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STOCHASTIC STABILITY OF GROUP FORMATION IN COLLECTIVE ACTION GAMES

Toshimasa Maruta[1] and Akira Okada[2]

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We present a game theoretic model of voluntary group formation in collective action and investigate its dynamic stability by applying the stochastic stability theory introduced by Young (1993) and Kandori, Mailath and Rob (1993). The heterogeneity of individuals' preferences yields the multiplicity of strict Nash equilibria including the global defection, the partial cooperation and the full cooperation. Global defection is never stochastically stable when participation costs are small. When the number of individuals with lower motivation to cooperate is larger than a critical level, partial cooperation is uniquely stochastically stable. Otherwise, the stochastic stability selects a version of risk dominant equilibrium. Full cooperation may be stochastically stable if there exists an individual whose incentive to free ride is not so strong.

Journal of Economic Literature Classification Numbers: C70, C72.

Keywords: Collective action, group formation, prisoner's dilemma, heterogeneous preferences, partial/full cooperation, stochastic stability.

[1] College of Economics, Osaka Prefecture University, Sakai, Osaka 599–8531 Japan. E-mail: *tosh@eco.osakafu-u.ac.jp*

[2] Institute of Economic Research, Kyoto University, Sakyo, Kyoto 606-8501 Japan. E-mail: *okada@kier.kyoto-u.ac.jp*

1 Introduction

The purpose of this paper is to present a game theoretic model of voluntary group in collective action and to consider dynamic stability of group formation by applying the stochastic evolutionary game theory introduced by Young (1993) and Kandori, Mailath and Rob (1993). The problem of collective action is formulated as a standard model of an n-person prisoner's dilemma game with heterogeneous players. We focus our analysis on how the heterogeneity of individuals' preferences affects the formation and dynamic stability of group in collective action.

In many real situations, individuals differ in their willingness to participate in a collective action. For example, some individuals are concerned very much with environmental pollution, and they are willing to contribute for anti-pollution movements even if they have a small number of followers. On the other hand, there are other types of individuals who are reluctant to participate in such a collective activity. They might contribute for anti-pollution only if a large number of people have already done so. In this paper, we consider the following questions. What kinds of groups are formed in the collective action problem with heterogeneous individuals? Does a group consist only of individuals with higher willingness for collective action, or does it include many types of individuals? If many kinds of groups are possible, which one is stable in the long run?

The process of group formation is modeled as a two-stage game. In the first stage, individuals decide independently to participate in a group or not. In the second stage, participants negotiate about collective action in their group. Cooperation must be agreed by all participants. If the agreement is reached, all members of the group take cooperative actions, bearing some group costs. Any non-member is allowed to free ride. If the agreement is not reached, the group is not formed and the noncooperative equilibrium of the n-person prisoner's dilemma prevails.

Every individual's incentive to cooperate is characterized by the minimum size of group in which participation can make him better off (even with bearing the group costs) than the noncooperative equilibrium of the prisoner's dilemma. In this paper, we call this positive integer the individual's *threshold of cooperation*. Individuals with smaller thresholds have higher motivation to cooperate. It is shown that a group is formed in the Nash equilibrium of the second stage game if and only if the group size exceeds every member's threshold of cooperation. Such a group is called *successful*.

By solving backward the two-stage game, the first stage game is reduced to the following n-person game in strategic form, which we call the *group formation game*. All individuals decide independently to participate in a group or not. The group is formed if and only if it is successful. The group formation game itself presents a model of collective action, which is different from the original n-person priosoner's dilemma. All individuals have binary choices (cooperate, participate, contribute, or not). Neither action dominates the other. The game has a payoff structure similar to the two-person chicken game. The discrete public good model by Palfrey and Rosenthal (1984) is a special case of it. In their model, there is a critical number of contributors for producing the public good, and all individuals have the same threshold of cooperation.

It is shown that the group formation game has many strict Nash equilibria. The non-participation is a strict Nash equilibrium when participation costs exist regardless of the success or failure of a group. In all other strict Nash equilibria, successful groups are formed with free riders possibly coexisting. An equilibrium group satisfies the property that every member is critical in the sense that her opting out makes the group unsuccessful. Equivalently, we can show that an equilibrium group has two stability property: No single member wants to opt out (internal stability), and no single outsider wants to join in (external stability).

In the second part of the paper, we apply the adaptive play model due to Young (1993, 1998) to the group formation game. Our main objective is to identify which group is stable in the long run when individuals play the group formation game in an adaptive way. To set the stage, we first show that the group formation game is acyclic. This implies from the result of Young (1993) that adaptive play (without mistakes) converges to a Nash equilibrium. We then examine the stochastic stability of Nash equilibria in the adaptive play with mistakes. To make the analysis transparent, we focus our attention on the group formation game with exactly two types of individuals. The first type consists of individuals who might cooperate even if some of the others do not. Specifically, it is assumed that they share a common threshold, and that it is strictly below the number of all individuals. The second type consists of individuals who have lower motivation to cooperate. Specifically, cooperation is optimal for a second type individual only if all the others do. In this game, there are three strict Nash equilibria: No cooperation, the full cooperation, and the partial cooperation, in which only the first types participate. In the partial cooperation equilibrium, the second types do not cooperate but benefit from the first types' cooperation. That is, they are free riding. In contrast, the first types never free ride in equilibrium. Because of this, we call the second types potential free riders.

It turns out that the long run equilibrium is either partial cooperation or full cooperation when the participation cost is small. There are two critical factors in determining the long run equilibrium: the number of potential free riders and the distribution of incentives among them. Interesting enough, they affect the outcome in a lexicographic way. We first show that if the number of potential free riders exceeds a critical level, then the partial cooperation is a unique long run equilibrium. In other words, when there are "large" number of potential free riders, the free riding equilibrium is the stable outcome regardless of the distribution of incentives among them. When the number of potential free riders is "small," in contrast, the distribution of incentives matters. Specifically, it is shown that the stochastic stability selects a (version of) risk dominant equilibrium (Harsanyi and Selten 1988) in such a case. The intuition behind this result is that, when there are not too many potential free riders, the group formation game can be regarded as a coordination game among them. Therefore the selection outcome is determined by risk dominance. The following version of risk dominance relation is relevant here. Since preferences are heterogeneous, the incentives to free ride differ, even among the potential free riders. In particular, there are the strongest and the weakest free riders, who have the largest and the smallest incentives to free ride, respectively. For each equilibrium, the Nash product of deviation losses is computed involving only the two distinguished individuals. An equilibrium risk dominates the other if the Nash product of the former exceeds that of the latter. In simple terms, the stable outcome is determined by the balance of the largest and the weakest incentive to free ride.

Since the seminal work by Olson (1965), the group formation in collective action has been extensively studied. The group size effect, argued by Olson, that larger groups are less successful in organizing collective action is not necessarily true in our model. The success of collective action critically depends upon the benefit and cost for each individual to participate. We show that the diversity of individual incentives to cooperation may enable the largest group in collective action easier than in a homogeneous case, while it causes the multiplicity of equilibrium groups. Although thresholds of cooperation play an important role in the analysis, our model is different in an important way from other "threshold" or "critical mass" models (Schelling

1978, Granovetter 1978, Oliver and Marwell 1988). These models presume a simple behavioral rule on the part of individuals, in that they participate in collective action if and only if the number of participants exceeds their thresholds. In contrast, we investigate group formation games in which individuals behave strategically. That is, even if the number of participants exceeds their thresholds, they do not necessarily participate, since the best response in such a situation might be to free ride. Our analysis is also related to a recent work by Diermeier and van Mieghem (2000) who study a dynamic stochastic process of collective action in Palfrey and Rosenthal's (1984) model of public goods. Working in a homogeneous population setup with a log-linear choice rule, they formulate the dynamic model of collective action as a a birth and death process and characterize its limit distribution.

