Optimal behavior of the support of the solutions to a class of degenerate parabolic systems

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Optimal Behavior of the Support of the Solutions to a Class of Degenerate Parabolic Systems

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Abstract

In this paper we deal with a class of quasilinear parabolic systems. We study a Cauchy problem in $\mathbb{R}^N$ with an initial datum in L^1. Sharp L^∞ estimates are proved. In the degenerate case, assuming that the initial datum has compact support, we prove the optimal speed of propagation of the support.

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1 Introduction

In this paper we consider the following Cauchy problem

(1.1)
$$\begin{cases} \dfrac{\partial u_j}{\partial t} = \sum_{i=1}^{N} \dfrac{\partial}{\partial x_i} \left(|\mathbf{U}|^{m-1} |\nabla \mathbf{U}|^{p-2} \dfrac{\partial}{\partial x_i} u_j \right), \ j = 1,..,l, \ \text{in } S_T = \mathbf{R}^N \times (0,T), \\ u_j(x,0) = u_{0j}(x), \ j = 1,..,l, \end{cases}$$

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where here and hereafter bold letters are standing for vectors of length l:

$$\mathbf{U} = (u_1, .., u_l), \quad l \geq 1, \quad |\nabla \mathbf{U}| := \left(\sum_{i=1}^{l} |\nabla u_i|^2 \right)^{1/2}.$$

Such kind of systems are named doubly nonlinear. The doubly nonlinear equations were introduced by Lions ([18]) and Kalashnikov ([14]) several decades ago. These equations, from one side, are used to model several physical phenomena, on the other side, they are a natural bridge between two of the main important quasilinear equations, i.e. porous medium and p-Laplacean (for references on these protype equations, see for instance ([22])). For these reasons many Authors studied these equations. Among them, we quote Tsustumi ([21]), Ivanov ([12]), ([13]), Porzio and Vespri ([19]), Ishige ([11]) and Fornaro and Sosio ([10]). In these papers the existence, the Hölder regularity and intrinsic Harnack properties of these solutions are proved. For further results concerning the qualitative properties of solutions of degenerate parabolic equation we refer the reader to the survey ([14]) and to the exaustive monographs ([7]) and ([22]).

Only recently it was discovered the importance of doubly nonlinear systems in modelling some physical phenomena (see, for instance, ([24]), ([25]), ([26]), ([27]) and ([28])). In ([5]), ([6]) and ([15]) the Authors use doubly nonlinear systems to describe the evolution of a fluid in non-Newtonian filtration of the water flow through the porous media, to describe non-equilibrium thermodynamics and to model semiconductors. In ([27]) (see also ([25]) and ([26])) it was shown that the system of porous medium equations (i.e. $p = 2$ in (1.1)) comes from Bean's critical-state model in the superconductivity theory.

For systems (1.1) with $m = 1$ and with more general structure let us quote ([8]) for the $C^{1,\alpha}$ regularity of the solutions.

To our knowledge, regularity estimates for the solutions of the Cauchy problem (1.1) (and even for systems with more general structure) are proved only in ([16]). In this paper the Authors prove sharp $L^1 - L^\alpha$ estimates for the solutions of (1.1) provided $u_i(x,t) \geq 0$, $i = 1, ., l$. Since there is no the maximum principle for (1.1) it is not clear when the nonnegativity assumption in latter case can be realized.

In this paper we drop this not easily verifiable assumption and we start to study in a sistematic way the regularity properties of the weak solutions of (1.1). More precisely, we focus our attention on the behaviour of the support of the solution in the degenerate case i.e. when

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$$\text{(1.2)} \qquad \begin{cases} p > 1 \\ p + m > 3 \end{cases}$$

We prove that the finite propagation of the support occurs exactly as in the case of the corresponding degenerate equations. More precisely, by adapting techniques introduced by Andreucci and Tedeev (see ([2]), ([3]) and ([4])), we are able to give an optimal estimate of the speed of the propagation of the support thanks to sharp L^∞ estimates.

We stress that we are able to prove sharp L^∞ estimates under more general conditions on m and p, i.e when

$$\text{(1.3)} \qquad \begin{cases} p > 1 \\ N(p + m - 3) + p > 0 \end{cases}$$

We recall that when $m + p < 3$ we are in the singular case (see, for instance, ([23]) for such a classification in the case of equations). When $m + p < 3$ and $N(p + m - 3) + p > 0$ we are in the supercritical case, while the case $m + p > 2$ and $N(p + m - 3) + p \leq 0$ is called subcritical case (see, for instance, ([23]) and ([9]) for such a classification in the case of equations). The case $m + p = 3$ was first considered by Trudinger ([20]) and for this reason, such equations are often called Trudinger's equations.

Theorem 1.1 *Let* $\mathbf{U}$ *be a weak solution of (1.1) in* S_∞*. Assume that (1.3) holds and* $\mathbf{U}_0 \in L^1(\mathbf{R}^N)$*. Then for any* $t > 0$

$$\text{(1.4)} \qquad \|\mathbf{U}(t)\|_\infty \leq C(N, l, m) t^{-\frac{N}{\lambda}} \|\mathbf{U}_0\|_1^{\frac{p}{\lambda}}$$

where $\lambda = N(p + m - 3) + p$ *is the Barenblatt exponent.*

Note that this result is sharp. Some explicit counterexamples are known in literature (see, for instance, ([7]) and ([22])) where it is shown that estimate (1.4) fails whenever $\lambda \leq 0$.

