehat{N}}(D_1)$.

So, we set $\mathbf{C}_H(\Xi(M)) = YD$, for a suitable complement D of X in H. Now set $|M/N| = p^t$ (note that, by Clifford's Theorem, t does not depend on $\lambda \in \widehat{N}^*$, as M/N is an irreducible Y-module and $Y \trianglelefteq H$), and let L/N be a complement for the irreducible Y-module M/N in P/N. Observe that, for $a, b \in P$ and $y \in Y$, we have

$$[a^y, b] = [a^y, b]^{y^{-1}} = [a, b^{y^{-1}}].$$

Therefore, for any $b \in M \setminus N$,

$$[L, M] = \langle [a, b^y] : a \in L,\ y \in Y \rangle = \langle [a^{y^{-1}}, b] : a \in L,\ y \in Y \rangle = [L, b]. \tag{1}$$

As a consequence, we get $|[L, M]| = |L|/|\mathbf{C}_L(b)| \leq |L/N| = p^{n-t}$; taking into account that $|N| = p^n$, this yields $|N/[L, M]| \geq p^t = |M/N|$.

Now, by Lemma 3.4, there are at least $p^{t/2}$ characters in $\widehat{N/[L, M]}$ that are not fully ramified in $M/[L, M]$. We claim that all the non-trivial characters in $\widehat{N/[L, M]}$ that are not fully ramified in $M/[L, M]$, are in $\Xi(M)$. In fact, given a $\nu \in \widehat{N/[L, M]}$ such that ν is not fully ramified in $M/[L, M]$, then (recalling Remark 3.3) there exists an element $b \in M \setminus N$ such that $[M, b] \leq \ker \nu$. As $[L, M] \leq \ker \nu$, then $[P, b] = [LM, b] \leq \ker \nu$. As in (1) (with P in place of L) one sees that $[P, b] = [P, M]$, and the claim is proved. In particular, we get $|\Xi(M)| \geq p^{t/2} - 1$.

Finally, set $d = |YD : \mathbf{C}_{YD}(M/N)|$. Since, by point (g), $YD/\mathbf{C}_{YD}(M/N)$ is an irreducible subgroup of $\Gamma(M/N)$, we have $|\mathbf{C}_{M/N}(D)| = p^{t/d}$, where $d = |D : C_D(M/N)|$. Noting that, by what observed before,

$$\mathcal{Z} = \{(Z_\lambda/N) \setminus \{1\} : \lambda \in \widehat{N}^* \text{ and } \mathbf{C}_{Z_\lambda/N}(D) > 1\}$$

is a partition of $\mathbf{C}_{P/N}(D) \setminus \{1\}$, we conclude that

$$|\mathcal{Z}| = \frac{|\mathbf{C}_{P/N}(D)| - 1}{p^{t/d} - 1} = \frac{p^{n/m} - 1}{p^{t/d} - 1}.$$

Since, as observed, $\Xi(Z_\lambda) \leq \mathbf{C}_{\widehat{N}}(D)$ for every Z_λ such that $(Z_\lambda/N) \setminus \{1\} \in \mathcal{Z}$, and $|\Xi(M)| \leq p^{t/2} - 1$, we deduce

$$p^{n/m} - 1 = |\mathbf{C}_{\widehat{N}}(D) \setminus \{1\}| \geq \frac{p^{n/m} - 1}{p^{t/d} - 1} \cdot (p^{t/2} - 1).$$

Hence $p^{t/2} - 1 \leq p^{t/d} - 1$, which implies $d = 2$ (so $|D|$ is even). In particular $p \neq 2$, thus $p^n - 1$ is an even number as well as $|D|$. But, by Lemma 3.5 the numbers $p^n - 1$ and $|D|$ must be coprime. This contradiction completes step (h). ∎

(j) *Final contradiction.*

Proof. By step (f), the graph $\Delta(G/N)$ is disconnected with connected components $\pi(X)$ and $\pi(D)$, where D is a complement for X in H.