The paper is organized as follows. Section 2 constructs the group formation game from the n-person prisoner's dilemma. Section 3 characterizes its strict Nash equilibria. Section 4 reviews the stochastic stability theory à la Young (1993), and then shows that the group formation game is acyclic. Section 5 analyses the stochastic stability of Nash equilibria in the group formation game with two types. Concluding remarks are given in Section 6. Appendix collects proofs that are omitted from the main body of the paper.

2 The Model

Consider an n-person prisoner's dilemma defined as follows. Let $N = \{1, 2, \cdots, n\}$ be the set of players. Every player $i \in N$ has two actions, C (cooperation) and D (defection). Player i's payoff is given by

$$u_i(a_i, h),\ a_i = C, D,\ h = 0, 1, \cdots, n-1,$$

where a_i is player i's action and h is the number of other players who select C. We make the following assumption.

Assumption 2.1. The payoff function of player i $(= 1, ..., n)$ satisfies:

(1) $u_i(D, h) > u_i(C, h)$ for every $h = 0, 1, \cdots, n-1$,

(2) $u_i(C, n-1) > u_i(D, 0)$,

(3) $u_i(C, h)$ and $u_i(D, h)$ are increasing in h.

This assumption is standard in the literature of an n-person prisoner's dilemma (Schelling 1978) except that players are "heterogeneous" in the sense that they have different payoff functions. The heterogeneity of players is critical to the analysis of this paper. Property (1) means that every player is better off by choosing defection than cooperation, regardless of other players' actions. This implies that every player has an incentive to free ride on others' cooperation. Thus, the action profile $(D, \cdots, D)$ is a unique Nash equilibrium of the game. On the other hand, property (2) says that if all players cooperate, they are all better off than at the Nash equilibrium. The Nash equilibrium is not Pareto optimal. Property (3) means that the more other players cooperate, the higher payoffs every player can receive, regardless of her action. The cooperative action by each player gives positive externality to all others' welfare.

The prisoner's dilemma game describes an anarchic state of nature in which players are free to choose their actions. In such a situation, a natural outcome of the game is the Nash equilibrium in which no players cooperate. There have been a huge body of literature which consider how self-interested individuals voluntarily cooperate in the prisoner's dilemma situations. To escape from the state of noncooperation, some suitable mechanisms for preventing opportunistic behavior are needed. The literature has considered the roles of diverse mechanisms such as morals, convention, norm, long term relationships, evolutionary selection, informal groups, organizations, law, etc. In this paper, we consider the voluntary creation of a group in which participants negotiate to cooperate.

The rule of group formation is defined as a two-stage game.

Participation decision stage: Every player i $(= 1, \ldots, n)$ decides independently whether or not to participate in a group. Participation takes small costs, say, for phone calls, mails and transportations. The participation cost is denoted by a small positive value ε_i (> 0). Let S be the set of all participants, and let $s = |S|$, where $|S|$ is the number of elements in set S. If $s = 0$ or $s = 1$, then no group is possible.[1]

Group negotiation stage: All participants negotiate about their cooperation according to the unanimity rule. They decide independently to accept or reject cooperation. The agreement of cooperation is reached if and only if all participants accept it. When the agreement of cooperation is reached, all participants choose cooperative actions with group costs allocated to them. The group costs (including participation cost ε_i) per capita is given by a real-valued

[1] When $s = 1$, the single participant has no incentive to cooperate in the prisoner's dilemma.

function $c(s)$ where s is the number of all participants. All non-participants are free to defect. When the agreement is not reached, all n individuals, both participants and non-participants, play the original prisoner's dilemma game.

The purpose of a group is to attain cooperation among its members. Since each member has an incentive to defect in the prisoner's dilemma, the group needs some suitable mechanism to enforce cooperation. The mechanism has various functions such as monitoring members' actions and punishing them for defection. Obviously, it is costly for group members to establish such an enforcement mechanism. In what follows, to keep our game model of group formation as simple as possible, we do not present a formal model of an enforcement mechanism in a group, but we represent it simply by a group cost function $c(s)$. Okada (1993) considered a related model of group formation in which group members negotiate for creating an enforcement institution.

Example 2.1. (Voluntary provision of a public good) There are n players i each with two actions, contributing a fixed amount m of money to producing a public good ($s_i = 1$), or not contributing ($s_i = 0$). Players decide their contributions independently. For an action profile $s = (s_1, ..., s_n)$, every player i receives $u_i(m \sum_{i=1}^{n} s_i) + m(1 - s_i)$. We assume that (1) the payoff function $u_i(s)$ of the total contribution $s = m \sum_{i=1}^{n} s_i$ is differentiable on the interval $[0, mn]$, (2) $0 < u_i'(s) < 1$ for all s, and (3) $u_i(mn) > u_i(0) + m$. Under these assumptions, it is easy to see that the game of voluntary contribution becomes an n-person prisoner's dilemma game.

We now consider a subgame perfect equilibrium of the two-stage game of group formation by the usual backward induction. First, we analyze the group negotiation stage. When a group of s members agree to cooperate, every member receives

$$v_i(C, s - 1) = u_i(C, s - 1) - c(s).$$

We call $v_i(C, s-1)$ the *group payoff* of player i where s is the number of members in the group. Concerning the group payoff, we assume the following property.

Assumption 2.2. For every $i \in N$, the group payoff $v_i(C, s - 1)$ of player i is monotonically increasing in s, and there exists a unique positive integer s_i ($2 \leq s_i \leq n$) such that

$$v_i(C, s_i - 2) < u_i(D, 0) < v_i(C, s_i - 1). \tag{2.1}$$

This assumption means that the property (Assumption 2.1) of the n-person prisoner's dilemma still holds true even if we replace the original cooperative payoff $u_i(C,h)$ with the group payoff $v_i(C,h)$. If Assumption 2.2 does not hold, the problem of group formation becomes rather trivial. For example, if $v_i(C, s-1) \leq u_i(D,0)$ for all $s \leq n$, then no players have incentive to participate in a group. The positive integer s_i in (2.1) shows the minimum size of a group in which member i can be better off than at the noncooperative equilibrium of the prisoner's dilemma. We call s_i player i's *threshold of cooperation*. Player i can benefit by cooperation when other $(s_i - 1)$ players also cooperate. In this sense, players with smaller thresholds have higher motivation to cooperate.

Definition 2.1. A subset S of N is called a *successful group* if $|S| \geq s_i$ for every $i \in S$.

The size of a successful group is greater than or equal to all members' thresholds of cooperation. By definition, every member of a successful group can receive higher payoff than the noncooperative payoff in the prisoner's dilemma. The naming of a successful group is explained by the following proposition.

Proposition 2.1. *In the group negotiation stage, an agreement for cooperation is reached by the members of a group in a Nash equilibrium if and only if the group is successful.*

Proof. Suppose that all s participants agree to cooperate. Then, every participant receives the group payoff $v_i(C, s-1)$. If any one member reject to cooperate, negotiations break down by the unanimity rule, and she receives the noncooperative payoff $u_i(D,0)$ in the prisoner's dilemma. Therefore, the agreement of cooperation in a group S is reached in a Nash equilibrium if and only if for all $i \in S$, $v_i(C, |S|-1) \geq u_i(D,0)$. From Assumption 2.2, this is equivalent to that the group is successful. $\square$

The proposition implies that in every successful group the agreement of cooperation is reached in a Nash equilibrium of the group negotiation stage. There exists, however, many "trivial" Nash equilibria leading to the disagreement. For example, any action profile where at least two participants reject to cooperate are such Nash equilibria. These trivial equilibria are peculiar to the unanimity rule where everyone has a veto power. We remark that from the viewpoint of every participant the action of agreement (weakly) dominates that of disagreement. By this reason, we consider only the Nash equilibrium leading to the agreement of cooperation in a successful group.

