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On Differentiability of Volume Time Functions

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Abstract. We show differentiability of a class of Geroch's volume functions on globally hyperbolic manifolds. Furthermore, we prove that every volume function satisfies a local anti-Lipschitz condition over causal curves, and that locally Lipschitz time functions which are locally anti-Lipschitz can be uniformly approximated by smooth time functions with timelike gradient. Finally, we prove that in stably causal space-times Hawking's time function can be uniformly approximated by smooth time functions with timelike gradient.

1. Introduction

In his classical work on domains of dependence and global hyperbolicity [14] Geroch showed how to construct a time function, namely a continuous function increasing over every future-directed causal curve, by considering a weighted volume of the chronological past of the point. This strategy was extended by Hawking [17,18] who proved that in a stably causal space-time it is possible to obtain a time function through suitable averages of Geroch's volume functions (see [8,13,25,29] and references therein for alternative constructions). Nevertheless, even in the globally hyperbolic case the differentiability properties of the resulting functions do not seem to have been properly understood so far.[1] The object of this note is to establish differentiability of a large class of volume functions under the hypothesis of global hyperbolicity, as well as smoothability of a class of time functions for stably causal space-times. Indeed, assuming global hyperbolicity, at the end of Sect. 3 below we prove:

Theorem 1.1. *Let $(\mathscr{M}, g)$ be a globally hyperbolic space-time with a $C^{2,1}$ metric. There exists a class of smooth functions $\varphi > 0$ such that the functions*

$$\tau^{\pm}_{\varphi}(p) = \int_{J^{\pm}(p)} \varphi \, \mathrm{d}\mu_g, \tag{1.1}$$

[1] The reader is referred to [8,28] for a review of the history of the problem.

where $d\mu_g$ is the volume element of g, are continuously differentiable with timelike gradient.

Following Geroch, we can now define $\tau_\varphi = \ln(\tau_\varphi^-/\tau_\varphi^+)$ so that if γ is an inextendible causal curve then $\tau_\varphi \circ \gamma$ is onto $\mathbb{R}$. As a consequence, the differentiability of τ_φ implies that the level sets of τ_φ are Cauchy spacelike hypersurfaces (thus not just acausal and Lipschitz).

In view of the analysis in [20] it is conceivable that the functions $\tau_\varphi^\pm$ are $C^{1,1}$, but we have not investigated the issue any further, as we will smooth out these functions in any case, see Corollary 5.6 below. The smoothing procedure will establish the existence of smooth Cauchy time functions in globally hyperbolic space-times. This result, already obtained in [6,7,13] by different means, plays a key role in the theory as it implies that Geroch's topological splitting [18] can actually be chosen smooth.

Some comments on the proof might be in order. We write a light-cone integral formula for a candidate derivative of $\tau_\varphi^\pm$. The integrand involves Jacobi fields which might be blowing up as one approaches the end of the interval of existence of the geodesic generators of

$$E^\pm(p) := J^\pm(p)\backslash I^\pm(p).$$

So the weighting function φ has to compensate for this, which provides one of the constraints on the set of admissible functions φ. In particular, we are going to introduce an auxiliary complete Riemannian metric h in order to control and obtain a sufficiently fast fall-off of φ at infinity.

An interesting feature of our candidate formula for the derivative of $\tau_\varphi^\pm$ is that it involves an integral on just $E^\pm(p)$, and not on the whole light cone issued from p. As a consequence, most of the pathological behavior connected with non-differentiability of the exponential map after conjugate points is avoided.

Now, the domain of integration that interests us is generated by lightlike geodesics that may be either complete or incomplete. The distinction between completeness and incompleteness is, however, rather unimportant since, without loss of generality, we may conformally rescale g so as to make all the null geodesics complete [2]. Next, there might exist generators that do not meet the cut-locus and which span an area region on $E^\pm(p)$ that does *not* vary continuously with p. (Example: Let (N,g) be any bounded globally hyperbolic subset of two-dimensional Minkowski space-time, with the weighting function φ in (1.1) equal to one. Let $q \in N$ and set $M = N\backslash J^-(q)$, with the induced metric. Then τ_φ^- is not differentiable at the boundary of the future of q; see Fig. 1.) It turns out that the discontinuous behavior illustrated by the example happens "close to infinity" in the complete auxiliary metric, where a fast fall-off of φ amends the problem.

In Sect. 4 we show that a local application of our integral formula gives a simple proof of the anti-Lipschitz character of $\tau_\varphi^\pm$ along causal curves. We also show that the anti-Lipschitz condition with respect to a space-time metric with wider light cones allows us to smooth the time function. This result is

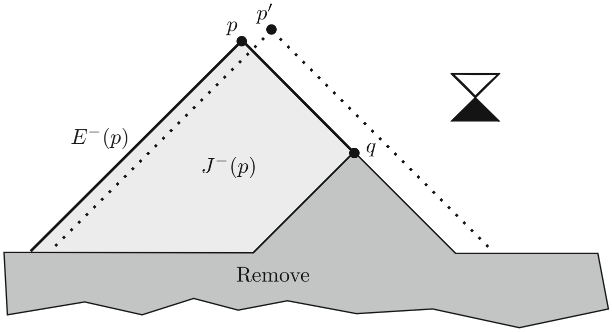

FIGURE 1. The volume of past light-cones is Lipschitz continuous but not differentiable at $\dot{J}^+(q)$

used to prove the smoothability of Hawking's time function in stably causal space-times, and to prove the existence of smooth Cauchy time functions in globally hyperbolic space-times, by taking advantage of the stability of global hyperbolicity. We also prove the equivalence between stable causality and the existence of a time function by taking advantage of the equivalence between the former property and K-causality.

Finally, in the last section we prove that we can dispense with the result on the stability of global hyperbolicity, and prove directly the smoothability of Geroch's Cauchy time functions, by showing that every Lipschitz and anti-Lipschitz time function is in fact anti-Lipschitz with respect to a space-time metric with wider light cones.

2. The Null Cut-Locus

To avoid ambiguities, we start by noting that our signature is $(-, + \cdots +)$, and that we use a convention in which the zero vector is *not* a null vector. Space-times of any dimension $n+1$, $n \geq 1$, are allowed, though it must be said that the case $n = 1$ is rather simpler than the remaining dimensions, as there are then no null conjugate points.

The proof of Theorem 1.1 will require some understanding of the null cut-locus. For this, we start by recalling some definitions and results from [3], see especially Sections 9.2 and 10.3 there.

Our space-time metric g will be $C^{2,1}$ throughout the paper. This condition assures the existence of convex neighborhoods, continuous differentiability of the exponential map and Lipschitzness of the Riemann tensor. These conditions can possibly be weakened.

The *past Lorentzian distance function*

$$d_L^- : \mathscr{M} \times \mathscr{M} \to [0, \infty]$$

is defined as

$$d_L^-(p,q) = \begin{cases} 0, & q \notin J^-(p); \\ \sup \int_\gamma \sqrt{-g(\dot\gamma,\dot\gamma)}, & \text{otherwise}, \end{cases} \tag{2.1}$$

where the sup is taken over all past-directed causal curves from p to q. In general smooth space-times the function d_L^- is lower semi-continuous [3], while for globally hyperbolic (smooth) space-times it is finite and continuous. Moreover, we observe that the arguments in [3,10] can be used to prove these results for C^2 metrics. (For more on the continuity properties of the Lorentzian distance function under low causality conditions see [23].)

Throughout this paper we choose once and for all a smooth complete Riemannian metric h on $\mathscr{M}$. We denote by $N_h^-\mathscr{M}$ the bundle of past-directed h-unit g-null vectors. A curve will be said to be *h-parametrized* if it is parametrized with arc-length measured with the metric h.

Sometimes, for simplicity, we shall speak of *geodesics* though strictly speaking we should speak of *pregeodesics*, namely when the curve is a geodesic up to the parametrization. We will say that γ is a *half-geodesic* if γ satisfies the geodesic equation and if $\gamma\colon [0,a) \to \mathscr{M}$ is maximally extended in the direction of increasing parameter. If γ is parametrized by h-arc length with respect to a complete Riemannian metric, then $a = \infty$ (see, e.g. [3,10,22]).

For any past-directed null half-geodesic $\gamma\colon [0,\infty) \to \mathscr{M}$ parametrized by h-arc length we set

$$t_-(\gamma) = \sup\left\{t \in \mathbb{R}^+ : d_L^-(\gamma(0),\gamma(t)) = 0\right\} \in [0,\infty]. \tag{2.2}$$

The points of $E^-(p)$ which are sufficiently close to p are connected to p by achronal geodesics starting at p. We now define a subset $\mathring{E}^-(p)$ through a union of lightlike geodesic segments as follows:

$$\mathring{E}^-(p) = \left\{\gamma(s) \mid \gamma(0) = p,\ s \in (0, t_-(\gamma))\right\}. \tag{2.3}$$

If $(\mathscr{M}, g)$ is globally hyperbolic and if $t_-(\gamma) < \infty$, then the point $\gamma(t_-(\gamma))$ is either conjugate to $\gamma(0)$ along γ and/or there exist two distinct null achronal geodesics from $\gamma(0)$ to $\gamma(t_-(\gamma))$ (see [3, Theorem 9.15], compare the arguments in the proof of Proposition 2.1). The points $\gamma(t_-(\gamma))$ are *end points* of generators of past light-cones [4]. The set of end points of past-directed generators starting at p is called *the past null cut-locus* of p. It is known, in space-time dimension $n+1$, that the past null cut-locus of p has vanishing n-dimensional measure within $\exp_p(\mathbb{R}(N_h^-)_p\mathscr{M})$. (This fact also follows from Fubini's theorem and Proposition 2.1 below; compare the proof of Lemma 3.1.) (The result is of course trivial in $1+1$ dimensions.)