Since step (h) yields $Y \leq \mathbf{C}_X(D)$, we have that $|Y|$ divides $p^{n/|D|}-1$. Now, $p^{n/|D|}-1$ is coprime to $k=(p^n-1)/(p^{n/|D|}-1)$ and, as k divides $|X|$ by (f), we see that k divides the order of $\emptyset X = \emptyset H \cap \Gamma_0(\widehat{N})$. Thus Lemma 3.5 yields that, for every $\lambda \in \widehat{N}$, a conjugate of D lies in $\mathbf{C}_H(\lambda)$. In particular, for every $\lambda \in \widehat{N} \setminus \{1\}$, we get $\mathbf{C}_H(\lambda) = Y \times D^x$ for a suitable $x \in X$.

Now, let $\chi \in \mathrm{Irr}(G)$ be such that χ_N has a non-trivial irreducible constituent λ, and let $\theta \in \mathrm{Irr}(P|\lambda)$ be an irreducible constituent of χ_P. Thus, $I_H(\theta) \leq \mathbf{C}_H(\lambda) = Y \times D^x$ (for some $x \in X$) is cyclic. So, as $\chi(1) = \psi(1)|G : I_G(\theta)|$ for a suitable $\psi \in \mathrm{Irr}(I_G(\theta)|\theta)$, we conclude that $\chi(1) = \theta(1)|H : I_H(\theta)|$ and hence, taking into account that $I_H(\theta)$ contains a conjugate of D by step (f), the prime divisors of $\chi(1)$ lie in $\{p\} \cup \pi(X)$. Therefore $\Delta(G)$ is disconnected, a contradiction. ■

The proof is now complete. ■

The following result, together with Remark 4.4 and Lemma 3.14 for what concerns the dimension n of the factors M_i, will yield Theorem A and, as a by-product, Theorem C. For this reason we do not include an independent proof for Theorem C, that could be obtained with a direct and much shorter argument.

Theorem 4.3. *Let G be a solvable group such that either $\Delta(G)$ is connected of diameter 3, or $\Delta(G)$ is disconnected. In the disconnected case, assume also that $F = \mathbf{F}(G)$ is non-abelian and that, whenever $\mathbf{O}_r(G)$ is non-abelian, the prime r is not an isolated vertex of $\Delta(G)$. Then the following conclusions hold.*

(a) *Let p be a prime. If $\mathbf{O}_p(G)$ is non-abelian, then it is a Sylow p-subgroup of G.*
(b) *There exists a unique prime p such that $P = \mathbf{O}_p(G)$ is non-abelian. Also, denoting by U the p-complement of F, we have $U \leq \mathbf{Z}(G)$.*
(c) *$\Delta(G/\gamma_3(P))$ is disconnected, and G/F is a non-nilpotent group whose Sylow subgroups are all cyclic. If c is the nilpotency class of P, all factors $M_1 = [P,G]/P'$ and $M_i = \gamma_i(P)/\gamma_{i+1}(P)$, for $2 \leq i \leq c$, are chief factors of G of the same order p^n. Moreover, for all $1 \leq i \leq c$, $G/\mathbf{C}_G(M_i)$ embeds in $\Gamma(p^n)$ as an irreducible subgroup.*

Proof. Let G be as in the assumptions and $F = \mathbf{F}(G)$. Observe that if $\Delta(G)$ is connected then, by Theorem 2.5, there exists a prime p such that $P = \mathbf{O}_p(G)$ is non-abelian (thus F is not abelian in any case). We argue by induction, and thus assume that G is a counterexample of minimal order.

If M is a normal subgroup of G such that $M \leq \mathbf{\Phi}(G)$, then the Fitting series of G/M is the image of the Fitting series of G under the natural homomorphism onto

G/M. Moreover, if $\mathbf{F}(G/M) = F/M$ is non-abelian and $\mathrm{V}(G) = \mathrm{V}(G/M)$, then all assumptions on G are inherited by G/M.