Given the Nash equilibrium of the group negotiation stage, the participation decision stage can be reduced to the following game. In the game, every player i in N choose simultaneously and independently either $\sigma_i = 1$ (participation) or $\sigma_i = 0$ (non-participation). Let $\Sigma_i = \{0, 1\}$ be the set of actions of player i, and let $\Sigma = \prod_{i \in N} \Sigma_i$ be the set of action profiles of n players. For an action profile $\sigma = (\sigma_1, \cdots, \sigma_n) \in \Sigma$, the set $S(\sigma)$ of participants is given by

$$S(\sigma) = \{i \in N | \sigma_i = 1\}.$$

The payoff $f_i(\sigma)$ of player i for an action profile $\sigma = (\sigma_1, \cdots, \sigma_n) \in \Sigma$ is defined as follows.

(i) When a group $S(\sigma)$ of participants is successful,

$$f_i(\sigma) = \begin{cases} v_i(C, |S(\sigma)| - 1) & \text{if } \sigma_i = 1, \\ u_i(D, |S(\sigma)|) & \text{if } \sigma_i = 0. \end{cases}$$

(ii) When $S(\sigma)$ is not successful,

$$f_i(\sigma) = \begin{cases} u_i(D, 0) - \varepsilon_i & \text{if } \sigma_i = 1, \\ u_i(D, 0) & \text{if } \sigma_i = 0. \end{cases}$$

where ε_i (> 0) is a participation cost for a group.

Formally, the reduced form of the participation decision stage is represented by an n-person game $\Gamma = (N, \{\Sigma_i, f_i\}_{i \in N})$ in strategic form. We call it the *group formation game*.

The group formation game Γ differs from the n-person prisoner's dilemma game in the following aspects. In the game Γ, every participant does not need to cooperate (he never cooperates in equilibrium) if the number of participants is not large enough to satisfy her threshold of cooperation. Neglecting a small participation cost ε_i, every participant can guarantee the noncooperative payoff in the prisoner's dilemma. This is not the case in the prisoner's dilemma game. If a player selects a cooperative action, she is free ridden by other defectors, and he may be worse off than the noncooperative equilibrium. In the group formation game, the action of non-participation does not dominate that of participation. A player can receive higher payoff by participation than by non-participation when participation is critical to the formation of a successful group.

Finally, we construct the group formation game Γ of the voluntary provision of a public good in Example 2.1. The game generalizes Palfrey and Rosenthal's (1984) model of discrete public good.

Example 2.2. (The group formation game in Example 2.1) When s players participate in a group in Example 2.1, the group payoff of player i is given by $u_i(sm) - c(s)$ where $c(s)$ is the group cost per member. Player i's threshold s_i of cooperation is given by the minimum integer s satisfying $u_i(sm) - c(s) > u_i(0) + m$. Every player i in N decide independently to participate in a group of contributors ($\sigma_i = 1$), or not to participate ($\sigma_i = 0$). For an action profile $\sigma = (\sigma_1, \cdots, \sigma_n)$, every player i's payoff $f_i(\sigma)$ is defined as follows.

(i) When the group of all participants is successful,

$$f_i(\sigma) = \begin{cases} u_i(sm) - c(s) & \text{if } \sigma_i = 1, \\ u_i(sm) + m & \text{if } \sigma_i = 0. \end{cases}$$

(ii) Otherwise,

$$f_i(\sigma) = \begin{cases} u_i(0) + m - \varepsilon & \text{if } \sigma_i = 1, \\ u_i(0) + m & \text{if } \sigma_i = 0. \end{cases}$$

Palfrey and Rosenthal's (1984) model of discrete public good is a special case of the group formation game defined above. The provision level of the public good is binary, and the public good is produced only if the number of contributors satisfy a critical level. All players have identical (linear) payoff functions for the public good, and thus their thresholds of cooperation are identical. The group cost $c(s)$ is equal to the participation cost ε for any number of participants.

3 The Nash Equilibria in the Group Formation Game

In this section, we characterize the set of Nash equilibria in the group formation game Γ. We first examine the best response structure of the game Γ. For an action profile $\sigma = (\sigma_1, \cdots, \sigma_n) \in \Sigma$, let σ_{-i} be the action profile obtained from σ by deleting σ_i. As usual, an action profile $\sigma = (\sigma_1, \cdots, \sigma_n)$ is sometimes denoted by $\sigma = (\sigma_{-i}, \sigma_i)$. Let $S(\sigma)$ be the set of participants in σ.

Definition 3.1. For an action profile $\sigma = (\sigma_{-i}, \sigma_i) \in \Sigma$ in Γ, player i's action σ_i is called a *best response* to σ if $f_i(\sigma_{-i}, \sigma_i) = \max_{\sigma'_i \in \Sigma_i} f_i(\sigma_{-i}, \sigma_i)$.

Definition 3.2. The *best response graph* V of Γ is a binary relation on the set of action profiles Σ such that, for every σ, $\sigma' \in \Sigma$, $(\sigma, \sigma') \in V$ if and only if $\sigma \neq \sigma'$ and there exists exactly one player i satisfying (i) $\sigma_{-i} = \sigma'_{-i}$ and (ii) σ'_i is a best response to σ for i. When $(\sigma, \sigma') \in V$, we write $\sigma \to \sigma'$ and call it an *edge* from σ to σ'.

The definition of the best response graph is due to Young (1993). It plays an important role in the analysis of stochastic stability of the Nash equilibrium as well as its existence.

Definition 3.3. For a successful group S, member i of S is called *critical* to S if $S - \{i\}$ is not successful.

No successful group can be sustained if any critical member opts out of it. The following proposition characterizes the best response graph of the group formation game Γ.

Proposition 3.1. *An edge of the best response graph V of the group formation game Γ must be one of the following types.*

(1) *When $S(\sigma)$ is a successful group, $\sigma = (\sigma_{-i}, 1) \to (\sigma_{-i}, 0)$ for all members i who are not critical to $S(\sigma)$.*

(2) *When $S(\sigma)$ is not a successful group, $\sigma = (\sigma_{-i}, 1) \to (\sigma_{-i}, 0)$ for all members i of S, and $\sigma = (\sigma_{-i}, 0) \to (\sigma_{-i}, 1)$ for all non-members i such that $S(\sigma) \cup \{i\}$ is a successful group.*

Proof. (1) Suppose that $S(\sigma)$ is a successful group and that member i is not critical. Since the group $S(\sigma) - \{i\}$ remains successful, we have

$$f_i(\sigma_{-i}, 1) = v_i(C, |S(\sigma)| - 1) < u_i(D, |S(\sigma)| - 1) = f_i(\sigma_{-i}, 0).$$

Therefore, $\sigma_i = 0$ is a best response to σ for all non-critical members i of $S(\sigma)$.

(2) Suppose that $S(\sigma)$ is not a successful group. Then, for all $i \in S$,

$$f_i(\sigma_{-i}, 1) = u_i(D, 0) - \varepsilon_i < u_i(D, 0) \leq f_i(\sigma_{-i}, 0),$$

where $f_i(\sigma_{-i}, 0)$ is equal to either $u_i(D, |S(\sigma)| - 1)$ or $u_i(D, 0)$, depending on whether the remaining group except player i is successful or not. For any non-member i such that $S(\sigma) \cup \{i\}$ is a successful group,

$$f_i(\sigma_{-i}, 0) = u_i(D, 0) < v_i(C, |S(\sigma)|) = f_i(\sigma_{-i}, 1).$$

Finally, it can be easily seen that there exist no other edges in the best response graph V given in the theorem. □

Proposition 3.1 reveals the best response structure of the group formation game. In a successful group, every non-critical member has an incentive to deviate from the group because, by doing so, she can free ride on cooperation by the group. In an unsuccessful group, every member has an incentive to deviate from the group for saving participation costs. Remark that a player outside an unsuccessful group has an incentive to join the group if her participation makes the group successful. By Proposition 3.1, we can characterize strict Nash equilibria in Γ.

Proposition 3.2. *The group formation game Γ has the following strict Nash equilibria $\sigma = (\sigma_1, \cdots, \sigma_n)$.*

(1) $\sigma = (0, \cdots, 0)$*, i.e.,* $S(\sigma) = \varnothing$.

