Mathematical Models for Fluids with Pressure-Dependent Viscosity Flowing in Porous Media

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Abstract

In this paper we study three filtration problems through porous media, assuming that the viscosity of the fluid depends on pressure. After showing that in this case Darcy's law is "formally" preserved (meaning that the formal relation remains unchanged except for viscosity that now depends on pressure), we focus on the following problems: Green–Ampt infiltration through a dry porous medium; the Dam problem; the Muskat problem. For each model (free boundary problems) we obtain explicit solutions that allow to quantify the detachment from the classical case, where with the word "classical" we mean that viscosity is taken constant

Keywords: Filtration, Darcy's Law, Pressure Dependent Viscosity,, Exact Solutions

1. Introduction

Fluids with viscosity depending on pressure have recently drawn a lot of attention from the scientific community. In the last decades a remarkable amount of experimental literature has been produced to support the claim that viscosity may vary with pressure (even if the fluid remains incompressible), proving that in many cases it is imperative to take such a dependence into account [8], [7], [3]. Of course many empirical models have been proposed [5], many of which take also into account the dependence on temperature.

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Concerning the empirical formula adopted (see, e.g., [19]) we indicate the linear law

$$\mu(P) = \alpha P, \qquad \text{or} \qquad \mu(P) = \mu_o(1 + \alpha P),$$

and the exponential (or Barus) law

$$\mu = \mu_o e^{\alpha P}.$$

Some other possibilities can be found, for instance, in [10], [11].

The flow of a fluid through a porous medium is typically described by means of the well known Darcy's law, which gives a linear relation between the discharge and the pressure gradient. Darcy's law can be derived through an homogenization procedure studying the Stokes flow at the micro-scale and then upscaling the system to the macro-scale (see, for instance, [12] and [4]). This derivation is well known when one deals with a viscous incompressible/compressible fluid, and it has been recently studied for the case of a fluid with pressure dependent viscosity [16]. We also refer the readers to [17] for the description of general thermodynamic framework (based on the criterion of maximal rate of entropy production) to obtain Darcy and Brinkman models and their generalizations.

In this paper we briefly outline how Darcy's law has to be modified in the case of a pressure-dependent viscosity and apply such modified version to some classical filtration problems. In particular, we begin by proving that "formally" Darcy's law remains the same, except for the fact that now viscosity is a function of pressure, namely

$$\mathbf{q} = -\frac{\mathbf{k}}{\mu(P)}(\nabla P - \mathbf{f}). \tag{1}$$

In section 2 we concisely outline the derivation of (1), referring the readers to the original paper [16]. The original part of this article can be found in sections 3–5, where we use (1) to study how three classical models: Green Ampt model (section 3); the Dam problem (section 4); the Muskat problem (section 5). We will show that explicit solutions can be found and illustrate the qualitative behavior of such solutions.

The use of (1) allows indeed to extend models that have been investigated using the classical Darcy's law to the case of pressure dependent viscosity. Indeed, an interesting application has been described in [18] and in [15]. Special flows of fluid with pressure depend viscosity (even not strictly related to filtration) have been recently studied in detail [20], [6].

2. Darcy's law for a fluid with pressure dependent viscosity

In this section we briefly recall how (1) can be obtained via homogenization. We refer the reader to [16] for all the details. Let $\Omega = \Omega_s \cup \Omega_\ell$ be the periodic cell consisting of a solid and a liquid part and let us suppose that the stress tensor in the fluid phase is given by $\mathbf{T} = -P\mathbf{I} + 2\mu(P)\mathbf{D}(\mathbf{u})$, where $\mathbf{u}$ is the velocity field,

$$\mathbf{D}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right), \tag{2}$$

is the symmetric part of the velocity gradient $\nabla\mathbf{u}$, P is pressure and μ (viscosity) is a smooth bounded function of P. Assuming incompressibility and creeping flow with no-slip on the solid boundary Γ_s we write the governing equations

$$\begin{cases} \nabla\cdot\mathbf{u} = 0, & \text{in } \ \Omega_\ell, \\ -\nabla P + \mu\Delta\mathbf{u} + 2\mu'\mathbf{D}[\nabla P] = -\mathbf{f}, & \text{in } \ \Omega_\ell, \\ \mathbf{u} = 0, & \text{on } \ \Gamma_s, \end{cases} \tag{3}$$

