Rational rates of uniform decay for strong solutions to a fluid-structure PDE system

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Abstract

In this work we investigate the uniform stability properties of solutions to a well-established partial differential equation (PDE) model for a fluid-structure interaction. The PDE system under consideration comprises a Stokes flow which evolves within a three-dimensional cavity; moreover, a Kirchhoff plate equation is invoked to describe the displacements along a (fixed) portion – say, Ω – of the cavity wall. Contact between the respective fluid and structure dynamics occurs on the boundary interface Ω. The main result in the paper is as follows: the solutions to the composite PDE system, corresponding to smooth initial data, decay at the rate of $O(1/t)$. Our method of proof hinges upon the appropriate invocation of a relatively recent resolvent criterion for polynomial decays of C_0-semigroups. While the characterization provided by said criterion originates in the context of operator theory and functional analysis, the work entailed here is wholly within the realm of PDE.

1 Introduction

1.1 The mathematical model, statement of the main result

In this paper we focus on the problem of deriving rational rates of uniform decay for a fluid-structure partial differential equation (PDE) system; this model has appeared repeatedly in the literature, in one form or another. (See e.g., [14], [19], [5].) This composite system of PDE describes the interactions of a viscous, incompressible fluid within a three-dimensional bounded domain $\mathcal{O}$ (the cavity), with an elastic dynamics displacing along a boundary interface Ω. More precisely, let the walled cavity within which the fluid evolves be denoted as $\mathcal{O}$, a bounded subset of $\mathbb{R}^3$. This bounded set $\mathcal{O}$ will have sufficiently smooth boundary $\partial\mathcal{O}$, with $\partial\mathcal{O} = \overline{S} \cup \overline{\Omega}$, and $S \cap \Omega = \emptyset$. In particular, $\partial\mathcal{O}$ has the following specific spatial configuration:

$$\Omega \subset \{x = (x_1, x_2, 0)\} \ , \quad S \subset \{x = (x_1, x_2, x_3) : x_3 \leq 0\} \ ;$$

see, e.g., the picture below.

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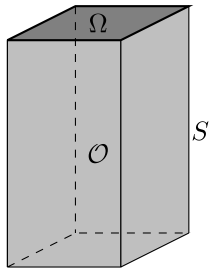

In consequence, if $\nu(x)$ denotes the exterior unit normal vector to $\partial\mathcal{O}$, then

$$\nu|_\Omega = [0,0,1]\,. \tag{1.1}$$

In addition, we assume throughout that the domain $\mathcal{O}$ and the boundary interface Ω satisfy the following assumption (where the picture above illustrates a geometrical configuration which is consistent with the case (G.2)):

Assumption 1.1 (Geometric Assumption). *The pair $[\mathcal{O},\Omega]$ is assumed to fall within one of the following classes:*

(G.1) *$\mathcal{O}$ is a convex domain with wedge angles $\leq \frac{2}{3}\pi$. Moreover, Ω has smooth boundary, and S is a piecewise smooth surface;*

(G.2) *$\mathcal{O}$ is a convex polyhedron having angles $\leq \frac{2}{3}\pi$, and so then Ω is a convex polygon with angles $\leq \frac{2}{3}\pi$.*

Remark 1.2. The reason for Assumption 1.1 is that, if $[\mathcal{O},\Omega]$ obey either (G.1) or (G.2), then for smooth initial data $[u_0,w_1,w_2]$ one is assured of sufficiently smooth solutions $[u,w,w_t]$ to the fluid-structure system (1.2) below. (See [4, Appendix], and the definition of associated fluid-structure generator $A_\rho\colon \mathbf{H}_\rho \to \mathbf{H}_\rho$ in (2.9)-(2.10) of Section 2.) Such higher regularity will justify the PDE multiplier computations, which are requisite for our derivation of rational decay rates.

With respect then to the assumed geometry, and with "rotational inertia parameter" $\rho \geq 0$, the PDE model is as follows, in solution variables $u(x,t) = [u^1(x,t),u^2(x,t),u^3(x,t)]$ and $w(x,t)$:

$$\begin{aligned}
&u_t - \Delta u + \nabla p = 0 && \text{in } \mathcal{O}\times(0,T) && \text{(1.2a)}\\
&\operatorname{div}(u) = 0 && \text{in } \mathcal{O}\times(0,T) && \text{(1.2b)}\\
&u = 0 && \text{on } S && \text{(1.2c)}\\
&u = [u^1,u^2,u^3] = [0,0,w_t] && \text{on } \Omega\,, && \text{(1.2d)}\\
&w_{tt} - \rho\Delta w_{tt} + \Delta^2 w = p|_\Omega && \text{in } \Omega\times(0,T) && \text{(1.2e)}\\
&w = \frac{\partial w}{\partial \nu} = 0 && \text{on } \partial\Omega && \text{(1.2f)}
\end{aligned}$$

with initial conditions

$$[u(0),w(0),w_t(0)] = [u_0,w_0,w_1] \in \mathbf{H}_\rho\,. \tag{1.3}$$

Here, the space of initial data $\mathbf{H}_\rho$ is defined as follows: Let

$$\mathcal{H}_f = \Big\{ f \in \mathbf{L}^2(\mathcal{O})\colon \operatorname{div}(f) = 0\,,\ f\cdot\nu|_S = 0 \Big\}\,; \tag{1.4}$$

and

$$V_\rho = \begin{cases} H_0^1(\Omega) \cap \hat{L}^2(\Omega) & \text{if } \rho > 0 \\ \hat{L}^2(\Omega) & \text{if } \rho = 0\,, \end{cases}$$

where

$$\hat{L}^2(\Omega) := \left\{ \omega \in L^2(\Omega) : \int_\Omega \omega \, d\Omega = 0 \right\}.$$

Therewith, we then set

$$\begin{aligned} \mathbf{H}_\rho = \Big\{ \big[f, h_0, h_1\big] \in \mathcal{H}_f \times \big[H_0^2(\Omega) \cap \hat{L}^2(\Omega)\big] \times V_\rho\,, \\ \text{with } \; f \cdot \nu|_\Omega = [0,0,f^3] \cdot [0,0,1] = h_1 \Big\}\,. \end{aligned} \tag{1.5}$$

As thus presented, the fluid PDE component of this fluid-structure dynamics consists of a three-dimensional incompressible Stokes flow which evolves within the walled cavity $\mathcal{O}$, in solutions variables $u(x,t)$ and $p(x,t)$, with u being the fluid velocity and p the pressure contraint (see (1.2a)-(1.2b)). As for the structural component: on the cavity wall portion Ω a fourth order plate equation of either Kirchhoff ($\rho > 0$) or Euler-Bernoulli ($\rho = 0$) type is invoked to describe the displacements along Ω; *clamped* boundary conditions are in place on $\partial\Omega$ (see (1.2e)-(1.2f)).

In addition, we note that for the fluid PDE component, the no-slip boundary condition is in play *only* on the wall S of the fluid container (see (1.2c)). In particular, there is a matching of velocities on Ω, by way of accomplishing the coupling betweeen the respective fluid and structure components (see (1.2d)). Moreover, the disparate dynamics are coupled via the Dirichlet boundary trace of the pressure; in particular, pressure variable p appears as a forcing term in the plate equation (1.2e) in Ω. We should also state that in general, fluid-structure PDE models with 'fixed' boundary interface Ω are physically relevant when operating under the assumption that these cavity wall displacements are small relative to the scale of the geometry; see [21].

We remind the reader that well-posedness for the initial/boundary value problem (1.2)-(1.3) when $\rho = 0$ (i.e., when the elastic equation is of Euler-Bernoulli type), was originally established in [19], by using Galerkin approximations. A novel proof of well-posedness pertaining to both cases $\rho = 0$ and $\rho > 0$, based upon the classical Lumer-Phillips Theorem (where the linear dynamics is governed by an appropriate operator A_ρ; that is, (2.9) below with domain (2.10)), as well as on a particular use of the Babuška-Brezzi Theorem (see, e.g., [29, p. 116]), has been recently given in [5]. The corresponding statement is as follows.

Theorem 1.3 (**Well-posedness [5]**). *The operator $A_\rho : \mathbf{H}_\rho \to \mathbf{H}_\rho$ defined by* (2.9)-(2.10) *generates a C_0-semigroup of contractions $\left\{e^{At}\right\}_{t\geq 0}$ on $\mathbf{H}_\rho$. Thus, given $[u_0, w_0, w_1] \in \mathbf{H}_\rho$, the weak solution $[u, w, w_t] \in C([0,T]; \mathbf{H}_\rho)$ of* (1.2)-(1.3) *is given by* (2.4).

Remark 1.4. The issues of well-posedness in the natural energy space, or of (local in time) existence, uniqueness and regularity of solutions have been the object of extensive investigation during the last decade, especially in the case of a well-recognized *nonlinear* FSI for the motion of an elastic body immersed in an incompressible fluid; see [11] and [31], along with their references. Variational and semigroup well-posedness for linearized versions of this model has also been considered in the literature; e.g., in [21] and [8], respectively. It is here important to emphasize that the FSI PDE models which are considered in the above-mentioned references are structurally quite different – in terms of their respective coupling mechanisms – than the present one under consideration. For the well-posedness and long-time behaviour analysis of other closely related fluid-plate PDE models, we refer the reader to [18] and [15].

The literature on FSI with moving interface is also extensive; see [20] and the recent [27], along with references therein. (For the sake of conciseness and consistency we omit any reference to FSI with compressible fluids.)

If one performs a simple energy method, which would commence by multiplying the fluid PDE (1.2a) by u and the structural PDE (1.2e) by w_t, and continue by integrating in time and space, one would find an underlying dissipation of the energy which governs the fluid-structure system. This dissipation comes solely from the gradient of the fluid component u. Given this fluid dissipation which propagates onto the entire fluid-structure PDE, an investigation of the stability properties for this coupled system would seem to be appropriate. We recall that uncoupled Stokes flow is governed by a uniformly decaying (and analytic) C_0-semigroup. On the other hand, uncoupled Kirchhoff plate dynamics exhibits a conservation of energy.

Thus, in the present work the long-time behaviour, as $t \to +\infty$, of the linear dynamics described by (1.2) is addressed, with focus on the more challenging case $\rho > 0$ (the elastic equation is of Kirchhoff type). When $\rho = 0$, uniform (exponential) stability of finite energy solutions holds true; this issue has been dealt with in [19], by using Lyapunov function arguments (in the *time domain*). A different proof of the aforesaid result has been subsequently given in [4], with a proof geared rather toward establishing certain sufficient resolvent estimates in the *frequency domain.* A similar 'frequency domain perspective' leads us in the case $\rho > 0$ to infer a weaker notion of uniform decay for the fluid-structure problem (1.2)-(1.3). In particular, the main result of this paper is the following stability result pertaining to *strong* solutions, which provides *rational* rates of decay.