(2) $S(\sigma)$ *is a successful group with every member critical to* $S(\sigma)$.

The proposition can be explained intuitively by an alternative definition of a Nash equilibrium of the group formation game Γ. A group of participants in the Nash equilibrium satisfy two stability properties:

Internal stability: No single member want to opt out of the group.

External stability: No single outsider want to join the group.

It is clear that the action profile $\sigma = (0, \cdots, 0)$ is a Nash equilibrium because no one is willing to cooperate unilaterally. When a group is not successful, the internal stability is violated because all participants want to opt out of the group for saving participation costs. When a group is successful, the external stability always holds because all non-participants have incentive to free ride. The internal stability implies that every participant is critical to the group. We remark that if there exists no participation costs, the action profile $\sigma = (0, \cdots, 0)$ is a non-strict Nash equilibrium because every player is indifferent to her decision of participation. In this case, action profiles leading to unsuccessful groups are non-strict Nash equilibria if no outsider's participation makes the group successful.

We next characterize a Nash equilibrium of the group formation game Γ in terms of players' thresholds of cooperation. For $S \subset N$ and $m = 2, \cdots, n$, we define $F_S(m)$ by the number of all members in S whose thresholds of cooperation are given by m. That is, $F_S(m) = |\{ i \in S \mid s_i = m \}|$. F_S represents the distribution of members of S in terms of thresholds of cooperation. Its definition implies the next lemma.

Lemma 3.1. *For $S \subset N$,*

(1) $F_S(2) + \cdots + F_S(|S|) \leq |S|$.

(2) *A group S is successful if and only if $F_S(2) + \cdots + F_S(|S|) = |S|$.*

Proposition 3.3. *A nonempty subset S of N is the set of participants in a Nash equilibrium of the group formation game Γ if and only if*

$$F_S(2) + \cdots + F_S(|S|) = |S| \quad \textit{and} \quad F_S(|S|) \geq 2.$$

Proof. From Proposition 3.2 and Lemma 3.1, it is sufficient to prove that every member of a successful group S is critical to S if and only if $F_S(|S|) \geq 2$. Suppose that $F_S(|S|) \geq 2$. For every $i \in S$, group $S - \{i\}$ is not successful because $F_{S-\{i\}}(|S|) \geq 1$. Thus, every member i of S is critical to S. If $F_S(|S|) = 1$, then a unique member i with $s_i = |S|$ is not critical to S because $S - \{i\}$ is a successful group. If $F_S(|S|) = 0$, all members j of S have thresholds s_j of cooperation with $s_j \leq |S| - 1$. Therefore, they are not critical to S. □

¿From the proposition, we can see how the heterogeneity of a society affects the group formation. When a society is homogeneous in the sense that all players have identical thresholds s ($2 \leq s \leq n$) of cooperation, the size of an equilibrium group is uniquely determined by the common threshold s. On the other hand, when a society is heterogeneous, there exist generally many Nash equilibria in the group formation game Γ. For example, if there exist at least two players who have thresholds s of cooperation for each integer $s = 2, \cdots, m$, then a successful group of every size s can be formed in a Nash equilibrium of Γ. The heterogeneity of a society causes the multiplicity of the Nash equilibrium in the group formation game.

The heterogeneity of a society also affects the efficiency (the number of cooperators) of a group as follows. In a homogeneous society, the largest group of n players can be formed under a stringent condition that all players' thresholds are equal to the number n of players, $s_1 = \cdots = s_n = n$. To put it differently, the full cooperation can be attained in a homogeneous society only if all players are "reluctant" to cooperate. In a heterogeneous society, the largest group may be sustained in equilibrium under a much weaker condition that $F_N(n) \geq 2$, that is, there are at least two players whose thresholds are n.

4 Stochastically Stability in Games

The analysis thus far shows that there are multiple equilibria in a group formation game. More specifically, there are three types of equilibria. First, the "global defection," in which no one cooperates, is always a strict equilibrium. Second, in games with heterogeneous thresholds, there are typically "partial cooperation" equilibria, in which some players cooperate but the others do not. Third, the "full cooperation," in which everyone cooperates, is also a strict equilibrium. Thus the question arises as to which type of equilibrium is most likely to prevail.

To tackle this problem, we adopt the stochastic equilibrium selection theory á la Young (1993). In this section, we briefly review the selection theory, and show that it is directly applicable to group formation games.

Let G be an n-person strategic form game, with the set of strategies A_i, $i = 1, \dots, n$. Given a positive integer m, let H be the m-fold direct product of $A = A_1 \times \cdots \times A_n$. We call an $h \in H$ a *state*. A state h is a sequence of strategy profiles with length m. H is the state space of Young's (1993) Markov chain.

Roughly speaking, the selection theory of Young (1993) works as follows. At each period, each player is given a set consists of k-strategy profiles ($k < m$). Let us call the set a *sample*. Profiles in the sample are randomly drawn (without replacement) from the current state h, which consists of m-most recently occurred profiles. In determining her strategy for that period, each player chooses a best response to her sample. There is a probability $\epsilon \geq 0$, however, that the player enters into an "experimentation" mode. In the experimentation mode, instead of playing a best response, she chooses her strategy randomly. This behavioral specification is called *adaptive play* (with or without mistakes according to $\epsilon > 0$ or $\epsilon = 0$). We make following assumptions. For each player i and each state h, every possible sample of size $k$from h has a positive probability to be drawn for i. In the adaptive play with mistakes ($\epsilon > 0$), given that i is in the experimentation mode, each possible strategy of i can be chosen with positive probability. Sample drawings, occurrence of experimentation modes, and random choices in them are all independent both across players and across periods. These assumptions make the adaptive play with mistakes an irreducible and aperiodic finite state Markov chain on H.

Thus, for each $\epsilon > 0$, there is a unique stationary distribution μ_ϵ on H. A state $h \in H$ is *stochastically stable* (Foster and Young 1990) if $\lim_{\epsilon \to 0} \mu_\epsilon(h) > 0$.

Recall the best response graph V of G (Definition 3.2). A finite sequence $a^1, \ldots, a^L$ in A is a *best response path* if $(a^l, a^{l+1}) \in V$ for every $l = 1, \ldots, L-1$. The game G is said to be *weakly acyclic* if, for every action profile $a \in A$, either a is a strict Nash equilibrium or there exists a best response path $a = a^1, a^2, \ldots, a^L$ such that a^L is a strict Nash equilibrium. For weakly acyclic games, the notion of stochastic stability leads to equilibrium selection as follows. A *recurrent class* of a finite state Markov chain is a nonempty set of states that is minimal with respect to the property that once the chain moves into the set, it stays within the set thereafter. Consider recurrent classes of the adaptive play without mistakes ($\epsilon = 0$). In general, a recurrent class contains multiple states. However, if the stage game is weakly acyclic, then there is a one to one correspondence between strict equilibria of G and recurrent classes of the adaptive play without mistakes. Specifically, let $NE = \{e^1, \ldots, e^J\}$ be the set of strict Nash equilibria in the weakly acyclic game G. Young (1993) shows that if sampling is sufficiently incomplete,[2] then recurrent classes are precisely $H^1, \ldots, H^J$, where $H^j = \{h^j\} = \{(e^j, \ldots, e^j)\}$. Moreover, Young (1993) shows that a stochastically stable state must belong to a recurrent class of the adaptive play without mistakes. Thus, for a weakly acyclic game G, a stochastically stable state is essentially a strict equilibrium of G. Let us say that a strict equilibrium of a weakly acyclic game is *stochastically stable* if the corresponding state is stochastically stable. If the stochastically stable equilibrium is unique, it is the one that is observed infinitely many more times than other equilibria in the long run, when the probability of mistakes is infinitely small. In this sense, the notion of stochastic stability gives rise to selection among strict equilibria.