where $\mathbf{f}$ represents the body force vector and where

$$\Big(2\mathbf{D}[\nabla P]\Big)_i = \sum_{j=1}^{3} \frac{\partial P}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Problem (3) is written for the macroscopic variable $\mathbf{x}$. We introduce the microscopic variable $\varepsilon\mathbf{y} = \mathbf{x}$, and write the classical expansion [1]

$$\mathbf{u} = \varepsilon^\alpha \sum_{k=0}^{\infty} \varepsilon^k \mathbf{u}^{(k)}(\mathbf{x},\mathbf{y}), \qquad P = \varepsilon^\beta \sum_{k=0}^{\infty} \varepsilon^k P^{(k)}(\mathbf{x},\mathbf{y}), \tag{4}$$

where $\mathbf{u}^{(k)}$ and $P^{(k)}$ are $\Omega-$periodic and α, β are parameters yielding physically meaningful solutions. Problem (3) can be reformulated as

$$\begin{cases} \varepsilon\nabla_x \cdot \mathbf{u} + \nabla_y \cdot \mathbf{u} = 0, & \text{in } \Omega_\ell, \\ -\varepsilon^2\nabla_x P - \varepsilon\nabla_y P + \mu[\varepsilon^2\Delta_x\mathbf{u} + \varepsilon\Delta_{xy}\mathbf{u} + \Delta_y\mathbf{u}] + & \\ +2\mu^{'}\Big\{\varepsilon^2\mathbf{D}_x[\nabla_x P] + \varepsilon(\mathbf{D}_x[\nabla_y P] + \mathbf{D}_y[\nabla_x P]) + \mathbf{D}_y[\nabla_y P]\Big\} = -\mathbf{f}, & \text{in } \Omega_\ell, \\ \mathbf{u} = 0, & \text{on } \Gamma_s. \end{cases} \tag{5}$$

Now we plug (4) into (5) and consider the leading order (the smallest integer values for α, β providing non-zero solutions are $\alpha = 2$, $\beta = 0$)

$$\begin{cases} \nabla_y \cdot \mathbf{u}^{(0)} = 0, & \text{in } \Omega_\ell, \\ \nabla_y P^{(0)} = 0, & \text{in } \Omega_\ell, \\ -\nabla_x P^{(0)} - \nabla_y P^{(1)} + \mu^{(0)}\Delta_y\mathbf{u}^{(0)} + & \\ 2\Big(\mu^{(0)}\Big)^{'}\Big\{\mathbf{D}_y(\mathbf{u}^{(0)})\underbrace{[\nabla_y P^{(0)}]}_{=0}\Big\} = -\mathbf{f}, & \text{in } \Omega_\ell, \end{cases} \tag{6}$$

where $\mu^{(0)}$ is $\mu(P^{(0)})$. We observe that $P^{(0)} = P^{(0)}(\mathbf{x})$ so that, at the leading order, the pressure does not depend on the microscopic coordinates. The system (6) reduces to

$$\begin{cases} \nabla_y \cdot \mathbf{u}^{(0)} = 0, & \text{in } \Omega_\ell, \\ -\nabla_x P^{(0)} - \nabla_y P^{(1)} + \mu^{(0)}\Delta_y\mathbf{u}^{(0)} + \mathbf{f} = 0, & \text{in } \Omega_\ell, \end{cases} \tag{7}$$

which is "formally" equivalent to the the system obtained when deriving Darcy's law in the case of a Newtonian fluid [12]. We now give a weak formulation of problem (6) and show how the modified Darcy's law can be derived. Let us consider the Hilbert space

$$\mathcal{H}(\Omega) = \left\{ \begin{array}{lll} \mathbf{w} = (w_1, w_2.w_3): & \mathbf{w} \in H^1(\Omega_\ell), & \mathbf{w}\ \ \Omega - \text{periodic} \\ \mathbf{w} = 0 \ \text{ on } \ \Gamma_s, & \nabla_y \cdot \mathbf{w} = 0. & \end{array} \right\},$$

endowed with the scalar product

$$<\mathbf{v},\mathbf{w}>_{\mathcal{H}(\Omega)}=\int_{\Omega_\ell}\Big(\nabla_y\mathbf{v}\cdot\nabla_y\mathbf{w}\Big)d\mathbf{y}=\int_{\Omega_\ell}\sum_{i=1}^{3}\Big(\nabla_y v_i\cdot\nabla_y w_i\Big)d\mathbf{y}.$$