Theorem 1.5 (Main result: Rational decay rates). *Let the rotational inertia parameter ρ be positive in* (1.2e). *Then for initial data $[u_0, w_0, w_1] \in \mathcal{D}(A_\rho)$, the corresponding solution $[u, w, w_t] \in C([0, T]; \mathcal{D}(A_\rho))$ of* (1.2)-(1.3) *satisfies the following decay rate for time t large enough:*

$$\left\| [u(t), w(t), w_t(t)] \right\|_{\mathbf{H}_\rho} \leq \frac{C}{t} \left\| [u_0, w_0, w_1] \right\|_{\mathcal{D}(A_\rho)} . \tag{1.6}$$

Remark 1.6. The primary virtue of this result is that it establishes uniform stability, along with explicit decay rates, for an actual FSI – namely, with pressure term actually present in the PDE model – with no added dissipation; see the detailed discussion in the next Section.

Remark 1.7. Exponential decay for solutions of the present FSI model does not seem likely: owing to the coupling mechanism of the disparate PDE dynamics, via the matching of structural and fluid velocities, control of the mechanical velocity solution variable in $H^1(\Omega)$-norm is quite problematic. By contrast, exponential decay of the FSI model for $\rho = 0$ is possible, inasmuch as the mechanical velocity solution component can be readily controlled in (energy) L^2-norm, via the Dirichlet trace of the dissipative fluid velocity; see [19] and [4]. Given the uniform decay rate of order $O(1/t^{1-\epsilon})$ which was obtained in [22] for a 'simplified' FSI model (see also [9]), the rational rate obtained in Theorem 1.5 appears optimal.

1.2 Background and further remarks

We should note that in principle, one might attempt to derive the rational decay estimate (1.6) by an analysis in the "time domain"; the associated energy method is in principle abstractly outlined, e.g., in [33, Theorem 3.2.2, p. 43]. However, the details of proof in the time domain would seem to be quite daunting; a technical insight on this issue is given below.

As it is well known, the energy/multipliers method underlies the pioneering contributions to the study of stabilization of *single* wave and plate equations, dating back to the seventies, as well as the fundamental work carried out by many authors during the eighties and nineties. (See the monographs [32], [30], [37, Vol. II] and their references.) The cornerstone work [36] laid the foundations for a fairly general method to derive decay rates for PDE systems, under the possible challenges of nonlinear localized or boundary damping in place, as well as nonlinear forces, with the wave equation as a prototype PDE model (we recall explicitly [41], [1] and the conclusive [45], beside [33, Section 3.2]).

It has been shown in the existing literature that intrinsically appropriate differential and/or operator theoretic multipliers can be useful in the task of attaining decay rates for the associated energies of certain, physically relevant, systems of *coupled* hyperbolic/parabolic PDE. Examples of coupled PDE systems which are amenable to such treatment include thermoelastic systems, where it was shown in [7] that an underlying dissipation, which emanates solely from the parabolic component, suffices to render the entire composite system uniformly exponentially stable. The aforesaid ideas and techniques have been adapted to pinpoint the stability properties of other composite systems of PDE, such as magneto-elastic or magneto-thermoelastic systems (see, e.g., [43]), structural acoustic models (in the absence or presence of thermal effects: see [3], [38]); and only very recently, certain FSI. It should be observed that the PDE analysis of both latter problems (acoustic-structure and fluid-structure interactions) give rise to significantly higher difficulties, these partly owing to the coupling which takes place through *boundary traces*.

While a thorough literature review would go much beyond the scope of the present Introduction, it is however important to emphasize that inasmuch as the subject of *well-posedness itself* of certain FSI has been an open problem until very recently, even in the case of linearized versions like the Stokes-Lamé system (see Remark 1.4), the topic of uniform stability of FSI is presently rich with open questions. With the exception of the aforementioned [19], along with [4] – each which deals with the same kind of FSI studied in the present work, and both providing a proof of exponential rates of uniform decay for finite energy solutions – and of this contribution, the studies on the stability properties of solutions to PDE systems which describe actual FSI, in the absence of any form of additional dissipation, are ongoing as conclusive answers are still lacking.

Most results available in the literature concern 'simplified' models comprising a heat equation (in place of the Navier-Stokes or the Stokes system) and a vectorial wave equation (in place of the system of elasticity), wherein the coupling of the two dynamics occurs through a boundary interface, via certain (flux) transmission conditions. Indeed, difficult technical hurdles are encountered already in the preliminary study of these systems of coupled heat-wave PDE. See [46], and [22] (the latter obtaining the sharper rates of decay $O(1/t^{1-\epsilon})$), whose analysis is based on the multiplier method in the time domain, and on the strength of requisite observability inequalities. In contrast, the results of [9] rely upon an analysis in the frequency domain, with an ultimate view of invoking the powerful resolvent criterion devised in [12]. The proper use of this abstract resolvent criterion entails the derivation of a series of nontrivial PDE estimates, for said simplified FSI. (The announced proof in [9] of the optimal rate $O(1/t)$ for the decay of strong solutions has been deferred to a separate, forthcoming paper.)

On the other hand, the recent study in [10] is carried out for a physically relevant FSI of the literature – i.e., with *pressure term present* –, where the geometry is that of a three-dimensional elastic body, immersed in a fluid occupying a three-dimensional space; see, e.g., [21]. The work in [10] actually yields *exponential decay* for the given FSI dynamics; however, to achieve this result, an *additional mechanical dissipation* is incorporated in the model.

We should also point out the papers [34, 35], which are devoted to the feedback stabilization problem for the true (*nonlinear*) model of the same FSI; see also the references therein. The reader is referred to the monograph [16, Chapters 5, 6, 11, 12] and to the recent paper [17] for well-posedness and asymptotic behaviour results/techniques provided for other flow and flow-plate interactions.

Aiming as we are to obtain sharp decay rate estimates for the solutions to the initial/boundary value problem (IBVP) (1.2)-(1.3), with the preliminary analysis of [4] in hand, which considered the case that the elastic wall dynamics is modeled via the Euler-Bernoulli equation (that is, when $\rho = 0$ in (1.2e)), we choose as in [4] to operate here in the "frequency domain". To wit, we will invoke an energy method with respect to a formally 'Laplace transformed' version of the system (1.2)-(1.3), with the ultimate objective of applying the sharp resolvent criterion in [12] (see a penultimate version of this resolvent criterion in [40]). As we mentioned above, such a frequency domain approach was previously utilized in [9], by way of establishing rational decays for a coupled heat-wave system.

We should state at the outset that one advantage which the frequency domain method enjoys over the time domain one, is that the former eventually allows for an adequate treatment of the pressure variable (as it appears as a forcing term in the plate equation (1.2e)). In particular, upon formally taking the Laplace Transform of (1.2)-(1.3), we will obtain a corresponding static fluid-structure system with frequency domain parameter β essentially replacing time variable t (see (4.4) below). Subsequently, one can then attempt to invoke classic Stokes Theory for (static) incompressible fluid flows.

Alternatively, if one were working directly with the time evolving fluid-structure system (1.2)-(1.3), by way of analyzing the pressure term $p(x,t)|_{\Omega}$, it seems likely to us that there would be the necessity of microlocalizing the fluid-structure system in order to obtain the required *a priori* estimates. Besides being quite technical in its own right, such a pseudo-differential approach might even be ultimately unavailing, inasmuch as there would be the issue of keeping a close track of the time dependent constants which would surely accumulate in the course of such a microlocal analysis. Hence, we are drawn instead to a frequency domain analysis which would ultimately appeal to the aforesaid resolvent criterion in [12], recorded as Theorem 3.1 in Section 3.

We finally note that uniform stability results for higher dimensional coupled PDE systems (namely, involving equations on n-dimensional manifolds, with $n > 1$), which are attained via frequency domain methods, are largely not available in the literature. We recall the recent polynomial decay result obtained in [25] for a complicated Mindlin-Timoshenko plate model, which depends upon a frequency domain approach and an argument by contradiction, with a view of invoking the aforesaid resolvent criterion in [12]; see also [26]. In general, those few higher dimensional results typically rest upon an argument by contradiction, in the style of [40], by way of establishing the requisite resolvent estimate in Theorem 3.1. By contrast, in the present paper we *explicitly* generate the necessary frequency domain estimate, as in [9] and [10].

We finally point out the recent work [2] where – in the case of a one-dimensional model – appropriate resolvent estimates are established in order to infer exponential or polynomial stability for a thermoelastic Timoshenko beam, with decay rates that are shown to be optimal. (The one dimensionality of the problem plays a key role in the proof of optimality.)

Outline of the paper. We conclude this Introduction with a brief overview of the paper's contents and organization. In Section 2 we introduce the functional and semigroup setup for the PDE problem. A PDE result pertaining to the fluid pressure, obtained in [6] and recorded here as Lemma 2.1, allows for a clear (and yet non trivial) definition of the linear operator which underlies the fluid-structure dynamics.

Section 3 is devoted to the spectral analysis for the dynamics operator, this being of intrinsic interest, as well as a prerequisite step for the proof of our main result, that is Theorem 1.5. The core of the proof of Theorem 1.5 is found in Section 4.

2 Semigroup framework

In this Section we recall from [6] the semigroup setup for the IBVP problem (1.2)-(1.3). Although the C_0-semigroup $e^{A_\rho t}$ underlying the linear evolution (1.2)-(1.3) was introduced in [19], a more explicit definition of the dynamics operator is given, based upon a certain identification of the fluid pressure p as the solution to an appropriate elliptic problem; see Lemma 2.1 below.

Let $A_D : \mathcal{D}(A_D) \subset L^2(\Omega) \to L^2(\Omega)$ be given by

$$A_D g = -\Delta g\,, \quad \mathcal{D}(A_D) = H^2(\Omega) \cap H_0^1(\Omega)\,. \tag{2.1}$$

If we subsequently make the denotation for all $\rho \geq 0$,

$$P_\rho = I + \rho A_D\,, \quad \mathcal{D}(P_\rho) = \begin{cases} \mathcal{D}(A_D) & \text{if } \rho > 0 \\ L^2(\Omega) & \text{if } \rho = 0 \end{cases}\,, \tag{2.2}$$

then the mechanical component (1.2e)-(1.2f) can be written as

$$P_\rho w_{tt} + \Delta^2 w = p|_\Omega \quad \text{ on } (0,T)\,.$$

Using that

$$\mathcal{D}(P_\rho^{1/2}) = \begin{cases} H_0^1(\Omega) & \text{if } \rho > 0 \\ L^2(\Omega) & \text{if } \rho = 0 \end{cases}\,,$$

(see [24]), we can endow the Hilbert space $\mathbf{H}_\rho$ with norm-inducing inner product

$$\big([f, \omega_0, \omega_1], [\tilde{f}, \tilde{\omega}_0, \tilde{\omega}_1]\big)_{\mathbf{H}_\rho} = (f, \tilde{f})_{\mathcal{O}} + (\Delta\omega_0, \Delta\tilde{\omega}_0)_\Omega + (P_\rho^{1/2}\omega_1, P_\rho^{1/2}\tilde{\omega}_1)_\Omega\,,$$

where $(\cdot,\cdot)_{\mathcal{O}}$ and $(\cdot,\cdot)_\Omega$ are the L^2-inner products on their respective geometries.

