Vincular pattern posets and the Möbius function of the quasi-consecutive pattern poset

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Abstract

We introduce vincular pattern posets, then we consider in particular the quasi-consecutive pattern poset, which is defined by declaring $\sigma \leq \tau$ whenever the permutation τ contains an occurrence of the permutation σ in which all the entries are adjacent in τ except at most the first and the second. We investigate the Möbius function of the quasi-consecutive pattern poset and we completely determine it for those intervals $[\sigma, \tau]$ such that σ occurs precisely once in τ.

1   Introduction

The study of patterns in permutations is one of the most active trends of research in combinatorics. The richness of the notion of permutation patterns is especially evident from its plentiful appearances in several very different disciplines, such as algebra, geometry, analysis, theoretical computer science, and many others. Even if it is arguably not possible to encompass all possible applications of this notion into a simple formal environment, it seems reasonable to assert that the single mathematical structure which best catches the concept of a pattern and allows us to express a great deal of results about it is the *permutation pattern poset*.

Given two permutations σ, τ, we say that $\sigma \leq \tau$ in the permutation pattern poset whenever there is an occurrence of σ in τ as a classical pattern. An extremely challenging open problem concerning the permutation pattern poset is the determination of its Möbius function. The problem, originally posed by Wilf [W], quite recently received much attention, and some partial results have been achieved [SV, ST, BJJS, MS, Sm1, Sm2]. However, a complete description of the Möbius function of the permutation pattern poset is not yet available. The same problem can be formulated for the *consecutive pattern poset*, where, by definition, $\sigma \leq \tau$ whenever σ appears in τ as a *consecutive* pattern. This poset is much easier than the classical one and, in particular, its Möbius function is now completely understood [BFS, SW].

We recall here that the Möbius function μ of a poset is an important element of the incidence algebra of that poset, i.e. a function mapping an interval of the poset into a scalar. More specifically, it is possible to give a recursive definition of μ as follows:

$$\begin{aligned} \mu(x,x) &= 1 \\ \mu(x,y) &= -\sum_{x \leq z < y} \mu(x,z), \quad \text{if } x \neq y. \end{aligned}$$

In particular, if $x \not\leq y$, then $\mu(x,y) = 0$. Using a duality argument, it is possible to show that the Möbius function of an interval can be equally computed "from top to bottom", according to the following formula (for $x \neq y$):

$$\mu(x,y) = - \sum_{x < z \leq y} \mu(z,y).$$

Both formulas for computing μ will be frequently used throughout the paper.

Consecutive and classical patterns are special (actually, extremal) cases of the more general notion of *vincular patterns*. An occurrence of a vincular pattern is an occurrence of that pattern in which entries are subject to certain adjacency conditions (see next section for a precise definition). Vincular patterns were introduced by Babson and Steingrímsson [BS] (under the name of generalized patterns), and constitute a vast intermediate continent between the two lands of consecutive patterns and classical patterns. Since the determination of the Möbius function is still open in the classical case and totally solved in the consecutive one, it is conceivable that, if one is able to define a reasonable poset structure on permutations depending on the type of vincular patterns under consideration, then the resulting class of permutation posets (which would somehow interpolate between the consecutive and the vincular posets) may shed light on the differences and on the analogies between the two extremal cases.

In the present paper we propose a definition of *vincular pattern poset of type A* (whose elements are vincular patterns, or dashed permutations, as defined in Section 2), where A is a suitable infinite matrix whose n-th row encodes the type of the pattern of length n belonging to the poset (by specifying which entries of the pattern have to be adjacent in the corresponding dashed permutation). This is done in Section 2, where we also address the problem of understanding in which cases (that is, for which matrices A) the partial order thus defined perfectly catches the notion of an A-vincular pattern; even if we are not able to give a complete characterization of such matrices, we find some partial results, and in particular we single out a special class of matrices for which everything works in the best possible way. In Section 3 we consider a single case, probably the closest to the consecutive one, in which all the entries of a pattern are required to be adjacent, except at most the first and the second. In Section 4 we address the problem of the computation of the Möbius function in this poset, which we call the *quasi-consecutive pattern poset*, and we completely determine the value $\mu(\sigma, \tau)$ when σ occurs precisely once in τ. In spite of its closeness with the consecutive case, it seems that, in the general case, the computation $\mu(\sigma, \tau)$ in the quasi-consecutive pattern poset is considerably more complicated. This is underlined, for instance, by the fact (hinted at in the last section) that the absolute value attained by the Möbius function seems to be unbounded, whereas in the consecutive pattern poset the only possible values are $-1, 0, 1$.

2 Vincular pattern containment orders

Denote with S the set of all finite permutations. Elements of S are represented in one-line notation, so that $\pi = \pi_1 \cdots \pi_n$ is the permutation of length n mapping i to π_i, for all $i \leq n$. A *dashed permutation* is a permutation in which some dashes are possibly inserted between any two consecutive elements. For instance, $5 - 13 - 42$ is a dashed permutation (of length 5). The *type* of a dashed permutation π of length n is the $(0,1)$-vector $r = (r_1, \ldots, r_{n-1})$ (having $n-1$ components) such that, for all $i \leq n-1$, $r_i = 0$ whenever there is no dash between π_i and π_{i+1}, and $r_i = 1$ whenever there is a dash between π_i and π_{i+1}. For example, the above dashed permutation $5-13-42$ has type $(1,0,1,0)$. We remark that, in different sources (see, for instance, [K]), this notion of type is expressed using a different formalism, namely by recording

the lengths of each interval of adjacent elements in the dashed permutation. For instance, the above dashed permutation is said to have type $(1, 2, 2)$.

Let π be a permutation of length n and ρ be a dashed permutation of length $k \leq n$. We say that π *contains an occurrence of the vincular pattern* ρ when there exists a subsequence $\pi_{i_1}, \ldots, \pi_{i_k}$ of elements of π which is order-isomorphic to ρ and such that π_{i_j} and $\pi_{i_{j+1}}$ appear consecutively inside π if there is no dash between ρ_j and ρ_{j+1}. In this case we also say that ρ is a vincular patter of π. Vincular patterns were introduced in [BS] (where they are called *generalized patterns*). For example, the permutation 51342 contains an occurrence of the vincular pattern $3 - 21$, which is witnessed by the elements $5, 4, 2$, whereas it doesn't contain any occurrence of the vincular pattern $32 - 1$.