We multiply equation $(7)_2$ by a test function $\mathbf{w}\in\mathcal{H}(\Omega)$ and integrate over Ω_ℓ

$$\Big(\nabla_x P^{(0)}-\mathbf{f}\Big)\cdot\int_{\Omega_\ell}\mathbf{w}d\mathbf{y}=\mu^{(0)}\int_{\Omega_\ell}\mathbf{w}\cdot\Delta_y\mathbf{u}^{(0)}d\mathbf{y}-\int_{\Omega_\ell}\mathbf{w}\cdot\nabla_y P^{(1)}d\mathbf{y}. \quad (8)$$

Using the divergence theorem, and recalling that $\mathbf{w}$ is periodic and that $\mathbf{w}=0$ on Γ_s, equation (8) can be rewritten as

$$\Big(\mathbf{f}-\nabla_x P^{(0)}\Big)\cdot\int_{\Omega_\ell}\mathbf{w}d\mathbf{y}=\mu^{(0)}\int_{\Omega_\ell}\nabla_y\mathbf{w}\cdot\nabla_y\mathbf{u}^{(0)}d\mathbf{y}.$$

Therefore we may reformulate the problem in a variational form: find $\mathbf{u}^{(0)}\in\mathcal{H}(\Omega)$ such that

$$\mu^{(0)}<\mathbf{u}^{(0)},\mathbf{w}>_{\mathcal{H}(\Omega)}=\Big(\mathbf{f}-\nabla_x P^{(0)}\Big)\cdot\int_{\Omega_\ell}\mathbf{w}d\mathbf{y},\qquad\forall\ \ \mathbf{w}\in\mathcal{H}(\Omega). \quad (9)$$

One can show that this problem admits a unique solution using Lax-Milgram lemma. Next, to derive Darcy's law 1 we consider the auxiliary problems

$$\begin{cases}\nabla_y\cdot\mathbf{u}_i=0, & \text{in }\ \Omega_\ell,\\ -\nabla_y m_i+\Delta_y\mathbf{u}_i+\mathbf{e}_i=0, & \text{in }\ \Omega_\ell, \qquad\qquad i=1,2,3,\\ \mathbf{u}_i=0 & \text{on }\ \Gamma_s,\end{cases} \quad (10)$$

where m_i is a Ω-periodic function and where $\mathbf{e}_i$ is the unit vector along y_i. The weak formulation of problem (10) is: find $\mathbf{u}_i$ such that

$$<\mathbf{u}_i,\mathbf{w}>=\mathbf{e}_i\cdot\int_{\Omega_\ell}\mathbf{w}\ \ d\mathbf{y},\qquad\forall\ \mathbf{w}\in\ \mathcal{H}(\Omega),\qquad i=1,2,3. \quad (11)$$

The existence and uniqueness of a solution is once again guaranteed by Lax-Milgram lemma. Now, let $\mathbf{u}^{(0)}$ be the solution of problem (9) and let $\mathbf{u}_i$ be the solutions of problem (11). One can easily prove that

$$\mathbf{u}^{(0)}=\frac{1}{\mu^{(0)}}\sum_{i=1}^{3}\left(f_i-\frac{\partial P^{(0)}}{\partial x_i}\right)\mathbf{u}_i.$$

Let us extend $\mathbf{u}^{(0)}$ and $\mathbf{u}_i$ to zero in the solid phase of Ω, define $\mathbf{q} = \frac{1}{|\Omega|}\int_{\Omega_\ell} \mathbf{u}^{(\mathbf{0})}\, d\mathbf{y}$, and

$$\mathbf{k}_i = \frac{1}{|\Omega|}\int_{\Omega_\ell} \mathbf{k}_i \, d\mathbf{y}. \tag{12}$$

Then, denoting with $\mathbf{k}$ the tensor with entries $(\boldsymbol{\kappa})_{i,j} = \mathbf{k}_i \cdot \mathbf{e}_j$ we get (1), namely

$$\mathbf{q} = -\frac{\mathbf{k}}{\mu(P)}\left(\nabla_x P - \mathbf{f}\right),$$

which is the "modified" Darcy's law (1) for a fluid whose viscosity depends on pressure.