Observe first that, if $\Delta(G)$ is disconnected, then G is as in (c) of Theorem 2.2; in particular, parts (a) and (b) are already known to be true.

Thus, as it concerns the proof of (a) and (b), we may assume that $\Delta(G)$ is connected. Let p be a prime such that $P = \mathbf{O}_p(G)$ is non-abelian. Setting $N = P'$, observe that $N \leq \mathbf{\Phi}(G)$. Then, in order to prove (a) and (b), we may assume that N is a minimal normal subgroup of G. In fact, if M is a normal subgroup of G such that $1 < M < N$, then $\mathbf{O}_p(G/M) = P/M$ is non–abelian and $\mathrm{V}(G) = \mathrm{V}(G/M)$, whence, as observed before, (a) and (b) hold in G/M. In particular, P is a Sylow p-subgroup of G, and so (a) holds in G. Also, if U is the p-complement of F, then UM/M is the p-complement of $\mathbf{F}(G/M)$; property (b) in G/M and normality of U in G yield $[U, G] \leq M \cap U = 1$, and (b) holds true in G.

Assuming thus N to be a minimal normal subgroup of G, we start proving claims (a) and (b) for G under the additional hypothesis that $\Delta(G/N)$ is disconnected. Then, again denoting by U the p-complement of F, Proposition 3.13 yields $P/N \not\leq \mathbf{Z}(G/N)$, and Lemma 2.3 ensures that F/N is abelian and UN/N is central in G/N. Thus, we get $[G, U] \leq N \cap U = 1$, and (b) is proved in G. Since Proposition 3.13 also ensures that $p \nmid |G/P|$, then (a) is achieved as well.

For the proof of (a) and (b), we may therefore assume that $\Delta(G/\mathbf{O}_p(G)')$ is connected for every prime p such that $\mathbf{O}_p(G)$ is non-abelian. Thus, let p be such a prime and, again, write $P = \mathbf{O}_p(G)$ and $N = P'$.

Suppose, by contradiction, that (a) does not hold in G, that is, p divides $|G/P|$. Then $\mathrm{V}(G/N) = \mathrm{V}(G)$ and, since $\Delta(G/N)$ is connected, $\mathrm{diam}(\Delta(G/N))$=3. In particular, F/N is non-abelian by Theorem 2.5, and therefore there exists a prime $q \neq p$ such that $\mathbf{O}_q(G/N)$ is non-abelian. Now, G/N satisfies the hypotheses of the Theorem and so, by choice of G, G/N satisfies (a), (b) and (c): in particular, P/N is central in G/N and all the Sylow subgroups of G/F are cyclic. Setting $R = \mathbf{O}_{p'}(F)$, and taking any θ in $\mathrm{Irr}(P)$, we have, as in the second paragraph of the proof of Proposition 3.13, that $\theta \times 1_R$ is an extension of θ to F, such that $I_G(\theta \times 1_R) = I_G(\theta)$. Since the Sylow subgroups of G/F are cyclic, $\theta \times 1_R$ (and therefore θ) extends to $I = I_G(\theta)$. We may then apply Lemma 3.12 and get the contradiction that p is a complete vertex of $\Delta(G)$. Hence, G satisfies (a).

We move next to (b). First, we prove the following claim: *if p and q are two different primes such that both $P = \mathbf{O}_p(G)$ and $Q = \mathbf{O}_q(G)$ are non-abelian, then the diameters of $\Delta(G/P')$ and $\Delta(G/Q')$ are both at most* 2. (Note that the hypothesis of this claim forces p and q to be adjacent in $\Delta(G)$.) In fact, assume that one of those graphs (which are connected by what proved before), $\Delta(G/P')$ say, has diameter 3. Then, arguing as in the last paragraph, we have that P/P' is central in G/P'. Let H be a p-complement of G, then $G = PH$ by (a) and H

centralizes P/P', hence H centralizes P by coprimality. Therefore, $G = P \times H$, so p is a complete vertex of $\Delta(G)$, which a contradiction. We thus conclude that the diameters of $\Delta(G/P')$ and $\Delta(G/Q')$ are both at most 2, as claimed. But in this situation, the only vertices of $\Delta(G)$ that may have distance 3 between each other turn out to be p and q, which on the other hand are clearly adjacent.