Let A be an infinite lower triangular $(0, 1)$-matrix, and denote with r_i the i-th row vector of A. Given $\pi \in S$, we say that $\rho \in S$ is an *occurrence of a vincular pattern of type A* (or an *occurrence of an A-vincular pattern*) in π when, given that ρ has length k, π contains an occurrence of ρ as a vincular pattern of type (given by the first $k - 1$ entries of) r_{k-1}. In this case, we also write $\rho \in_A \pi$. If π does not contain any occurrence of an A-vincular pattern ρ, that is $\rho \notin_A \pi$, we say that π *avoids ρ as an A-vincular pattern*. The most studied special cases are obtained when all the entries of A below and on the main diagonal are equal to 1 and when A is the null matrix. In the former case we recover the notion of classical pattern, whereas in the latter one we get the notion of consecutive pattern.

In some cases, the notion of an A-vincular pattern can be described by means of a suitable partial order. Given $\pi, \rho \in S$, having lengths n and $n - 1$ respectively, we say that π *covers* ρ whenever ρ appears as an A-vincular pattern in π. In this case, we write $\rho \prec_A \pi$, or simply $\rho \prec \pi$ when A is clear from the context. The transitive and reflexive closure of this covering relation is a partial order which will be called the *A-vincular pattern containment order*, and the resulting poset will be called the *A-vincular pattern poset*. When σ is less than or equal to τ in the A-vincular pattern poset, we will write $\sigma \leq_A \pi$ or simply $\sigma \leq \pi$, when no confusion is likely to arise.

It is not difficult to realize that, in the two special cases mentioned above, the resulting posets are well known. When A is the lower triangular matrix having all 1's below and on the main diagonal we obtain the classical pattern poset, whereas when A is the null matrix we get the consecutive pattern poset. We observe that, in these two cases, the partial order relation $\leq_A$ coincides with the binary relation $\in_A$, that is $\sigma \leq_A \tau$ if and only if $\sigma \in_A \tau$. However this is not always true, for the relation $\in_A$ is not transitive in general. In this direction, it would be interesting to characterize all matrices A for which the partial order relation (rather than the covering one) is directly defined in terms of occurrences of A-vincular patterns. More precisely, it would be nice to have a characterization of those matrices A such that $\sigma \leq_A \tau$ if and only if $\sigma \in_A \tau$. Such matrices A are those for which the structure of the A-pattern poset perfectly describes the notion of an A-vincular pattern. Unfortunately, we have not been able to find such a characterization yet, so we leave it as our first open problem. We have however some partial results, which we are going to illustrate below.

The first thing we observe is that in general, in the above conjectured equivalence, $\sigma \in_A \tau$ does not imply $\sigma \leq_A \tau$, nor does $\sigma \leq_A \tau$ imply $\sigma \in_A \tau$, as shown by the following two examples.

Examples.

1. Let A be the (lower triangular) matrix all of whose elements are 0, except for row 3, which is $r_3 = (0, 1, 0)$. If $\sigma = 1234$ and $\tau = 342156$, then $\sigma \in_A \tau$ (there is precisely one occurrence, in the subsequence 3456), however $\sigma \not\leq_A \tau$ (any A-vincular pattern

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of τ of length 5 has to be consecutive in τ). Notice that this example can be easily generalized to a matrix A whose n-th row has only one 1 and all successive rows are identically 0.

2. If A is such that $r_4 = (1, 0, 0, 0)$ and $r_5 = (0, 0, 0, 0, 1)$, then it is immediate to verify that $31524 \prec_A 361524$ and $361524 \prec_A 3615274$; thus we have $31524 \leq_A 3615274$, but $31524 \notin_A 3615274$, since the unique occurrence is of type $(1, 0, 0, 1)$.

If we restrict to the class of infinite lower triangular matrices having constant columns, we can show that one of the previous implications holds. Observe that, in this case, we can completely describe A using the (infinite) vector a whose i-th component a_i is the unique nonzero value appearing in column i of A. In particular, we modify the notations accordingly, by writing $\sigma \in_a \tau$ and $\sigma \leq_a \tau$ in place of $\sigma \in_A \tau$ and $\sigma \leq_A \tau$, respectively.

Proposition 2.1 *If $\sigma \in_a \tau$, then $\sigma \leq_a \tau$.*

Proof. Fix an occurrence of σ in τ as an a-vincular pattern. Let π be the smallest consecutive pattern of τ containing such an occurrence. Then $\pi \leq_a \tau$: this follows from the more general fact that, given any matrix A, for any consecutive pattern ρ' of a permutation ρ, we have $\rho' \leq_A \rho$ (starting from ρ, repeatedly remove either the first or the last element until getting to ρ'). To conclude the proof we now need to find a (descending) chain of coverings from π to σ in the a-vincular pattern poset. To this aim, partition the elements of the fixed occurrence of σ in π into blocks of consecutive elements of π of maximal length, and suppose that such lengths are $\alpha_1, \alpha_2, \ldots, \alpha_r$. This implies that the vector a certainly has a 1 in positions $\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_r$. Therefore we can remove the leftmost element of π not belonging to the selected occurrence of σ, which is the $(\alpha_1 + 1)$-th element of π, thus obtaining a permutation covered by π in the a-vincular pattern poset. We can repeat this operation until the first two blocks of σ become adjacent, thus obtaining a chain of coverings in the a-vincular pattern poset from π down to a permutation $\overline{\pi}$. Now observe that, starting from $\overline{\pi}$, we can certainly remove its leftmost element not belonging to the selected occurrence of σ to obtain a covering, since it is the $(\alpha_1 + \alpha_2 + 1)$-th element of π (and a has a 1 in position $\alpha_1 + \alpha_2$). This argument can be repeated, by simply observing that, each time we remove the leftmost element of a permutation of the chain not belonging to the highlighted occurrence of σ, we create a new covering in the a-vincular pattern poset. We can thus complete the chain of coverings from τ to π to a chain of covering from τ to σ, which is what we need to conclude that $\sigma \leq_a \tau$. ∎

However, also in this special case, the reverse implication does not hold, as the following example clarifies.