3. Application to the Green-Ampt infiltration model

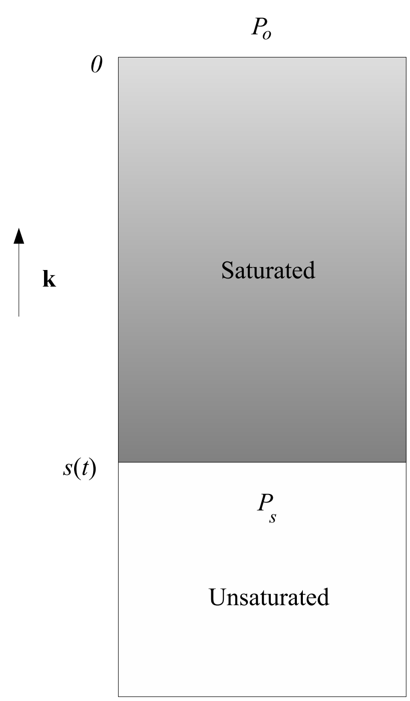

Figure 1: The Green Ampt model

As a first example we study how the use of relation (**??**) modifies the classical Green-Ampt infiltration model [**?**]. For the sake of simplicity we limit ourselves to consider the one dimensional setting depicted in Fig. 1, assuming that a dry medium occupying the region $z < 0$ is penetrated by a fluid supplied at the surface $z = 0$. We assume that the motion is driven by gravity (i.e. $\mathbf{f} = -\rho g \mathbf{k}$) and by a pressure difference applied between the injection surface $z = 0$, and the wetting front $z = s(t)$, namely $\Delta P = P|_{z=0} - P|_{z=s} = P_o - P_s$, with $P_o > P_s > 0$. We assume that the permeability tensor is diagonal with constant entries κ and that the viscosity is a smooth function of pressure. Supposing that everything depends only on the coordinate z, the mass conservation gives $\dfrac{\partial q}{\partial z} = 0$, with

$$q(t) = -\frac{\kappa}{\mu(P)} \left(\frac{\partial P}{\partial z} + \rho g \right), \tag{13}$$

q being the discharge. Denoting by ϕ the porosity of the medium, we impose that the wetting front velocity $\dot{s}(t)$, coincides with the molecular velocity, i.e. $\dot{s}\phi = q$. Hence the problem for the pressure becomes

$$\begin{cases} \dfrac{\partial P}{\partial z} = -\left(\dfrac{\mu(P)\ q}{\kappa} + \rho g \right), & 0 < z < s(t), \\ \\ P|_{z=0} = P_o. \end{cases} \tag{14}$$

Integration entails

$$z = \int_{P(z)}^{P_o} \frac{\kappa\, dy}{q\mu(P) + \rho g \kappa}.$$

If we now impose $P|_{z=s} = P_s$, we get

$$s = \int_{P_s}^{P_o} \frac{\kappa\, dy}{\dot{s}\ \phi\mu(P) + \rho g \kappa}, \tag{15}$$

which, together with $s(0) = 0$ defines the nonlinear ODE for the wetting front $s(t)$ (which is actually a free boundary).

Remark 1. *If we neglect gravity we get*

$$\frac{d}{dt}\left(s^2\right) = \int_{P_s}^{P_o} \frac{\kappa\, dy}{\phi\ \mu(P)} = const., \quad \Rightarrow \quad s(t) \propto -\sqrt{t}.$$

∎

Remark 2. *Assuming $\mu = const.$, we retrieve the classical Green-Ampt model. Indeed integration of (15) entails*

$$\dot{s} = \frac{1}{s}\left[\frac{\kappa(P_o - P_s)}{\phi\mu}\right] - \frac{\kappa\rho g}{\phi\mu}.$$

$\blacksquare$

An interesting case is the one in which the dependence of viscosity on pressure is exponential, namely Barus law $\mu = \mu_o e^{\alpha P}$. In this case (15) becomes

$$\frac{s}{\kappa} = \int_{P_s}^{P_o} \frac{dy}{\dot{s}\ \phi\mu_o\ e^{\alpha P} + \rho g\kappa},$$

whose integration yields

$$s\ \rho g = \left[y - \frac{1}{\alpha}\ \ln\ |\dot{s}\ \phi\mu_o\ e^{\alpha P} + \rho g\kappa|\right]_{P_s}^{P_o}.$$

We find

$$|\dot{s}\ \phi\mu_o\ e^{\alpha P_s} + \rho g\kappa| = |\dot{s}\ \phi\mu_o\ e^{\alpha P_o} + \rho g\kappa| e^{\alpha\rho g s - \alpha P_o + \alpha P_s},$$

or equivalently

$$\dot{s} = \left(\frac{\rho g\kappa}{\phi\mu_o e^{\alpha P_s}}\right)\ \left[\frac{\pm e^{\alpha\rho g s} e^{\alpha(P_s - P_o)} - 1}{1 \mp e^{\alpha\rho g s}}\right]. \tag{16}$$