As already written, the main result regards optimal estimates about the speed of propagation of the support of the solution:

Theorem 1.2 *Let* **U** *be a weak solution of (1.1) in* S_∞ *and supp* $\mathbf{U}_0 \subset B_{R_0}(0) = \{|x| < R_0\}$. *Assume that (1.2) holds and* $\mathbf{U}_0 \in L^1(\mathbf{R}^N)$. *Then for any* $t > 0$

(1.5)
$$
\begin{cases}
Z(t) = \inf \left\{ r > 0 : |\mathbf{U}(t)| = 0, \ x \in \mathbf{R}^N \backslash B_r(0) \right\} \\[2mm]
\leq 4R_0 + C(N, l, m) t^{\frac{1}{\lambda}} \|\mathbf{U}_0\|_1^{\frac{p+m-3}{\lambda}}.
\end{cases}
$$

Note that this result is sharp. in fact the support of the Barenblatt solutions exibits exactly this behaviour (see, for instance, ([7]) and ([22]).

Furher generalizations.
The results of the paper can be extended to a parabolic systems with measurable coefficients of the form:

(1.6) $\quad \dfrac{\partial u_j}{\partial t} = \displaystyle\sum_{k,\mu=1}^{N} \dfrac{\partial}{\partial x_k} \left(a_{k\mu}(x,t) \, |\mathbf{U}|^{m-1} \, |\nabla \mathbf{U}|^{p-2} \, \dfrac{\partial}{\partial x_\mu} u_j \right), \ \ j = 1, .., l$

where $a_{k\mu}(x,t)$ are measurable functions satisfying the conditions

(1.7) $\qquad\qquad\qquad \Lambda^{-1} \xi^2 \leq a_{k\mu}(x,t) \xi_k \xi_\mu \leq \Lambda \xi^2, \ \ \Lambda \geq 1$

2 Preliminary lemmata

The proof of Theorem 1.1 is based on some preliminary lemmata:

Lemma 2.1 *Let* $U \in W^{1,2}(\mathbb{R}^N)$ *than for any* $\varepsilon \geq 0$

(2.1) $\qquad\qquad\qquad \displaystyle\sum_{j=1}^{l} \sum_{k=1}^{N} u_{jx_k} \dfrac{\partial}{\partial x_k} \left(\dfrac{u_j}{\left(\varepsilon^2 + |\mathbf{u}|^2 \right)^{1/2}} \right) \geq 0$

Proof.
Let us develop the calculations:

$$
\dfrac{\partial}{\partial x_k} \left(\dfrac{u_j}{\left(\varepsilon^2 + |\mathbf{u}|^2 \right)^{1/2}} \right) = \dfrac{u_{jx_k} \left(\varepsilon^2 + |\mathbf{u}|^2 \right) - \sum_{s=1}^{l} u_s u_{sx_k} u_j}{\left(\varepsilon^2 + |\mathbf{u}|^2 \right)^{3/2}}
$$

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Therefore we have that

$$\sum_{j=1}^{l}\sum_{k=1}^{N} u_{jx_k} \frac{\partial}{\partial x_k}\left(\frac{u_j}{\left(\varepsilon^2+|\mathbf{u}|^2\right)^{1/2}}\right) = \frac{\sum_{j=1}^{l}\sum_{k=1}^{N} u_{jx_k}^2\left(\varepsilon^2+|\mathbf{u}|^2\right)-\frac{1}{4}\sum_{k=1}^{N}(\sum_{j=1}^{l}(u_j^2)_{x_k})^2}{\left(\varepsilon^2+|\mathbf{u}|^2\right)^{3/2}}$$

Now

$$\sum_{k=1}^{N}\sum_{j=1}^{l} u_{jx_k}^2\left(\varepsilon^2+|\mathbf{u}|^2\right) > \sum_{k=1}^{N}\sum_{j=1}^{l} u_{jx_k}^2\,|\mathbf{u}|^2$$

and estimate (2.1) comes from the fact that by Cauchy-Schwartz

$$\sum_{j=1}^{l} u_{jx_k}^2\,|\mathbf{u}|^2 \geq (\sum_{j=1}^{l} u_{jx_k} u_j)^2 \quad \square$$

The next Lemma says that the L^1 norm of the solution is bounded

Lemma 2.2 *Let* $\mathbf{U}(x,t)$ *be a solution of equation (1.1), then for a.e.* $t>0$

$$(2.2) \qquad \int_{\mathbf{R}^N} |\mathbf{U}(x,t)|\,dx \leq \int_{\mathbf{R}^N} |\mathbf{U}_0(x)|\,dx$$

Proof. Consider the family of solutions

$$(2.3) \quad \begin{cases} \dfrac{\partial u_i^{(n)}}{\partial t} = \text{div}\left(\left|\mathbf{U}^{(n)}\right|^{m-1}\left|\nabla\mathbf{U}^{(n)}\right|^{p-2}\nabla u_i^{(n)}\right) & \text{in } B_n(0)\times(0,\infty), \\ u_i^{(n)} = 0 \ \text{ on } \ \partial B_n(0)\times(0,\infty), \\ u_i^{(n)}(x,0) = u_{0i}^{(n)}(x), \end{cases}$$