Example. Let $a = (0, 1, 0, 0, \ldots)$ (that is, the vector whose unique nonzero component is the second one). If $\sigma = 123$ and $\tau = 51423$, then $123 \prec_a 4123 \prec_a 51423$, and so $\sigma \leq_a \tau$; on the other hand, $\sigma \notin_a \tau$, since in the unique occurrence of σ in τ the first two elements are not consecutive.

The reason why the above counterexample holds is essentially that in a there is a 1 preceded by a 0. If this does not happen, we are able to prove the following proposition.

Proposition 2.2 *Suppose that the vector a has the first k components equal to 1 and all the remaining components equal to 0. Then $\sigma \in_a \tau$ if and only if $\sigma \leq_a \tau$.*

Proof. We only need to prove that $\sigma \leq_a \tau$ implies that $\sigma \in_a \tau$. Observe that, if the length of σ is $\leq k + 1$, then the thesis becomes trivial. So, from now on, we will tacitly assume that

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σ is sufficiently long. We proceed by induction on the difference between the length of τ and the length of σ. If σ and τ have the same length, then clearly $\sigma = \tau$. Moreover, if the length of τ is one more than the length of σ, then the definitions of $\in_a$ and $\leq_a$ coincide. Now suppose that the assertion holds for pairs of permutations whose lengths differ by less than n, and let σ and τ be permutations whose lengths differ by exactly n. Since $\sigma \leq_a \tau$, there is a chain of coverings $\tau = \rho_0 \succ_a \rho_1 \succ_a \cdots \succ_a \rho_{n-1} \succ_a \rho_n = \sigma$. By inductive hypothesis, $\rho_{n-1} \in_a \tau$. Due to our assumptions on a, this means that there is an occurrence of ρ_{n-1} in τ all of whose elements must appear consecutively except at most the first $k+1$. Since $\sigma \prec_a \rho_{n-1}$, σ is obtained from ρ_{n-1} by removing either one of the first $k+1$ elements or the last one. In all cases, it is easy to check that the occurrence of σ in τ, resulting from the selected occurrence of ρ_{n-1} in τ, is made of consecutive elements of τ, except at most the first $k+1$. This is equivalent to saying that $\sigma \in_a \tau$, as desired. ∎

3   The quasi-consecutive pattern poset

In the rest of the paper we will deal with a special case, which turns out to be particularly manageable, due to its closeness to the consecutive case.

Let A be the infinite lower triangular $(0,1)$-matrix for which an entry is equal to 1 if and only if it belongs to the first column. So the first lines of A are as follows:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since A has constant columns, according to the notations introduced in the previous section, it corresponds to the infinite vector $a = (1, 0, 0, 0, \ldots)$ having 1 only in the first position. Therefore an a-vincular pattern σ is interpreted as a dashed permutation having only one dash, which is placed between the first two elements of σ. Due to the last result of the previous section, in this special case, we have that $\sigma \leq_a \tau$ if and only if $\sigma \in_a \tau$. In other words, we can assert that $\sigma \leq_a \tau$ when there is an occurrence of σ in τ whose first two elements are possibly not consecutive, whereas all the remaining elements have to appear consecutively in τ. For this reason, we will say that σ is a *quasi-consecutive pattern* of τ, and the resulting poset will be called the *quasi-consecutive pattern poset*. As an example, the permutation 432516 contains the quasi-consecutive pattern 231, since the entries 351 show an occurrence of the vincular pattern $2 - 31$. When a permutation τ does not contain the quasi-consecutive pattern σ, we say that τ *avoids* the quasi-consecutive pattern σ. For instance, the preceding permutation 432516 avoids the quasi-consecutive pattern 123. Observe that 432516 contains other instances of vincular patterns obtained by inserting dashes in the permutation 123 (for example, there are occurrences of both $12 - 3$ and $1 - 2 - 3$). To have an idea of how intervals look in the quasi-consecutive pattern poset, we refer the reader to subsequent figures.

The quasi-consecutive pattern poset has many similarities with the consecutive pattern poset studied in [BFS]. Clearly it has a slightly more complicated structure, which would be interesting to investigate in detail. A first structural property of such a poset is recorded in the next proposition.

Proposition 3.1 *For any $\tau \in S$, τ covers at most three permutations, which are obtained by removing either the first, the second or the last entry of τ (and suitably rearranging the remaining ones).*

Proof. If τ has length n, a permutation ρ of length $n-1$ covered by τ can appear as an occurrence either consecutive or not. In the former case, ρ has to appear either as a prefix or a suffix of τ; thus ρ is obtained by removing either the first or the last entry of τ. In the latter case, the only possibility is that the first two entries of ρ are not consecutive in τ; thus ρ is obtained from τ by removing the second entry. ∎

The property proved in the above proposition gives an important feature of the quasi-consecutive poset, which will be very useful in the next section.

4 On the Möbius function of the quasi-consecutive pattern poset

In this section, which contains the main results of our paper, we address the problem of the computation of the Möbius function of the quasi-consecutive pattern poset. Specifically, given $\sigma, \tau \in S$, with $\sigma \leq \tau$, we want to compute $\mu(\sigma, \tau)$. We will completely solve the case of a single occurrence, which turns out to be not a trivial one, and leave the general case as an open problem.

Throughout this section, we will frequently write permutations with a dash between the first entry and the second entry. This is done in order to emphasize the fact that they have to be interpreted as specific vincular patterns in the quasi-consecutive pattern poset. Moreover, as we already started to do in the previous paragraph, the partial order relation of the quasi-consecutive pattern poset will be denoted simply by $\leq$ (instead of $\leq_a$), as no other partial orders will appear from now on.

We start by giving some partial results, depending on whether τ covers precisely i permutations, for $i = 1, 2$. Recall that a *monotone permutation* is a permutation which is either increasing or decreasing (so that, for any n, the monotone permutations are simply $12 \cdots n$ and $n(n-1)\cdots 1$).