In (16) the "plus" or "minus" sign must be specified. We take the plus sign in the denominator since it gives a solution with non singular $\dot{s}(0)$. Hence, setting $\sigma = e^{\alpha(P_s - P_o)} < 1$, $\gamma = \rho\alpha$, and $\omega = \dfrac{\rho g\kappa}{\phi\mu_o e^{\alpha P_s}}$, we have this Cauchy problem

$$\begin{cases} \dot{s} = -\omega\left[\dfrac{\sigma e^{\gamma s} + 1}{1 + e^{\gamma s}}\right], \\ \\ s(0) = 0. \end{cases} \tag{17}$$

Integration of (17) yields

$$\omega t = -s + \frac{1}{\gamma}\left(1 - \frac{1}{\sigma}\right)\ \ln\left(\frac{1 + \sigma e^{\gamma s}}{1 + \sigma}\right),$$

which defines t as a function of s. For $|s| \ll 1$ (i.e. when the penetration front is sufficiently close to $z = 0$) we may expand to get

$$s \approx -\frac{\sigma+1}{2}\omega t,$$

meaning that the free boundary initially grows linearly in t.

4. Application to the Dam problem

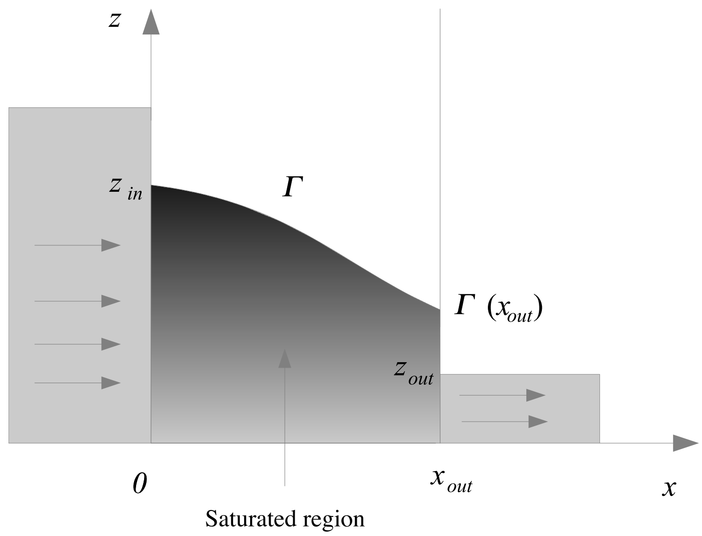

Figure 2: The Dam problem

In this section we use (1) to study the classical rectangular Dam problem [2]. Referring to Fig. 2 we assume that a fluid with pressure dependent viscosity is flown through a porous strip $[0, x_{out}] \times [0, \infty)$. The saturated and unsaturated domains in the strip are separated by a sharp interface Γ. Pressure is set to zero at the dry boundaries and it is assumed to be hydrostatic on those parts of the boundary in touch with the fluid. The so-called "phreatic" surface Γ is supposed of the form $z = f(x)$. The discharge $\mathbf{q}$ is assumed to be tangential on Γ and on $z = 0$. In this case all the variables depend on x and z and the problem is bi-dimensional. We suppose that the viscosity is of the form $\mu = \mu_o e^{\alpha P}$ (the linear case is simply obtained taking the first order approximation).

Remark 3. *Assuming incompressible flow $\nabla \cdot \mathbf{q} = 0$, we notice that the modified Darcy's law implies that the equation for P is no longer the Laplace equation (pressure is no longer an harmonic function).* ∎

To study the Dam problem we introduce the function

$$\pi = -\int_{P^*}^{P} \frac{\kappa}{\mu(y)} dy < 0, \qquad \frac{d\pi}{dP} = -\frac{\kappa}{\mu(P)} < 0,$$

where P^* is a reference pressure. Of course the function π is invertible and we have

$$\mathbf{q} = \nabla \pi - \frac{\kappa \rho g}{\mu} \mathbf{e}_z,$$

so that

$$\Delta \pi = \rho g \left(\frac{1}{\mu} \frac{d\mu}{dP} \right) \frac{\partial \pi}{\partial z} = \rho g \left[\frac{d}{dy} \ln \mu \right] \Big|_{y=P(\pi)} \frac{\partial \pi}{\partial z}.$$