Thus, we have proved that there exists a unique prime p such that $\mathbf{O}_p(G)$ is non-abelian. It remains to show that the p-complement U of F is central in G.

We claim that every irreducible character of U extends to its inertia subgroup in G. Since U is an abelian normal subgroup of G, this is certainly the case if U admits a complement. Otherwise, $N_0 = U \cap \mathbf{\Phi}(G) \neq 1$. In this case, $\mathbf{F}(G/N_0)$ is clearly non-abelian and $\mathrm{V}(G/N_0) = \mathrm{V}(G)$. Thus G/N_0 inherits our assumptions and, by choice of G, the Sylow subgroups of G/F are cyclic, a fact ensuring that also in this case every irreducible character of U extends to its inertia subgroup in G.

We are therefore in a position to apply Lemma 3.10 and get the desired conclusion unless p has distance at most 2 from every other vertex of $\Delta(G)$. But this would force (as we know that is $\Delta(G/N)$ connected) $\Delta(G/N)$ to have diameter 3, which is against Theorem 2.5, because the Fitting subgroup of G/N is abelian. The proof that (b) holds in G is complete.

Finally, we prove that (c) holds in G. Let U be the p-complement of F (thus $U \leq \mathbf{Z}(G)$ by part (b)); since $\mathbf{F}(G/U) = F/U = PU/U$ is non-abelian, the graph $\Delta(G/U)$ has the same set of vertices as $\Delta(G)$, and therefore G/U inherits the assumptions. Furthermore, the projection $G \to G/U$ induces a G-isomorphism $P \to PU/U$, and $\gamma_i(PU/U) = \gamma_i(P)U/U$.

We claim that, by choice of G, U is trivial. In fact, suppose $U \neq 1$; then, by induction on the order of the group, the conclusions concerning the factors M_i and the actions of the groups $G/\mathbf{C}_G(M_i)$ on them are easily achieved, and we also get that $\Delta(G/\gamma_3(P)U)$ is disconnected. Now, since $U \leq \mathbf{Z}(G)$ and $\gamma_3(P) \leq \mathbf{\Phi}(G)$, we have $\mathbf{F}(G/\gamma_3(P)U) = F/\gamma_3(P)U$. As the Sylow p-subgroup of this nilpotent factor group is non-abelian, we get $\mathrm{V}(G/\gamma_3(P)U) = \mathrm{V}(G)$, whence $G/\gamma_3(P)U$ is a group as in (c) of Theorem 2.2. In particular, G/F is a non-nilpotent group whose Sylow subgroups are all cyclic, and every irreducible character of $\mathbf{F}(G/\gamma_3(P)) = F/\gamma_3(P)$ extends to its inertia subgroup in $G/\gamma_3(P)$. We are then in a position to apply Lemma 2.4 to obtain the identity $\Delta(G/\gamma_3(P)) = \Delta(G/\gamma_3(P)U)$, and so $\Delta(G/\gamma_3(P))$ is disconnected, as wanted. Thus, $U = 1$.

We now observe that we may further reduce to the case $\gamma_3(P) = 1$. In fact, suppose $\gamma_3(P) \neq 1$; then, setting $\overline{G} = G/\gamma_3(P)$ and adopting the bar convention, we have $\mathrm{V}(\overline{G}) = \mathrm{V}(G)$ and the group $\overline{G}$ satisfies our hypotheses. Hence, the choice of G yields that $\Delta(\overline{G})$ is disconnected, and the following conclusions follow: G/F

is a non-nilpotent group whose Sylow subgroups are all cyclic, M_1 and M_2 are G-chief factors of the same order p^n, and $G/\mathbf{C}_G(M_i)$ embeds in $\Gamma(p^n)$ as an irreducible subgroup for $i \in \{1,2\}$. Finally, as $G/\gamma_3(P)$ is now as in (c) of Theorem 2.2, the prime p turns out to be adjacent in $\Delta(G)$ to every prime divisor of $|\mathbf{F}(H)|$ (it cannot therefore be adjacent to all vertices in $|H/\mathbf{F}(H)|$, as otherwise it would be a complete vertex in $\Delta(G)$). An application of Lemma 3.14 yields now what is left of claim (c) for G.