We note here the necessity, as there was in [19], for imposing that the wave initial displacement and velocity each have zero mean average. To see this: invoking the boundary condition (1.2c)-(1.2d) and the fact that the normal vector ν coincides with $[0,0,1]$ on Ω , we have then by Green's formula, that for all $t \geq 0$,

$$\int_\Omega w_t(t)\,dx = \int_\Omega u^3(t)\,dx = \int_{\partial\mathcal{O}} u(t)\cdot\nu\,d\sigma = 0\,. \tag{2.3}$$

And so we have necessarily,

$$\int_\Omega w(t)\,dx = \int_\Omega w_0\,dx \quad \text{for all } t \geq 0\,.$$

This accounts for the choice of the structural finite energy space components for $\mathbf{H}_\rho$, in (1.5).

Well-posedness of the IBVP (1.2)-(1.3) has been fully discussed in [5] and [6, Appendix] for both cases $\rho > 0$ and $\rho = 0$. The proof of well-posedness provided in [5] hinges upon demonstrating the existence of a modeling semigroup $\{e^{A_\rho t}\}_{t\geq 0} \subset \mathcal{L}(\mathbf{H}_\rho)$, for an appropriate generator $A_\rho : \mathbf{H}_\rho \to \mathbf{H}_\rho$. Subsequently, by means of this family, the solution to (1.2)-(1.3), for initial data $[u_0, w_0, w_1] \in \mathbf{H}_\rho$, will then of course be given via the relation

$$\begin{bmatrix} u(t) \\ w(t) \\ w_t(t) \end{bmatrix} = e^{A_\rho t} \begin{bmatrix} u_0 \\ w_0 \\ w_1 \end{bmatrix} \in C([0,T]; \mathbf{H}_\rho)\,. \tag{2.4}$$

We recall here that the particular choice here of generator $A_\rho : \mathbf{H}_\rho \to \mathbf{H}_\rho$ is dictated by the following consideration, whose proof is given for the reader's convenience.

Lemma 2.1 ([5]). *If $p(t)$ is a viable pressure variable for* (1.2)-(1.3)*, then pointwise in time $p(t)$ necessarily satisfies the following boundary value problem, for $[u(t), w(t)]$ "smooth enough":*

$$\begin{cases} \Delta p = 0 & \text{in } \mathcal{O} \\ \frac{\partial p}{\partial \nu} = \Delta u \cdot \nu\big|_S & \text{on } S \\ \frac{\partial p}{\partial \nu} + P_\rho^{-1} p = P_\rho^{-1}\Delta^2 w + \Delta u^3\big|_\Omega & \text{on } \Omega\,. \end{cases} \tag{2.5}$$

Proof. To show that p is harmonic in Ω, we take the divergence of both sides of (1.2a) and use the divergence free condition in (1.2b). Moreover, dotting both sides of (1.2a) with the unit normal vector ν, and then subsequently taking the resulting trace on S will yield the boundary condition in (2.5) that pertains to S. (Implicitly, we are also using the fact that $u = 0$ on S.)

Finally, we consider the particular geometry which is in play (with $\nu = [0, 0, 1]$ on Ω). Using the equation (1.2e) and the boundary condition (1.2d), we have on Ω

$$\begin{aligned} P_\rho^{-1}\Delta^2 w &= -w_{tt} + P_\rho^{-1} p\big|_\Omega \\ &= -\frac{d}{dt}(0, 0, w_t) \cdot \nu + P_\rho^{-1} p\big|_\Omega \\ &= -\left[u_t \cdot \nu\right]_\Omega + P_\rho^{-1} p\big|_\Omega \\ &= -\left[\Delta u \cdot \nu\right]_\Omega + \frac{\partial p}{\partial \nu}\Big|_\Omega + P_\rho^{-1} p\big|_\Omega\,, \end{aligned}$$

which gives the boundary condition in (2.5) that pertains to Ω. □

The boundary value problem (BVP) (2.5) can be solved through the agency of appropriate harmonic extensions from the boundary of $\mathcal{O}$, that are the 'Robin-Neumann' maps R_ρ and $\tilde{R}_\rho$ defined by

$$R_\rho g = f \iff \Big\{\Delta f = 0 \ \text{ in } \mathcal{O}\,,\ \frac{\partial f}{\partial \nu} = 0 \text{ on } S\,,\ \frac{\partial f}{\partial \nu} + P_\rho^{-1} f = g \ \text{ on } \Omega\Big\}\,;$$

$$\tilde{R}_\rho g = f \iff \Big\{\Delta f = 0 \ \text{ in } \mathcal{O}\,,\ \frac{\partial f}{\partial \nu} = g \text{ on } S\,,\ \frac{\partial f}{\partial \nu} + P_\rho^{-1} f = 0 \text{ on } \Omega\Big\}\,.$$

By the Lax-Milgram Theorem, we have

$$R_\rho \in \mathcal{L}\big(H^{-1/2}(\Omega), H^1(\mathcal{O})\big)\,, \quad \tilde{R}_\rho \in \mathcal{L}\big(H^{-1/2}(S), H^1(\mathcal{O})\big)\,. \tag{2.6}$$

(We are also using implicitly the fact that P_ρ^{-1} is positive definite, self-adjoint on Ω.)

Therewith, the pressure variable $p(t)$, as necessarily the solution of (2.5), can be written pointwise in time as

$$p(t) = G_{\rho,1} w(t) + G_{\rho,2} u(t)\,, \tag{2.7}$$

where $G_{\rho,1}$ and $G_{\rho,2}$ are defined as follow:

$$G_{\rho,1} w = R_\rho(P_\rho^{-1}\Delta^2 w)\,; \tag{2.8a}$$

$$G_{\rho,2} u = R_\rho(\,\Delta u^3\big|_\Omega) + \tilde{R}_\rho(\,\Delta u \cdot \nu|_S)\,. \tag{2.8b}$$

The above relations suggest the following choice for the generator $A_\rho : \mathbf{H}_\rho \to \mathbf{H}_\rho$. We set

$$A_\rho \equiv \begin{bmatrix} \Delta - \nabla G_{\rho,2} & -\nabla G_{\rho,1} & 0 \\ 0 & 0 & I \\ P_\rho^{-1} G_{\rho,2}\big|_\Omega & -P_\rho^{-1}\Delta^2 + P_\rho^{-1} G_{\rho,1}\big|_\Omega & 0 \end{bmatrix} \tag{2.9}$$

with domain

$$\mathcal{D}(A_\rho) = \Big\{ [u, w_1, w_2] \in \mathbf{H}_\rho : \; u \in \mathbf{H}^2(\mathcal{O}) \, ; \quad w_1 \in \mathcal{S}_\rho \, , \; w_2 \in H_0^2(\Omega) \, , \\ u = 0 \text{ on } S \, , \quad u = (0, 0, w_2) \text{ on } \; \Omega \Big\} \, , \tag{2.10}$$

where the space $\mathcal{S}_\rho$ in (2.10) is as follows:

$$\mathcal{S}_\rho := \begin{cases} H^3(\Omega) \cap H_0^2(\Omega) \, , & \rho > 0 \\ H^4(\Omega) \cap H_0^2(\Omega) \, , & \rho = 0 \, . \end{cases}$$

The fact that the operator A_ρ defined by (2.9)–(2.10) is the infinitesimal generator of a C_0-semigroup $e^{A_\rho t}$ on the space $\mathbf{H}_\rho$, for any given $\rho \geq 0$, has been shown in [5]. (See also [4, Appendix], where a detailed proof of higher regularity of strong solutions is produced.) Thus, well-posedness of the IBVP (1.2)-(1.3) in the function space $\mathbf{H}_\rho$ is an immediate consequence (see Theorem 1.3).

Remark 2.2. Note that if $[u, w_1, w_2] \in \mathcal{D}(A_\rho)$, then $\Delta u \cdot \nu|_{\partial\mathcal{O}} \in \mathbf{H}^{-1/2}(\partial\mathcal{O})$, and so $A_\rho[u, w_1, w_2]$ is indeed well-defined (note in particular the entries 1-1 and 3-1 of matrix A_ρ and (2.8b)). This is so, as $u \in \mathbf{H}^2(\mathcal{O})$, and Δu is divergence free. Indeed, in general we have the classic basic trace estimate for any function $\mu \in \mathbf{L}^2(\mathcal{O})$ with $\mathrm{div}(\mu) \in L^2(\mathcal{O})$:

$$\|\Delta\mu \cdot \nu|_{\partial\mathcal{O}}\|_{H^{-1/2}(\mathcal{O})} \leq C \left(\|\mu\|_{\partial\mathcal{O}} + \|\mathrm{div}(\mu)\|_{\mathcal{O}} \right) \tag{2.11}$$

(see e.g., [44, Theorem 1.2, p. 7]).

In what follows, we will have need of the following regularity result for solutions of Stokes system on Ω, in the scenarios (G.1) or (G.2) of Assumption 1.1. (As Ω is a bounded Lipschitz domain, then the well known elliptic regularity results in [44] do not ostensibly apply.)

Lemma 2.3 (Lemma 3.1 in [4])**.** *With $[\mathcal{O}, \Omega]$ obeying the Assumption 1.1 – including the flatness of Ω and (G.1)-(G.2) – we consider the following inhomogeneous Stokes problem, with parameter $\lambda \geq 0$:*

$$\begin{aligned} \lambda u - \Delta u + \nabla p &= u^* && \textit{in } \mathcal{O} \\ \mathrm{div}(u) &= 0 && \textit{in } \mathcal{O} \\ u|_S &= [0, 0, 0] && \textit{on } S \\ u|_\Omega &= [0, 0, w] && \textit{in } \Omega, \end{aligned} \tag{2.12}$$

where data $[u^*, w] \in \mathbf{L}^2(\mathcal{O}) \times H_0^{3/2+\epsilon}(\Omega)$, with $\epsilon > 0$, and w satisfying the compatibility condition $\int_\Omega w \, d\Omega = 0$. Then one has the following regularity estimate for the solution pair $[u, p]$:

$$\|u\|_{\mathbf{H}^2(\mathcal{O})} + \|p\|_{H^1(\mathcal{O}) \cap [L^2(\mathcal{O})/\mathbb{R}]} \leq C_\lambda \|[u^*, w]\|_{\mathbf{L}^2(\mathcal{O}) \times H_0^{3/2+\epsilon}(\Omega)} \, . \tag{2.13}$$

Hereafter the parameter of rotational inertia ρ will be assumed to be positive. The case $\rho = 0$ was treated in [4].