For a $C^{2,1}$ metric g, the set $\mathring{E}^-(p)$ is a $C^{1,1}$ null hypersurface, indeed it is an immersion by the local injectivity of the exponential map away from conjugate points, and it is an embedding because there are no self-intersections as we remove the points of the light cones behind the cut points and the cut points themselves. In a globally hyperbolic space-time

$$E^-(p) = \overline{\mathring{E}^-(p)}. \tag{2.4}$$

We can parameterize the set of all maximally extended past-directed half-geodesics by the initial positions and h-normalized tangent vector at the starting point. In other words, such half-geodesics are in one-to-one correspondence with vectors in $N_h^-\mathscr{M}$. This induces in the obvious way a topology on the set of half-geodesics. We have:

Proposition 2.1. *Let* $(\mathscr{M}, g)$ *be globally hyperbolic with a twice-differentiable metric. Then the map* $t_-\colon N_h^-\mathscr{M} \to [0,\infty]$ *is continuous.*

Proof. The proof is adapted from that of the corresponding result in Riemannian geometry given in [9, Prop. 5.4], compare [21, Vol. II, p. 99]. Let $\gamma_i\colon [0,\infty) \to \mathscr{M}$ be a sequence of past-directed null half-geodesics parametrized by h-arc length such that $\gamma_i(0) \to \gamma(0)$ and $\dot\gamma_i(0) \to \dot\gamma(0)$ as $i \to \infty$. By continuous dependence on initial conditions of solutions of the geodesic equation it holds that $\gamma_i(t) \to \gamma(t)$ as $i \to \infty$ for each $t > 0$. We split the rest of the proof into two steps:

1. Upper semi-continuity of t_-: Let $t > 0$ and assume that there exists infinitely many i such that $d_L^-(\gamma_i(0), \gamma_i(t)) = 0$. Since $\gamma_i(t) \to \gamma(t)$ as $i \to \infty$, and $d_L^-\colon \mathscr{M} \times \mathscr{M} \to [0,\infty]$ is continuous, it follows that $d_L^-(\gamma(0), \gamma(t)) = 0$. Hence $t_-(\gamma) \geq t$. Therefore, $t_-(\gamma) \geq \limsup_{i\to\infty} t_-(\gamma_i)$, as required. (Note that there was nothing to prove when $t_-(\gamma) = \infty$.)

2. Lower semi-continuity of t_-: We need to show that $t_-(\gamma) \leq \liminf_{i\to\infty} t_-(\gamma_i)$. We assume that $\liminf_{i\to\infty} t_-(\gamma_i) < +\infty$, otherwise there is nothing to prove. Let

$$t > \liminf_{i\to\infty} t_-(\gamma_i) =: \ell,$$

then there exists a subsequence, also denoted $\{\gamma_i\}$, such that $t_-(\gamma_i) < t$ for all i with $t_-(\gamma_i) \to \ell < t$ as $i \to \infty$. Since $t_-(\gamma_i) < t$, we deduce that $\gamma_i(t) \in I^-(\gamma_i(0))$, and therefore there exist *timelike* half-geodesics σ_i parametrized by h-arc length such that $\gamma_i(0) = \sigma_i(0)$ and $\gamma_i(t) = \sigma_i(\widetilde{t}_i)$ for some $\widetilde{t}_i \in (0,\infty)$.

Passing to another subsequence if needed, by global hyperbolicity there exists a causal past-directed half-geodesic σ such that $\sigma_i(s) \to \sigma(s)$ for $s \in [0,\infty)$, with the sequence $\{\widetilde{t}_i\}_{i\in\mathbb{N}}$ convergent, and $\sigma_i(\widetilde{t}_i) \to \gamma(t)$. If σ and γ are distinct, it follows that $t \geq t_-(\gamma)$, as desired. If σ and γ coincide but t is larger than or equal to the distance to the first conjugate point of $\gamma(0)$ along γ we again obtain $t \geq t_-(\gamma)$, and we are done.

It remains to consider the possibility that σ and γ coincide and that t is smaller than the h-distance (possibly infinite, if $t_-(\gamma) = \infty$) to the first conjugate point along γ of $\gamma(0)$. For $X \in T\mathscr{M}$ let γ_X denote a half-geodesic such that $\dot\gamma_X(0) = X$. By continuity of $\det \exp_*$, there exists a neighborhood $\mathscr{O}$ of $\dot\gamma(0)$ in $T\mathscr{M}$ such that for every causal vector $X \in \mathscr{O}$ and every past-directed half-geodesic $s \mapsto \gamma_X(s)$ the h-distance along γ_X to the first conjugate point of $\gamma_X(0)$ is larger than t. This contradicts $t_-(\gamma_i) < t$ thus this case does not really apply. Hence, t_- is lower semi-continuous, and thus also continuous, as desired. □

3. The Derivative of $\tau_\varphi^\pm$

In this section we assume that $(\mathscr{M}, g)$ is a globally hyperbolic space-time and show that the functions $\tau_\varphi^\pm$, as defined in Eq. (1.1), are differentiable for suitably chosen φ. We consider only τ_φ^-, the result for τ_φ^+ follows by changing time-orientation. We will always assume that φ is continuous and non-negative. We start by assuming that φ has compact support.

Let $p \in \mathscr{M}$, and let

$$\gamma\colon \mathbb{R} \to \mathscr{M} \tag{3.1}$$

be any future-directed, timelike h-arc length parametrized curve passing through $p = \gamma(0)$. Choose a g-orthonormal frame at p, and parallel-propagate the frame along γ. This defines g-orthonormal frames $\{e_\mu(s)\}$ at $\gamma(s)$. We will say that g-geodesics at different points of γ are *pointing in the same direction* if the frame components of their initial velocities in the frame $\{e_\mu(s)\}$ coincide, i.e. if their tangent vectors at γ are parallel transports of each other along γ. Then, for each generator of $\mathring{E}^-(p)$, we may associate a family of half-geodesics, parametrized by $s \in \mathbb{R}$, that emanate from the point $\gamma(s)$ with initial tangent vector pointing in the same direction as the chosen generator. Thus, points on neighboring light-cones with vertices on γ can be obtained by flowing along the associated Jacobi fields. This explains the construction that follows.

Let $\tau \mapsto \Gamma_s(\tau)$ be any past-directed affinely parametrized null half-geodesic starting at $\gamma(s)$, where $\tau \in [0, \tau_-(s))$, with $\Gamma_s(\tau_-(s))$ the cut point of Γ_s. Its tangent vector $\frac{\mathrm{d}}{\mathrm{d}\tau}$ at $\gamma(s)$ is extended all over γ through parallel transport, i.e.

$$\frac{D}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}\tau} = 0,$$

over γ. Taking this parallel tangent vector field as initial data in the geodesic equations, the definition of Γ_s is extended to different values of s. It is well known that by the local injectivity of the exponential map away from conjugate points, $\tau_-(s)$ is lower semi-continuous, thus the pairs (s, τ), $s \neq 0$, for which Γ is defined form an open set. The mapping $\Gamma(s, \tau) = \Gamma_s(\tau)$ is generated by past-directed lightlike half-geodesics with initial endpoint $\gamma(s)$ at γ and is really an embedding (surface). Indeed, two geodesics generators relative to different values of s cannot intersect, namely it cannot be $\Gamma_{s'}(\tau') = \Gamma_s(\tau)$ for $s' > s$, otherwise it would be possible to go from $\Gamma_{s'}(\tau')$ to $\gamma(s')$ with a timelike curve in contradiction with the fact that $\Gamma_{s'}(\tau')$ stays before the cut point of $\Gamma_{s'}$. Similarly, the image of Γ cannot develop focusing points, for this would imply that a certain Jacobi field X to be introduced in a moment vanishes, a fact which we prove to be impossible.

Since (s, τ) provide coordinates over the image of Γ, we have that

$$T := \Gamma_* \partial_\tau \quad \text{and} \quad X := \Gamma_* \partial_s$$

commute near γ, thus

$$\frac{D}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}s} = 0$$

over γ. Now, observe that X is the variational field of $\Gamma_s(\tau)$, where the longitudinal curves are geodesics. Thus it is a Jacobi field whose value at $\gamma(s)$ is $\dot{\gamma}(s)$, while its first covariant derivative vanishes thanks to the mentioned commutation relation. It is interesting to observe that, since X is Jacobi, for every fixed s we have that $\tau \mapsto g(X, \frac{\mathrm{d}}{\mathrm{d}\tau})$ is an affine function of τ over $\Gamma_s(\tau)$, a fact which, given the initial conditions, implies that $g(X, \frac{\mathrm{d}}{\mathrm{d}\tau})$ is a constant whose value can be inferred from its value at the tip $\gamma(s)$. In particular, since $\dot{\gamma}$ is future-directed and timelike, $g(X, \frac{\mathrm{d}}{\mathrm{d}\tau})$ is positive over Γ_s so, as anticipated, there cannot be focusing points due to the variation of coordinate s, although, of course, each individual light cone for fixed s might develop a conjugate point at or after the cut point and hence outside the restricted τ-domain $[0, \tau_-(s))$. We conclude that the image of Γ is really a surface. Observe that since X does not vanish it can even be defined at the cut point. However, by changing the generator ending at the endpoint (and Γ) one would get a different value of X. This fact will play no significant role in what follows due to the fact that the set of cut points has negligible measure.

So far X has been defined over the surface defined by the mapping $(s, t) \mapsto \Gamma(s, \tau)$. By taking generators of $\mathring{E}^-(\gamma(s))$ with starting tangent vector having different components with respect to the base $\{e_\mu\}$ we obtain a vector field defined over $\cup_s \mathring{E}^-(\gamma(s))$.