Proposition 4.1 *If τ covers precisely one permutation, then $[\sigma, \tau]$ is a chain, for any $\sigma \leq \tau$.*

Proof. If τ covers precisely one permutation, then the same permutation is obtained by removing any of the three allowed elements of τ. In particular, if removing the first or the last entry of τ results in the same permutation, necessarily τ has to be monotone. So σ has to be monotone too, and in this case $[\sigma, \tau]$ is clearly a chain. ∎

Thus, if τ covers precisely one permutation, then $\mu(\sigma, \tau) = 1$ when $\sigma = \tau$, $\mu(\sigma, \tau) = -1$ when τ covers σ, and $\mu(\sigma, \tau) = 0$ in all the other cases.

Proposition 4.2 *Let $\tau = a_1 - a_2 \cdots a_n$. If τ covers precisely two permutations, then there are two distinct possibilities:*

- *a_1 and a_2 are consecutive integers;*

- *either $\tau = 1n(n-1)\cdots 32$ or $\tau = n12\cdots(n-1)$.*

Proof. If τ covers precisely two permutations, then there exist two elements in the set $\{a_1, a_2, a_n\}$ such that removing either of them from τ gives the same permutation. It cannot happen that removing either a_1 or a_n yields the same permutation, since otherwise τ would be monotone, and so it would cover only one permutation, which is not the case. If removing either a_1 or a_2 yields the same permutation, then each of the remaining entries of τ has to be either bigger or smaller than both a_1 and a_2. Thus a_1 and a_2 must be consecutive integers. Finally, if removing either a_2 or a_n yields the same permutation, then the subpermutation determined by the elements $a_2, \ldots, a_n$ is monotone and a_1 must be either smaller or bigger than such elements. Since τ itself cannot be monotone, then a_1 is either 1 or n, and so either $\tau = 1n(n-1)\cdots 32$ or $\tau = n12\cdots(n-1)$. ∎

Observe that, if τ covers precisely two permutations and a_1 and a_2 are not consecutive integers, then we can determine the structure of the interval $[\sigma, \tau]$, and so its Möbius function. Indeed, suppose that any occurrence of σ in τ does not involve a_1. Then necessarily σ is monotone. Thus a permutation in $[\sigma, \tau]$ is uniquely determined by a (possibly empty) set of entries to remove from the end of τ and by choosing if a_1 has to be removed or not. For these reasons, it is not difficult to realize that, if we denote with k the difference between the length of τ and the length of σ, then $[\sigma, \tau] \simeq \mathbf{2} \times \mathbf{k}$, where $\mathbf{i}$ denotes the chain of the first i positive integers. Therefore, in this situation, the computation of the Möbius function can be easily carried out:

$$\mu(\sigma, \tau) = \begin{cases} 1, & \text{if } \sigma \text{ has length } n; \\ -1, & \text{if } \sigma \text{ has length } n-1; \\ 1, & \text{if } \sigma \text{ has length } n-2; \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $\sigma = 1$ does not fit into the previous case, nevertheless using a completely analogous argument we get to the same formula for $\mu(\sigma, \tau)$.

On the other hand, if there is an occurrence of σ which involves a_1, then necessarily every occurrence of σ involves a_1. In this case, it is easily seen that either σ is not monotone or σ has length 2. Thus $[\sigma, \tau]$ is a finite chain, whose Möbius function is trivial to compute.

4.1 The case of one occurrence

Let $\sigma \leq \tau = a_1 - a_2 \cdots a_n$ and suppose that σ occurs precisely once in τ. We will be concerned with the computation of $\mu(\sigma, \tau)$ in this particular case, which we will be able to solve completely.

We will distinguish several cases, depending on the fact that the single occurrence of σ in τ does or does not involve the three elements a_1, a_2 and a_n.

A first obvious consideration is that σ cannot involve all of the three above elements, unless $\sigma = \tau$, and in this last case of course $\mu(\sigma, \tau) = 1$.

The second possibility is that two of the three elements a_1, a_2, a_n are involved in the unique occurrence of σ in τ. There are of course three distinct cases; however in any of them the interval $[\sigma, \tau]$ is a chain. For instance, if σ involves a_1 and a_n (and not a_2), then σ is (isomorphic to) $a_1 - a_k a_{k+1} \cdots a_n$, for some $k > 2$, and it is immediate to see that any $\rho \in [\sigma, \tau]$ is obtained by starting from τ and repeatedly removing the second element of the resulting permutation until we get to ρ. This is of course the only possible way to remove elements from τ and remain inside $[\sigma, \tau]$ (a_1 and a_n cannot be removed, being part of the unique occurrence of σ in τ). This means that the interval $[\sigma, \tau]$ is a chain. The remaining cases can be dealt with in a completely analogous way. Thus, we can conclude that, in all these cases, $\mu(\sigma, \tau) = 0$, unless σ has length $n-1$, in which case $\mu(\sigma, \tau) = -1$.

We next examine the case in which only one among a_1, a_2 and a_n is involved in the unique occurrence of σ in τ. There are three distinct cases to consider.

Proposition 4.3 *Suppose that a_1 occurs in σ (whereas a_2 and a_n do not). Then $\mu(\sigma, \tau) = 0$, unless $[\sigma, \tau]$ has rank 2 (in which case $\mu(\sigma, \tau) = 1$).*

Proof. In this case σ is (isomorphic to) $a_1 - a_k a_{k+1} \cdots a_{k+h}$, for some k, h with $2 < k < n$ and $0 \leq h < n - k$. If $\rho \in [\sigma, \tau]$, then ρ is (isomorphic to) $a_1 - a_i a_{i+1} \cdots a_{k+h+j}$, for suitable i and j, and so it is uniquely determined by two intervals of elements $\{a_2, \ldots a_{i-1}\}$ and $\{a_{k+h+j+1}, \ldots a_n\}$ to be removed from τ in a well-specified order. Therefore $[\sigma, \tau]$ is a *grid*, i.e. it is isomorphic to a product of two chains, whose lengths (which of course depend on the values of i and j) are ≥ 1 (this is due to the fact that σ does not involve a_2 and a_n). Thus $\mu(\sigma, \tau) = 0$, unless the two chains both have length 1. Clearly, in the latter case $\mu(\sigma, \tau) = 1$. ∎

Proposition 4.4 *Suppose that a_2 occurs in σ (whereas a_1 and a_n do not). Then $\mu(\sigma, \tau) = 0$, unless $[\sigma, \tau]$ has rank 2 (in which case $\mu(\sigma, \tau) = 1$).*

Proof. The argument is completely analogous to the one used in the above proposition. The only difference here is that the roles of a_1 and a_2 have to be swapped. ∎

Notice that the two cases considered so far are essentially equivalent to the case of a single occurrence in the consecutive pattern poset. Indeed, using the notation introduced in [BFS], it is not too difficult to realize that $[\sigma, \tau]$ and $[{}'\sigma, {}'\tau]$ are order isomorphic.