3 Spectral analysis

In order to establish the sharp estimate of the decay rates for the solutions of the PDE system, we will, as we said, use a powerful *frequency domain* criterion by Borichev and Tomilov, which for the readers' convenience is recorded below.

Theorem 3.1 ([12]). *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a Hilbert space H with generator A, such that $i\mathbb{R} \subset \varrho(A)$. Then, for fixed $\alpha > 0$ the following are equivalent:*

$$(i) \quad R(is;A) = O(|s|^{\alpha}), \qquad |s| \to \infty; \tag{3.1}$$
$$(ii) \quad \|T(t)x\|_H = o(t^{-1/\alpha})\|x\|_{\mathcal{D}(A)}, \qquad t \to +\infty.$$

To apply the above result, we need as a preliminary to show that the imaginary axis belongs to the resolvent set $\varrho(A_\rho)$ of the dynamics operator A_ρ. The present Section is entirely devoted to this objective.

3.1 $\lambda = 0$ is in the resolvent set $\varrho(A_\rho)$

We begin our analysis by showing that the dynamics operator A_ρ is *boundedly invertible* on the state space $\mathbf{H}_\rho$, for $\rho > 0$.

Proposition 3.2. *The generator $A_\rho : \mathcal{D}(A_\rho) \subset \mathbf{H}_\rho \to \mathbf{H}_\rho$ is boundedly invertible on $\mathbf{H}_\rho$. Namely, $\lambda = 0$ is in the resolvent set of A_ρ.*

Proof. Given data $[\mu^*, \omega_1^*, \omega_2^*] \in \mathbf{H}_\rho$, we look for $[\mu, \omega_1, \omega_2] \in \mathcal{D}(A_\rho)$ which solves

$$A_\rho \begin{bmatrix} \mu \\ \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} \mu^* \\ \omega_1^* \\ \omega_2^* \end{bmatrix}. \tag{3.2}$$

Component-wise, we obtain then from (2.9) the coupled system of operator theoretic equations

$$\begin{cases} \Delta\mu - \nabla G_2\mu - \nabla G_1\omega_1 = \mu^* \\ \omega_2 = \omega_1^* \\ P_\rho^{-1}G_2\big|_\Omega \mu - P_\rho^{-1}\Delta^2\omega_1 + P_\rho^{-1}G_1\big|_\Omega \omega_1 = \omega_2^*, \end{cases}$$

where we denoted, in short, by G_i the linear operators $G_{\rho,i}$, $i = 1, 2$, defined in (2.8a) and (2.8b), respectively. We invoke now the definition of the G_i in (2.8), set

$$\pi_0 = G_1\omega_1 + G_2\mu, \tag{3.3}$$

and consider the definition of the fluid-structure generator (2.9)-(2.10). Then, finding $[\mu, \omega_1, \omega_2] \in \mathcal{D}(A_\rho)$ which solves the abstract equation (3.2) is equivalent to finding $\{[\mu, \omega_1, \omega_2], \pi_0\} \in \mathcal{D}(A_\rho) \times H^1(\mathcal{O})$ which solves

$$\Delta\mu - \nabla\pi_0 = \mu^* \quad \text{in } \mathcal{O} \tag{3.4a}$$
$$\text{div}\mu = 0 \quad \text{in } \mathcal{O} \tag{3.4b}$$
$$\mu = 0 \quad \text{on } S \tag{3.4c}$$
$$\mu = (0, 0, \omega_2) \quad \text{on } \Omega \tag{3.4d}$$
$$\omega_2 = \omega_1^* \quad \text{in } \Omega \tag{3.4e}$$
$$P_\rho^{-1}\Delta^2\omega_1 - P_\rho^{-1}\pi_0\big|_\Omega = -\omega_2^* \quad \text{in } \Omega \tag{3.4f}$$
$$\omega_1 = \frac{\partial\omega_1}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{3.4g}$$

(i) The Plate Velocity. From (3.4e), the velocity component ω_2 is immediately resolved.

(ii) The Fluid Velocity. We next consider the Stokes system (3.4a)-(3.4d). From (3.4e) and (3.4d) it follows that $\mu|_{\partial\mathcal{O}}$ satisfies

$$\int_{\partial\mathcal{O}} \mu\cdot\nu\, d\sigma = \int_{\Omega} \big[0,0,\mu_3\big]^T\cdot\nu\, d\sigma = \int_{\Omega}\omega_2\, d\sigma = \int_{\Omega}\omega_1^*\, d\sigma = 0 \tag{3.5}$$

(as $[\mu^*,\omega_1^*,\omega_2^*]\in\mathbf{H}_\rho$). Since this compatibility condition is satisfied and data $\{\mu^*,\omega_1^*\}\in\mathbf{L}^2(\mathcal{O})\times H_0^2(\Omega)$, we can find a unique (fluid and pressure) pair $(\mu,q_0)\in\big[\mathbf{H}^2(\mathcal{O})\cap\mathcal{H}_f\big]\times\mathbf{H}^1(\mathcal{O})/\mathbb{R}$ which solves

$$\begin{cases}\Delta\mu-\nabla q_0=\mu^* & \text{in } \mathcal{O}\\ \operatorname{div}(\mu)=0 & \text{in } \mathcal{O}\\ \mu=(0,0,\omega_1^*) & \text{in } \Omega,\quad \mu=0 \text{ in } S.\end{cases} \tag{3.6}$$

Moreover, one has the estimate

$$\|\mu\|_{\mathbf{H}^2(\mathcal{O})\cap\mathcal{H}_f}+\|q_0\|_{\mathbf{H}^1(\mathcal{O})/\mathbb{R}}\le C\left[\|\mu^*\|_{\mathcal{H}_f}+\|\omega_1^*\|_{H_0^2(\Omega)}\right]; \tag{3.7}$$

see [44, Theorem 2.4 and Remark 2.5] and Lemma 2.3 above.

(iii) The Mechanical Displacement. Subsequently, we consider the equations (3.4f)-(3.4g) pertaining to the (plate) component ω_1. By Lax-Milgram and either (i) classical elliptic theory (see [39]) if structural geometry Ω obeys assumption (G.1), or (ii) the "polygonal" regularity result in [13] if structural geometry Ω obeys assumption (G.2), we then have the following: there exists a solution $\widehat{\omega}_1\in H^3(\Omega)\cap H_0^2(\Omega)$ to the boundary value problem

$$\begin{cases}\Delta^2\widehat{\omega}_1=q_0|_\Omega-P_\rho\omega_2^* & \text{in } \Omega\\ \widehat{\omega}_1=\frac{\partial\widehat{\omega}_1}{\partial\nu}=0 & \text{on } \partial\Omega\,,\end{cases} \tag{3.8}$$

where q_0 is the (pressure) variable in (3.6); moreover, the following estimate holds true:

$$\begin{aligned}\|\widehat{\omega}_1\|_{H^3(\Omega\cap H_0^2(\Omega)} &\le C\,\|q_0|_\Omega+P_\rho\omega_2^*\|_{H^{-1}(\Omega)}\\ &\le C\,\|q_0|_\Omega\|_{H^{1/2}(\Omega)}+\|P_\rho\omega_2^*\|_{H^{-1}(\Omega)}\\ &\le C\,\|[\mu^*,\omega_1^*,\omega_2^*]\|_{\mathbf{H}_\rho}\,.\end{aligned} \tag{3.9}$$

(In the last inequality we have also invoked Sobolev trace theory and (3.7)).

Now if, as in [19], we let $\mathbb{P}$ denote the orthogonal projection of $H_0^2(\Omega)$ onto $H_0^2(\Omega)\cap\hat{L}^2(\Omega)$ (orthogonal with respect to the inner product $[\omega,\tilde{\omega}]\to(\Delta\omega,\Delta\tilde{\omega})_\Omega$), then one can readily show that its orthogonal complement $I-\mathbb{P}$ can be characterized as

$$(I-\mathbb{P})H_0^2(\Omega)=\operatorname{span}\Big\{\overline{\omega}:\ \Delta^2\overline{\omega}=1 \text{ in } \Omega\,,\ \overline{\omega}=\frac{\partial\overline{\omega}}{\partial\nu}=0 \text{ on } \partial\Omega\Big\}; \tag{3.10}$$

see [19, Remark 2.1, p. 1639].
With these projections, we then set

$$\omega_1:=\mathbb{P}\widehat{\omega}_1\,,\qquad \pi_0:=q_0-\Delta^2(I-\mathbb{P})\widehat{\omega}_1\,; \tag{3.11}$$

therefore, by (3.8) and $\widehat{\omega}_1=\mathbb{P}\widehat{\omega}_1+(I-\mathbb{P})\widehat{\omega}_1$, we will have that ω_1 solves (3.4f)-(3.4g). (And of course since π_0 and q_0 differ only by a constant, then the pair (μ,π_0) also solves (3.4a)-(3.4d).)

Thus, in view of classical elliptic theory (when Ω obeys G.1), or the polygonal regularity result in [13] (when Ω obeys G.2), (3.7) and ((3.9), we obtain the estimate

$$\begin{aligned}\|\omega_1\|_{H^3(\Omega)\cap H_0^2(\Omega)\cap\hat{L}^2(\Omega)} + \|\pi_0\|_{H^1(\mathcal{O})} \leq \\ \leq C\left(\|\Delta^2(I-\mathbb{P})\widehat{\omega}_1\|_{L^2(\Omega)}\|q_0\|_{H^1(\mathcal{O})/\mathbb{R}} + \|P_\rho\omega_2^*\|_{H^{-1}(\Omega)}\right) \\ \leq C\,\|[\mu^*,\omega_1^*,\omega_2^*]\|_{\mathbf{H}_\rho}\,,\end{aligned} \tag{3.12}$$

where implicity we are also using the fact that $\Delta^2(I-\mathbb{P}) \in \mathcal{L}(H_0^2(\Omega),\mathbb{R})$, by the Closed Graph Theorem.