In particular, since $\mu = \mu_o e^{\alpha P}$, we have

$$\pi = \frac{\kappa}{\mu_o \alpha} \left[e^{-\alpha P} - 1 \right],$$

and

$$\Delta \pi = \rho g \alpha \frac{\partial \pi}{\partial z}. \tag{18}$$

Remark 4. *If we take $\mu = \mu_o(1 + \alpha P)$ and $P^* = 0$ then*

$$1 + \alpha P = \exp\left(-\frac{\alpha \mu_o \pi}{\kappa} \right),$$

and

$$\Delta \pi = \left(\frac{\rho g \kappa}{\mu_o} \right) \cdot \frac{\partial}{\partial z} \left[\exp\left(\frac{\pi \alpha \mu_o}{\kappa} \right) \right] \tag{19}$$

∎

We now write the problem for the variable π. We rescale the variables as

$$\tilde{P} \alpha^{-1} = P, \qquad \left(\frac{\kappa}{\alpha \mu_o} \right) \tilde{\pi} = \pi, \qquad \tilde{\mathbf{q}} \left(\frac{\kappa \rho g}{\mu_o} \right) = \mathbf{q},$$

$$\frac{1}{\rho g \alpha} \tilde{z} = z, \qquad \frac{1}{\rho g \alpha} \tilde{x} = x.$$

Neglecting the tildes, the problem is the following (see [2] for all the details)

$$\begin{cases} \Delta\pi = \dfrac{\partial \pi}{\partial z}, & x \in (0, x_{out}), \quad 0 < z < f(x), \\ \pi(0,z) = e^{-(z_{in}-z)} - 1, & z \in [0, z_{in}], \\ \pi(x_{out},z) = e^{-(z_{out}-z)} - 1, & z \in [0, z_{out}], \\ \pi(x_{out},z) = 0, & z \in [z_{out}, f(x_{out})], \\ \dfrac{\partial \pi}{\partial z} = \pi + 1, & z = 0, \quad x \in [0, x_{out}], \\ \pi(x, f(x)) = 0, & x \in [0, x_{out}], \\ \mathbf{q} \cdot \mathbf{n}|_{\Gamma} = [\nabla\pi - \mathbf{e}_z] \cdot [-f', 1] = 0, & \text{on } \Gamma. \end{cases} \tag{20}$$

Remark 5. *Notice that condition (20)$_5$ comes from taking $\mathbf{q}\cdot\mathbf{k} = 0$ on $z = 0$.* ∎

Remark 6. *Existence and uniqueness of problem (20) can be proved following [2] with minor changes.* ∎

We are interested in estimating the overall discharge

$$\mathcal{Q} = \int_0^{f(x)} \mathbf{q} \cdot \mathbf{i} dz,$$

when $\mu = \mu(P)$. To this aim we take an interval $[x_1, x_2] \subset [0, x_{out}]$ and consider the region $E = \{(x,z) : \ x \in (x_1, x_2), \quad z \in (0, f(x))\}$. We have

$$\int_E \nabla \cdot \mathbf{q} \, dS = \int_{\partial E} \mathbf{q} \cdot \mathbf{n} \, d\sigma = 0,$$

so that $\int_0^{f(x_1)} \mathbf{q} \cdot \mathbf{e}_x \, dz = \int_0^{f(x_2)} \mathbf{q} \cdot \mathbf{e}_x \, dz$. Therefore

$$\mathcal{Q} x_{out} = \int_0^{x_{out}} d\xi \left(\int_0^{f(\xi)} \frac{\partial \pi}{\partial x} dz \right) = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3$$

where

$$\mathcal{I}_1 = \int_{f(x_{out})}^{z_{in}} d\xi \left(\int_0^{f^{-1}(\xi)} \frac{\partial \pi}{\partial x} dx \right),$$

$$\mathcal{I}_2 = \int_{z_{out}}^{f(x_{out})} d\xi \left(\int_0^{x_{out}} \frac{\partial \pi}{\partial x} dx \right),$$

$$\mathcal{I}_3 = \int_0^{z_{out}} d\xi \left(\int_0^{x_{out}} \frac{\partial \pi}{\partial x} dx \right).$$