$i=1,..,l$ with $\mathbf{U}_0^{(n)} = (u_{01}^{(n)}(x),..,u_{0l}^{(n)}(x))$ smooth enough and such that $\left\|\mathbf{U}_0^{(n)}-\mathbf{U}_0\right\|_{L^1(\mathbf{R}^N)} \to 0$ as $n\to\infty$. Then, by reasoning as in ([1]), it is possible to prove that, thanks to the monotonicity of the operator, the solution of (1.1), is the weak limit of $\mathbf{U}^{(n)}$. So it is enough to prove that for any $n\geq 1$ and $\varepsilon = n^{-\sigma}$ with $\sigma \geq N$

$$(2.4) \qquad \int_{B_n(0)}\left(\varepsilon^2+\left|\mathbf{U}^{(n)}(x,t)\right|^2\right)^{1/2}dx \leq \int_{B_n(0)}\left(\varepsilon^2+\left|\mathbf{U}_0^{(n)}(x)\right|^2\right)^{1/2}dx$$

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for any $t > 0$. Multiply both sides of (2.3) by

$$\frac{u_i^{(n)}}{\left(\varepsilon^2 + \left|\mathbf{U}^{(n)}\right|^2\right)^{1/2}},$$

integrate by parts over $B_n(0)$ and add up with respect the index i, to get

(2.5)
$$\begin{cases} \dfrac{d}{dt} \displaystyle\int_{B_n(0)} \left(\varepsilon^2 + \left|\mathbf{U}^{(n)}\right|^2\right)^{1/2} dx \\[3mm] = -\displaystyle\int_{B_n(0)} \sum_{j=1}^{l} \sum_{k=1}^{N} \left(\left|\mathbf{U}^{(n)}\right|^{m-1} \left|\nabla\mathbf{U}^{(n)}\right|^{p-2} \nabla u_{jx_k}^{(n)}\right) \nabla(u_{jx_k}^{(n)} \left(\varepsilon^2 + \left|\mathbf{U}^{(n)}\right|^2\right)^{-1/2}) dx. \end{cases}$$

and therefore by (2.1) we get

$$\frac{d}{dt} \int_{B_n(0)} \left(\varepsilon^2 + \left|\mathbf{U}^{(n)}\right|^2\right)^{1/2} dx \leq 0 \qquad \square$$

Remark.
Let us sketch the proof of Lemma 2.2 in the general case.
We claim that

$$a_{k\mu}(x,t) \frac{\partial}{\partial x_\mu} u_j \frac{\partial}{\partial x_k} \left(\frac{u_j}{\left(\varepsilon^2 + \left|\mathbf{U}\right|^2\right)^{1/2}}\right) \geq 0$$

Indeed the left hand side of this inequality is equal to

$$\frac{a_{k\mu}(x,t)\frac{\partial}{\partial x_\mu}u_j \left(\frac{\partial}{\partial x_k}u_j((\varepsilon^2 + |\mathbf{U}|^2) - u_j u_\alpha \frac{\partial u_\alpha}{\partial x_k}\right)}{\left(\varepsilon^2 + |\mathbf{U}|^2\right)^{3/2}}.$$

Next using the Cauchy inequality for the positive defined quadratic forms:

$$a_{k\mu}\zeta_k\eta_\mu \leq \left(a_{k\mu}\zeta_k\zeta_\mu\right)^{1/2} \left(a_{k\mu}\eta_k\eta_\mu\right)^{1/2}$$

and choosing

$$\zeta_k = \frac{u_j}{|\mathbf{U}|} u_{jx_k}, \ \eta_\mu = \frac{u_\alpha}{|\mathbf{U}|} u_{\alpha x_\mu}$$

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we get the result □

Define $C(h_1, h_2) = h_1/(h_1 - h_2)$.

Lemma 2.3 . *Assume that assumptions of Theorem 1.1 hold. Define $D_j = \mathbf{R}^N \times (t_j, t)$, $j = 1, 2$, $0 < t_2 < t_1 < t$. Then for all $h_1 > h_2 > 0$ we have for any $s > 0$ so large such that $s - 2 + m - p > 0$*

(2.6)
$$
\begin{cases}
\displaystyle \sup_{t_1 < \tau < t} \int_{\mathbf{R}^N} \left(|\mathbf{U}|^2 - h_1\right)_+^{(s+1)/2} dx + \iint_{D_1} \left(|\mathbf{U}|^2 - h_1\right)_+^{\frac{s+m-2-p}{2}} \left|\nabla \left(|\mathbf{U}|^2 - h_1\right)_+\right|^p dx d\tau \\
\displaystyle \leq \gamma C(h_1, h_2)^{p+m-1}(t_1 - t_2)^{-1} \iint_{D_2} \left(|\mathbf{U}|^2 - h_2\right)_+^{(s+1)/2} dx d\tau.
\end{cases}
$$

Proof. For the sake of simplicity, assume that $\mathbf{U}$ is the strong solution of (1.1) so it is possible to work pointwise. Assume for the moment $s \geq 1$. Multiply both sides of (1.1) by $u_i \left(|\mathbf{U}|^2 - h_2\right)_+^{\frac{s-1}{2}} \zeta^p(\tau)$, $i = 1, .., l$, where $\zeta(\tau)$ is a smooth cutoff function of $(0, t)$, such that $\zeta = 1$ for $t_1 \leq \tau \leq t$, $\zeta = 0$ out of $0 < \tau \leq t_2$, $|\zeta_\tau| \leq c(t_1 - t_2)^{-1}$ integrate by parts and sum with the respect the index i to get