(v) Resolution of the Pressure. At this point we invoke the estimate (2.11) in Remark 2.2, and (3.7), to have the following trace regularity for the fluid velocity in (3.6):

$$\begin{aligned}\left\|\Delta\mu\cdot\nu\right\|_{H^{-1/2}(\partial\mathcal{O})} &\leq C\left[\|q_0\|_{H^1(\mathcal{O})} + \|\mu^*\|_{\mathbf{L}^2(\mathcal{O})}\right] \\ &\leq C\left[\|\mu^*\|_{\mathcal{H}_f} + \|\omega_1^*\|_{H_0^2(\Omega)}\right].\end{aligned} \tag{3.13}$$

To use this estimate: the pressure variable π_0 of problem (3.2) – given explicitly in (3.11) – solves *a fortiori*

$$\begin{cases}\Delta\pi_0 = 0 & \text{in } \mathcal{O} \\ \frac{\partial\pi_0}{\partial\nu} = \Delta\mu\cdot\nu|_S & \text{on } S \\ \frac{\partial\pi_0}{\partial\nu} + P_\rho^{-1}\pi_0 = P_\rho^{-1}\Delta^2\omega_1 + \Delta\mu^3\big|_\Omega & \text{on } \Omega.\end{cases} \tag{3.14}$$

We justify the previous assertion. Applying the divergence operator to both sides of (3.4a) and using $\text{div}\mu = \text{div}\mu^* = 0$, we obtain that π_0 is harmonic in $\mathcal{O}$. Next, dotting both sides of (3.4a) with repect to the normal vector, and subsequently taking the boundary trace on the portion S, we get the corresponding boundary condition in (3.14). (Implicitly we are also using $\mu^*\cdot\nu|_S = 0$, as $[\mu^*,\omega_1^*,\omega_2^*] \in \mathbf{H}_\rho$).

Finally, since $\mu^*\cdot\nu|_\Omega = \omega_2^*$, as $[\mu^*,\omega_1^*,\omega_2^*] \in \mathbf{H}_\rho$, from (3.4f) it follows that

$$\begin{aligned}P_\rho^{-1}\pi_0\big|_\Omega &= \omega_2^* + P_\rho^{-1}\Delta^2\omega_1 \\ &= \Delta\mu\cdot\nu|_\Omega - \nabla\pi_0\cdot\nu|_\Omega + P_\rho^{-1}\Delta^2\omega_1,\end{aligned}$$

which gives the boundary condition on Ω.
Necessarily then, the pressure term must be given by the expression

$$\pi_0 = G_1\omega_1 + G_2\mu \in H^1(\mathcal{O})\,, \tag{3.15}$$

with the well-definition of the right hand side ensured by (3.13).

Finally, we collect: the fluid variable μ as the solution to (3.6) with the estimate (3.7), the respective structure and pressure variables ω_1, ω_2 and π_0 given by (3.4e) and (3.11), along with the estimate (3.12) (and where $\widehat{\omega}_1$ is defined by (3.8)); (3.15) characterizes the pressure π_0 in terms of the variables ω_1 and μ. This shows that the solution of (3.4) actually belongs to $\mathcal{D}(A_\rho)$. In short, $0 \in \varrho(A_\rho)$, which concludes the proof. □

3.2 $\lambda = i\beta$ is in the resolvent set $\varrho(A)$

Let us recall the expression of the dynamics operator semigroup A_ρ in (2.9). In straightforward fashion, one can then compute the associated adjoint operator $A_\rho^*\colon \mathcal{D}(A_\rho^*) \subset \mathbf{H}_\rho \to \mathbf{H}_\rho$ to be

$$A_\rho^* \equiv \begin{bmatrix}\Delta - \nabla G_{\rho,2} & \nabla G_{\rho,1} & 0 \\ 0 & 0 & -I \\ P_\rho^{-1}G_{\rho,2}|_\Omega & P_\rho^{-1}\Delta^2 - P_\rho^{-1}G_{\rho,1}|_\Omega & 0\end{bmatrix}, \tag{3.16}$$

with $\mathcal{D}(A_\rho^*) = \mathcal{D}(A_\rho)$. The above operator will be utilized in the proof of the following result.

Proposition 3.3. *Let $\sigma(A_\rho)$ be the spectrum of the dynamics operator A_ρ defined by* (2.9)-(2.10). *Then $i\mathbb{R} \cap \sigma(A_\rho) = \emptyset$.*

Proof. Let $\sigma_p(A_\rho)$, $\sigma_r(A_\rho)$, $\sigma_r(A_\rho)$ denote, respectively, the point, continuous and residual spectrum of the operator A.
1. (*Point spectrum*) We aim at showing that $i\mathbb{R} \cap \sigma_p(A_\rho) = \emptyset$. Given $\beta \in \mathbb{R} \setminus \{0\}$, we consider the equation

$$A_\rho \begin{bmatrix} \mu \\ \omega_1 \\ \omega_2 \end{bmatrix} = i\beta \begin{bmatrix} \mu \\ \omega_1 \\ \omega_2 \end{bmatrix} \tag{3.17}$$

for some $[\mu, \omega_1, \omega_2] \in \mathcal{D}(A_\rho)$. Moreover, we set

$$\pi_0 \equiv G_{\rho,1}(\omega_1) + G_{\rho,2}(\mu)\,. \tag{3.18}$$

Taking the inner product of both sides, and subsequently integrating by parts, then it follows

$$\begin{aligned}
i\beta \left\| \begin{bmatrix} \mu \\ \omega_1 \\ \omega_2 \end{bmatrix} \right\|^2_{\mathbf{H}_\rho} &= \left(A_\rho \begin{bmatrix} \mu \\ \omega_1 \\ \omega_2 \end{bmatrix}, \begin{matrix} \mu \\ \omega_1 \\ \omega_2 \end{matrix} \right) \\
&= (\Delta\mu - \nabla\pi_0, \mu)_{\mathcal{O}} + (\Delta\omega_2, \Delta\omega_1)_\Omega + (-\Delta^2\omega_1 + \pi_0|_\Omega, \omega_2)_\Omega = \\
&= (\pi_0|_\Omega (0,0,1), (0,0,\omega_2))_\Omega - (\nabla\mu, \nabla\mu)_{\mathcal{O}} + \left\langle \frac{\partial\mu}{\partial\nu}, \mu \right\rangle_\Omega - \langle \pi_0\nu, \mu\rangle_\Omega + \\
&\qquad + (\Delta\omega_2, \Delta\omega_1)_\Omega + (\nabla\Delta\omega_1, \nabla\omega_2)_\Omega = \\
&= (\Delta\omega_2, \Delta\omega_1)_\Omega - (\Delta\omega_1, \Delta\omega_2)_\Omega - (\nabla\mu, \nabla\mu)_{\mathcal{O}} + \\
&\qquad + \left(\begin{bmatrix} \partial_{x_3}\mu^1 \\ \partial_{x_3}\mu^2 \\ \partial_{x_3}\mu^3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \mu^3 \end{bmatrix} \right)_\Omega,
\end{aligned} \tag{3.19}$$

or

$$i\beta \left\| \begin{bmatrix} \mu \\ \omega_1 \\ \omega_2 \end{bmatrix} \right\|^2_{\mathbf{H}_\rho} = -\,\|\nabla\mu\|^2_{\mathcal{O}} - 2i\,\mathrm{Im}(\Delta\omega_1, \Delta\omega_2)_\Omega\,; \tag{3.20}$$

whence we obtain (after using Poincaré's Inequality)

$$\mu = 0 \quad \text{in } \mathcal{O}\,. \tag{3.21}$$

In turn, the boundary condition $\mu = (0, 0, \omega_2)$ on Ω, intrinsic to elements of $\mathcal{D}(A_\rho)$, yields as well

$$\omega_2 = 0 \quad \text{in } \Omega\,. \tag{3.22}$$

And furthermore, the second component relation in (3.17), combined with the appearance of A_ρ in (2.9), yield $i\beta\omega_1 = \omega_2$. Hence for $\beta \neq 0$,

$$\omega_1 = 0 \quad \text{in } \Omega\,. \tag{3.23}$$

The relations (3.21), (3.22) and (3.23) give the conclusion that $i\beta$ is not an eigenvalue of A_ρ.

2. (*Residual spectrum*) We aim at showing that $i\mathbb{R}\cap\sigma_r(A_\rho)=\emptyset$. Given $\beta\in\mathbb{R}\backslash\{0\}$, if $i\beta$ is in the residual spectrum of A_ρ, then necessarily $i\beta$ is in the point spectrum of $A_\rho^*: D(A_\rho^*)\subset\mathbf{H}_\rho\to\mathbf{H}_\rho$; see e.g., [23, p. 127]. In this case, given the appearance and the domain of A_ρ^* in (3.16), we proceed *verbatim* along the lines of Step 1. to deduce that $i\mathbb{R}\cap\sigma_r(A_\rho)=\emptyset$.

3. (*Continuous spectrum*) This is by far the most challenging part of the proof. To make the inference that $i\mathbb{R}$ has empty intersection with the continuous spectrum, it is enough to show that $i\mathbb{R}$ does not intersect with the *approximate spectrum*; see e.g., [23, p. 128].

To this end, let $\beta\in\mathbb{R}\setminus\{0\}$ be given. If $i\beta$ is in the approximate spectrum of A_ρ, then by definition there exists a sequence

$$\left\{\begin{bmatrix}\mu_n\\ \omega_{1,n}\\ \omega_{2,n}\end{bmatrix}\right\}_{n=1}^{\infty}\subset\quad\mathcal{D}(A_\rho)\text{ such that for all }n:\quad\left\|\begin{bmatrix}\mu_n\\ \omega_{1,n}\\ \omega_{2,n}\end{bmatrix}\right\|_{\mathbf{H}_\rho}=1$$

$$\text{and}\quad\begin{bmatrix}\mu_n^*\\ \omega_{1,n}^*\\ \omega_{2,n}^*\end{bmatrix}=(i\beta-A_\rho)\begin{bmatrix}\mu_n\\ \omega_{1,n}\\ \omega_{2,n}\end{bmatrix}\text{ satisfies }\left\|\begin{bmatrix}\mu_n^*\\ \omega_{1,n}^*\\ \omega_{2,n}^*\end{bmatrix}\right\|_{\mathbf{H}_\rho}<\frac{1}{n}. \tag{3.24}$$

We consider therewith the relation

$$(i\beta-A_\rho)\begin{bmatrix}\mu_n\\ \omega_{1,n}\\ \omega_{2,n}\end{bmatrix}=\begin{bmatrix}\mu_n^*\\ \omega_{1,n}^*\\ \omega_{2,n}^*\end{bmatrix}. \tag{3.25}$$