As the Riemann tensor is Lipschitz, and since X satisfies the Jacobi equation, by using the dependence on initial conditions of first order ODEs we have that the vector field X is Lipschitz, a fact to be used below.

Let $\mathscr{U}$ be any relatively compact domain containing the support of φ. Consider the map $\hat{L}\colon T\mathscr{M} \to \mathbb{R}$ defined as

$$Z \mapsto \int_{\mathring{E}^-(\pi(Z))\cap\mathscr{U}} \varphi\, L(Z)\rfloor \mathrm{d}\mu_g,$$

where $L(Z)$ is the Jacobi field X over $E^-(\pi(Z))$ obtained as the solution of the Jacobi equation over each generator by imposing, (a) $L(Z) = Z$ at $\pi(Z)$, and (b) a vanishing derivative at $\pi(Z)$ in the direction of the affinely parametrized generator: $\frac{D}{d\tau} L(Z)|_{\tau=0} = 0$.

The field $L(Z)$ is linear in Z and hence at each point depends continuously on Z. Moreover, by Proposition 2.1 the sets $\partial(\mathring{E}^-(p)) \cap \mathscr{U}$, when non-empty, are continuous radial graphs which vary continuously with p. As the domain of integration $\mathring{E}^-(\pi(Z)) \cap \mathscr{U}$ and the integrand depend continuously on Z we conclude that $\hat{L}(Z)$ is continuous. In particular, if $Z(p)$ is a continuous vector field, then $\hat{L}(Z(p))$ is continuous in p.

We have the following:

Lemma 3.1. *Let $(\mathscr{M}, g)$ be globally hyperbolic with a $C^{2,1}$ metric g. Suppose that φ is smooth and compactly supported. Then τ_φ^- is differentiable and for every $Z \in T\mathscr{M}$ we have*

$$Z(\tau_\varphi^-) = \hat{L}(Z). \tag{3.2}$$

Proof. Let us first assume that Z is future-directed timelike and set $p = \pi(Z)$. Let γ be a parametrized future-directed inextendible timelike curve such that Z is the tangent vector at $p = \gamma(0)$. Since φ is supported in $\mathscr{U}$, formula (1.1) can be rewritten as

$$\tau_\varphi^-(q) = \int_{J^-(q)\cap\mathscr{U}} \varphi \, \mathrm{d}\mu_g, \tag{3.3}$$

for each $q \in \mathscr{M}$. We want to calculate the derivative of $\tau_\varphi^- \circ \gamma(s)$ with respect to s at $s = 0$, showing in the course of the calculation that this derivative exists.

Denote by X the Jacobi field induced from $\dot\gamma$ as explained above.

Suppose, first, that X is Lipschitz in a neighborhood of the support of φ. Then, at least for small s, $J^-(\gamma(s)) \cap \mathscr{U}$ is obtained by flowing $J^-(\gamma(0)) \cap \mathscr{U}$ along X. It is then standard that $\tau_\varphi^- \circ \gamma$ is differentiable near $s = 0$, with

$$Z(\tau_\varphi^-) = \frac{\mathrm{d}(\tau_\varphi^- \circ \gamma)}{\mathrm{d}s}(0) = \int_{J^-(p)\cap\mathscr{U}} L_X[\varphi \mathrm{d}\mu_g]. \tag{3.4}$$

We can now use the identity $L_X = di_X + i_X d$ in the integral above. The term $i_X d$ gives a vanishing contribution since $\varphi d\mu_g$ has already maximum degree as a differential form, while the former contribution can be integrated according to Stokes' theorem for Lipschitz fields on domains with Lipschitz boundaries [12, Section 5.8] to give

$$Z(\tau_\varphi^-) = \int_{J^-(p)\cap\mathscr{U}} L_X[\varphi \mathrm{d}\mu_g] = \int_{\mathring{E}^-(p)\cap\mathscr{U}} \varphi \, X\rfloor \mathrm{d}\mu_g, \tag{3.5}$$

as desired.

However, X will not be Lipschitz in general. In fact, in general X will not even extend by continuity to the null cut set. In such cases we proceed as follows: let $\Sigma(s)$ be the set at which $\dot J^-(\gamma(s))$ fails to be a C^2-manifold. It can be useful to recall that $\Sigma(s) = \gamma(s) \cup \Sigma'(s) \cup \Sigma''(s)$, where [1] (compare [11] for a proof of pseudoconvexity of acausal boundaries, as needed to apply [1]), in space-time dimension $n+1$, $\Sigma'(s)$ is included in a rectifiable $(n-1)$-manifold, and $\Sigma''(s)$ has vanishing $(n-1)$-dimensional Hausdorff measure. Let

$$\Sigma = \cup_s \Sigma(s).$$

It is well known that $\Sigma \cap \mathscr{U}$ has zero $(n+1)$-dimensional Lebesgue measure, but we give the argument for completeness. For this, let C^- denote the past light cone in Minkowski space-time minus its vertex. Let

$$\varPhi \colon \mathbb{R} \times C^- \to \mathscr{M}$$

be the map which to a point $(s, X) \in \mathbb{R} \times C^-$ associates $\exp(X)$, where X is viewed as a vector in $T_{\gamma(s)}\mathscr{M}$ using the construction in the paragraph following (3.1). Then $\varPhi$ is a locally Lipschitz map from $\mathbb{R} \times C^-$ to $\mathscr{M}$.

For each s consider the inverse image $\varPhi^{-1}(\Sigma(s))$. Now, every null geodesic in C^- intersects the set $\varPhi^{-1}(\Sigma(s))$ at at most one point. Fubini's theorem with respect to the measure μ_n induced on C^- from the Lebesgue measure using the flat metric on $\mathbb{R}^{n+1}$ shows that $\mu_n(\varPhi^{-1}(\Sigma(s))) = 0$. This implies that

$\cup_s \Phi^{-1}(\Sigma(s))$ is measurable on $\mathbb{R} \times C^-$ with respect to the product measure $\lambda^1 \times \mu_n$. Using Fubini's theorem again we obtain

$$(\lambda^1 \times \mu_n)\big(\cup_s \Phi^{-1}(\Sigma(s))\big) = 0,$$

where λ^1 is the Lebesgue measure on $\mathbb{R}$. Since $\cup_s \Sigma(s)$ is the image by the locally Lipschitz map Φ of $\cup_s \Phi^{-1}(\Sigma(s))$, we conclude that

$$\mu_g(\Sigma) = \mu_g(\cup_s \Sigma(s)) = 0, \tag{3.6}$$

where μ_g is the usual metric measure on $\mathscr{M}$.

Using global hyperbolicity it is pretty easy to show that every point $p \in I^-(\gamma)$ belongs to one and only one set $E^-(\gamma(s))$, $s \in \mathbb{R}$. Thus $I^-(\gamma) = \exp(\mathbb{R}N_h|_\gamma)$ where $N_h|_\gamma$ is the past h-unit lightlike bundle over γ. However, the image of the star domain in which this exponential map is a local diffeomorphism is $I^-(\gamma)\backslash\Sigma$, which must be open by local injectivity, thus Σ is closed in the topology of $I^-(\gamma)$ (the argument is analogous to that used in [21], Sect. VII.7 vol II, to show that the cut point set is closed). As a consequence for any chosen interval $[\underline{s}, \overline{s}]$, $\bar\Sigma = \cup_{s\in[\underline{s},\overline{s}]}\Sigma(s)$ is closed.

Let d denote the distance in $\mathscr{M}$ from the set $(\operatorname{supp}\phi) \cap \bar\Sigma$ with respect to our auxiliary complete Riemannian metric h. The set $(\operatorname{supp}\phi)\cap\bar\Sigma$ is closed and compact which implies that d is Lipschitz. Let $f\colon \mathbb{R} \to \mathbb{R}$ denote any smooth non-decreasing function which vanishes on $(-\infty, 1/2]$ and equals one on $[1,\infty)$. Set

$$\phi_\epsilon = f(d/\epsilon).$$

Then ϕ_ϵ is Lipschitz, vanishes in a neighborhood of $\bar\Sigma$, and

$$\forall p \notin \bar\Sigma \quad \phi_\epsilon(p) \to_{\epsilon\to 0} 1.$$

Since the vector field X, $X = L(Z)$, $Z = \dot\gamma(s)$, $s \in [s_1, s_2] \subset (\underline{s}, \overline{s})$, is Lipschitz on the support of $\phi_\epsilon\varphi$ and this support does not intersect $\bar\Sigma$, we have by the result already established

$$\tau^-_{\phi_\epsilon\varphi}(\gamma(s_2)) - \tau^-_{\phi_\epsilon\varphi}(\gamma(s_1)) = \int_{s_1}^{s_2}\left(\int_{\mathring{E}^-(\gamma(s))} \phi_\epsilon\varphi\, L(Z)\rfloor d\mu_g\right)\mathrm{d}s. \tag{3.7}$$

From the dominated convergence theorem we have

$$\tau^-_{\phi_\epsilon\varphi}(s) \to_{\epsilon\to0} \tau^-_\varphi(s), \qquad \int_{\mathring{E}^-(\gamma(s))} \phi_\epsilon\varphi\, L(Z)\rfloor \mathrm{d}\mu_g \to_{\epsilon\to0} \int_{\mathring{E}^-(\gamma(s))} \varphi\, L(Z)\rfloor d\mu_g.$$

Passing to the limit $\epsilon \to 0$ in (3.7) we obtain

$$\tau^-_\varphi(\gamma(s_2)) - \tau^-_\varphi(\gamma(s_1)) = \int_{s_1}^{s_2}\left(\int_{\mathring{E}^-(\gamma(s))} \varphi\, L(Z)\rfloor \mathrm{d}\mu_g\right)\mathrm{d}s. \tag{3.8}$$

It follows from Lebesgue's continuity theorem that the integrand is a continuous function of s. Our derivative formula for timelike Z immediately follows.