(2.7)
$$
\begin{cases}
\displaystyle \frac{1}{s+1} \int_{\mathbf{R}^N} \zeta^p \left(|\mathbf{U}|^2 - h_2\right)_+^{\frac{s+1}{2}} dx \\
\displaystyle + \sum_{i=1}^{l} \iint_{D_2} \zeta^p \left(|\mathbf{U}|^{m-1} |\nabla \mathbf{U}|^{p-2} \nabla u_i\right) \nabla \left(u_i \left(|\mathbf{U}|^2 - h_2\right)_+^{\frac{s-1}{2}}\right) dx d\tau \\
\displaystyle = -\frac{p}{s+1} \iint_{D_2} \zeta^{p-1} \zeta_\tau \left(|\mathbf{U}|^2 - h_2\right)_+^{\frac{s+1}{2}} dx d\tau.
\end{cases}
$$

We have

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$$\begin{cases} \sum_{i=1}^{l} \left(|\mathbf{U}|^{m-1} |\nabla \mathbf{U}|^{p-2} \nabla u_i \right) \nabla \left(u_i \left(|\mathbf{U}|^2 - h_2 \right)_+^{\frac{s-1}{2}} \right) \\ = |\mathbf{U}|^{m-1} |\nabla \mathbf{U}|^p \left(|\mathbf{U}|^2 - h_2 \right)_+^{\frac{s-1}{2}} \\ + \frac{s-1}{4} |\mathbf{U}|^{m-1} |\nabla \mathbf{U}|^{p-2} \left(\nabla |\mathbf{U}|^2 \right)^2 \left(|\mathbf{U}|^2 - h_2 \right)_+^{\frac{s-1}{2}-1} \\ \geq |\mathbf{U}|^{m-1} |\nabla \mathbf{U}|^p \left(|\mathbf{U}|^2 - h_2 \right)_+^{\frac{s-1}{2}}. \end{cases}$$

Note that when $|\mathbf{U}|^2 > h_1$ we have

(2.8)

$$|\mathbf{U}|^2 = |\mathbf{U}|^2 - h_2 + h_2 \frac{h_1 - h_2}{h_1 - h_2} \leq (|\mathbf{U}|^2 - h_2) + h_2 \frac{|\mathbf{U}|^2 - h_2}{h_1 - h_2} = C(h_1, h_2) \left(|\mathbf{U}|^2 - h_2 \right).$$

Moreover

(2.9) $$\left| \nabla(|\mathbf{U}|^2 - h_1) \right|^p = |\nabla(|\mathbf{U}|)|^p \, 2^p \, |\mathbf{U}|^p.$$

Noting that

$$|\mathbf{U}|^{m-1} |\nabla \mathbf{U}|^p \left(|\mathbf{U}|^2 - h_2 \right)_+^{\frac{s-1}{2}} \geq |\mathbf{U}|^{m-1} |\nabla |\mathbf{U}||^p \left(|\mathbf{U}|^2 - h_2 \right)_+^{\frac{s-1}{2}}$$

by (2.9), we have

$$|\mathbf{U}|^{m-1} |\nabla \mathbf{U}|^p \left(|\mathbf{U}|^2 - h_2 \right)_+^{\frac{s-1}{2}} \geq 2^{-p} \left(|\mathbf{U}|^2 - h_2 \right)_+^{\frac{s-1}{2}} |\mathbf{U}|^{m-1-p} \left| \nabla(|\mathbf{U}|^2 - h_1) \right|^p$$

Therefore, if $m - 1 - p < 0$ by (2.8)

(2.10) $$\begin{cases} |\mathbf{U}|^{m-1} |\nabla \mathbf{U}|^p \left(|\mathbf{U}|^2 - h_2 \right)_+^{\frac{s-1}{2}} \\ \geq 2^{-p} C(h_1, h_2)^{-\frac{m-1-p}{2}} \left(|\mathbf{U}|^2 - h_1 \right)_+^{\frac{s-2+m-p}{2}} \left| \nabla(|\mathbf{U}|^2 - h_1) \right|^p \end{cases}$$

If $m - 1 - p \geq 0$ we have that $|\mathbf{U}|^{m-1-p} \geq |(\mathbf{U} - h_1)_+|^{m-1-p}$ and again we can prove (2.10).
Combining (2.7), (2.10) and by the definition of the cut-off ζ, we get (2.6).
If $-1 < s < 1$, for each $n \in N$ consider the test function

$$\psi_n = u_i [(|\mathbf{U}|^2 - h_2)_+^{\frac{s-1}{2}} \wedge n] \zeta^p(\tau).$$

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Letting $n \to \infty$ one can get (2.6) $\square$

Remark.
Note that the proof of Lemma 2.3 works also in the more general case of equation (1.6) $\square$

In the sequel we need an interpolation inequality and the celebrated Gagliardo-Nirenberg inequality (for both these results, see, for instance, Chapter 0 of [9]).