Then $\gamma_3(P) = 1$. Suppose, by contradiction, that there exists $M \trianglelefteq G$ with $1 < M < P'$. Then the graph $\Delta(G/M)$ is disconnected: in fact, the group G/M satisfies our hypotheses and therefore, setting $\overline{G} = G/M$ and adopting again the bar convention, we have that $\Delta(\overline{G}/\gamma_3(\overline{P}))$ is disconnected; but $\gamma_3(\overline{P})$ is trivial, so $\Delta(\overline{G}/\gamma_3(\overline{P})) = \Delta(G/M)$. Now the factor group G/M must be as in Theorem 2.2(c), hence G fulfills the assumptions of Lemma 3.14, an application of which yields that $P' = \gamma_2(P)/\gamma_3(P)$ is minimal normal in $G/\gamma_3(P) = G$, against the assumption on M.

In conclusion, P' is a minimal normal subgroup of G and G satisfies the hypotheses of Proposition 4.2, so all the desired conclusions follow. Hence, the proof that G also satisfies (c) is complete, but this is a contradiction by the choice of G. This completes the proof of the theorem. ∎

Remark 4.4. Let G be a solvable group such that $\Delta(G)$ has diameter 3. Then, assuming the notation and the conclusions of Theorem A, $G/\gamma_3(P)$ is a group as in part (c) of Theorem 2.2, and so $\Delta(G/\gamma_3(P))$ (which has the same vertices of $\Delta(G)$) is disconnected with components $\pi_1 = \{p\} \cup \pi(\mathbf{F}_2(G)/\mathbf{F}(G))$ and $\pi_2 = \pi(G/\mathbf{F}_2(G))$. Thus, both π_1 and π_2 induce complete subgraphs of $\Delta(G)$. Note that $|\pi_2| \geq 2$, as otherwise $\Delta(G)$ would have a complete vertex. Also, $\Delta(G/P')$ is a disconnected graph with components subgraphs $\pi_1 \setminus \{p\}$ and π_2.

Now, we have that $G = PH$, with P a p-group, $\gamma_3(P) \neq 1$, and $H/\mathbf{C}_H(P)$ embeds in $\Gamma(p^n)$. So, setting $d = |G/\mathbf{F}_2(G)|$, an application of Lemma 3.5 yields that d divides n, $(p^n - 1)/(p^{n/d} - 1)$ divides $|\mathbf{F}_2(G)/\mathbf{F}(G)|$, $(p^n)^3$ divides $|P|$ and d is coprime to $p^n - 1$. As a consequence, on one hand [10, Theorem 5.1] yields $|\pi_1 \setminus \{p\}| \geq 2^{|\pi_2|} - 1$. On the other hand, we easily get $2 \notin \pi_2$; in particular, as $|\pi_2| \geq 2$, we also get that n is divisible by two odd primes, as stated in part (d) of Theorem A.

Also, we note that $d_G(p, v) \leq 2$ for every $v \in \mathrm{V}(G)$. In fact, assume that $d_G(p, v) = 3$ and let $p - t - r - v$ be a path in $\Delta(G)$; then we get $t \in \pi_1$, $r \in \pi_2$ and, given $\chi \in \mathrm{Irr}(G)$ such that $tr \mid \chi(1)$, the prime p does not divide $\chi(1)$. But now χ is in fact in $\mathrm{Irr}(G/P')$ and hence t is adjacent to r in $\Delta(G/P')$, a contradiction.