It remains to prove the formula for any vector $S \in T\mathscr{M}$, namely let us prove

$$\tau^-_\varphi(\exp(S\epsilon)) - \tau^-_\varphi(\pi(S)) = \epsilon\hat L(S) + o(\epsilon).$$

Let $p=\pi(S)$ and let $T\in T_p\mathscr{M}$ be a future-directed timelike vector such that $T+S$ is future-directed timelike. By continuity we can find a small normal coordinate neighborhood, with coordinates $\{x^i\}$, such that the vectors $(S^i+T^i)\partial_i$ and $T^i\partial_i$ (constant components) are timelike over the neighborhood. Then

$$\begin{aligned}\tau^-_\varphi(S^i\epsilon)-\tau^-_\varphi(0) &= \tau^-_\varphi(S^i\epsilon)-\tau^-_\varphi((S^i+T^i)\epsilon))+\tau^-_\varphi((S^i+T^i)\epsilon)-\tau^-_\varphi(0)\\ &= -\hat L|_{S^i\epsilon}(T^i\partial_i)\epsilon+\hat L|_0(S+T)\epsilon+o(\epsilon)\\ &= -\hat L|_0(T)\epsilon+\hat L|_0(T+S)\epsilon+[\hat L|_0(T)-\hat L|_{S^i\epsilon}(T^i\partial_i)]\epsilon+o(\epsilon)\\ &= \hat L|_0(S)\epsilon+o(\epsilon),\end{aligned}$$

where we used the continuity of $p\mapsto \hat L|_p$ to infer that the term in square brackets vanishes as $\epsilon\to 0$. □

The identity $Z(\tau^-_\varphi)=\hat L(Z)$ and the continuity of $\hat L$ imply that τ^-_φ is continuously differentiable.

Remark 3.2. Both for our purposes here and those of next section, we note that if τ is a continuously differentiable function such that $C(\tau)>0$ for every future-directed causal vector C then $-\nabla\tau$ is future-directed and timelike. Indeed, $C(\tau)=g(\nabla\tau,C)$ and $g(Y,C)$ is positive for every future-directed causal vector C if and only if Y is past-directed and timelike as can be easily checked in an orthogonal base at the point.

As such, Remark 3.2 implies that $\nabla\tau^-_\varphi(p)$ is past-directed and timelike provided $E^-(p)$ intersects the interior of the support of φ, namely the open set $V=\{x:\varphi(x)>0\}$. Indeed, if γ is a causal curve the integrand in $\hat L(\dot\gamma)$ reads

$$\varphi\, X\rfloor \mathrm{d}\mu_g=\varphi\, g\left(X,\frac{\mathrm{d}}{\mathrm{d}\tau}\right)\mathrm{d}A\,\mathrm{d}\tau,$$

where X in the Jacobi field induced from $\dot\gamma$, dA is the area element transverse to the generators of $E^-(\gamma(s))$. In order to show that the integral is positive recall, from above, that $g(X,\frac{d}{d\tau})$ is constant over $\Gamma_s(\tau)$. Hence it coincides with its value at the tip $\gamma(s)$, where it is $g(\dot\gamma,\frac{d}{d\tau})$. As γ is causal, and $\frac{d}{d\tau}$ is null and past-directed, this scalar product is positive unless $\dot\gamma(s)$ is null and $\frac{d}{d\tau}$ is proportional to it. However, as the integral involves all directions, for $n\geq 2$ this exceptional null generator does not affect the positivity of the integral, as it has vanishing measure within $\mathring{E}^-(\gamma(s))\cap\mathscr{U}$. The conclusion does not change for $n=1$ since the integral would be the sum of the non-negative contribution from two lightlike geodesic segments and only one of those can vanish.

Stated in another way, if $E^-(\gamma(s))$ intersects V, since V is open we can always find a generator of $E^-(\gamma(s))$ not aligned with X at $\gamma(s)$ and intersecting V. The integral in a neighborhood of this generator gives a positive contribution. Thus either $E^-(\gamma(s))$ does not intersect V and $\nabla\tau^-_\varphi(p)$ vanishes, or $E^-(\gamma(s))$ intersects V and $\nabla\tau^-_\varphi(p)$ is timelike and past directed.

Thus, we have proved:

Lemma 3.3. *In globally hyperbolic space-times the functions $\tau_\varphi^\pm$ are continuously differentiable with timelike or vanishing gradient for all continuous compactly supported non-negative functions φ.* □

However, τ_φ^- is zero on $\mathscr{M}\setminus J^+(\mathrm{supp}(\varphi))$, so it is not a time function there. Similarly, τ_φ^- is constant near every point p such that $\mathrm{supp}(\varphi) \subset I^-(p)$. So, a little more work is needed to construct a differentiable time function:

Let $\{B_{p_i}(r_i)\}_{i\in\mathbb{N}}$ be any locally finite covering of $\mathscr{M}$ with open h-balls centered at p_i with h-radius $r_i \le 1$. Let φ_i be a partition of unity associated with this covering. Let $\tau_{\varphi_i}^-$ be the associated (continuously differentiable) volume functions. Define

$$c_i = \sup_{q\in\mathscr{M}} \tau_{\varphi_i}^-(q), \qquad C_i = 1 + c_i + \sup_{p\in B(p_1,i)} |D\tau_{\varphi_i}^-|_h < \infty.$$

For any sequence $D_i \ge C_i$ set

$$\varphi = \sum_i \frac{1}{2^i D_i}\varphi_i \tag{3.9}$$

(in what follows the reader can simply assume that $D_i = C_i$, the point of introducing the D_is is to make it clear that any sequence $\{D_i\}$ with $D_i \ge C_i$ leads to a differentiable time function). Consider the function

$$\tau_\varphi^-(p) = \int_{J^-(p)} \varphi \,\mathrm{d}\mu_g = \sum_i \frac{1}{2^i D_i}\tau_{\varphi_i}^-. \tag{3.10}$$

Let K be a compact subset of $\mathscr{M}$, there exists $n \in \mathbb{N}$ such that $K \subset B_{p_1}(n)$. Then

$$\begin{aligned}\sup_{q\in B_{p_1}(n)} &\sum_{i=1}^\infty \frac{1}{2D_i}|D\tau_{\varphi_i}^-|_h \\ &\le \underbrace{\sum_{i=1}^n \frac{1}{2^i D_i}\sup_{q\in B_{p_1}(n)} |D\tau_{\varphi_i}^-|_h(q)}_{<\infty} + \sum_{i=n+1}^\infty \frac{1}{2^i D_i}\underbrace{\sup_{q\in B_{p_1}(n)} |D\tau_{\varphi_i}^-|_h(q)}_{\le C_i} \\ &< \infty.\end{aligned} \tag{3.11}$$

This shows that the series defining τ_φ^- converges in C^1 norm on every compact set, resulting in a differentiable function. Since each $\tau_{\varphi_i}^-$ has timelike or vanishing gradient, with $d\tau_{\varphi_i}^-$ non-vanishing on the interior of $\mathrm{supp}(\tau_{\varphi_i}^-)$, the timelikeness of $\nabla\tau_\varphi^-$ readily follows, and Theorem 1.1 is proved. □

4. Smoothing Anti-Lipschitz Time Functions

In this section we first show that the volume time functions of the previous section are locally anti-Lipschitz, a property to be defined shortly, and then that any time function which shares the anti-Lipschitz property with respect to a metric with wider light cones can be smoothed. These results are then applied to prove the existence of smooth time functions in stably causal space-times, and smooth Cauchy time functions in globally hyperbolic space-times. Finally, using the equivalence between stable causality and K-causality we prove that the existence of a time function implies the existence of a smooth one.

We begin with a simple lemma.

Lemma 4.1. *Let $(\mathscr{M}, g)$ be a strongly causal space-time. The following two conditions on a function $\tau^+ : \mathscr{M} \to \mathbb{R}$, respectively $\tau^- : \mathscr{M} \to \mathbb{R}$, are equivalent:*

(i) *for every point $p \in \mathscr{M}$ there exists a relatively compact neighborhood $\mathscr{O}_p$ of p and a constant $C_p > 0$ so that for every h-parametrized past-directed (resp. future-directed) causal curve γ with image in $\mathscr{O}_p$ we have, for all $s_2 \geq s_1$,*

$$\tau^{\pm}(\gamma(s_2)) - \tau^{\pm}(\gamma(s_1)) \geq C_p(s_2 - s_1), \tag{4.1}$$

(ii) *for every compact set K there is a constant $C_K > 0$ such that for every h-parametrized past-directed (resp. future-directed) causal curve γ with image in K, τ^+ (resp. τ^-) satisfies, for all $s_2 \geq s_1$,*

$$\tau^{\pm}(\gamma(s_2)) - \tau^{\pm}(\gamma(s_1)) \geq C_K(s_2 - s_1). \tag{4.2}$$

Clearly, both conditions imply that $\mp\tau^{\pm}$ is a time function.

Proof. $(ii) \implies (i)$. Just take the relatively compact neighborhood to be the interior $\mathring{K}$ of any compact set K so that p belongs to $\mathring{K}$.