Lemma 2.4 *Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of equi-bounded positive numbers satisfying the recursive inequalities*

$$Y_n \leq Cb^n Y_{n+1}^{1-\alpha}$$

where $C, b > 1$ and $\alpha \in (0,1)$ are given constants. Then

$$Y_0 \leq (2Cb^{\frac{1-\alpha}{\alpha}})^{\frac{1}{\alpha}}.$$

Lemma 2.5 *(see also [17])*
Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of equi-bounded positive numbers satisfying the recursive inequalities

$$Y_n \leq Cb^n Y_{n+1}^{1+\alpha}$$

where $C, b > 1$ and $\alpha > 0$) are given constants. Then, if

$$Y_0 \leq C^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}}$$

then $Y_n \to 0$ when $n \to +\infty$.

Lemma 2.6 *Let $u \in W^{1,p}(\mathbb{R}^N)$ with $p > 1$. Let $0 < \mu < q < \frac{pN}{N-p}$ if $p < N$ and $0 < \mu < q$ otherwise*

$$(2.11) \qquad \|u\|_q \leq C\|Du\|_p^{\alpha}\|u\|_{\mu}^{1-\alpha}$$

where C is a constant depending only by N, p, q, μ and

$$\alpha = \Big(\frac{1}{\mu} - \frac{1}{q}\Big)\Big(\frac{1}{N} - \frac{1}{p} + \frac{1}{\mu}\Big)^{-1}.$$

Note that in general the Gagliardo-Nirenberg is stated with the assumption that $1 < \mu < q$. But the proof works also in the weaker assumption $0 < \mu < q$, even if, in general, the space $L^{\mu}(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$ are no longer Banach spaces.

3 Proof of Theorem 1.1

The idea is to prove that there exists a $p > 1$ such that $\left(|\mathbf{U}|^2 - a\right)_+^{\frac{1}{2}} \in L^p(\mathbb{R}^N)$ where a is a suitable positive constant. Then, by applying a De Giorgi iterative scheme, starting from this estimate we are able to prove that $|\mathbf{U}|$ is bounded.

Consider two positive times $0 < \tau_2 < \tau_1$ and two positive constants $0 < a_2 < a_1$. Let $\vartheta_n = \tau_2 + (\tau_1 - \tau_2)2^{-n}$, $k_n = a_2 + (a_1 - a_2)2^{-n}$, $n \geq 0$. Apply Lemma 2.3 with $t_1 = \vartheta_n$, $t_2 = \vartheta_{n+1}$; $h_1 = k_n$, $h_2 = k_{n+1}$. Choose $s > 0$ such that $s + m - p - 2 > 0$.
Note that

$$
\begin{cases}
\left| \nabla \left(|\mathbf{U}|^2 - k_{2n+1}\right)_+^{\frac{s+m+p-2}{2p}} \right|^p \\
= \left(\dfrac{s+m+p-2}{p}\right)^p \left(|\mathbf{U}|^2 - k_{2n+1}\right)_+^{\frac{s+m-2-p}{2}} \left| \nabla \left(|\mathbf{U}|^2 - k_{2n+1}\right)_+ \right|^p.
\end{cases}
$$

Then we have

(3.1)
$$
\begin{cases}
\displaystyle \sup_{\vartheta_{2n} < \tau < t} \int_{\mathbb{R}^N} \left(|\mathbf{U}|^2 - k_{2n+1}\right)_+^{\frac{s+1}{2}} dx + \iint_{S_{2n+1}} \left| \nabla \left(|\mathbf{U}|^2 - k_{2n+1}\right)_+^{\frac{s+m+p-2}{2p}} \right|^p dx d\tau \\
\leq c 2^n (\tau_1 - \tau_2)^{-1} \iint_{S_{2n+2}} \left(|\mathbf{U}|^2 - k_{2n+2}\right)_+^{\frac{s+1}{2}} dx d\tau
\end{cases}
$$

where $S_n = \mathbb{R}^N \times (\vartheta_n, t)$. Let z_n be a smooth cut-off function such that $0 \leq z_n \leq 1$, $z_n = 1$ in S_{2n}, $z_n = 0$ outside of S_{2n+1}, $0 \leq z_{n\tau} \leq 2^{2n+1}(\tau_1 - \tau_2)^{-1}$. We have

$$
\iint_{S_{2n+1}} \left| \nabla \left(|\mathbf{U}|^2 - k_{2n}\right)_+^{\frac{s+m+p-2}{2p}} z_n \right|^p dx d\tau \leq \iint_{S_{2n+1}} \left| \nabla \left(|\mathbf{U}|^2 - k_{2n+1}\right)_+^{\frac{s+m+p-2}{2p}} \right|^p dx d\tau.
$$

Thus defining

$$
v_n = \left(|\mathbf{U}|^2 - k_{2n}\right)_+^{\frac{s+m+p-2}{2p}} z_n
$$

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we get from (3.1) with $q = \dfrac{p(s+1)}{s+m+p-2}$

$$(3.2) \quad \sup_{\vartheta_{2n} < \tau < t} \int_{\mathbf{R}^N} v_n^q dx + \iint_{S_{2n+1}} |\nabla v_n|^p \, dx d\tau \leq c 2^{2n} (\tau_1 - \tau_2)^{-1} \iint_{S_{2n+2}} v_{n+1}^q dx d\tau.$$

Note that if $m+p > 3$ we have $q < p$ and therefore we are in the degenerate case. Moreover if the Barenblatt exponent $\lambda = N(m+p-3)+p$ is positive, we can use the Gagliardo-Nirenberg inequality (see Lemma 2.6). Therefore

$$(3.3) \quad \int_{\mathbf{R}^N} v_{n+1}^q dx \leq c \left(\int_{\mathbf{R}^N} |\nabla v_{n+1}|^p \, dx \right)^{\alpha q/p} \left(\int_{\mathbf{R}^N} v_{n+1}^\mu dx \right)^{(1-\alpha)q/\mu}.$$

Where $\mu = \dfrac{p}{s+m-2+p}$.