Finally, we sketch the proof that Lewis' example in [9] is of the smallest possible order. In fact, as observed above, $|G|$ is a multiple of

$$(p^n)^3 \cdot \frac{p^n - 1}{p^{n/d} - 1} \cdot d \, .$$

The smallest value of such an integer is attained for $p = 2$ and $n = d = 15$, that is $2^{45} \cdot (2^{15} - 1) \cdot 15$, which is precisely the order of Lewis' group.

We conclude with a proof of Corollary B.

Proof of Corollary B. In the setting of Theorem A, we have that G/F is a group whose Sylow subgroups are all cyclic. This implies that G/F is a metacyclic group and, since it is not nilpotent, the desired conclusion follows. ∎

References

[1] B. Beisiegel, *Semi-extraspezielle p-Gruppen*, Math. Z. **156** (1977), 247–254.

[2] S. Dolfi, *On independent sets in the class graph of a finite group*, J. Algebra **303** (2006), 216–224.

[3] W. Gröbner, *Matrizenrechnung*, München, 1956.

[4] H. Heineken, *Gruppen mit kleinen abelschen Untergruppen*, Arch. Math. **29** (1977), 20–31.

[5] B. Huppert, *Endliche Gruppen I*, Springer, Berlin, 1983.

[6] B. Huppert, N. Blackburn, *Finite groups II*, Springer, Berlin, 1982.

[7] I.M. Isaacs, *Character Theory of Finite Groups*, AMS Chelsea Publishing, 2006.

[8] M.L. Lewis, *An overview of graphs associated with character degrees and conjugacy class sizes in finite groups*, Rocky Mountain J. Math. **38** (2008), 175–211.

[9] M.L. Lewis, *A solvable group whose character degree graph has diameter* 3, Proc. Amer. Math. Soc. **130** (2002), 625–630.

[10] M.L. Lewis, *Solvable groups whose degree graphs have two connected components*, J. Group Theory **4** (2001), 255–275.

[11] M.L. Lewis, D.L. White, *Diameters of degree graphs of nonsolvable groups. II.* J. Algebra **312** (2007), 634–649.

[12] O. Manz, T.R. Wolf, *Representations of Solvable Groups*, Cambridge University Press, Cambridge, 1993.

[13] C.P. Morresi Zuccari, *Character graphs with no complete vertices*, J. Algebra **353** (2012), 22–30.

[14] P.P. Palfy, *On the character degree graph of solvable groups. II. Disconnected graphs*, Studia Scient. Math. Hungarica **38** (2001), 339–355.

[15] P.P. Palfy, *On the character degree graph of solvable groups. I. Three primes*, Period. Math. Hungar **36** (1998), 61–65.

[16] J.Z. Zhang, *A note on character degrees of finite solvable groups*, Comm. Algebra **28** (2000), 4249–4258.

[17] E. Warning, *Bemerkungen zur vorstehenden Arbeit von Herrn Chevalley*, Abh. Math. Sem. Univ. Hamburg **11** (1936), 76–83.

Carlo Casolo, Dipartimento di Matematica e Informatica U. Dini,
Università degli Studi di Firenze, viale Morgagni 67/a, 50134 Firenze, Italy.
E-mail address: carlo.casolo@unifi.it

Silvio Dolfi, Dipartimento di Matematica e Informatica U. Dini,
Università degli Studi di Firenze, viale Morgagni 67/a, 50134 Firenze, Italy.
E-mail address: dolfi@math.unifi.it

Emanuele Pacifici, Dipartimento di Matematica F. Enriques,
Università degli Studi di Milano, via Saldini 50, 20133 Milano, Italy.
E-mail address: emanuele.pacifici@unimi.it

Lucia Sanus, Departament d'Àlgebra, Facultat de Matemàtiques,
Universitat de València, 46100 Burjassot, València, Spain.
E-mail address: lucia.sanus@uv.es