What follows is the formal procedure to identify stochastically stable equilibria of a weakly acyclic game G. Let $h = (a^1, \ldots, a^m) \in H$ and $h' = (b^1, \ldots, b^m) \in H$ be two states. A state h' is a *successor* of h if $b^l = a^{l+1}$ for $l = 1, \ldots, m-1$. Let h' be a successor of h. Note that any sample of size k from h given to player i, namely, a k-length subsequence of $(a^1, \ldots, a^m)$, determines a probability distribution on A_{-i}, as its empirical frequency. Denote $b^m = (b^m_i, b^m_{-i})$. The strategy $b^m_i \in A_i$ chosen by i is a *mistake* in the transition from h to h' if h has no sample of size k such that b^m_i is a best response to the empirical frequency of that sample. For every

[2]Let L_G is the maximum length of all such best response sequences. The sampling is sufficiently incomplete if $k \leq m/(L_G + 2)$.

h, $h' \in H$, define *resistance* $r(h, h')$ *of the transition from* h *to* h' as follows.

$$r(h,h') = \begin{cases} \text{the total number of mistakes} \\ \text{in the transition from } h \text{ to } h', & \text{if } h' \text{ is a successor of } h, \\ \\ \infty, & \text{otherwise.} \end{cases}$$

A sequence of states $\omega = (h^1, \ldots, h^L)$ with $h^l \in H$ $(1 \leq l \leq L)$ is called a *path from* h^1 *to* h^L if h^{l+1} is a successor of h^l for $l = 1, \ldots, L-1$. Define *resistance* $r(\omega)$ *of the path* ω by $r(\omega) = \sum_{l=1}^{L-1} r(h^l, h^{l+1})$. In words, $r(\omega)$ is the total number of mistakes that the path ω contains. Note that a path $(h^1, \ldots, h^L)$ can be considered to be a sequence of action profiles, as opposed to a sequence of states. That is, the m action profiles that consist of h^1, followed by the "rightmost" profiles of the successive states $h^2, \ldots, h^L$. In other words, any sequence of action profiles with length exceeding m determines a path.

For each $e \in NE$, let $h(e)$ be the corresponding state: $h(e) = (e, \ldots, e)$. In stochastic stability analysis, the notion of resistance between equilibrium states is crucial. For two different equilibria e, $e' \in NE$, define *resistance from* e *to* e' by

$$r(e, e') = \min \left\{ r(\omega) \mid \omega \text{ is a path from } h(e) \text{ to } h(e') \right\}.$$

In words, $r(e, e')$ is the minimum number of mistakes that is sufficient for allowing a path from e to e'.

A binary relation T on NE is an *e-tree* if (i) $(e, e') \notin T$ for every $e' \in NE$; (ii) for every $e' \neq e$, there are $e^{\iota 1}, \ldots, e^{\iota L} \in NE$ with $e^{\iota 1} = e'$ and $e^{\iota L} = e$ such that $(e^{\iota l}, e^{\iota l+1}) \in T$ for every $l = 1, \ldots, L-1$. Given an e-tree T, we define *stochastic potential* $\rho(T)$ of T by

$$\rho(T) = \textstyle\sum_{(e', e'') \in T} r(e', e'').$$

Young (1993) shows that a strict Nash equilibrium e of G is stochastically stable if and only if for every e'-tree T' there is an e-tree T such that $\rho(T) \leq \rho(T')$.

It remains to show that the selection theory of Young (1993) is directly applicable to the group formation game. A best response path $a^1, \ldots, a^L$ is called a *cycle* if $a^1 = a^L$. The stage game G is said to be *acyclic* if no best response path is a cycle. Since the stage game G is finite, if G is acyclic, then it is weakly acyclic.

Proposition 4.1. *The group formation game* Γ *is acyclic.*

Proof. Let $\sigma = (\sigma_1, \cdots, \sigma_n)$ be any action profile in Γ, and let $S(\sigma)$ be the set of participants at σ. The following two cases are possible.

Case 1. $S(\sigma)$ is a successful group:

It follows from Proposition 3.1 that any possible edge $\sigma \to \sigma'$ from σ must have the form of $\sigma' = (\sigma_{-i}, 0)$ for some $i \in S$ such that $S(\sigma)$ is successful. This means that any best response path starting at σ never return to σ, and thus that there is no cycle including σ.

Case 2. $S(\sigma)$ is not a successful group:

It follows from Proposition 3.1 that any possible edge $\sigma \to \sigma'$ from σ must be one of the following two types: (i) $\sigma' = (\sigma_{-i}, 0)$ for some $i \in S$, and (ii) $\sigma' = (\sigma_{-i}, 1)$ for some $i \notin S$ such that $S \cup \{i\}$ is a successful group. It can be seen from the proof in case 1 that, once subcase (ii) happens, all vertices σ'' in a best response path following σ' are associated with successful groups $S(\sigma'')$. This implies that there is no cycle including σ in subcase (ii). Now assume that subcase (i) holds. Consider an edge $\sigma \to \sigma' = (\sigma_{-i}, 0)$ for $i \in S$. If $S(\sigma')$ is a successful group, the same arguments as in subcase (ii) can be applied. Therefore, it suffices us to consider only a best response path $\sigma = \sigma^0, \sigma^1, \sigma^2. \cdots, \sigma^m$ such that for all $k = 1, \cdots, m,\ S(\sigma^k) = S(\sigma^{k-1}) - \{j\}$ for some $j \in S(\sigma^{k-1})$ and $S(\sigma^k)$ is not a successful group. Clearly, such a best response path never returns to σ. Thus, there exists no cycle including σ in subcase (i), either. $\square$

5 Equilibrium Selection in Group Formation Games

In the group formation game with heterogeneous thresholds, there are three types of equilibria: the global defection, the partial cooperation, and the full cooperation. Which equilibrium is most likely to prevail? Since the group formation game is acyclic, we can identify, in principle, which equilibrium is the most stable in the sense of stochastic stability. In practice, however, it turns out to be quite complex to evaluate the relevant resistances for general group formation games. Thus we turn to a specific class of them, which consists of the *group formation games with two types of players*.

5.1 The Group Formation Game with Two Types of Players

A group formation game with two types of players is defined as follows. Recall that the set of players is $N = \{1, \ldots, n\}$. Let us partition N into two sets, $N_1 = \{1, \ldots, n_1\}$ and

$N_2 = \{n_1 + 1, \ldots, n\}$. The size of N_1 and N_2 are n_1 and $n_2 = n - n_1$, respectively. We assume that n_1, $n_2 \geq 2$. N_1 and N_2 represent two types of players as follows. Assume that

$$s_i = n_1 \text{ for every } i \in N_1, \text{ and } s_i = n = n_1 + n_2 \text{ for every } i \in N_2,$$

where s_i is player i's threshold of cooperation, defined in Section 2. For a player in N_1, it may be optimal to cooperate when just $n_1 - 1$ others do. In contrast, a player in N_2 has lower motivation to cooperate, in that only when all the other $n_1 + n_2 - 1$ players cooperate, it becomes advantageous to herself to follow suit.

The assumption of two types drastically simplify the analysis in many ways but it still possesses essential characters of the group formation game. Most important, it follows from Proposition 3.3 that there are exactly three strict Nash equilibria.

Proposition 5.1. *In a group formation game with two types, there are exactly three strict Nash equilibria. They are*

$$e^1 = (0, \ldots, 0), \quad e^2 = (\overbrace{1, \ldots, 1}^{n_1}, \overbrace{0, \ldots, 0}^{n_2}), \quad \textit{and} \quad e^3 = (1, \ldots, 1).$$

In words, e^1 is the global defection, e^2 is the partial cooperation, and e^3 is the full cooperation. Concerning the best response structure of this game, notice the crucial difference between the two types. For a player in N_1, there are no situations in which she receives more by defecting than the global defection payoff. In other words, no member in N_1 has an incentive to free ride. In contrast, when all members in N_1 cooperate, a player in N_2 possibly earns more by defecting than the global defection payoff. That is, she has an incentive to free ride. In this sense, members of N_2 are potential free riders. As a result, their best response structure is more intricate than that of players in N_1. For this reason, it is behavior and payoff of potential free riders that become most critical to subsequent analysis. Note that members in N_2 are in fact free riding in the partial cooperation equilibrium.