Define

$$(3.4) \quad\quad\quad A := \frac{\alpha q}{p}, \quad\quad B := \frac{(1-\alpha)q}{\mu}$$

Note that $A < 1$ if $\lambda = N(m+p-3)+p > 0$.
Integrating in time and applying Young's inequalities to (3.3), we obtain

$$\begin{cases} \displaystyle\iint_{S_{2n+2}} v_{n+1}^q dx d\tau \\[3em] \leq \left(\displaystyle\iint_{S_{2n+2}} |\nabla v_{n+1}|^p \, dx d\tau \right)^{A} \left(\displaystyle\int_{\vartheta_{2n+2}}^t \left(\displaystyle\int_{\mathbf{R}^N} v_{n+1}^\mu dx \right)^{\frac{B}{1-A}} d\tau \right)^{1-A} \end{cases}$$

Therefore we get

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(3.5)
$$
\begin{cases}
\displaystyle\iint\limits_{S_{2n+2}} v_{n+1}^q \, dx \, d\tau \\[2em]
\displaystyle \leq c(t - \vartheta_{2n+2})^{1-A} \left(\iint\limits_{S_{2n+2}} |\nabla v_{n+1}|^p \, dx \, d\tau\right)^A \sup_{\vartheta_{2n+2} < \tau < t} \left(\int_{\mathbf{R}^N} v_{n+1}^\mu(\tau) \, dx\right)^B
\end{cases}
$$

Note that by Lemma 2.2

$$
\sup_{\vartheta_{2n+2} < \tau < t} \int_{\mathbf{R}^N} v_{n+1}^\mu(\tau) \, dx \leq \int_{\mathbf{R}^N} |\mathbf{U}_0| \, dx
$$

so the quantity on the right in (3.5) is finite.
Combining (3.2) and (3.5), we get

(3.6)
$$
\begin{cases}
\displaystyle J_n := \sup_{\vartheta_{2n} < \tau < t} \int_{\mathbf{R}^N} v_n^q \, dx + \iint\limits_{S_{2n+1}} |\nabla v_n|^p \, dx \, d\tau \\[2em]
\displaystyle \leq b_1^n (\tau_1 - \tau_2)^{-1} (t - \vartheta_{2n+2})^{1-A} \left(\sup_{\vartheta_{2n+2} < \tau < t} \int_{\mathbf{R}^N} v_{n+1}^\mu(\tau) \, dx\right)^B J_{n+1}^A
\end{cases}
$$

where b_1 a constant greater than 1.
Thus by the iterative Lemma 2.4 we get

(3.7)
$$
\begin{cases}
\displaystyle \sup_{\tau_1 < \tau < t} \int_{\mathbf{R}^N} \left(|\mathbf{U}|^2 - a_2\right)_+^{\frac{s+1}{2}} dx \\[2em]
\displaystyle \leq c(\tau_1 - \tau_2)^{-\frac{1}{1-A}} (t - \tau_2) \left(\sup_{\tau_2 < \tau < t} \int_{\mathbf{R}^N} \left(|\mathbf{U}|^2 - a_1\right)_+^{\frac{1}{2}} dx\right)^{\frac{B}{1-A}}
\end{cases}
$$

and this implies the higher integrability of $\left(|\mathbf{U}|^2 - a_2\right)_+^{\frac{1}{2}}$.

To prove the boundness of $|\mathbf{U}|$ we have to apply a DeGiorgi iterative scheme.
Let $t_j = t/2(1 - 2^{-j})$, $k_j = k(1 - 2^{-j-1})$, and $\bar{k}_j = (k_j + k_{j+1})/2$.

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Then in (3.7) plug $\tau_1 = t_{j+1}$, $\tau_2 = t_j$, $a_2 = \overline{k}_j$, $a_1 = k_j$, to get

$$
(3.8) \qquad
\begin{cases}
\displaystyle \sup_{t_{j+1}<\tau<t} \int_{\mathbf{R}^N} \left(|\mathbf{U}|^2 - \overline{k}_j\right)_+^{\frac{s+1}{2}} dx \\[2em]
\displaystyle \leq cb_2^j t^{-\frac{A}{1-A}} \left(\sup_{t_j<\tau<t} \int_{\mathbf{R}^N} \left(|\mathbf{U}|^2 - k_j\right)_+^{1/2} dx \right)^{1+\frac{B+A-1}{1-A}}
\end{cases} .
$$

Note that, as $s > 0$ and $\lambda = N(m+p-3)+p > 0$, we have $B + A > 1$.
Denote

$$
M_{j+1} = \sup_{t_{j+1}<\tau<t} \int_{\mathbf{R}^N} \left(|\mathbf{U}|^2 - k_{j+1}\right)_+^{1/2} dx.
$$