$(i) \implies (ii)$. Since $(\mathscr{M}, g)$ is strongly causal, each point $p \in K$ belongs to a relatively compact open causally convex set $\hat{\mathscr{O}}_p \subset \mathscr{O}_p$, thus there is a finite subcovering of K, $\{\hat{\mathscr{O}}_{p_j} : j = 1, \ldots, n\}$. Since no causal curve can enter a causally convex set twice, (ii) holds with $C_K := \min_j C_{p_j}$. □

We shall say that τ is *locally ($\pm$-g-)anti-Lipschitz* if it satisfies (i) or (ii) above. Clearly, this property is independent of the Riemannian metric h used, as two different Riemannian metrics are Lipschitz equivalent over compact sets. (In space-times which are not strongly causal, one could use e.g. (4.1) as a definition of anti-Lipschitz in general, but this generality will not be needed in what follows.)

Remark 4.2. Observe that a past volume function can be discontinuous and yet locally anti-Lipschitz, e.g. remove a past inextendible timelike geodesic, including the future endpoint, from a strip $(-1, 1) \times \mathbb{R}$ of Minkowski $1 + 1$ space-time with coordinates (t, x).

Proposition 4.3. *Let $\tau^- : \mathscr{M} \to \mathbb{R}$ be continuously differentiable. Then τ^- has past-directed timelike gradient if and only if it is locally anti-Lipschitz.*

There is evidently a time-dual version of Proposition 4.3.

Proof. Suppose that τ^- is anti-Lipschitz. Let X be a h-normalized future-directed causal vector at p, and let $\gamma(s)$ be a causal curve with tangent X at $p = \gamma(0)$. Taking the limit $s \to 0$ of the anti-Lipschitz condition we find

$X(\tau^-) \geq C_K > 0$ where K is a compact neighborhood of p. Since X is arbitrary, using Remark 3.2 we infer that $\nabla\tau$ is past-directed and timelike.

Conversely, let us assume that τ^- has past-directed timelike gradient, and let K be a compact set. Let us observe that $d\tau = g(\nabla\tau, \cdot)$. Let $g_{\|} = \frac{1}{g(\nabla\tau,\nabla\tau)}\, d\tau^2$, and let $g_\perp$ be a quadratic form such that $g = g_\perp + g_{\|}$. Let $\tilde{g} = \alpha g_\perp + g_{\|} = \alpha g + (1-\alpha)g_{\|}$, with $\alpha < 1$. For α sufficiently close to 1, $\tilde{g}$ is Lorentzian over K with light cones wider than those of g, and moreover $\tilde{g}(\nabla\tau, X) = g(\nabla\tau, X)$ for $X \in TK$. Let $X \in TK$ be any h-normalized future-directed causal vector, then $\tilde{g}(X,X) < 0$ and there is $C_K > 0$ such that

$$X(\tau^-) = g(\nabla\tau^-, X) = \tilde{g}(\nabla\tau^-, X) \geq [\tilde{g}(\nabla\tau^-, \nabla\tau^-)\tilde{g}(X,X)]^{1/2} \geq C_K,$$

where in the last inequality we used the compactness of the bundle of h-normalized causal vectors in TK. From here the anti-Lipschitz condition follows upon integration in h-arc length s. □

Corollary 4.4. *Let $(\mathscr{M}, g)$ be globally hyperbolic. The continuously differentiable function τ_φ^- of* Theorem 1.1*, with $\varphi > 0$, is locally anti-Lipschitz.*

Actually we can prove something more. We shall need a simple preliminary result:

Lemma 4.5. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a non-decreasing function such that for every $s \in \mathbb{R}$,*

$$\liminf_{\epsilon\to 0^+} \frac{1}{\epsilon}[f(s+\epsilon) - f(s)] \geq C \geq 0,$$

then if $s_2 - s_1 \geq 0$, we have $f(s_2) - f(s_1) \geq C(s_2 - s_1)$.

Proof. The assumption for $s = s_1$ tells us that there is a maximal right-neighborhood $[s_1, \tilde{s})$ such that for every $s \in [s_1, \tilde{s})$, we have $f(s) - f(s_1) \geq C(s - s_1)$. Let us show that $\tilde{s}$ must be infinite and thus that $s_2 \in [s_1, \tilde{s})$. For, if not, taking the limit for $s \to \tilde{s}$ of $f(s) - f(s_1) \geq C(s - s_1)$, and using the fact that f is non-decreasing, we obtain $f(\tilde{s}) - f(s_1) \geq C(\tilde{s} - s_1)$. But from the assumption applied to $\tilde{s}$, there is $\hat{s} > \tilde{s}$ such that for $s \in [\tilde{s}, \hat{s})$, $f(s) - f(\tilde{s}) \geq C(s - \tilde{s})$, which summed to the previous equation gives $f(s) - f(s_1) \geq C(s - s_1)$, for $s \in [s_1, \hat{s})$, showing that $\tilde{s}$ was not maximal, a contradiction. □

Theorem 4.6. *Let $(\mathscr{M}, g)$ be past-distinguishing where g is $C^{2,1}$, and let τ_φ^- of* (1.1) *be defined through a continuous function $0 < \varphi \in L^1(\mathscr{M})$. Then τ_φ^- is locally anti-Lipschitz.*

We emphasize that τ_φ^- might not be continuous without further hypotheses, compare Remark 4.2.

Proof. We just need to show that for every $p \in \mathscr{M}$ there is a neighborhood U, and a positive continuous function $c : U \to (0, +\infty)$ such that if γ is a future-directed h-arc length parametrized causal curve in U, then for every $q = \gamma(s)$ we have

$$\liminf_{\epsilon\to 0^+} \frac{1}{\epsilon}[\tau_\varphi^-(\gamma(s+\epsilon)) - \tau_\varphi^-(\gamma(s))] \geq c(q) > 0.$$

By Lemma 4.5, τ_φ^- would be anti-Lipschitz on that open subset of U for which $c(q) > c(p)/2$ (with anti-Lipschitz constant $C = c(p)/2$), and hence, given the arbitrariness of p, τ_φ^- would be locally anti-Lipschitz.

Now, observe that we can find $r > 0$, sufficiently small, so that the ball $B_h(p,r)$ is contained in a past-distinguishing neighborhood contained in a convex neighborhood contained in a globally hyperbolic neighborhood, so that for every $q \in B_h(p,r)$, the intersection of $\dot{J}^-(q)$ with the ball of radius r, $B_h(p,r)$, is a smooth null hypersurface except at the tip q. Let ϕ_p be a smooth non-negative "cut-off" function such that $\phi_p \le 1$, $\phi_p(q) = 1$ for $q \in B_h(p,r/2)$ and with support in $U := B_h(p,r)$. Let

$$c(q) := \inf_{X(q)} \int_{B_h(p,r)\cap \dot{J}^-(q)} \phi_p\, \varphi X \rfloor \, \mathrm{d}\mu_g,$$

where X is the already introduced Jacobi field which depends only on the h-normalized future-directed causal vector $X(q)$ in a continuous way. As already explained, the integrand is non-negative when the formula is rewritten in terms of the coordinate-Lebesgue measure, and the integral is positive and continuous. By construction, $c(q)$ is then continuous and positive.

Finally, observe that for γ and q as above, if we set

$$f_q(s) := \int_{B_h(p,r)\cap J^-(\gamma(s))} \phi_p\, \varphi \, \mathrm{d}\mu_g,$$

then since ϕ_p is supported in a globally hyperbolic neighborhood we can apply the formula of the previous section $\frac{d}{ds} f_q(s) = \int_{B_h(p,r)\cap \dot{J}^-(q)} \phi_p\, \varphi X \rfloor \, \mathrm{d}\mu_g \ge c(q)$ and

$$\begin{aligned} \tau_\varphi^-(s+\epsilon) - \tau_\varphi^-(s) &= \int_{J^-(\gamma(s))\setminus J^-(q)} \varphi \, \mathrm{d}\mu_g \ge \int_{B_h(p,r)\cap J^-(\gamma(s))\setminus J^-(q)} \phi_p\, \varphi \, \mathrm{d}\mu_g \\ &= f_q(s+\epsilon) - f_q(s), \end{aligned}$$

from which we obtain the desired conclusion. □

Remark 4.7. In the proof above we used the derivative formula for the volume function which we obtained in the previous section. In this application we are working in a convex neighborhood contained in a globally hyperbolic neighborhood since the argument of the integral includes a cut-off function ϕ_p. In the current setting the proof of the derivative formula is in fact much simpler as there are no focusing or cut points to p in the supports of ϕ_p.

We recall that $\hat{g} \succ g$ means that the causal cone of g is contained in the timelike cone of $\hat{g}$ at all points in space-time. If we can find $\hat{g}$ such that $(\mathscr{M}, \hat{g})$ is causal, then we say that $(\mathscr{M}, g)$ is stably causal. We also recall that a *Cauchy time function* is a time function onto $\mathbb{R}$ whose level sets are intersected (precisely) once by every inextendible causal curve. A space-time admits a Cauchy time function if and only if it is globally hyperbolic [14,18].

Theorem 4.8. *Let $(\mathscr{M}, g)$ be a stably causal space-time with a continuous metric g, and let τ be a time function on $\mathscr{M}$. Moreover, suppose that*

(*) *there exists a metric $\hat{g} \succ g$ such that τ is locally $\hat{g}$-anti-Lipschitz.*

Then for every function $\alpha : \mathscr{M} \to (0, +\infty)$ there exists a smooth g-time function $\hat{\tau}$, with g-timelike gradient, such that $|\hat{\tau} - \tau| < \alpha$. As a consequence, if τ is Cauchy we can choose $\hat{\tau}$ Cauchy (take α bounded).