It proves useful to parameterize players' payoffs. For every $i \in N_1$, define

$$a_i = v_i(C, n_1 - 1), \quad c_i = u_i(D, 0) - \varepsilon_i, \quad d_i = u_i(D, 0).$$

a_i is the equilibrium payoff in e^2, the partial cooperation. A unilateral deviation by an $i \in N_1$ from e^2 results in d_i, which is equal to the equilibrium payoff in e^1, the global defection. A unilateral deviation by an $i \in N_1$ from e^1 results in c_i. Note that $a_i > d_i > c_i$. For every $i \in N_2$, let

$$a_i = v_i(C, n-1), \quad c_i = u_i(D, 0) - \varepsilon_i, \quad d_i = u_i(D, 0), \quad f_i = u_i(D, n_1).$$

For player $i \in N_2$, a_i is the equilibrium payoff in e^3, the full cooperation. A unilateral deviation by an $i \in N_2$ from e^3 results in d_i, which is equal to the equilibrium payoff in e^1. A unilateral deviation by an $i \in N_2$ from e^1 results in c_i. f_i is the free riding payoff. Note that $f_i > d_i > c_i$ and $a_i > d_i$.

5.2 Evaluating Resistances

In order to identify the stochastically stable equilibrium, we invoke the "tree analysis" described in Section 4. To do so, we need to evaluate the resistances. Recall that the resistance $r(e, e')$ is a positive integer such that any path from an equilibrium e to another equilibrium e' contains at least $r(e, e')$ mistakes, and that there is such a path with exactly $r(e, e')$ mistakes. There are two ways to evaluate the resistance. First, to evaluate it from above, it suffices to construct a path from e to e'. $r(e, e')$ never exceeds the number of mistakes that the constructed path contains. Second, to evaluate it from below, it proves useful to consider an *exiting path* from the originating equilibrium and its *first exitors*.

Definition 5.1. Given an equilibrium state $h(e) = (e, \ldots, e)$, an *exiting path* from e is a path[3] of action profiles from $h(e)$ to another state h that contains a profile σ in which some player $i \in N$ plays a best response σ_i different from e_i. For an existing path $(e, \cdots, e, \sigma^1, \cdots, \sigma^T)$ from e, a player $i^* \in N$ is called a *first exitor* if i^* plays a best response that differs from the equilibrium e for the first time during the path.

For example, a path from $h(e^2)$ is an exiting path if it contains σ such that $\sigma_i = 1$ for some $i \in N_2$ and this choice is a best response. Thus in this case, the path contains a sample to which $i \in N_2$ optimally chooses 1. If $i^* \in N$ who chooses $\sigma_{i^*}^{\tau^*}$ at date τ^* is a first exitor of an exiting path from $h(e)$, then for every $i \in N$ (including i^*) and every date $1 \leq \tau < \tau^*$ any action σ_i^τ is a mistake whenever $\sigma_i^\tau \neq e_i$. Note that a first exitor need not be unique. Any path from $h(e^i)$ to $h(e^j)$ is an exiting path from e^i, but not vice versa.

[3] Taken as a sequence of action profiles, as opposed to a sequence of states.

In a group formation game with two types, the best response structure of a member of N_1 differs from that of a member of N_2. For example, the action 1 (participation) is a best response for $i \in N_2$ only to e^3_{-i}, but $j \in N_1$ can optimally choose 1 to e^2_{-j} as well as to e^3_{-j}. Thus we need to distinguish exiting paths accordingly. Recall that $r(\omega)$ is the number of mistakes that a path ω contains. For each equilibrium e, define

$$r_i(e, N_1) = \min \{ r(\omega) \mid \omega \text{ is an exiting path from } e, \text{ with } i \in N_1 \text{ as its first exitor} \},$$

and $r_i(e, N_2)$ analogously. Define further that

$$r(e) = \min \left\{ \min_{i \in N_1} r_i(e, N_1), \min_{i \in N_2} r_i(e, N_2) \right\},$$

which we call the *exit resistance* of e. It is clear that $r(e, e') \geq r(e)$ for any $e' \neq e$. Moreover, if there is a path from e to e' with exactly $r(e)$ number of mistakes, then $r(e, e') = r(e)$.

Let us start with the global defection equilibrium, e^1, in which every player chooses 0. Consider an exiting path from e^1 with a player $i \in N_1$ as its first exitor. Any such path (a sequence of action profiles with length more than m) from e^1 contains a sample (a k-length subsequence of the path) to which i can optimally choose 1. Recall that $i \in N_1$ optimally plays action 1 either to the action profile e^2_{-i} or to e^3_{-i}. To any other action profile, 0 is the unique best response. Thus any such sample must contain a sufficient number of e^2_{-i} or e^3_{-i}. Since we are interested in the minimum number of mistakes, fix a sample that contains e^2_{-i}, but not e^3_{-i}. Now, how many e^2_{-i} is needed? If i chooses 1, she earns the partial cooperation payoff a_i against e^2_{-i}. Against other profiles, she receives the noncooperative payoff d_i, minus the participation cost ε_i. On the other hand, she can assure d_i if she chooses 0. Now it is clear that if the cost is sufficiently small, then just one e^2_{-i} suffices. In fact, in our model ε_i represents only the participation cost for negotiation, as opposed to the maintenance cost for cooperation. Thus it is natural to assume that it is indeed "small." This leads us to conclude that $r_i(e^1, N_1) = n_1 - 1$. A similar argument shows that $r_i(e^1, N_2) = n - 1$. Thus we have $r(e^1) = n_1 - 1$.

For the partial cooperation equilibrium e^2, we need more careful treatment. It is useful to introduce some terminology. Given an action profile σ and a player $i \in N$, σ_{-i} is called a *subprofile*. Any subprofile σ_{-i} such that $\sigma_{-i} \notin \{e^2_{-i}, e^3_{-i}\}$ is called a *disequilibrium subprofile*.[4] Note that just one mistake is enough to turn e^2_{-i} or e^3_{-i} into a disequilibrium subprofile.

[4]In particular, e^1_{-i} is a disequilibrium subprofile.

Consider $r_i(e^2, N_2)$ first. In e^2, every $i \in N_2$ free rides by choosing 0. To evaluate $r_i(e^2, N_2)$, we look for the minimum number of mistakes that is enough to rationalize $i \in N_2$ to switch to 1. Fix a player $i \in N_2$ and assume that i is a first exitor in an exiting path from e^2. For $i \in N_2$, e^3_{-i} is the only subprofile to which she optimally plays action 1. The best response to e^2_{-i} is 0, and by so choosing, she earns the free riding payoff, f_i. The unique best response to a disequilibrium subprofile is also 0, which results in the global defection payoff, d_i. Against what kind of sample does player $i \in N_2$ optimally choose 1? There are two kinds of samples to be distinguished. First, consider a sample that arises as follows. On day 1 (say), all players $j \in N_2$ $(j \neq i)$ happen to make mistakes simultaneously. And from date 2 on, up to the point where a sufficient number of e^3_{-i}s accumulate, the simultaneous mistakes occur consecutively. This yields a sample that consists of e^2_{-i} and e^3_{-i}. The incentive to free ride is present, and it is directly countered by the sufficient number of e^3_{-i}. Player i's best response to the sample is 1 if

$$sa_i + (k - s)c_i \geq sd_i + (k - s)f_i,$$

where s ($k - s$, resp.) is the number of e^3_{-i} (e^2_{-i}, resp.) in the sample. Therefore the sufficient number of e^3_{-i} turns out to be at least $\alpha_i k$, where

$$\alpha_i = \left(\frac{f_i - c_i}{a_i - d_i + f_i - c_i} \right).$$

For this type of exit to happen, at least $(n_2 - 1)\alpha_i k$ mistakes are required. An exit of this kind is called an *exit via direct transition*. Figure 1 exhibits a path in which an exit via direct transition occurs. In this and similar figures that follow, an action by mistake is indicated by an asterisk, as 1^*.