Recalling the boundary conditions of (20) we get

$$\mathcal{Q} x_{out} = \int_0^{z_{in}} \left[1 - e^{-(z_{in}-z)}\right] dz - \int_0^{z_{out}} \left[1 - e^{-(z_{out}-z)}\right] dz,$$

which yields the nondimensional discharge

$$\mathcal{Q} x_{out} = (z_{in} - z_{out}) + e^{-z_{in}} - e^{-z_{out}}.$$

In dimensional variables the above becomes

$$\frac{\mathcal{Q} \mu_o \alpha x_{out}}{\kappa} = (z_{in} - z_{out}) + e^{-\rho g \alpha z_{in}} - e^{-\rho g \alpha z_{out}}.$$

We notice that expanding around $\alpha \approx 0$ and taking the leading order we get

$$\frac{\mathcal{Q} \mu_o \alpha x_{out}}{\kappa} = \frac{z_{in}^2 - z_{out}^2}{2},$$

which is the expression for the discharge when viscosity is constant [2].

5. The Muskat problem

In this section we consider the extension of the so-called Muskat problem, that is the one of two immiscible fluids flowing in a porous medium where one is displacing the other [14]. In particular, we consider the one-dimensional problem for two fluids with pressure dependent viscosities. Referring to Fig. 3 we suppose that the regions occupied by the two fluids are $\Omega_1 \cup \Omega_2$ where $\Omega_1 = [0, s(t)]$, $\Omega_2 = [s(t), L]$ and where $x = s(t)$ is the separating interface which is not a known (free boundary). Assuming that the flow is driven

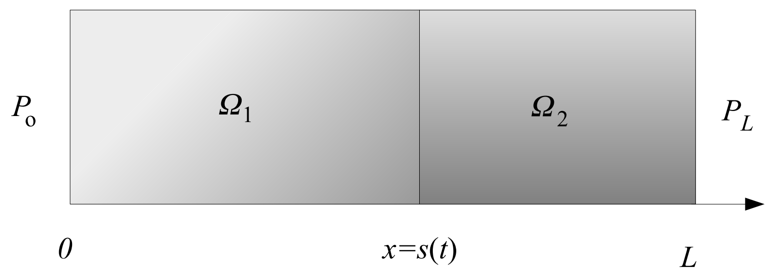

Figure 3: The Muskat problem

by a constant pressure difference $\Delta P = P_o - P_L > 0$ applied at the lateral boundaries $x = 0,\ x = L$ we write

$$\begin{cases} q_1 = -\dfrac{\kappa}{\mu_1(P)}\dfrac{\partial P}{\partial x}, \quad x \in \Omega_1, \\ \\ \dfrac{\partial q_1}{\partial x} = 0, \quad \Rightarrow \qquad q_1 = q_1(t), \end{cases} \qquad \begin{cases} q_2 = -\dfrac{\kappa}{\mu_2(P)}\dfrac{\partial P}{\partial x}, \quad x \in \Omega_2, \\ \\ \dfrac{\partial q_2}{\partial x} = 0, \quad \Rightarrow \qquad q_2 = q_2(t). \end{cases}$$

On the free boundary $x = s(t)$ we impose the the continuity of pressure and the continuity of flux

$$q_1(t) = q_2(t) = \phi\dot{s}, \qquad\qquad P|_{s^+} = P|_{s^-} = P^*$$

where $\phi \in [0,1]$ is, as usual, the medium porosity and $P^* = P(s(t))$ is unknown. It is easy to check that

$$q_i(t) = -\frac{\partial}{\partial x}\int_{P^*}^{P}\frac{\kappa}{\mu_i(y)}dy, \qquad i = 1, 2,$$

so that integration between x and s yields

$$\phi\dot{s}(s - x) = \int_{P^*}^{P}\frac{\kappa}{\mu_i(y)}dy, \qquad i = 1, 2.$$

Imposing the conditions for pressure on the lateral boundary we get

$$\phi\dot{s}s = \int_{P^*}^{P_o}\frac{\kappa}{\mu_1(y)}dy, \qquad \phi\dot{s}(s - L) = \int_{P^*}^{P_L}\frac{\kappa}{\mu_2(y)}dy. \tag{21}$$