The last case is by far the most challenging one.

Proposition 4.5 *Suppose that a_n occurs in σ (whereas a_1 and a_2 do not). Then $\mu(\sigma, \tau) = 0$, unless $[\sigma, \tau]$ has rank 2. In this last case, if a_1 and a_2 are consecutive integers, then $\mu(\sigma, \tau) = 0$, otherwise $\mu(\sigma, \tau) = 1$.*

Proof. It is convenient to distinguish two cases, depending on whether σ appears as a consecutive pattern in τ or not.

If σ is not a consecutive pattern of τ, then it is not difficult to realize that $[\sigma, \tau]$ has only one atom, which can be obtained in the following way: take the unique occurrence of σ in τ and add to it the element of τ immediately to the left of its consecutive part. Observe that, in this case, if σ has length m, then $n - m \geq 3$ (since a_1 and a_2 do not occur in σ and σ is not consecutive in τ, so there must be at least one element of τ, which is not in the unique occurrence of σ, between the first and the second element of σ), and so $\mu(\sigma, \tau) = 0$.

If σ is a consecutive pattern of τ, then σ is (isomorphic to) $a_k a_{k+1} \cdots a_n$, where $2 < k \leq n$. Figure 1 shows an instance of this situation. Denote with $\tau^{(1)}$ the permutation obtained from τ by removing a_1 (and of course suitably renaming the remaining elements). Similarly, denote with $\tau^{(2)}$ the permutation obtained from τ by removing a_2 and with $\tau^{(1,2)}$ the permutation obtained from τ by removing both a_1 and a_2. Finally, let

$$C^{(1)} = \{\rho \in [\sigma, \tau] \mid \rho < \tau^{(1)}, \rho \not\leq \tau^{(1,2)}\},$$
$$C^{(2)} = \{\rho \in [\sigma, \tau] \mid \rho < \tau^{(2)}, \rho \not\leq \tau^{(1,2)}\}.$$

We start by observing that, in this case, $\tau^{(1)}$ and $\tau^{(2)}$ are the only coatoms of $[\sigma, \tau]$. So, if $\tau^{(1)} = \tau^{(2)}$ (i.e., a_1 and a_2 are consecutive integers), then clearly $\mu(\sigma, \tau) = 0$. Thus, in what follows we will suppose that $\tau^{(1)} \neq \tau^{(2)}$.
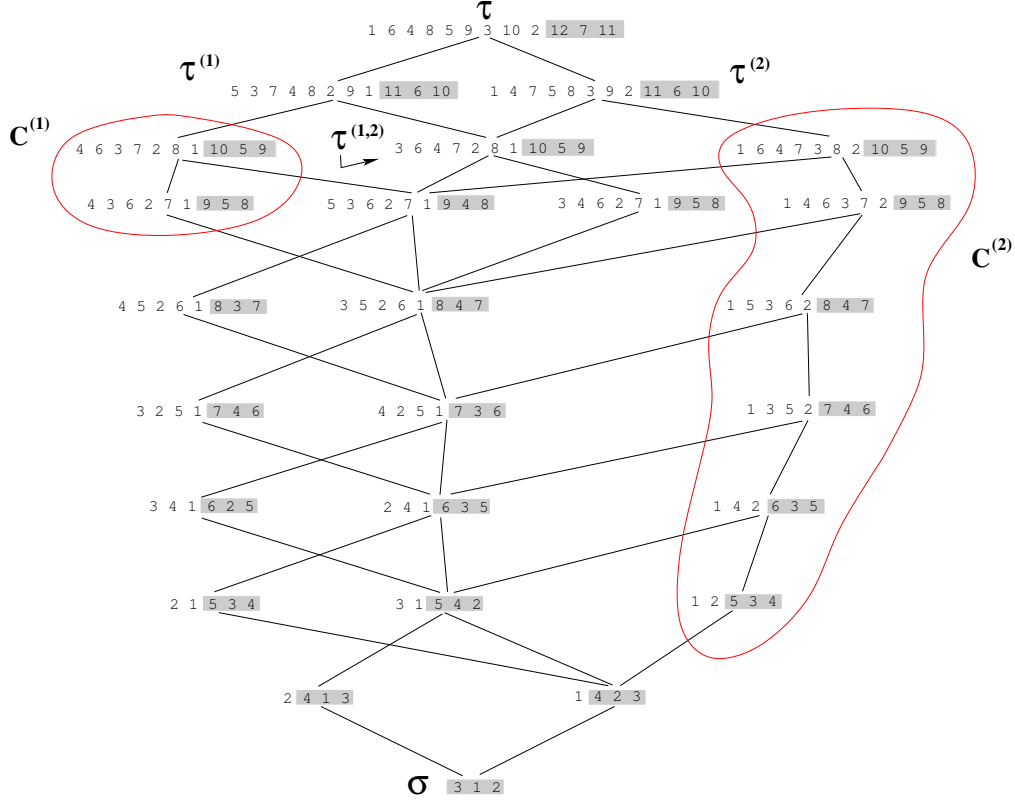
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Figure 1: A single occurrence of a pattern at the end of a permutation.

We next show that $C^{(1)}$ and $C^{(2)}$ are chains. Indeed, any $\rho \in C^{(2)}$ can be obtained from $\tau^{(2)}$ by repeatedly removing the second element of the resulting permutation, so ρ is uniquely determined by the set of consecutive elements of τ which have been removed. An analogous argument shows that also $C^{(1)}$ is a chain.