(Figure 1 appears about here.)

There is another type of exit from e^2. Consider a sample which consists of $k - 1$ disequilibrium subprofiles, together with just one e^3_{-i}. For example, this sample arises as follows. Since originating equilibrium is e^2, on date 0 $i \in N_2$ has the sample that entirely consists of e^2_{-i}. From day 1 to day k, at least one player $j \neq i$ makes a mistake, and on just one of these dates all players $j \in N_2$ $(j \neq i)$ happen to make mistakes simultaneously. On date $k + 1$, the incentive to free ride disappears. As a result and similarly to the argument applied to $r(e^1)$, the best response to the sample is 1 if the participation cost is small enough. For such an event to happen, the total number of mistakes required is $k - 1 + (n_2 - 1)$. Let us call this kind of exit an *exit via indirect transition*. Such an exit is depicted in Figure 2.

(Figure 2 appears about here.)

Now it is clear that n_2, the number of potential free riders, matters. If $n_2 = 2$, the required number of mistakes in an exit via direct transition never exceeds that in an exit via indirect transition. Thus $r_i(e^2, N_2)$ is given by the number of mistakes in the direct transition. For three or more players, however, it may well be the case that indirect transition arises with smaller number of mistakes. Intuitively, $r_i(e^2, N_2)$ is given by the direct transition if n_2 is "small," but when it is "large" it is given by the indirect transition. Note that the indirect transition with three or more players requires more than k mistakes ($k - 1 + (n_2 - 1) > k$). It is worth emphasizing that the large/small distinction becomes relevant because we allow three or more players in N_2. The distinction reflects a salient feature of games with more than two players.

Consider $r_i(e^2, N_1)$ next. In e^2, every $i \in N_1$ takes 1. Thus we look for the minimum number of mistakes required for an optimal switch by i to 0. For $i \in N_1$, 0 is a best response only to disequilibrium subprofiles. Since the originating state is $h(e^2)$, however, the initial sample given to i entirely consists of e^2_{-i}. Thus each e^2_{-i} should be replaced by a disequilibrium subprofile. Specifically, when the participation cost ε_i is small enough, the argument given to $r_i(e^1, N_1)$ also applies here. Namely, all of the e^2_{-i}s need to be replaced by disequilibrium subprofiles. This can happen, for example, on each day from date 1 to date k at least one player $j \neq i$ makes a mistake. On date $k + 1$, a sample that contains only disequilibrium subprofiles is available for i, which allows her to switch optimally to 0. Thus $r_i(e^2, N_1) = k$. We also call this type of exit an exit via indirect transition. To summarize, we have $r(e^2) = r_i(e^2, N_1) = k$ when n_2 is large and $r(e^2) \leq r_i(e^2, N_2) < k$ when n_2 is small.

For exits from the full cooperation equilibrium e^3, we also need a large/small distinction of n_2. There is an important difference to note, however. $r_i(e^3, N_2)$ never exceeds k even if n_2 is "large," since 0 is a best response to the sample that consists entirely of disequilibrium subprofiles.

These observations lead to the following definition.

Definition 5.2. The *incentive ratio* of player $i \in N_2$ is the fraction

$$\eta_i = \frac{a_i - d_i}{f_i - d_i}.$$

The population size n_2 of N_2 is *large* (*small*, resp.) *to exit from* e^2 *for player* $i \in N_2$ if

$$n_2 - 2 \geq \eta_i \quad (n_2 - 2 < \eta_i, \text{ resp.}).$$

n_2 is *large to exit from* e^2 if it is large to exit from e^2 for every $i \in N_2$. Otherwise, n_2 is *small to exit from* e^2.

Similarly, n_2 is *large* (*small*, resp.) *to exit from* e^3 *for player* $i \in N_2$ if

$$n_2 - 2 \geq \frac{1}{\eta_i} \quad \left(n_2 - 2 < \frac{1}{\eta_i}, \text{ resp.}\right).$$

n_2 is *large to exit from* e^3 if it is large to exit from e^3 for every $i \in N_2$. Otherwise, n_2 is *small to exit from* e^3.

Compared to the global defection, every $i \in N_2$ receives a larger payoff in both the partial cooperation and the full cooperation. The incentive ratio η_i measures the relative magnitude of the payoff advantages of the full cooperation equilibrium and the partial cooperation equilibrium. It plays a crucial role in the rest of the analysis. Define

$$\alpha = \min_{i \in N_2} \left(\frac{f_i - c_i}{a_i - d_i + f_i - c_i} \right) \quad \text{and} \quad \beta = \min_{i \in N_2} \left(\frac{a_i - d_i}{a_i - d_i + f_i - c_i} \right).$$

Denote by $\lceil z \rceil$ the minimum integer greater or equal to a real number z.

Lemma 5.1. *Assume that ε_i is sufficiently small and k is sufficiently large.*

(1) *If n_2 is large to exit from e^2, then $r(e^2) = k$.*

(2) *If n_2 is small to exit from e^2, then $r(e^2) = \min_{i \in N_2} r_i(e^2, N_2)$ and*

$$(n_2 - 1)\alpha k \leq r(e^2) \leq (n_2 - 1)\lceil \alpha k \rceil < k.$$

(3) *If n_2 is large to exit from e^3, then $r(e^3) = k$.*

(4) *If n_2 is small to exit from e^3, then $r(e^3) = \min_{i \in N_2} r_i(e^3, N_2)$ and*

$$(n_2 - 1)\beta k \leq r(e^3) \leq (n_2 - 1)\lceil \beta k \rceil < k.$$

The lemma confirms that the large/small distinction works as desired. In Lemma 5.1, αk approximates the required number of repetitions of e^2_{-i} in an exit via direct transition, and βk does so for e^3_{-i}. Lemma 5.1 is proved in Appendix. We proceed to evaluate resistances.

Lemma 5.2.

(1) $r(e^1, e^2) \leq n_1$ *and* $r(e^1, e^2) < r(e^1, e^3)$.

(2) $r(e^2, e^1) = r(e^2)$.

(3) $r(e^3, e^1) = r(e^3)$.

(4) *If* n_2 *is large to exit from* e^2*, then* $r(e^2, e^3) > k$.

The exit resistance $r(e)$ gives a lower bound for the resistance $r(e, e')$. By constructing a path from e to e', on the other hand, we get an upper bound for it. Consider the global defection equilibrium e^1. Starting from state $h(e^1)$, suppose that players $i = 1, \ldots, n_1$ simultaneously choose action 1 by mistake on, say, date 1. As we saw earlier, just one e^2_{-i} is enough for $i \in N_1$ to switch optimally to 1, provided the participation cost is sufficiently small. Therefore every $i \in N_1$ can play action 1 optimally from date 2 on. See Figure 3. Thus $r(e^1, e^2) \leq n_1$. Moreover, one can show that $r(e^1, e^3) > r(e^1, e^2)$. Thus we have Lemma 5.2.(1).

(Figure 3 appears about here.)