In PDE terms, each $[\mu_n,\omega_{1,n},\omega_{2,n}]$ satisfies the following problem:

$$i\beta\mu_n-\Delta\mu_n+\nabla p_n=\mu_n^*\quad\text{in }\mathcal{O} \tag{3.26a}$$
$$\operatorname{div}(\mu_n)=0\quad\text{in }\mathcal{O} \tag{3.26b}$$
$$\mu_n=0\quad\text{on }S \tag{3.26c}$$
$$\mu_n=(0,0,\omega_{2,n})\quad\text{on }\Omega \tag{3.26d}$$
$$i\beta\omega_{1,n}-\omega_{2,n}=\omega_{1,n}^*\quad\text{in }\Omega \tag{3.26e}$$
$$i\beta\omega_{2,n}+P_\rho^{-1}\Delta^2\omega_{1,n}-P_\rho^{-1}p_n\big|_\Omega=\omega_{2,n}^*\quad\text{in }\Omega \tag{3.26f}$$
$$\omega_{1,n}\big|_\Omega=\frac{\partial\omega_{1,n}}{\partial\nu}\Big|_\Omega=0\quad\text{on }\partial\Omega \tag{3.26g}$$

where, for each n, the associated pressure term is given by

$$p_n=G_1\omega_{1,n}+G_2\mu_n\,. \tag{3.27}$$

Multiplying both members of the expression (3.25) by $[\mu_n,\omega_{1,n},\omega_{2,n}]$ and integrating by parts gives

$$\|\nabla\mu_n\|_{\mathcal{O}}^2=\left(\begin{bmatrix}\mu_n^*\\ \omega_{1,n}^*\\ \omega_{2,n}^*\end{bmatrix},\begin{bmatrix}\mu_n\\ \omega_{1,n}\\ \omega_{2,n}\end{bmatrix}\right)_{\mathbf{H}_\rho}-i\beta\left\|\begin{bmatrix}\mu_n\\ \omega_{1,n}\\ \omega_{2,n}\end{bmatrix}\right\|_{\mathbf{H}_\rho}^2-2i\operatorname{Im}(\Delta\omega_{1,n},\Delta\omega_{2,n})_\Omega\,.$$

We have then from (3.24) that

$$\mu_n\longrightarrow 0\quad\text{(strongly) in }\mathbf{H}^1(\mathcal{O}). \tag{3.28}$$

In turn, from the boundary condition (3.26d) and the Sobolev Embedding Theorem, we have

$$\|\omega_{2,n}\|_{H^{1/2}(\Omega)} = \|\mu_n^3\|_{H^{1/2}(\Omega)} \leq C\,\|\mu_n\|_{\mathbf{H}^1(\mathcal{O})}\,,$$

whence

$$\omega_{2,n} \longrightarrow 0 \quad \text{(strongly) in } H^{1/2}(\Omega)\,. \tag{3.29}$$

At this point, we invoke the unique decomposition

$$p_n = c_n + q_n\,, \tag{3.30}$$

where for each n,

$$c_n = \text{constant}\,; \quad q_n \in L^2(\mathcal{O})/\mathbb{R}\,. \tag{3.31}$$

Then, from the known finite energy well-posedness of Stokes flow – see, e.g., Theorem 2.4 and Remark 2.5 of [44] – we have from (3.26a)-(3.26c)

$$\begin{aligned}\|q_n\|_{L^2(\mathcal{O})/\mathbb{R}} &\leq C\left(\|i\beta\mu_n\|_{\mathbf{L}^2(\mathcal{O})} + \|\mu_n\|_{\mathbf{H}^{1/2}(\partial\mathcal{O})} + \|\mu_n^*\|_{\mathbf{L}^2(\mathcal{O})}\right)\\ &\leq C_\beta\left(\|\mu_n\|_{\mathbf{H}^1(\mathcal{O})} + \|\mu_n^*\|_{\mathbf{L}^2(\mathcal{O})}\right)\,;\end{aligned} \tag{3.32}$$

whence we obtain from (3.28) and (3.24),

$$q_n \longrightarrow 0 \quad \text{strongly in } L^2(\mathcal{O})\,. \tag{3.33}$$

Moreover, since each q_n is harmonic *a fortiori*, we have available the boundary trace estimate

$$\begin{aligned}\|q_n|_{\partial\mathcal{O}}\|_{H^{-1/2}(\partial\mathcal{O})} &\leq C\,\|q_n\|_{L^2(\mathcal{O})}\\ &\leq C_\beta\,\left(\|\mu_n\|_{\mathbf{H}^1(\mathcal{O})} + \|\mu_n^*\|_{\mathcal{H}_{\mathrm{f}}}\right)\end{aligned} \tag{3.34}$$

(see e.g., [6, Proposition 1]; in attaining the second estimate we have also used (3.32)); appealing again to (3.28) and (3.24) we then have

$$q_n|_{\partial\mathcal{O}} \longrightarrow 0 \quad \text{strongly in } H^{-1/2}(\mathcal{O})\,. \tag{3.35}$$

Now using the decomposition (3.30) in the structural equation (3.26f), we have for all n,

$$c_n = -q_n|_\Omega + \Delta^2\omega_{1,n} + i\beta P_\rho\omega_{2,n} - P_\rho\omega_{2,n}^*\,,$$

and so a measurement in the $H^{-2}(\Omega)$-topology gives

$$\begin{aligned}|c_n|\,\|1\|_{H^{-2}(\Omega)} &= \left\| -q_n|_\Omega + \Delta^2\omega_{1,n} + i\beta P_\rho\omega_{2,n} - P_\rho\omega_{2,n}^*\right\|_{H^{-2}(\Omega)}\\ &\leq C_\beta\left(\|q_n|_\Omega\|_{H^{-1/2}(\Omega)} + \|\omega_{1,n}\|_{H_0^2(\Omega)} + \|\omega_{2,n}\|_{L^2(\Omega)} + \left\|\omega_{2,n}^*\right\|_{D(P_\rho^{1/2})}\right)\,.\end{aligned} \tag{3.36}$$

Combining (3.24), (3.35) and (3.29) with (3.36) we achieve the conclusion that

$$\{c_n\}_{n\geq 1} \quad \text{is uniformly bounded in } n.$$

Hence, there is a subsequence of constants – still denoted as $\{c_n\}_{n\geq 1}$ – which satisfies for some $\tilde{c}$,

$$c_n \to \tilde{c} \quad \text{(strongly) in } \mathbb{C}. \tag{3.37}$$

We now turn our attention to the mechanical system (3.26f)-(3.26g), that is

$$\begin{cases} \Delta^2\omega_{1,n} = p_n|_\Omega - i\beta P_\rho\,\omega_{2,n} + P_\rho\,\omega^*_{2,n} & \text{in } \Omega \\ \omega_{1,n} = \dfrac{\partial\omega_{1,n}}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

By way of looking at this sequence of boundary value problems, let us invoke the realization $\mathcal{A}$ of the biharmonic operator, defined by $\mathcal{A}\varphi := \Delta^2\varphi$, $\varphi \in \mathcal{D}(\mathcal{A}) = H^4(\Omega) \cap H^2_0(\Omega)$. Then we have abstractly

$$\mathcal{A}\omega_{1,n} = c_n + q_n|_\Omega - i\beta P_\rho\,\omega_{2,n} + P_\rho\,\omega^*_{2,n} \in \left[\mathcal{D}(\mathcal{A}^{1/2})\right]',$$

where $\mathcal{D}(\mathcal{A}^{1/2}) = H^2_0(\Omega)$ from [24].

Applying the inverse $\mathcal{A}^{-1} \in \mathcal{L}(L^2(\Omega), \mathcal{D}(\mathcal{A}))$ to both sides of the above equality gives

$$\omega_{1,n} = \mathcal{A}^{-1}c_n + \mathcal{A}^{-1}\big(q_n|_\Omega - i\beta P_\rho\,\omega_{2,n} + P_\rho\omega^*_{2,n}\big) \in \mathcal{D}(\mathcal{A}^{1/2})\,. \tag{3.38}$$

(In arriving at this conclusion, we are implicitly using the fact that $\mathcal{D}(\mathcal{A}^{1/2}) \subset \mathcal{D}(P_\rho)$.) Subsequently we can then pass to the limit in (3.38) (meanwhile using (3.37), (3.35), (3.29) and (3.24)) so as to have

$$\tilde{\omega} = \lim_{n\to\infty} \omega_{1,n} = \lim_{n\to\infty} \mathcal{A}^{-1}\,c_n = \mathcal{A}^{-1}\tilde{c}\,. \tag{3.39}$$

Thus, this (structural) limit must satisfy

$$\Delta^2\tilde{\omega} = \tilde{c} \quad \text{in } \Omega, \qquad \tilde{\omega} = \frac{\partial\tilde{\omega}}{\partial\nu} = 0 \quad \text{on } \partial\Omega. \tag{3.40}$$

Now since $\omega_{1,n} \in H^2_0(\Omega) \cap \left[\hat{L}^2(\Omega)\right]$ for every n, then so is strong limit $\tilde{\omega}$. But from (3.40) and the characterization (3.10), we have also that $\tilde{\omega} \in \left[H^2_0(\Omega) \cap \hat{L}^2(\Omega)\right]^\perp$. Thus,

$$\lim_{n\to\infty} \omega_{1,n} = 0\,. \tag{3.41}$$

Finally, from (3.26e),

$$\omega_{2,n} = i\beta\omega_{1,n} - \omega^*_{1,n}\,;$$

whence we obtain with (3.24) and (3.41),

$$\lim_{n\to\infty} \omega_{2,n} = 0 \text{ in } \mathcal{D}(P_\rho^{1/2}). \tag{3.42}$$

The limits in (3.41) and (3.42), combined with the one in (3.28), now contradict the fact from (3.24) that

$$\big\|[\mu_n, \omega_{1,n}, \omega_{2,n}]\big\|_{\mathbf{H}_\rho} = 1 \qquad \forall n\,.$$

Since $\beta \in \mathbb{R} \setminus \{0\}$ was arbitrary, we conclude that the approximate spectrum of A_ρ does not intersect with $i\mathbb{R}$. $\square$

4 Proof of Theorem 1.5 (Main result)

Here we will utilize Theorem 3.1 (see [12, Theorem 2.4]) in the case currently being considered; namely, $\rho > 0$, so that rotational forces are accounted for in the fluid-structure PDE dynamics. By way of ultimately invoking the aforesaid resolvent criterion, we consider arbitrary data

$[\mu^*, \omega_1^*, \omega_2^*] \in \mathbf{H}_\rho$, and the corresponding pre-image $[\mu, \omega_1, \omega_2] \in \mathcal{D}(A_\rho)$ which solve the following relation for given $\beta \in \mathbb{R}$:

$$(i\beta - A_\rho) \begin{bmatrix} \mu \\ \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} \mu^* \\ \omega_1^* \\ \omega_2^* \end{bmatrix} \in \mathbf{H}_\rho \, . \tag{4.1}$$

With respect to this relation, the proof of Theorem 1.5 will be established if we derive the following estimate for $|\beta|$ sufficiently large (and a positive constant C):

$$\left\| \begin{bmatrix} \mu \\ \omega_1 \\ \omega_2 \end{bmatrix} \right\|_{\mathbf{H}_\rho} \leq C\,|\beta| \left\| \begin{bmatrix} \mu^* \\ \omega_1^* \\ \omega_2^* \end{bmatrix} \right\|_{\mathbf{H}_\rho} ; \tag{4.2}$$

this is the frequency domain estimate (3.1) with $\alpha = 1$.