Moreover, we have that $C^{(1)} \cap C^{(2)} = \emptyset$. Indeed, if we had $C^{(1)} \cap C^{(2)} \neq \emptyset$, then any $\rho \in C^{(1)} \cap C^{(2)}$ would be order isomorphic to both $a_1 a_r a_{r+1} \cdots a_n$ and $a_2 a_r a_{r+1} \cdots a_n$, for a suitable r. We wish to show that, in this situation, a_1 and a_2 have to be consecutive integers. Indeed, suppose that a_1 and a_2 are not consecutive and, w.l.o.g., that $a_1 < a_2$. First of all, for all i such that $r \leq i \leq n$, it cannot be $a_1 < a_i < a_2$, otherwise $a_1 a_r a_{r+1} \cdots a_n$ and $a_2 a_r a_{r+1} \cdots a_n$ would not be order isomorphic. So there should exist an element a such that $a_1 < a < a_2$, which appears before a_r in τ. But then $a a_r a_{r+1} \cdots a_n$ would be order isomorphic to ρ, which is not possible, since otherwise $\rho \leq \tau^{(1,2)}$. Therefore a_1 and a_2 have to be consecutive integers, which is not true (remember that we are supposing that $\tau^{(1)} \neq \tau^{(2)}$). We can thus conclude that $C^{(1)} \cap C^{(2)} = \emptyset$, as desired.

Our next goal is to prove that, for all $\rho \in C^{(1)} \cup C^{(2)}$, $\mu(\rho, \tau) = 0$. In fact, if $\alpha \in [\rho, \tau]$, then $\alpha \not\leq \tau^{(1,2)}$, since otherwise we have $\rho \leq \tau^{(1,2)}$, which is not true. So, if for instance $\rho \in C^{(1)}$, then we have $\alpha \in C^{(1)} \cup \{\tau^{(1)}, \tau\}$. Therefore $[\rho, \tau]$ is a chain having at least three elements (choose $\alpha = \tau^{(1)}$), whence $\mu(\rho, \tau) = 0$.

Finally, we are now in a position to prove that, for all $\rho < \tau^{(1,2)}$, $\mu(\rho, \tau) = 0$. Indeed, if ρ is covered by $\tau^{(1,2)}$, then necessarily $[\rho, \tau] = \{\rho, \tau^{(1,2)}, \tau^{(1)}, \tau^{(2)}, \tau\}$, and it is immediate to see that $\mu(\rho, \tau) = 0$. Instead, if $\rho < \tau^{(1,2)}$ is not covered by $\tau^{(1,2)}$, then $[\rho, \tau]$ contains the same five permutations listed above as well as a set X of permutations less than $\tau^{(1,2)}$ (but "closer" than ρ to $\tau^{(1,2)}$) and (possibly) a set Y of permutations contained in $C^{(1)} \cup C^{(2)}$. Using an

inductive argument (on the distance from $\tau^{(1,2)}$), we can show that, for all permutations $\alpha \in X$, $\mu(\alpha, \tau) = 0$; moreover, we already know (from the previous paragraph) that, for all $\alpha \in Y$, $\mu(\alpha, \tau) = 0$. Thus the only $\alpha \in [\rho, \tau]$ such that $\mu(\alpha, \tau) \neq 0$ are $\tau^{(1,2)}, \tau^{(1)}, \tau^{(2)}, \tau$, and so we can immediately conclude that $\mu(\rho, \tau) = 0$.

Since of course $\sigma \leq \tau^{(1,2)}$, if the rank of $[\sigma, \tau]$ is greater than 2, then $\sigma < \tau^{(1,2)}$ and so $\mu(\sigma, \tau) = 0$. Otherwise, if a_1 and a_2 are consecutive integers, then $\tau^{(1)} = \tau^{(2)}$, and $[\sigma, \tau]$ is a 3-elements chain, so that $\mu(\sigma, \tau) = 0$. If instead a_1 and a_2 are not consecutive, then $[\sigma, \tau]$ is the product of two 2-elements chains, whence $\mu(\sigma, \tau) = 1$. ∎

The last case to be considered is when none of a_1, a_2 and a_n belongs to the unique occurrence of σ in τ. It will be convenient to distinguish two cases, depending on whether the occurrence of σ is consecutive or not.

Proposition 4.6 *If σ does not occur consecutively in τ, and does not involve a_1, a_2 and a_n, then $\mu(\sigma, \tau) = 0$.*

Proof. Denote with π_1 the permutation isomorphic to the smallest consecutive pattern in τ containing σ and with π_2 the permutation isomorphic to the smallest suffix of τ containing σ.

We start by observing that, for all $\rho \in [\sigma, \tau]$, we have $\rho \leq \pi_2$ or $\rho \geq \pi_1$. Indeed, if $\rho \not\leq \pi_2$, then the leftmost element of an occurrence of ρ in τ must correspond to an entry of τ which appears on the left of σ. Thus necessarily all the other elements of the occurrence of ρ have to appear consecutively in τ and to contain σ, and this implies that $\rho \geq \pi_1$.

We next give more precise information on the structure of the interval $[\sigma, \tau]$ (see also Figure 2).

- $[\pi_1, \pi_2]$ is a chain: this follows from the fact that π_1 occurs only once in π_2, and the first two elements of π_2 belong to such an occurrence.

- $[\sigma, \pi_1]$ is a chain: this is analogous to the above one, since σ occurs only once in π_1 and the first and last elements belong to such an occurrence.