Proof. Consider $p \in \mathscr{M}$, let x^μ be local coordinates near p, and let $\mathscr{C}_{g,p} \subset T_pM$ denote the collection of g-causal vectors at p. By continuity, there exists $\epsilon(p) > 0$ so that for all q, q' in a relatively compact coordinate ball $B_p(3\epsilon(p))$ of radius $3\epsilon(p)$ centered at p and for all vectors $X(q) = X^\mu(q)\partial_\mu|_q \in \mathscr{C}_{g,q}$ the vector $X(q') := X^\mu(q)\partial_\mu|_{q'} \in T_{q'}\mathscr{M}$, with coordinate components $X^\mu(q')$ at q' equal to its coordinate components $X^\mu(q)$ at q, is $\hat{g}$-timelike at q'. The constant ϵ can be chosen so small that if $X \in T_q\mathscr{M}$ and $Y \in T_{q'}\mathscr{M}$ are two non-zero vectors on $TB_p(3\epsilon(p))$ such that $X^\mu(q) = Y^\mu(q')$, then the ratio of their h-norms belongs to $[1/2, 2]$.

Let $\{\mathscr{O}_i := B_{p_i}(\epsilon_i)\}_{i\in\mathbb{N}}$ be a locally finite covering of $\mathscr{M}$ by such balls. Let φ_i be a partition of unity subordinate to the cover $\{\mathscr{O}_i\}_{i\in\mathbb{N}}$. Choose some $0 < \eta_j < \epsilon_j$. In local coordinates on $\mathscr{O}_j$ let τ_j be defined by convolution with an even non-negative function χ, supported in the coordinate ball of radius one, with integral one:

$$\tau_j(x) = \begin{cases} \frac{1}{\eta_j^{n+1}} \int_{B_{p_j}(3\epsilon_j)} \chi\left(\frac{y-x}{\eta_j}\right) \tau(y)\, d^{n+1}y, & x \in B_{p_j}(2\epsilon_j); \\ 0, & \text{otherwise.} \end{cases}$$

We define the smooth function

$$\hat{\tau} := \sum_j \varphi_j \tau_j.$$

The non-vanishing terms at each point are finite in number, and $\hat{\tau}$ converges pointwise to τ as we let the constants η_j converge to zero. The idea is to control the constants η_j to get the desired properties for $\hat{\tau}$.

Let $x \in \mathscr{O}_j = B_{p_j}(\epsilon_j)$, and let $X = X^\mu\partial_\mu$ be any g-causal vector at $x \in \mathscr{O}_j$, of h-length one, then the curve $x^\mu(s) = x^\mu + X^\mu s$ is $\hat{g}$-timelike as long as it stays within $B_{p_j}(3\epsilon_j)$. We observe that s is not the h-arc length parametrization of the curve, however from our choice of ϵ we have $\inf_{\overline{\mathscr{O}_j}} |X^\mu\partial_\mu|_h \geq 1/2$, thus for $s_2 > s_1$ it holds that $(s_2 - s_1) < 2(t_2 - t_1)$, where t is the h-arc length parametrization. Let $2C_j$ be the $\hat{g}$-anti-Lipschitz constant over $\overline{\mathscr{O}_j}$.

We write:

$$\begin{aligned} &\hat{\tau}(x(s)) - \hat{\tau}(x) \\ &\quad = \underbrace{\sum_j \big(\varphi_j(x(s)) - \varphi_j(x)\big)\tau_j(x(s))}_{=:I(s)} + \underbrace{\sum_j \varphi_j(x)\big(\tau_j(x(s)) - \tau_j(x)\big)}_{=:II(s)}. \end{aligned}$$

We have at $x \in \mathscr{O}_j$,

$$\begin{aligned} &\lim_{t\to 0} \frac{II(s(t))}{t} = \frac{1}{|X(x)|_h} \lim_{s\to 0} \frac{II(s)}{s} \\ &= \lim_{s\to 0} \frac{1}{s} \sum_k \varphi_k(x)\big(\tau_k(x + Xs) - \tau_k(x)\big) \end{aligned}$$

$$= \lim_{s\to 0} \frac{1}{s} \sum_k \frac{\varphi_k(x)}{\eta_k^{n+1}} \int_{B_0(\epsilon_k)} \chi\left(\frac{z}{\eta_k}\right) \big(\underbrace{\tau(x+Xs+z)-\tau(x+z)}_{\geq C_k s}\big)\, d^{n+1}z$$

$$\geq \sum_k \varphi_k(x) C_k \geq \min_{k\,:\,\mathscr{O}_k\cap\mathscr{O}_j\neq\emptyset} C_k =: B_j > 0,$$

where the constant B_j does not depend on the set of constants $\{\eta_i\}$.

For every j let

$$R_j := \sup_{k\,:\,\mathscr{O}_k\cap\mathscr{O}_j\neq\emptyset} \sup_{x\in\overline{\mathscr{O}_j}} |\nabla^h \varphi_k(x)|_h,$$

let N_j be the number of distinct sets $\mathscr{O}_k$ which have non-empty intersection with $\mathscr{O}_j$, and let us choose η_j so small that

$$\sup_{x\in\overline{\mathscr{O}_j}} |\tau(x)-\tau_j(x)| < \min_{\ell:\mathscr{O}_\ell\cap\mathscr{O}_j\neq\emptyset} \left\{ \frac{1}{N_\ell} \inf_{\overline{\mathscr{O}_\ell}} \alpha,\ \frac{B_\ell}{2N_\ell R_\ell} \right\}.$$

Let χ_k be the characteristic function of $\mathscr{O}_k$, so that $\varphi_k \leq \chi_k$. The sets $\mathscr{O}_j$ and $\overline{\mathscr{O}_j}$ intersect the same sets of the covering $\{\mathscr{O}_i\}$, which are N_j in number, thus

$$\begin{aligned}
&\sup_{x\in\overline{\mathscr{O}_j}} \sum_{k:\mathscr{O}_k\cap\mathscr{O}_j\neq\emptyset} [\chi_k(x)|\tau(x)-\tau_k(x)|] \\
&\quad\leq \sum_{k:\mathscr{O}_k\cap\mathscr{O}_j\neq\emptyset} \sup_{x\in\overline{\mathscr{O}_k}} |\tau(x)-\tau_k(x)| \\
&\quad\leq \sum_{k\,:\,\mathscr{O}_k\cap\mathscr{O}_j\neq\emptyset} \frac{B_j}{2R_jN_j} = \frac{B_j}{2R_j}.
\end{aligned}$$

Then at $x\in\mathscr{O}_j$,

$$\begin{aligned}
&\left|\lim_{t\to 0}\frac{I(s(t))}{t}\right| = \left|\lim_{s\to 0}\frac{I(s)}{s}\right| = \left|\lim_{s\to 0}\sum_k \frac{\varphi_k(x(s))-\varphi_k(x)}{s}\,\tau_k(x(s))\right| \\
&\quad= \left|\sum_k X\big(\varphi_k(x)\big)\tau_k(x)\right| = \left|\sum_k X\big(\varphi_k(x)\big)\big[\tau(x)-\big(\tau(x)-\tau_k(x)\big)\big]\right| \\
&\quad= \Bigg|\underbrace{X\Big(\sum_k\varphi_k(x)\Big)}_{=X(1)=0}\tau(x) - \sum_k X\big(\varphi_k(x)\big)(\tau(x)-\tau_k(x))\Bigg| \\
&\quad\leq \sum_k \left|X\big(\varphi_k(x)\big)\right| |\tau(x)-\tau_k(x)| = \sum_{k\,:\,\mathscr{O}_k\cap\mathscr{O}_j\neq\emptyset} \left|X\big(\varphi_k(x)\big)\right| |\tau(x)-\tau_k(x)| \\
&\quad\leq R_j \sum_{k:\mathscr{O}_k\cap\mathscr{O}_j\neq\emptyset} \chi_k(x)|\tau(x)-\tau_k(x)| \leq \frac{B_j}{2}.
\end{aligned}$$

Hence, for every $x\in\mathscr{O}_j$ and every g-causal vector $X\in T_x\mathscr{M}$ of h-length one, there exists a constant $B_j/2$ such that we have

$$X(\hat\tau) \geq B_j/2. \tag{4.3}$$

In particular, $\hat{\tau}$ is a differentiable function which is strictly increasing along any g-causal curve. By Remark 3.2, the g-gradient of $\hat{\tau}$ is everywhere g-timelike. Finally, for every $x \in \mathscr{M}$, there is some j such that $x \in \mathscr{O}_j$, hence

$$|\tau(x) - \hat{\tau}(x)| = |\sum_k \varphi_k(x)[\tau(x) - \tau_k(x)]| \leq \sum_{k:\mathscr{O}_k \cap \mathscr{O}_j \neq \emptyset} \sup_{x \in \overline{\mathscr{O}_k}} |\tau(x) - \tau_k(x)|$$
$$\leq \sum_{k:\mathscr{O}_k \cap \mathscr{O}_j \neq \emptyset} \frac{1}{N_j} \inf_{\overline{\mathscr{O}_j}} \alpha \leq \alpha(x) \sum_{k:\mathscr{O}_k \cap \mathscr{O}_j \neq \emptyset} \frac{1}{N_j} = \alpha(x).$$

Note that the smoothness of $\hat{\tau}$ depends only upon the smoothness of $\mathscr{M}$, regardless of the smoothness of the metric.