Using the definition of $A_\rho : \mathcal{D}(A_\rho) \subset \mathbf{H}_\rho \to \mathbf{H}_\rho$, this gives

$$\begin{aligned} i\beta\mu - \Delta\mu + \nabla G_{\rho,1}\omega_1 + \nabla G_{\rho,2}\mu &= \mu^* && \text{in } \mathcal{O} \\ i\beta\omega_1 - \omega_2 &= \omega_1^* && \text{on } \Omega \\ i\beta\omega_2 + P_\rho^{-1}\,\Delta^2\omega_1 - P_\rho^{-1}G_{\rho,1}\omega_1|_\Omega - P_\rho^{-1}G_{\rho,2}\mu|_\Omega &= \omega_2^* && \text{on } \Omega. \end{aligned}$$

Upon a rearrangement and setting pressure variable

$$\pi \equiv G_{\rho,1}\omega_1 + G_{\rho,2}\mu \, , \tag{4.3}$$

we then have

$$\begin{aligned} i\beta\mu - \Delta\mu + \nabla\pi &= \mu^* && \text{in } \mathcal{O} \\ \omega_2 &= i\beta\omega_1 - \omega_1^* && \text{on } \Omega \\ -\beta^2\omega_1 - i\beta\omega_1^* + P_\rho^{-1}\Delta^2\omega_1 - P_\rho^{-1}\pi|_\Omega &= \omega_2^* && \text{on } \Omega. \end{aligned}$$

We have then following (static) fluid-structure PDE system:

$$\begin{aligned} i\beta\mu - \Delta\mu + \nabla\pi &= \mu^* && \text{in } \mathcal{O} && \text{(4.4a)} \\ \operatorname{div}(\mu) &= 0 && \text{in } \mathcal{O} && \text{(4.4b)} \\ \mu &= 0 && \text{on } S && \\ \mu = [\mu^1, \mu^2, \mu^3] &= [0, 0, i\beta\omega_1 - \omega_1^*] && \text{on } \Omega && \text{(4.4c)} \\ \omega_2 &= i\beta\omega_1 - \omega_1^* && \text{in } \Omega && \text{(4.4d)} \\ -\beta^2\,P_\rho\omega_1 + \Delta^2\omega_1 - \pi|_\Omega &= P_\rho\,\omega_2^* + i\beta P_\rho\,\omega_1^* && \text{in } \Omega && \text{(4.4e)} \\ \omega_1|_{\partial\Omega} = \left.\frac{\partial\omega_1}{\partial n}\right|_{\partial\Omega} &= 0 && \text{on } \partial\Omega. && \text{(4.4f)} \end{aligned}$$

Step 1. (An estimate for the fluid gradient) Let us return to the resolvent equation (4.1). It is easily seen that an integration by parts gives the following static dissipation relation:

$$\begin{aligned} &\left((i\beta - A_\rho) \begin{bmatrix} \mu \\ \omega_1 \\ \omega_2 \end{bmatrix}, \begin{bmatrix} \mu \\ \omega_1 \\ \omega_2 \end{bmatrix} \right)_{\mathbf{H}_\rho} \\ = \;\; & i\beta \left\| \begin{bmatrix} \mu \\ \omega_1 \\ \omega_2 \end{bmatrix} \right\|_{\mathbf{H}_\rho}^2 + \|\nabla\mu\|_{\mathcal{O}}^2 + 2i\mathrm{Im}(\Delta\omega_1, \Delta\omega_2)_\Omega \, ; \end{aligned}$$

see (3.19)-(3.20). Combining this with relation (4.1), we then have

$$i\beta \left\| \begin{bmatrix} \mu \\ \omega_1 \\ \omega_2 \end{bmatrix} \right\|_{\mathbf{H}_\rho}^2 + \|\nabla\mu\|_{\mathcal{O}}^2 + 2i\,\mathrm{Im}(\Delta\omega_1, \Delta\omega_2)_\Omega =$$

$$= \left((i\beta - A_\rho) \begin{bmatrix} \mu \\ \omega_1 \\ \omega_2 \end{bmatrix}, \begin{bmatrix} \mu \\ \omega_1 \\ \omega_2 \end{bmatrix} \right)_{\mathbf{H}_\rho} = \left(\begin{bmatrix} \mu^* \\ \omega_1^* \\ \omega_2^* \end{bmatrix}, \begin{bmatrix} \mu \\ \omega_1 \\ \omega_2 \end{bmatrix} \right)_{\mathbf{H}_\rho},$$

whence we obtain

$$\|\nabla\mu\|_{L^2(\mathcal{O})}^2 = \mathrm{Re} \left(\begin{bmatrix} \mu^* \\ \omega_1^* \\ \omega_2^* \end{bmatrix}, \begin{bmatrix} \mu \\ \omega_1 \\ \omega_2 \end{bmatrix} \right)_{\mathbf{H}_\rho}. \tag{4.5}$$

Step 2. (Control of the β-mechanical displacement in a lower topology) Using the fluid Dirichlet boundary condition in (4.4c) we have

$$i\beta\omega_1 = \mu^3\big|_\Omega + \omega_1^*.$$

We estimate this expression by invoking in sequence, the Sobolev Embedding Theorem, Poincaré's inequality and (4.5). Through these means we then obtain

$$\begin{aligned} \|\beta\omega_1\|_{H^{1/2}(\Omega)} &\le \left\|\mu^3\big|_\Omega + \omega_1^*\right\|_{H^{1/2}(\Omega)} \\ &\le C\left(\|\nabla\mu\|_{L^2(\mathcal{O})} + \|\omega_1^*\|_{H_0^2(\Omega)}\right) \\ &\le C\left(\sqrt{\left| \mathrm{Re} \left(\begin{bmatrix} \mu^* \\ \omega_1^* \\ \omega_2^* \end{bmatrix}, \begin{bmatrix} \mu \\ \omega_1 \\ \omega_2 \end{bmatrix} \right)_{\mathbf{H}_\rho} \right|} + \|\omega_1^*\|_{H_0^2(\Omega)} \right). \end{aligned} \tag{4.6}$$

Step 3. (Control of the mechanical displacement) We multiply both sides of the mechanical equation in (4.4e) by $\overline{\omega_1}$ and integrate. This gives the relation

$$\left(\Delta^2\omega_1, \omega_1\right)_{L^2(\Omega)} = \beta^2\left\|P_\rho^{1/2}\omega_1\right\|_{L^2(\Omega)}^2 + (\pi|_\Omega, \omega_1)_\Omega + (P_\rho\omega_2^* + i\beta P_\rho\omega_1^*, \omega_1)_{L^2(\Omega)}. \tag{4.7}$$

(3.i) To handle the first term on the right hand side of (4.7), we invoke Poincaré's Inequality, thereby obtaining

$$\beta^2\left\|P_\rho^{1/2}\omega_1\right\|_{L^2(\Omega)}^2 = \beta^2\left(\|\omega_1\|_{L^2(\Omega)}^2 + \rho\left\|\nabla\omega_1\right\|_{L^2(\Omega)}^2\right) \le C_\rho\beta^2\left\|\nabla\omega_1\right\|_{L^2(\Omega)}^2. \tag{4.8}$$

Now,

$$\begin{aligned} \beta^2\left\|\nabla\omega_1\right\|_{L^2(\Omega)}^2 &= \beta\left(\nabla\omega_1, \beta\nabla\omega_1\right)_{L^2(\Omega)} = \beta\left\langle\nabla\omega_1, \beta\nabla\omega_1\right\rangle_{H^{1/2}(\Omega)\times H^{-1/2}(\Omega)} \\ &\le C\,|\beta|\,\|\omega_1\|_{H^{3/2}(\Omega)}\,\|\beta\omega_1\|_{H^{1/2}(\Omega)}. \end{aligned}$$

Subsequently, interpolating between $H^2(\Omega)$ and $H^{1/2}(\Omega)$ with interpolation parameter $\theta = 1/3$ (see e.g. [39], or [42]), we obtain

$$\begin{aligned} \beta^2\left\|\nabla\omega_1\right\|_{L^2(\Omega)}^2 &\le C\,|\beta|^{2/3}\,\|\beta|^{1/3}|\omega_1\|_{H^{3/2}(\Omega)}\|\beta\omega_1\|_{H^{1/2}(\Omega)} \\ &\le C\,|\beta|^{2/3}\left[\|\omega_1\|_{H^2(\Omega)}^{2/3}\,\|\beta\omega_1\|_{H^{1/2}(\Omega)}^{1/3}\right]\|\beta\omega_1\|_{H^{1/2}(\Omega)}. \end{aligned}$$

Via Young's inequality, with conjugate exponents 3 and 3/2, we then have

$$\beta^2 \big\|\nabla\omega_1\big\|^2_{L^2(\Omega)} \le C\,\|\omega_1\|^{2/3}_{H^2(\Omega)}\,|\beta|^{2/3}\,\|\beta\omega_1\|^{4/3}_{H^{1/2}(\Omega)} \le \epsilon\|\omega_1\|^2_{H^2(\Omega)} + C_\epsilon|\beta|\,\|\beta\omega_1\|^2_{H^{1/2}(\Omega)}\,;$$

subsequently reinvoking the estimate (4.6), we then have for $|\beta| > 1$,

$$\begin{aligned}
\beta^2\big\|\nabla\omega_1\big\|^2_{L^2(\Omega)} &\le \epsilon\|\omega_1\|^2_{H^2(\Omega)} + C_\epsilon\,|\beta|\left(\left\|\mathrm{Re}\left(\begin{bmatrix}\mu^*\\ \omega_1^*\\ \omega_2^*\end{bmatrix}, \begin{bmatrix}\mu\\ \omega_1\\ \omega_2\end{bmatrix}\right)_{\mathbf{H}_\rho}\right\| + \|\omega_1^*\|^2_{H_0^2(\Omega)}\right)\\
&\le 2\,\epsilon\left\|\begin{bmatrix}\mu\\ \omega_1\\ \omega_2\end{bmatrix}\right\|^2_{\mathbf{H}_\rho} + C_\epsilon|\beta|^2\left\|\begin{bmatrix}\mu^*\\ \omega_1^*\\ \omega_2^*\end{bmatrix}\right\|^2_{\mathbf{H}_\rho}.
\end{aligned} \tag{4.9}$$

Applying the obtained estimate (4.9) to the right hand side of (4.8) yields now

$$\beta^2\big\|P_\rho^{1/2}\omega_1\big\|^2_{L^2(\Omega)} \le C_\rho\epsilon\left\|\begin{bmatrix}\mu\\ \omega_1\\ \omega_2\end{bmatrix}\right\|^2_{\mathbf{H}_\rho} + C_\epsilon\,|\beta|^2\left\|\begin{bmatrix}\mu^*\\ \omega_1^*\\ \omega_2^*\end{bmatrix}\right\|^2_{\mathbf{H}_\rho}. \tag{4.10}$$

(3.ii) To handle the second term on the right hand side of (4.7), we observe that since $[\mu,\omega_1,\omega_2] \in \mathbf{H}_\rho$, then in particular

$$\int_\Omega \omega_1\, d\Omega = 0\,.$$

In consequence, one has wellposedness of the following boundary value problem (see [44, Theorem 2.4 and Remark 2.5]):

$$\begin{cases} -\Delta\psi + \nabla q = 0 & \text{in } \mathcal{O}\\ \mathrm{div}\,(\psi) = 0 & \text{in } \mathcal{O}\\ \psi|_S = 0 & \text{on } S\\ \psi|_\Omega = \left(\psi^1,\psi^2,\psi^3\right)\big|_\Omega = (0,0,\omega_1) & \text{on } \Omega\,, \end{cases} \tag{4.11}$$

with the estimate

$$\big\|\nabla\psi\big\|_{\mathbf{L}^2(\mathcal{O})} + \|q\|_{L^2(\mathcal{O})} \le C\,\|\omega_1\|_{H^{1/2}(\Omega)} \tag{4.12}$$

(implicitly, we are also using Poincaré inequality).