Since

$$
\int_{\mathbf{R}^N} \left(|\mathbf{U}|^2 - \overline{k}_j\right)_+^{\frac{s+1}{2}} dx \geq (k_{j+1} - \overline{k}_j)^s \int_{\mathbf{R}^N} \left(|\mathbf{U}|^2 - k_{j+1}\right)_+^{1/2} dx
$$

from (3.8) one gets

$$
M_{j+1} \leq cb_3^j t^{-\frac{A}{1-A}} k^{-s} M_j^{1+\frac{A+B-1}{1-A}}.
$$

Noting that $\dfrac{Ns}{\lambda} = \dfrac{A}{1-A}$ and $\dfrac{ps}{\lambda} = \dfrac{A+B-1}{1-A}$, we have that

$$
M_{j+1} \leq cb_3^j t^{-\frac{Ns}{\lambda}} k^{-s} M_j^{1+\frac{ps}{\lambda}}.
$$

It follows from Lemma 2.5 that $M_j \to 0$ as $j \to \infty$ if

$$
(3.9) \qquad\qquad\qquad k = Ct^{-\frac{N}{\lambda}} \widetilde{M}_0^{\frac{p}{\lambda}}
$$

where $\widetilde{M}_0(t) = \displaystyle\sup_{t/4<\tau<t} \int_{\mathbf{R}^N} |\mathbf{U}(t,x)| \, dx$.

Therefore, by Lemma 2.2, (3.9) holds true if

$$
k = Ct^{-\frac{N}{\lambda}} \left(\int_{\mathbf{R}^N} |\mathbf{U_0}(x)| \, dx \right)^{\frac{p}{\lambda}}
$$

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and this implies (1.4) □

Remark.

Note that the proof of Theorem 1.1 works exactly in the more general case of equation (1.6) □

4 Proof of Theorem 1.2.

At the beginning the proof is similar to the one of Lemma 2.3.

Consider the sequence $r_n = 2r(1-2^{-n-1})$, $n = 0, 1, .., r > 2R_0$, $\bar{r}_n = \dfrac{(r_n + r_{n+1})}{2}$, $\Omega_n = R^N \backslash B_{r_n}(0)$, $\overline{\Omega}_n = R^N \backslash B_{\bar{r}_n}(0)$. Let $\eta_n(x)$ be a sequence of cutoff functions satisfying:

$\eta_n(x) = 0$ for $x \in B_{r_n}(0)$, $\eta_n(x) = 1$ for $x \in \overline{\Omega}_n$, $|\nabla \eta_n| \le c 2^n r^{-1}$.

Let θ a positive constant that will be fixed later. Then multiplying both sides of (1.1) by $\eta_n^p |\mathbf{U}|^{\theta-1} u_j$, $i = 1, .., l$ and assume $\theta \ge 1$. By integrating over S_t, we obtain

(4.1)
$$
\begin{cases}
\dfrac{1}{\theta+1} \int\limits_{\mathbf{R}^N} \eta_n^p |\mathbf{U}|^{\theta+1} \, dx + \iint\limits_{S_t} \eta_n^p |\mathbf{U}|^{m-1} |\nabla \mathbf{U}|^{p-2} \nabla u_j \nabla \left(|\mathbf{U}|^{\theta-1} u_j \right) dx d\tau \\
= -p \iint\limits_{S_t} \eta_n^{p-1} |\mathbf{U}|^{m-1} |\nabla \mathbf{U}|^{p-2} \nabla u_j \nabla \eta_n |\mathbf{U}|^{\theta-1} u_j dx d\tau.
\end{cases}
$$

Noting that

$$
\begin{cases}
|\nabla \mathbf{U}|^{p-2} \nabla u_j \nabla \left(|\mathbf{U}|^{\theta-1} u_j \right) = |\mathbf{U}|^{\theta-1} |\nabla \mathbf{U}|^{p-2} \left(|\nabla \mathbf{U}|^2 + (\theta-1) |\nabla |\mathbf{U}||^2 \right) \\
\ge \theta |\mathbf{U}|^{\theta-1} |\nabla \mathbf{U}|^p.
\end{cases}
$$

and that, by the Cauchy inequality,

$$
\begin{cases}
\eta_n^{p-1} |\mathbf{U}|^{m-1} |\nabla \mathbf{U}|^{p-2} \nabla u_j \nabla \eta_n |\mathbf{U}|^{\theta-1} u_j \\
\le \frac{\theta}{2p} \eta_n^p |\mathbf{U}|^{m+\theta-2} |\nabla \mathbf{U}|^p + \gamma |\nabla \eta_n|^p |\mathbf{U}|^{p+m+\theta-2}
\end{cases}
$$

estimate (4.1) yields

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$$
(4.2) \quad
\begin{cases}
\sup_{0<\tau<t} \int_{\overline{\Omega}_n} |\mathbf{U}|^{\theta+1} \, dx + \int_0^t \int_{\overline{\Omega}_n} \eta_n^p \, |\mathbf{U}|^{m+\theta-2} \, |\nabla \mathbf{U}|^p \, dx d\tau \\[3mm]
\leq c \dfrac{2^{pn}}{r^p} \int_0^t \int_{\Omega_n \backslash \overline{\Omega}_n} |\mathbf{U}|^{p+m+\theta-2} \, dx d\tau.
\end{cases}
$$

If $0 < \theta < 1$ for each $m \in \mathbb{N}$ consider $\eta_n^p (|\mathbf{U}|^{\theta-1} \wedge m) u_j$, repeat the previous argument and pass to the limit when $m \to \infty$ to get again (4.2).