For the last claim, since τ is Cauchy it is onto $\mathbb{R}$ thus the same holds for $\hat{\tau}$, and since each constant slice $\hat{\tau}^{-1}(t)$ is contained in $\tau^{-1}([t-1, t+1])$, namely between the Cauchy hypersurfaces $\tau^{-1}(t-1)$ and $\tau^{-1}(t+1)$, the level-set $\hat{\tau}^{-1}(t)$ is also a Cauchy hypersurface, and thus $\hat{\tau}$ is Cauchy. □

In a distinguishing space-time the functions τ_φ^- and τ_φ^+, though increasing over future-directed (resp. past-directed) causal curves, might be only upper semi-continuous and thus might fail to be time functions. Indeed, they are continuous if and only if the space-time is causally continuous [16,26]. Under the weaker notion of stable causality Hawking was able to construct a time function averaging Geroch's volume functions for wider metrics [17,18], as follows: suppose that (M, g) is stably causal, so that there is $\tilde{g} \succ g$ such that $(M, \tilde{g})$ is causal. Without loss of generality we can assume $\tilde{g}$ to be C^2. Let

$$g_\lambda = (1 - \frac{\lambda}{3})g + \frac{\lambda}{3}\tilde{g}, \qquad \lambda \in [0, 3].$$

Clearly, $g_0 = g$, $g_3 = \hat{g}$ and if $\lambda_1 < \lambda_2$ then $g_{\lambda_1} \prec g_{\lambda_2}$. In particular, for each λ, (M, g_λ) is causal.

Let μ be a finite measure, e.g. $d\mu = \varphi \, \mathrm{d}\mu_{g_0}$, and let us define the Geroch's volume functions

$$\tau_\lambda^-(p) = \int_{J_{g_\lambda}^-(p)} \mathrm{d}\mu.$$

Hawking considers the average

$$\tau_H^-(p) = \int_1^2 \tau_\lambda^-(p) \, \mathrm{d}\lambda,$$

and proves that this function is indeed a time function.

The next result with its corollary provides the simplest proof that stably causal space-times admit smooth time functions, and that, in fact, they can be chosen to approximate Hawking's time (previous existence proofs did not establish this approximation property [7]). This result was announced long ago by Seifert [29] (with a not-entirely-transparent proof) and has been used by Hawking and Ellis [18, Prop. 6.4.9] (who referred to Seifert's original doctoral thesis). Our approach is quite close in spirit to Seifert's original work. We emphasize that Seifert's article contains many important ideas. In particular, Seifert was the first to recognize the role of the local anti-Lipschitz condition (Seifert speaks of *uniform* time functions).

Theorem 4.9. *Let $(\mathscr{M}, g)$ be a stably causal space-time with a $C^{2,1}$ metric g. For every function $\alpha\colon \mathscr{M} \to (0, +\infty)$ there exists a smooth time function τ^-_α, with timelike gradient, such that $|\tau^-_H - \tau^-_\alpha| < \alpha$.*

Proof. According to Theorem 4.8 we need only to prove that τ^-_H is locally anti-Lipschitz with respect to $g_{1/2}$. As we chose $\hat{g}$ to be $C^{2,1}$ (this can always be done), we have that g_λ is $C^{2,1}$ with respect to $x \in \mathscr{M}$ and C^∞ with respect to λ. We wish to prove that for $p \in \mathscr{M}$ the functions $\tau^-_\lambda(q)$ are anti-Lipschitz over a neighborhood $V \ni p$, with anti-Lipschitz constants C_λ that can be chosen to depend continuously on λ. If so, since every h-parametrized $g_{1/2}$-causal curve is a g_λ-causal curve for $\lambda \in [1, 2]$, τ^-_H would be anti-Lipschitz over V with anti-Lipschitz constant not smaller than $C := \int_1^2 C_\lambda \, \mathrm{d}\lambda > 0$. Indeed, for $s_2 \geq s_1$,

$$\begin{aligned}\tau^-_H(\gamma(s_2)) - \tau^-_H(\gamma(s_1)) &= \int_1^2 [\tau^-_\lambda(\gamma(s_2)) - \tau^-_\lambda(\gamma(s_1))] \, \mathrm{d}\lambda \\ &\geq \int_1^2 C_\lambda(s_2 - s_1) \, \mathrm{d}\lambda = C(s_2 - s_1).\end{aligned}$$

The fact that C_λ is continuous in λ follows immediately from continuity in λ of the function $c_\lambda(q)$ mentioned in Theorem 4.6: indeed, this function reads

$$c_\lambda(q) := \inf_{X^{(\lambda)}(q)} \int_{B_h(p,r) \cap \dot{J}^-_{g_\lambda}(q)} \phi_p \, \varphi X^{(\lambda)} \rfloor \, \mathrm{d}\mu_g,$$

where $X^{(\lambda)}$ is the Jacobi field obtained by solving the g_λ-Jacobi equation. The results on the dependence with respect to the initial conditions and parameters of the theory of ordinary differential equations assure that this function is continuous [15]. □

Corollary 4.10. *Every stably causal space-time endowed with a continuous metric g admits a smooth time function with timelike gradient.*

Proof. Any stably causal C^0 metric g admits some smooth $\hat{g} \succ g$ such that $(\mathscr{M}, \hat{g})$ is stably causal. This result holds for g continuous, see [13] (alternatively, the result might be obtained following [5] and adapting some steps to the low differentiability case wherever required). But any smooth time function for $(\mathscr{M}, \hat{g})$ is a smooth time function for $(\mathscr{M}, g)$, which by Remark 3.2 has timelike gradient with respect to both metrics. □

We can use a strategy quite similar to that followed above to prove existence of smooth Cauchy time functions in globally hyperbolic space-times. This result was also announced by Seifert [18,29] who provided a non-transparent argument. A first detailed proof appeared in [6,7].

Theorem 4.11. *Every globally hyperbolic space-time $(\mathscr{M}, g)$ where g is continuous admits a smooth Cauchy time function with timelike gradient.*

Proof. Recall that global hyperbolicity is stable [5,13], in the sense that it is possible to find a smooth metric $\hat{g} \succ g$ such that $(\mathscr{M}, \hat{g})$ is globally hyperbolic. So let $\hat{g} \succ g$ be such that $(\mathscr{M}, \hat{g})$ is globally hyperbolic and $\hat{g}$ is smooth. Geroch's time functions τ_φ^+ and τ_φ^- for the space-time $(\mathscr{M}, \hat{g})$ are locally anti-Lipschitz with respect to $\hat{g}$. As a consequence $\tau = \ln(\tau_\varphi^-/\tau_\varphi^+)$ is also locally anti-Lipschitz with respect to $\hat{g}$ for some choice of φ (for τ_φ^+ can be chosen continuously differentiable and hence locally Lipschitz by Theorem 1.1). Since τ is Cauchy for $(\mathscr{M}, \hat{g})$ it is also Cauchy for $(\mathscr{M}, g)$. The claim follows from the last statement of Theorem 4.8. □

An alternative proof, that does not invoke the stability of global hyperbolicity, will be given in the next section.

We end this section by proving that existence of a time function implies existence of a smooth one with timelike gradient. This result was first proved in [7,28] by different methods. Let us recall that K^+ is the smallest closed and transitive relation which contains the causal relation J^+. A space-time is said to be K-causal if K^+ is a partial order [30]. A self-contained proof of the equivalence between K-causality and stable causality can be found in [24].

Theorem 4.12. *Let (M, g) be any space-time with a C^2 metric g. The following conditions are equivalent:*

(a) (M, g) *admits a time function,*
(b) (M, g) *admits a smooth time function with timelike gradient,*
(c) (M, g) *is stably causal.*

Proof. The implication (c) $\Longrightarrow$ (b) is given by Corollary 4.10. The implication (b) $\Longrightarrow$ (a) is obvious. Finally, in order to prove that (a) $\Longrightarrow$ (c), we recall that any space-time which admits a time function is K-causal [25, Lemma 4-(b)], and K-causality coincides with stable causality [24]. □

The previous result probably holds already for continuous metrics g since the proofs given in [24,25] do not seem to depend in any essential way on the differentiability of the metric, but we have not attempted to check all details of this.

As shown in [25] one could go in the other direction, namely use any independently obtained proof of the implication (a) $\Longrightarrow$ (b) to show the equivalence between K-causality and stable causality.

5. Extending the Anti-Lipschitz Property to Wider Metrics

The anti-Lipschitz condition with respect to a wider metric $\hat{g} \succ g$ is the key ingredient to our Theorem 4.8 on uniform approximation. Assuming a local Lipschitz condition, we can infer this property from the anti-Lipschitz condition with respect to g. This allows us to smooth directly the differentiable time functions $\tau_\varphi^\pm$ of $(\mathscr{M}, g)$ obtained in Sect. 2.

We shall need the following lemma:

Lemma 5.1. *Let (Q, h) be a Riemannian space, f a locally Lipschitz function, and let $\gamma\colon [0,1] \to Q$ be an injective C^2 curve. We can find another C^2 curve $\alpha\colon [0,1] \to Q$, arbitrarily close to γ in C^2 norm, such that the differential df of f exists almost everywhere on the image of α, $f \circ \alpha$ is almost everywhere differentiable, $\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \alpha) = \mathrm{d}f[\dot\alpha]$ almost everywhere on $[0,1]$ and $f(\alpha(1)) - f(\alpha(0)) = \int_0^1 \mathrm{d}f[\dot\alpha]\mathrm{d}t$.*

Proof. Let us introduce coordinates $x^1, \ldots, x^n$ in a neighborhood of $\gamma([0,1])$ in such a way that $\gamma(t) = (t, 0, \ldots, 0)$, and let P be a coordinate parallelepiped of sides $1, 2\epsilon, \ldots, 2\epsilon$, and Lebesgue-coordinate volume $V = (2\epsilon)^{n-1}$ around γ. By Rademacher's theorem *df* exists almost everywhere, that is, it exists on a measurable subset S of P and $\int_P \chi_S = V$ where χ_S is the characteristic function of S, and furthermore $\mathrm{d}f = (\partial_1 f, \ldots, \partial_n f)$, that is, wherever $\mathrm{d}f$ exists, the partial derivatives also exist (see e.g. [19]). However, by Fubini's theorem $V = \int_P \chi_S = \int_{-\epsilon}^{\epsilon} \mathrm{d}x^n \cdots \int_{-\epsilon}^{\epsilon} \mathrm{d}x^2 \int_0^1 \mathrm{d}x^1 \chi_S$ which proves that for almost all segments parallel to the x^1-axis we have $\int_0^1 \mathrm{d}x^1 \chi_S = 1$, that is $\mathrm{d}f$ exists almost everywhere on almost every segment $\alpha(t) = (t, x^2, \ldots, x^n)$ parallel to the image of γ. But clearly, wherever *df* exists on α, $\partial_1 f = \mathrm{d}f[e_1] = \mathrm{d}f[\dot\alpha]$. Using $\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \alpha) = \partial_1 f$ we obtain $\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \alpha) = \mathrm{d}f[\dot\alpha]$ almost everywhere on $[0,1]$. Finally $f \circ \alpha$ is the composition of a C^2 function with a locally Lipschitz function, thus locally Lipschitz and hence absolutely continuous, from which the last identity follows. □

We can now prove that the light cones can be opened preserving the local anti-Lipschitz condition on the time function.