Remark 7. *If we suppose that* $\mu_1,\ \mu_2$ *are constant, then eliminating* P^* *from (21) we get*

$$\dot{s}[s(\mu_1-\mu_2)+L\mu_2]=\frac{\Delta P\kappa}{\phi},$$

whose integration with the initial datum $s_o\in(0,L)$ *entails*

$$\frac{s^2-s_o^2}{2}(\mu_1-\mu_2)+L\mu_2(s-s_o)=\frac{\Delta P\kappa}{\phi}t, \tag{22}$$

which is the classical solution of the Muskat problem. Notice that, from (22), we can obtain the time at which fluid 1 has completely displaced fluid 2, namely

$$t_L=\frac{\phi}{2\Delta P\kappa}\Big[\mu_2(L-s_o)^2+\mu_1(L^2-s_o^2)\Big].$$

∎

Let us now consider

$$\mu_1=\beta_1 e^{\alpha_1 P},\qquad \mu_2=\beta_2 e^{\alpha_2 P}.$$

In this case integration of (21) yields

$$\frac{\beta_1\phi\dot{s}s}{\kappa}=\frac{1}{\alpha_1}\left[e^{-\alpha_1 P^*}-e^{-\alpha_1 P_o}\right], \tag{23}$$

$$\frac{\beta_2\phi\dot{s}s}{\kappa}=\frac{1}{\alpha_2}\left[e^{-\alpha_2 P^*}-e^{-\alpha_2 P_L}\right], \tag{24}$$

Eliminating P^* between (23) and (24) we get

$$\frac{\alpha_2}{\alpha_1}\ln\left[\frac{\beta_1\alpha_1\phi\dot{s}s}{\kappa}+e^{-\alpha_1 P_o}\right]=\ln\left[\frac{\beta_2\alpha_2\phi\dot{s}(s-L)}{\kappa}+e^{-\alpha_2 P_L}\right],$$

which provides the nonlinear fully implicit ODE for s, namely

$$\begin{cases}\left[\dfrac{\beta_1\alpha_1\phi\dot{s}s}{\kappa}+e^{-\alpha_1 P_o}\right]^{\alpha_2}-\left[\dfrac{\beta_2\alpha_2\phi\dot{s}(s-L)}{\kappa}+e^{-\alpha_2 P_L}\right]^{\alpha_1}=0,\\[2ex] s(0)=s_o\in(0,L).\end{cases} \tag{25}$$

In Fig. 4 we have plotted the solution of (25) for different value of the

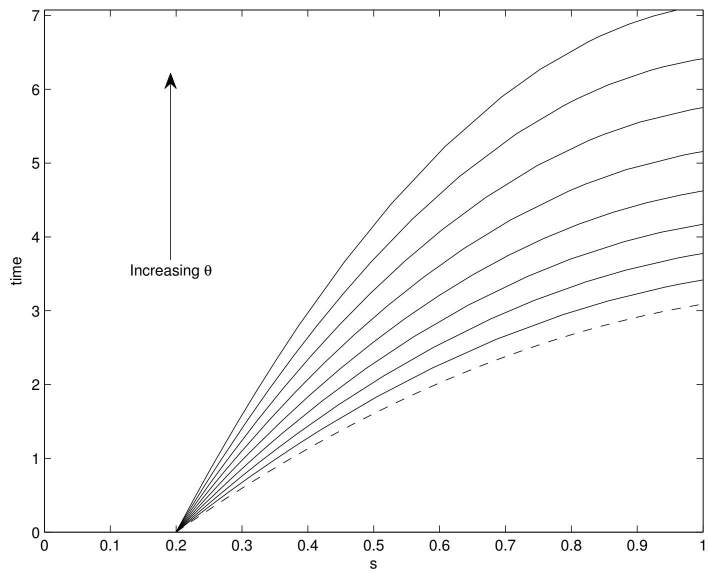

Figure 4: Plot of solution of (25) for increasing

parameters. The plot shows the adimensional solution obtained rescaling space with L and time with

$$T = \frac{\beta_1 \phi L^2}{\kappa P_o}.$$

Solutions are displayed for different values of the ratio $\theta = \alpha_2/\alpha_1$, in particular we have taken $\theta \in [1, 4]$. The dashed line denotes the solution of the linear case (22), obtained with the same rescaling. As one can see, an increase in θ (which means that fluid 2 becomes more viscous) implies a longer time for completely displacing fluid.

6. Conclusions and perspectives

In this paper we have shown some applications of the modified Darcy's law (1). This law has been rigorously obtained via homogenization [16]. We have studied some classical filtration problems to show how the dependence on pressure affects the qualitative behavior of the solution. In particular we have focussed on the Green-Ampt model, the Dam problem and Muskat problem. Of course many others problem can be studied using this modified version of Darcy's law.

In particular we mention the Buckley-Leverett transport model for a two immiscible fluid flow in porous media. In this case the problem (that can be solved autonomously in the saturation when discharge and viscosity are constant, see [13]) becomes much more complicated. We plan to devote a future paper to this problem.

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