Let $\zeta_n(x)$ be such that $\zeta_n = 0$ for $x \in \overline{\Omega}_n$ and $\zeta_n = 1$ for $x \in \Omega_{n+1}$. Let $v_n = |\mathbf{U}|^{(p+m+\theta-2)/p} \zeta_n^s$. Then, from (4.2), we get

$$
(4.3) \quad
\begin{cases}
\sup_{0<\tau<t} \int_{\mathbf{R}^N} v_n^q dx + \int_0^t \int_{\mathbf{R}^N} |\nabla v_n|^p \, dx d\tau \\[3mm]
\leq c \dfrac{2^{2n}}{r^p} \int_0^t \int_{\mathbf{R}^N} v_n^p dx d\tau
\end{cases}
$$

with $q = \dfrac{p(1+\theta)}{p+m+\theta-2}$. Note that if $m + p > 3$ then $q < p$.

Using the Nirenberg-Gagliardo inequality

$$
\int_{\mathbf{R}^N} v_{n-1}^p dx \leq c \left(\int_{\mathbf{R}^N} |\nabla v_{n-1}|^p \, dx \right)^{\alpha} \left(\int_{\mathbf{R}^N} v_{n-1}^q dx \right)^{p(1-\alpha)/q}
$$

where

$$
(4.4) \quad \alpha = \frac{N(p+m-3)}{N(p+m-3)+p(1+\theta)}
$$

is given by Lemma 2.6.
By integrating the prevous inequality in time, we obtain

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$$(4.5) \quad \begin{cases} \displaystyle\int\limits_0^t \int\limits_{\mathbf{R}^N} v_{n-1}^p \, dx d\tau \leq ct^{1-\alpha} \left(\int\limits_0^t \int\limits_{\mathbf{R}^N} |\nabla v_{n-1}|^p \, dx d\tau \right)^\alpha \\ \times \left(\displaystyle\sup_{0<\tau<t} \int\limits_{\mathbf{R}^N} v_{n-1}^q(\tau, x) dx \right)^{p(1-\alpha)/q} \end{cases}$$

Therefore, combining (4.3) and (4.5), we get

$$\begin{cases} I_n = \displaystyle\sup_{0<\tau<t} \int\limits_{\mathbf{R}^N} v_n^q \, dx + \int\limits_0^t \int\limits_{\mathbf{R}^N} |\nabla v_{n-1}|^p \, dx d\tau \\ \leq \gamma \dfrac{b^n}{r^p} t^{1-\alpha} I_{n-1}^{1+(1-\alpha)(\frac{p}{q}-1)} \end{cases}$$

Then, by iterative Lemma 2.5, we conclude that $I_n \to 0$ as $n \to \infty$ provided

$$(4.6) \qquad \frac{t^{1-\alpha}}{r^p} I_0^{(1-\alpha)(\frac{p}{q}-1)} \leq \varepsilon$$

where $\varepsilon = \varepsilon(m, N, \theta)$ is small enough. Notice that by (4.3) we have

$$I_0 \leq \frac{c}{r^p} \int\limits_0^t \int\limits_{\mathbf{R}^N} |\mathbf{U}|^{p+m+\theta-2} \, dx d\tau \leq \frac{c}{r^p} \int\limits_0^t (\| \, |\mathbf{U}(\tau)| \|_\infty^{p+m+\theta-3} \int\limits_{\mathbf{R}^N} |\mathbf{U}(\tau, x)| \, dx) d\tau$$

By (2.2)

$$(4.7) \qquad I_0 \leq \frac{c}{r^p} \| \, |\mathbf{U}_0| \|_1 \int\limits_0^t \| \, |\mathbf{U}(\tau)| \|_\infty^{p+m+\theta-3} \, d\tau.$$

Choose θ so small that

$$0 < 1 - \frac{N(p+m+\theta-3)}{\lambda}.$$

By (1.4) and (4.7)

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(4.8)

$$I_0 \leq \frac{\gamma}{r^p} \parallel |\mathbf{U}_0| \parallel_1^{1+\frac{p(p+m+\theta-3)}{\lambda}} \int\limits_0^t \tau^{-\frac{N(p+m+\theta-3)}{\lambda}} d\tau = \frac{\gamma t^{1-\frac{N(p+m+\theta-3)}{\lambda}}}{r^p} \parallel |\mathbf{U}_0| \parallel_1^{1+\frac{p(p+m+\theta-3)}{\lambda}}.$$

Plugging now (4.8) in (4.6) and recalling (4.4), we deduce that $|\mathbf{U}| = 0$ out of $|x| \leq 4R_0 + c \parallel |\mathbf{U}_0| \parallel_1^{\frac{p+m-3}{\lambda}} t^{\frac{1}{\lambda}}$. $\square$

Remark.
Note that the proof of Theorem 1.2 works exactly in the more general case of equation (1.6) . $\square$

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