Theorem 5.2. *Let $(\mathscr{M}, g)$ be a stably causal space-time with a continuous metric g, and let τ be a time function on $\mathscr{M}$. If τ is locally Lipschitz and locally g-anti-Lipschitz then the condition (*) of* Theorem 4.8 *holds, that is, there exists a metric $\hat g \succ g$ such that τ is locally $\hat g$-anti-Lipschitz.*

Proof. Let us suppose that τ is locally Lipschitz and satisfies the anti-Lipschitz condition on g-causal curves parametrized by h-arc length, that is, for every compact set K we can find $C(K) > 0$ such that

$$\tau(\gamma(s_2)) - \tau(\gamma(s_1)) \geq C(K)(s_2 - s_1). \tag{5.1}$$

Let X be a h-normalized g-causal vector, then taking the limit of this formula we find

$$d\tau[X] \geq C(K)$$

wherever τ is classically differentiable on K, hence almost everywhere on K. We wish to prove that the inequality $\mathrm{d}\tau[Y] \geq C(K)/2$ holds wherever $\mathrm{d}\tau$ exists on K, where Y is a h-normalized $\hat g_K$-causal vector for $\hat g_K \succ g$ sufficiently close to g on K. Unfortunately, we cannot use a continuity argument because $\mathrm{d}\tau$ exists only almost everywhere, and is not necessarily continuous.

Suppose that $\tilde{g}_K \succ g$ is so close to g that for any h-normalized $\tilde{g}_K$-causal vector Y we can find a h-normalized g-causal vector X such that $\|Y - X\| \le C(K)/(2L)$, where L is the Lipschitz constant of τ in K (clearly, $\tilde{g}_K$ exists by a compactness argument). We have

$$\begin{aligned}|\mathrm{d}\tau[X] - \mathrm{d}\tau[Y]| &= \left|\lim_{t\to 0}\left[\frac{\tau(p+Xt)-\tau(p)}{t} - \frac{\tau(p+Yt)-\tau(p)}{t}\right]\right| \\ &\le \lim_{t\to 0}\left|\frac{\tau(p+Xt)-\tau(p+Yt))}{t}\right| \le L\|X-Y\| \le C(K)/2\end{aligned}$$

which implies $d\tau[Y] \ge C(K)/2$. Let $\hat{g}_K$ be such that $g \prec \hat{g}_K \prec \tilde{g}_K$, and let $t_1 > t_0$. Then given a $\hat{g}_K$-causal h-parametrized curve $\gamma(t)$, we have by the previous lemma $\tau(\alpha(t_1)) - \tau(\alpha(t_0)) \ge \frac{1}{2}C(K)(t_1 - t_0)$ over a h-parametrized $\tilde{g}_K$-causal curve α which we can take arbitrarily close to γ. Using the continuity of τ we obtain $\tau(\gamma(t_1)) - \tau(\gamma(t_0)) \ge \frac{1}{2}C(K)(t_1 - t_0)$.

Let K_i be a countable sequence of compact sets such that $K_i \subset \mathrm{Int}K_{i+1}$, $\cup_i K_i = \mathscr{M}$, and let $\hat{g}_i$ be the metric just found for the compact set $A_i = K_i \backslash \mathrm{Int}K_{i-1}$. By making suitable point-dependent convex combinations of $\hat{g}_i$ with g, we can find $\hat{g} \succ g$ such that $\hat{g} \prec \hat{g}_i$ over every A_i. Clearly, τ is locally $\hat{g}$-anti-Lipschitz, which finishes the proof. □

The following result, pointed out to us by A. Fathi (private communication), turns out to be useful:

Proposition 5.3. *Let $(\mathscr{M}, g)$ be a space-time admitting a time function τ, then for every $\epsilon > 0$ there is a locally g-anti-Lipschitz time function $\tilde{\tau}$ such that $|\tau - \tilde{\tau}| < \epsilon$.*

Proof. By Theorem 4.12 there is a smooth time function with timelike gradient $t : \mathscr{M} \to \mathbb{R}$. Let $\tau_\epsilon = \tau + \epsilon \tanh t$ so that $|\tau - \tau_\epsilon| < \epsilon$. Since t is locally g-anti-Lipschitz (Proposition 4.3), so is τ_ϵ. □

Given a locally Lipschitz time function τ, we can use Proposition 5.3 to deform τ to a time function which is both locally Lipschitz and anti-Lipschitz. By Theorems 5.2 and 4.8 we conclude:

Corollary 5.4. *Let $(\mathscr{M}, g)$ be a stably causal space-time with a continuous metric g, and let τ be a time function on $\mathscr{M}$. If τ is locally Lipschitz then for every ϵ there exists a smooth g-time function τ_ϵ, with g-timelike gradient, such that $|\tau_\epsilon - \tau| < \epsilon$. If τ is Cauchy, then τ_ϵ is also Cauchy.* □

Remark 5.5. Under the hypotheses of the corollary, if τ is further known to be locally anti-Lipschitz then one likewise concludes that for every function $\alpha\colon \mathscr{M} \to (0, +\infty)$ there exists a smooth g-time function τ_α, with g-timelike gradient, such that $|\tau_\alpha - \tau| < \alpha$.

Recall that in [13] smooth time functions are constructed on stably causal space-times, by first constructing Lipschitz ones. Corollary 5.4 gives an alternative justification of the last step of the Fathi–Siconolfi construction. We also note that the hypothesis that τ is Lipschitz is not necessary for the conclusion of Corollary 5.4, as any time functions can be approximated by locally Lipschitz ones (A. Fathi, private communication).

We can now apply directly Theorem 4.8 to the continuously differentiable volume function with timelike gradient obtained in Sect. 2:

Corollary 5.6. *In a globally hyperbolic space-time Geroch's Cauchy time function $\tau = \ln(\tau^-_\varphi/\tau^+_\varphi)$ is continuously differentiable with timelike gradient for some φ, and moreover, for such choice of φ and for every function $\alpha\colon \mathscr{M} \to (0,+\infty)$ there exists a smooth Cauchy time function with timelike gradient, say τ_α, such that $|\tau - \tau_\alpha| < \alpha$.*

Proof. The analysis of Sect. 2 shows that we can choose φ so as to make τ^-_φ and τ^+_φ continuously differentiable (and hence Lipschitz). Geroch's original argument proves that τ is Cauchy. Theorem 4.6 proves that τ^-_φ and τ^+_φ are locally g-anti-Lipschitz and Theorem 5.2 proves that for arbitrarily chosen positive functions $\Delta\tau^\pm(x)$, there are smooth time functions with timelike gradient τ^-_α and $-\tau^+_\alpha$, such that $|\tau^-_\varphi - \tau^-_\alpha| < \Delta\tau^-(x)$, and $|\tau^+_\varphi - \tau^+_\alpha| < \Delta\tau^+(x)$.

Let $x, y > 0$ and $z = \ln(x/y)$ then $z(a', b') - z(a, b) = \int_{a;y=b}^{a'} \mathrm{d}x \frac{\partial z}{\partial x} + \int_{b;x=a'}^{b'} \mathrm{d}y \frac{\partial z}{\partial y}$, and using the facts that $\frac{\partial z}{\partial x} = \frac{1}{x}$, $-\frac{\partial z}{\partial y} = \frac{1}{y}$ are decreasing, $|z(a', b') - z(a, b)| \le \sup(\frac{1}{a'}, \frac{1}{a})|a' - a| + \sup(\frac{1}{b'}, \frac{1}{b})|b' - b|$.

Define $\tau_\alpha = \ln(\tau^-_\alpha/\tau^+_\alpha)$, we then have at every point $x \in \mathscr{M}$,

$$|\tau - \tau_\alpha| \le \frac{\Delta\tau^-}{\tau^-_\varphi - \Delta\tau^-} + \frac{\Delta\tau^+}{\tau^+_\varphi - \Delta\tau^+},$$

wherever the denominators are positive. Choosing $\Delta\tau^\pm \le \min\left(\frac{1}{4}\alpha, \frac{1}{2}\right)\tau^\pm_\varphi$ we obtain

$$|\tau(x) - \tau_\alpha(x)| \le \min(\alpha(x), 2)$$

(alternatively, this formula can be proved using the local Lipschitz and local anti-Lipschitz character of $\tau^\pm_\varphi$ to show that τ has the same properties). In particular, since the right-hand side is bounded, τ_α is onto $\mathbb{R}$, and since every level set of τ_α is contained in $\tau^{-1}([a, b])$, that is, it stays between two Cauchy hypersurfaces $\tau^{-1}(a)$, $\tau^{-1}(b)$, τ_α is Cauchy. □

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