With this solution variable ψ of (4.11) in hand, we now address the second term on the right hand side of (4.7). Since the normal vector ν equals $(0,0,1)$ on Ω (and as the fluid variable μ is divergence free), we have

$$\begin{aligned}
(\pi|_\Omega,\omega_1)_\Omega &= -\left(\frac{\partial\mu}{\partial\nu}, \begin{bmatrix}0\\0\\ \omega_1\end{bmatrix}\right)_{\mathbf{L}^2(\Omega)} + \left(\pi|_\Omega\,\nu, \begin{bmatrix}0\\0\\ \omega_1\end{bmatrix}\right)_{\mathbf{L}^2(\Omega)}\\
&= -\left(\frac{\partial\mu}{\partial\nu},\psi\right)_{\mathbf{L}^2(\partial\mathcal{O})} + \left(\,\pi|_\Omega\,\nu,\psi\right)_{\mathbf{L}^2(\partial\mathcal{O})},
\end{aligned} \tag{4.13}$$

after invoking the boundary conditions in (4.11).

The use of Green's Identities and the Stokes system in (4.11) then gives

$$\begin{aligned}(\pi|_\Omega,\omega_1)_\Omega &= -\Big(\frac{\partial\mu}{\partial\nu},\psi\Big)_{\mathbf{L}^2(\partial\mathcal{O})} + \big(\pi|_\Omega\nu,\psi\big)_{\mathbf{L}^2(\partial\mathcal{O})}\\ &= -\big(\Delta\mu,\psi\big)_{\mathbf{L}^2(\mathcal{O})} - \big(\nabla\mu,\nabla\psi\big)_{\mathbf{L}^2(\mathcal{O})} + \big(\nabla\pi,\psi\big)_{\mathbf{L}^2(\mathcal{O})}\\ &= -i\beta\,(u,\psi)_{\mathbf{L}^2(\mathcal{O})} - \big(\nabla u,\nabla\psi\big)_{\mathbf{L}^2(\mathcal{O})} + \big(u^*,\psi\big)_{\mathbf{L}^2(\mathcal{O})}\,.\end{aligned}$$

Estimating this right hand side by means of Poincaré Inequality, we then have for $|\beta|>1$,

$$\big|(\pi|_\Omega,w_1)_\Omega\big| \le C\,|\beta|\,\|\nabla\psi\|_{\mathbf{L}^2(\mathcal{O})}\big(\|\nabla u\|_{\mathbf{L}^2(\mathcal{O})}+\|u^*\|_{\mathbf{L}^2(\mathcal{O})}\big)\,; \tag{4.14}$$

and subsequently refining this inequality by means of (4.5), (4.12) and (4.6), we establish

$$\begin{aligned}\big|(\pi|_\Omega,w_1)_\Omega\big| &\le C\,|\beta|\,|\omega_1\|_{H^{1/2}(\Omega)}\left(\left|\mathrm{Re}\left(\begin{bmatrix}\mu^*\\ \omega_1^*\\ \omega_2^*\end{bmatrix},\begin{bmatrix}\mu\\ \omega_1\\ \omega_2\end{bmatrix}\right)_{\mathbf{H}_\rho}\right|^{1/2} + \|\mu^*\|_{\mathbf{L}^2(\mathcal{O})}\right) \le\\
&\le C\left(\left|\mathrm{Re}\left(\begin{bmatrix}\mu^*\\ \omega_1^*\\ \omega_2^*\end{bmatrix},\begin{bmatrix}\mu\\ \omega_1\\ \omega_2\end{bmatrix}\right)_{\mathbf{H}_\rho}\right|^{1/2} + \|\omega_1^*\|_{H_0^2(\Omega)}\right)\cdot\\
&\quad\cdot\left(\left|\mathrm{Re}\left(\begin{bmatrix}\mu^*\\ \omega_1^*\\ \omega_2^*\end{bmatrix},\begin{bmatrix}\mu\\ \omega_1\\ \omega_2\end{bmatrix}\right)_{\mathbf{H}_\rho}\right|^{1/2} + \|\mu^*\|_{\mathbf{L}^2(\mathcal{O})}\right) =\\
&= C\left\{\left|\mathrm{Re}\left(\begin{bmatrix}\mu^*\\ \omega_1^*\\ \omega_2^*\end{bmatrix},\begin{bmatrix}\mu\\ \omega_1\\ \omega_2\end{bmatrix}\right)_{\mathbf{H}_\rho}\right| + \left|\mathrm{Re}\left(\begin{bmatrix}\mu^*\\ \omega_1^*\\ \omega_2^*\end{bmatrix},\begin{bmatrix}\mu\\ \omega_1\\ \omega_2\end{bmatrix}\right)_{\mathbf{H}_\rho}\right|^{1/2}\|\mu^*\|_{L^2(\mathcal{O})}+\right.\\
&\quad\left.+\left|\mathrm{Re}\left(\begin{bmatrix}\mu^*\\ \omega_1^*\\ \omega_2^*\end{bmatrix},\begin{bmatrix}\mu\\ \omega_1\\ \omega_2\end{bmatrix}\right)_{\mathbf{H}_\rho}\right|^{1/2}\|\omega_1^*\|_{H_0^2(\Omega)} + \|\omega_1^*\|_{H_0^2(\Omega)}\|\mu^*\|_{L^2(\mathcal{O})}\right\}.\end{aligned}$$

Multiple applications of the inequality $|a\,b|\le \epsilon a^2 + C_\epsilon b^2$ yield now

$$\big|(\pi|_\Omega,w_1)_\Omega\big| \le \epsilon\left\|\begin{bmatrix}\mu\\ \omega_1\\ \omega_2\end{bmatrix}\right\|_{\mathbf{H}_\rho}^2 + C_\epsilon\left\|\begin{bmatrix}\mu^*\\ \omega_1^*\\ \omega_2^*\end{bmatrix}\right\|_{\mathbf{H}_\rho}^2 . \tag{4.15}$$

(3.iii) It remains to handle the third term on the right hand side of (4.7). By way of estimate (4.9) we have readily for $|\beta|>1$

$$\begin{aligned}\big|\big(P_\rho\omega_2^*+i\beta P_\rho\omega_1^*,\omega_1\big)_{L^2(\Omega)}\big| &= \big|\big(\omega_2^*+i\beta\omega_1^*,\omega_1\big)_{L^2(\Omega)} + \rho\big(\nabla[\omega_2^*+i\beta\omega_1^*],\nabla\omega_1\big)_{L^2(\Omega)}\big|\\ &\le C_\rho\,|\beta|\,\big\|\nabla\omega_1\big\|_{L^2(\Omega)}\big(\big\|\nabla\omega_1^*\big\|_{L^2(\Omega)}+\big\|\nabla\omega_2^*\big\|_{L^2(\Omega)}\big)\\ &\le 2\,\epsilon\left\|\begin{bmatrix}\mu\\ \omega_1\\ \omega_2\end{bmatrix}\right\|_{\mathbf{H}_\rho}^2 + C_\epsilon|\beta|^2\left\|\begin{bmatrix}\mu^*\\ \omega_1^*\\ \omega_2^*\end{bmatrix}\right\|_{\mathbf{H}_\rho}^2 .\end{aligned} \tag{4.16}$$

Applying now (4.10), (4.15), and (4.16) to the right hand side of (4.7), and using the fact that ω_1 satisfies *hinged* boundary conditions, we then have

$$\begin{aligned}\|\Delta\omega_1\|^2_{L^2(\Omega)} &= \left(\Delta^2\omega_1,\omega_1\right)_{L^2(\Omega)}\\ &\le \epsilon\left(C_\rho+3\right)\left\|\begin{bmatrix}\mu\\ w_1\\ w_2\end{bmatrix}\right\|^2_{\mathbf{H}_\rho} + C_\epsilon\,|\beta|^2\left\|\begin{bmatrix}\mu^*\\ w_1^*\\ w_2^*\end{bmatrix}\right\|^2_{\mathbf{H}_\rho}.\end{aligned} \tag{4.17}$$

Step 4. (Control of the mechanical velocity) Via the resolvent relation $\omega_2 = i\beta\omega_1 - \omega_1^*$ we have

$$\|\omega_2\|_{H^1(\Omega)} \le \|\beta\omega_1\|_{H^1(\Omega)} + \|\omega_1^*\|_{H^1(\Omega)} \le C\,\|\beta\nabla\omega_1\|_{L^2(\Omega)} + \|\omega_1^*\|_{H^1(\Omega)}\,,$$

after again using Poincaré Inequality. Applying (4.9) once more, we attain

$$\|\omega_2\|^2_{H^1(\Omega)} \le \epsilon\,C\left\|\begin{bmatrix}\mu\\ \omega_1\\ \omega_2\end{bmatrix}\right\|^2_{\mathbf{H}_\rho} + C_\epsilon\,|\beta|^2\left\|\begin{bmatrix}\mu^*\\ \omega_1^*\\ \omega_2^*\end{bmatrix}\right\|^2_{\mathbf{H}_\rho}. \tag{4.18}$$

To finish the proof of Theorem 1.5, we collect (4.5), (4.17) and (4.18). This gives the following conclusion: the solution of the resolvent equation satisfies, for $|\beta|>1$, the estimate

$$\left\|\begin{bmatrix}\mu\\ \omega_1\\ \omega_2\end{bmatrix}\right\|^2_{\mathbf{H}_\rho} \le \epsilon\,C\left\|\begin{bmatrix}\mu\\ \omega_1\\ \omega_2\end{bmatrix}\right\|^2_{\mathbf{H}_\rho} + C_\epsilon\,|\beta|^2\left\|\begin{bmatrix}\mu^*\\ \omega_1^*\\ \omega_2^*\end{bmatrix}\right\|^2_{\mathbf{H}_\rho},$$

which gives the estimate (4.2), for $\epsilon>0$ small enough. This concludes the proof of Theorem 1.5.
□

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