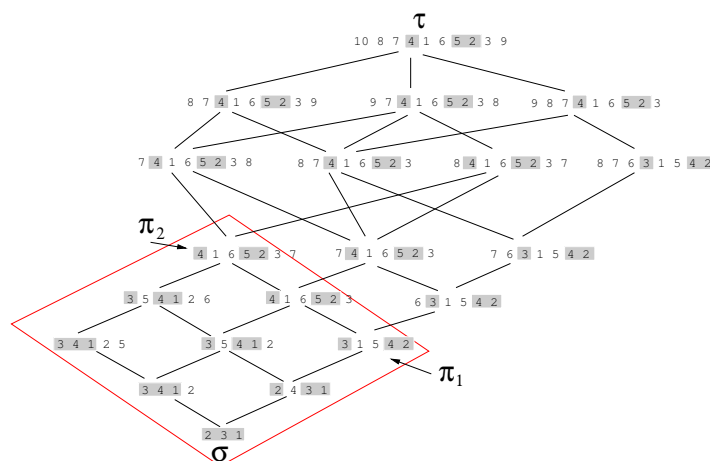- $[\sigma, \pi_2]$ is a product of two nonempty chains: this follows from the proof of Proposition 4.3.

Figure 2: The unique occurrence of σ in each permutation of $[\sigma, \tau]$ is highlighted.

Therefore we have

$$\mu(\sigma, \tau) = -\sum_{\sigma \le \rho < \tau} \mu(\sigma, \rho) = -\sum_{\rho \in [\sigma, \pi_2]} \mu(\sigma, \rho) - \sum_{\rho \notin [\sigma, \pi_2]} \mu(\sigma, \rho).$$

It is immediate to see that, in the last expression, the first sum is 0. As for the second sum, we observe that each $\rho \ge \pi_1$ (and so in particular each $\rho \notin [\sigma, \pi_2]$) lies either above all the four elements of $[\sigma, \pi_2]$ having nonzero Möbius function, or above two of them, namely σ and one of the two atoms of $[\sigma, \tau]$. In both cases, an inductive argument (on the difference between the length of ρ and the length of π_1) shows that, for all $\rho \notin [\sigma, \pi_2]$, $\mu(\sigma, \rho) = 0$. From all the above considerations it then follows that $\mu(\sigma, \tau) = 0$. ∎

Proposition 4.7 *If σ occurs consecutively in τ, and does not involve a_1, a_2 and a_n, then $\mu(\sigma, \tau) = 0$, unless $[\sigma, \tau]$ has rank 3, the unique occurrence of σ in τ contains a_{n-1} and τ covers three elements in $[\sigma, \tau]$. In such a case, we have $\mu(\sigma, \tau) = -1$.*

Proof. Denote with π the permutation isomorphic to the smallest suffix of τ containing the unique occurrence of σ and with η the permutation isomorphic to the prefix of τ of length $n - 1$.

Using an argument analogous to that of Proposition 4.6, we observe that, for every $\rho \in [\sigma, \tau]$, we have $\rho \le \eta$ or $\rho \ge \pi$. Indeed, if $\rho \not\le \eta$, then an occurrence of ρ necessarily contains both the last element of τ and the unique occurrence of σ, so that ρ contains π (i.e. $\rho \ge \pi$).

Since π and η are clearly incomparable, the two intervals $[\sigma, \eta]$ and $[\pi, \tau]$ constitute a partition of $[\sigma, \tau]$. Therefore concerning the Möbius function of $[\sigma, \tau]$ we have:

$$\mu(\sigma, \tau) = -\sum_{\pi \le \rho < \tau} \mu(\sigma, \rho).$$

Suppose first that $[\sigma, \pi]$ has rank at least 2. This situation is illustrated in Figure 3. Observe that, in this case, $\mu(\sigma, \pi) = 0$, since σ is a prefix of π, and so $[\sigma, \pi]$ is a chain of length at least 2. Given $\rho \in [\pi, \tau]$, suppose now that $\mu(\sigma, \alpha) = 0$, for all $\pi \le \alpha < \rho$. Notice that ρ is made by an element of τ followed by a suffix of τ containing σ. Thus, by removing the rightmost element of ρ, we obtain a permutation β contained in η (observe that β is a coatom of $[\sigma, \rho]$). Moreover, the reader can immediately see that β and π are incomparable and that, for all $\alpha \in [\sigma, \rho]$, either $\alpha \le \beta$ or $\alpha \ge \pi$. So we have a partition of $[\sigma, \rho]$ and we can conclude that, for all $\rho \in [\pi, \tau]$,

$$\mu(\sigma, \rho) = -\sum_{\sigma \le \alpha < \rho} \mu(\sigma, \alpha) = -\sum_{\pi \le \alpha < \rho} \mu(\sigma, \alpha) = 0,$$

which is enough to assert that $\mu(\sigma, \tau) = 0$.

Otherwise, i.e. when $[\sigma, \pi]$ has rank 1, we observe that the two intervals $[\sigma, \eta]$ and $[\pi, \tau]$ both fall into the scopes of Proposition 4.5. Indeed, there is a unique occurrence of σ and π in η and τ respectively, in both cases at the end of the permutation. Each permutation $\gamma \in [\pi, \tau]$ covers exactly one permutation $\gamma' \in [\sigma, \eta]$ (just remove the last entry of γ, which is its only entry which can be removed in order to get a permutation lying below η). This fact will now be used to describe the structure of $[\sigma, \tau]$.

Concerning the interval $[\pi, \tau]$, we have two possibilities: either τ covers one element or two elements. In the former case, necessarily η covers only one element in $[\sigma, \eta]$; in the latter case, η may cover either one or two elements. We thus have a total of three cases, which are illustrated in Figure 4.
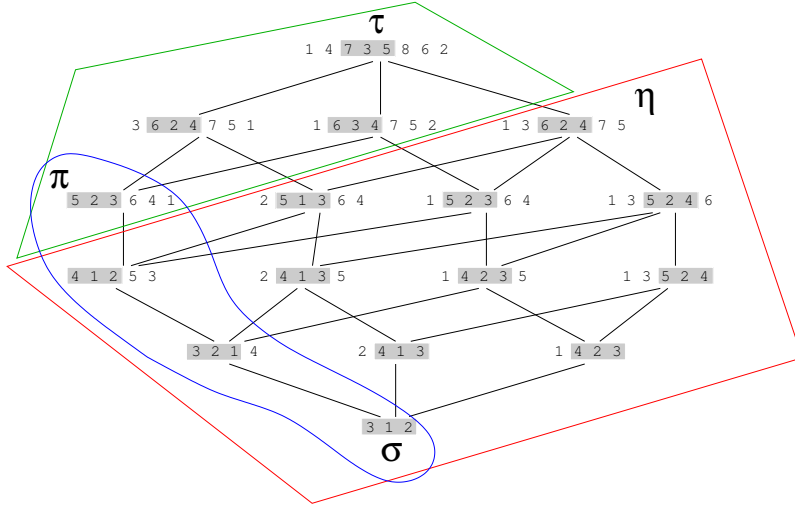
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Figure 3: A single occurrence of a consecutive pattern not involving the three critical entries of a permutation (that is, the first two and the last one).
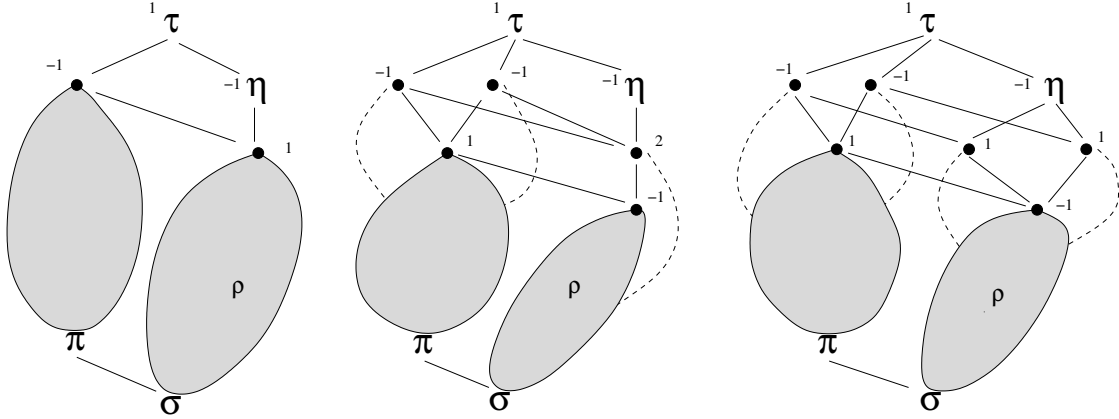

Figure 4: The Möbius function is computed from top to bottom. Dashed lines denote chains. In each of the three cases, there are covering relations between elements of the two shaded regions, as well as between elements of two corresponding chains, under the map denoted with $'$ in the proof of Proposition 4.7.

We first observe that, when $[\sigma, \tau]$ has rank 3, the second and the third cases depicted in Figure 4 correspond exactly to the exceptional case of the statement of the proposition, and it is immediate to observe that $\mu(\sigma, \tau) = -1$. Instead, in the first case we have $\mu(\sigma, \tau) = 0$. So from now on we suppose that the rank of $[\sigma, \tau]$ is at least 4.

If we compute the Möbius function of $[\sigma, \tau]$ from top to bottom, we can restrict ourselves to compute it in each element of $[\sigma, \eta]$, that is:

$$\mu(\sigma, \tau) = - \sum_{\sigma < \rho \leq \eta} \mu(\rho, \tau).$$

It is now not too difficult to realize that, in each of the three possible cases, for any $\rho \in [\sigma, \eta]$ except that for the top three levels of the interval (i.e. η, its coatoms and the elements covered by such coatoms), the value of $\mu(\rho, \tau)$ is 0. Indeed, denote with Δ the set of all permutations

in $[\rho, \tau] \setminus \{\rho\}$ other than those whose Möbius function is explicitly indicated in Figure 4. Then,

$$\mu(\rho, \tau) = -\sum_{\delta \in \Delta} \mu(\delta, \tau) - \sum_{\substack{\xi > \rho \\ \xi \notin \Delta}} \mu(\xi, \tau).$$

We can assume inductively that each summand of the first sum is equal to 0; as far as the second sum is concerned, the reader is referred again to Figure 4 to get convinced that it is equal to 0 in all cases as well. Thus, in particular, $\mu(\sigma, \tau) = 0$. ∎

Summing up all the results obtained so far in this section, we then have the following theorem, which completely solves the problem of the computation of the Möbius function of the quasi-consecutive pattern poset in the case of one occurrence.

Theorem 4.1 *Suppose that σ occurs exactly once in $\tau = a_1 - a_2 \cdots a_n$ as a quasi-consecutive pattern. Then $\mu(\sigma, \tau) = 0$, unless one of the following cases hold:*

- *$\sigma = \tau$, in which case $\mu(\sigma, \tau) = 1$;*

- *σ is covered by τ, in which case $\mu(\sigma, \tau) = -1$.*

- *$[\sigma, \tau]$ has rank 2 and σ involves a_1 but not a_2 and a_n, in which case $\mu(\sigma, \tau) = 1$ (the same holds when a_1 and a_2 are swapped).*

- *$[\sigma, \tau]$ has rank 2, σ involves a_n but not a_1 and a_2, and a_1 and a_2 are not consecutive integers, in which case $\mu(\sigma, \tau) = 1$.*

- *$[\sigma, \tau]$ has rank 3, σ involves a_{n-1} but not a_1, a_2 and a_n, σ occurs consecutively in τ and τ covers three elements in $[\sigma, \tau]$, in which case $\mu(\sigma, \tau) = -1$.*

5   Further work

The study of vincular pattern posets, which has been initiated in the present paper, is of course very far from being completed.

From the point of view of vincular pattern posets in general, the main open problem, already stated in Section 2, is perhaps that of characterizing those matrices A for which $\sigma \leq_A \tau$ if and only if $\sigma \in_A \tau$.

Concerning the main topic investigated here, namely the Möbius function of the quasi-consecutive pattern poset, a lot of work has still to be done. The case of one occurrence which we have completely solved in the previous section seems not to be really representative of the general case. For instance, the absolute value of μ can be different from 0 and 1 (a simple example is given by the interval $[12, 2413]$, whose Möbius function equals 2), and there is computational evidence [St] that $|\mu(\sigma, \tau)|$ is actually unbounded. Despite its closeness with the consecutive case, this fact shows that the quasi-consecutive case can sometimes be very similar to the classical (unrestricted) case. Another conjecture suggested by [St] is the following: if τ is the direct sum of some copies of σ, then $\mu(\sigma, \tau) = 1$.

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