By definition, if there is a path from e to e' with exactly $r(e)$ number of mistakes, then $r(e, e') = r(e)$. Consider the partial cooperation equilibrium e^2, and recall the path that implements an exit by $i \in N_2$ via direct transition (Figure 1). In the path, all players in N_1 keep best responding. It is players $j \in N_2$ ($j \neq i$) that makes mistakes. While they are making mistakes, i keeps playing 0 optimally. Therefore, no e^3_{-j} appears in the path. Thus there is no chance for j to switch to 1. Consequently, after i optimally switches to 1, her consecutive choices of 1 give rise to an accumulation of disequilibrium subprofiles for $j \neq i$. This directs, in particular, players in N_1 to switch to 0. In this way, the optimal switch to 1 by $i \in N_2$ does not lead to e^3, but to e^1. Therefore $r(e^2, e^1) = r(e^2)$. Similar arguments apply, not only for indirect transitions out of e^2, but also for transitions out of the full cooperation equilibrium e^3. Thus we have Lemma 5.2.(2) and (3). As a result, we have $r(e^2, e^1) \leq r(e^2, e^3)$ and $r(e^3, e^1) \leq r(e^3, e^2)$. This result suggests that when the adaptive play moves from the partial cooperation to the full cooperation, or the other way around, it is easier to follow the indirect path, in which the play moves first to the global defection equilibrium, and then to the destination equilibrium. In the next section, this intuition turns out to be correct.

Lemma 5.2.(4) is a consequence of the fact that $r_i(e^2, N_2)$ exceeds k when n_2 is large to exit from e^2. Lemma 5.2 is proved in Appendix.

The analysis so far results in two types of evaluations of resistances. First, the resistances out of e^2 or e^3 are evaluated in terms of k. Specifically, they are at most k, and some of them are less than k only if n_2 is small. Second, the resistances out of e^1 are independent of k, and are evaluated in terms of n_1. Having made no assumption concerning the relative magnitude of k and n_1, there is no way to compare the two types of resistances. We focus our analysis on the situation in which the sample size k is much larger than the group size n_1. In other words, we restrict attention to games with "medium" number of players.

Lemma 5.3. *For sufficiently large k, $r(e^1, e^2) < \min\{r(e^2, e^1), r(e^3, e^1)\}$.*

Proof. By Lemma 5.2.(1), $r(e^1, e^2) \leq n_1$. If the sample size k is large enough so that

$$k > \max_{i \in N_2} \left\{ \frac{n_1(a_i - d_i + f_i - c_i)}{f_i - c_i}, \frac{n_1(a_i - d_i + f_i - c_i)}{a_i - d_i} \right\},$$

then we have $n_1 < \min\{r(e^2), r(e^3)\}$ by Lemma 5.1. By definition of the exit resistance, $r(e^\iota) \leq r(e^\iota, e^1)$, where $\iota = 2, 3$. Therefore $n_1 < \min\{r(e^2, e^1), r(e^3, e^1)\}$. □

We are now ready to derive equilibrium selection results.

5.3 Equilibrium Selection

Let us say that a tree (weakly, resp.) *dominates* another if the stochastic potential of the former is strictly less than (less than or equal to, resp.) that of the latter. In the group formation game, there are nine trees to consider. They are shown in Figure 4. In what follows, we assume that the sample size k is sufficiently large and participation costs ε_i $(i \in N_2)$ are sufficiently small.

(Figure 4 appears about here.)

Lemma 5.4. *Minimum tree is either T_4, T_5, T_7, or T_8. Moreover, T_5 weakly dominates T_4.*

Proof. By Lemma 5.2.(1), T_4 and T_7 dominate T_6 and T_9, respectively. By Lemma 5.3, T_4, T_5, and T_7 dominate T_1, T_2, and T_3, respectively. Finally, T_5 weakly dominates T_4 by Lemma 5.2.(3). □

The first result is an immediate consequence of Lemma 5.4.

Theorem 5.1. *The global defection equilibrium is not stochastically stable.*

If the set N_1 of players is "medium" sized, the resistance from the global defection to the partial cooperation becomes the smallest among the all resistances. That is, the global defection is the easiest one to flow out from. Therefore it is not stochastically stable. It follows that the stochastically stable equilibrium must be either partial cooperation or full cooperation. To identify the selection outcome, it turns out to be critical whether or not n_2 is large to exit from e^2. In what follows, let us say n_2 is large if n_2 is large to exit from e^2.

Theorem 5.2. *If n_2 is large, then the partial cooperation equilibrium is uniquely stochastically stable.*

Proof. By Lemma 5.4, it suffices to show that T_5 dominates both T_7 and T_8. It follows from Lemma 5.1.(3), (4), and Lemma 5.2.(3) that $r(e^3, e^1) \leq k$. If n_2 is large to exit from e^2, then $r(e^2, e^3) > k$ by Lemma 5.2.(4). Thus $r(e^2, e^3) > r(e^3, e^1)$. Therefore T_5 dominates T_7. If n_2 is large to exit from e^2, then $r(e^2, e^1) \geq k$ by Lemma 5.1.(1). Thus $r(e^2, e^1) \geq r(e^3, e^1)$. This inequality and Lemma 5.2.(1) together imply that

$$r(e^2, e^1) + r(e^1, e^3) > r(e^3, e^1) + r(e^1, e^2).$$

Therefore T_5 dominates T_8. □

In words, when the number of potential free riders exceeds a critical level, the partial cooperation equilibrium is stable. There are two ways to read the assumption of the theorem

$$n_2 - 2 \geq \max_{i \in N_2} \frac{a_i - d_i}{f_i - d_i},$$

which in turn suggest two interpretations of the theorem, respectively. First, given the incentive ratios of the potential free riders, the theorem states that the free riding equilibrium is the unique stable outcome when there are sufficient number of them. Second, given the number of potential free riders, the stronger the incentive to free ride, the smaller the incentive ratio.

The theorem tells us, quite naturally, when the incentive to free ride is sufficiently strong, the free riding equilibrium is likely to be observed in the long run. Notice that if $f_i > a_i$ for every $i \in N_2$, the theorem applies for every $n_2 \geq 3$. Since we allow heterogeneous preferences, however, some players may well have large incentive ratios. In such a case, the assumption of the theorem becomes harder to be satisfied.

Technically, the result comes roughly as follows. When n_2 is large it follows that the resistance from the partial cooperation e^2 to the full cooperation e^3 is greater than k, and that the resistance from the partial cooperation e^2 to the global defection e^1 is exactly k. The resistance from e^3 to e^1 is, on the other hand, less than or equal to k. By the first consequence, the e^3 tree $e^1 \to e^2 \to e^3$ is dominated by the e^2 tree $e^3 \to e^1 \to e^2$. By the second, the other e^3 tree $e^2 \to e^1 \to e^3$ is dominated by the same e^2 tree. Thus the minimum tree is $e^3 \to e^1 \to e^2$.

When the number of potential free riders is smaller than the critical level, it turns out that the stochastically stable outcome is determined by a variant of risk dominance relation (Harsanyi and Selten 1988). Before stating the result, let us introduce the risk dominance relation relevant here.

Assume that all players in N_2 expects that the game will be played according to either the full cooperation equilibrium or the partial cooperation equilibrium, but they are not certain about which equilibrium will prevail. Suppose that each player i in N_2 expects that the partial cooperation equilibrium is played with probability t, and the full cooperation equilibrium with probability $1-t$. If she participates in a group, she receives expected payoff $td_i + (1-t)a_i$ (neglecting small participation costs ε_i). If she does not participate, she receives expected payoff $tf_i + (1-t)d_i$. Then, it is optimal for her to stay at the full cooperation equilibrium if $t < \frac{\eta_i}{1+\eta_i}$. Thus, $\min_{i \in N_2} \frac{\eta_i}{1+\eta_i}$ can be interpreted as the maximum level of risk that all players in N_2 can take in staying at the full cooperation equilibrium. Similarly, $\min_{i \in N_2} \frac{1}{1+\eta_i}$ can be interpreted as the maximum level of risk that all players in N_2 can take in staying at the partial cooperation equilibrium. Specifically, if

$$\min_{i \in N_2} \frac{\eta_i}{1+\eta_i} > \min_{i \in N_2} \frac{1}{1+\eta_i}, \tag{5.1}$$

then the full cooperation equilibrium is more "robust" than the partial cooperation equilibrium in the risk consideration. In this case, following the spirit of Harsanyi and Selten (1988), we say that the full cooperation equilibrium *risk dominates* the partial cooperation equilibrium. We are now ready to present the final result. The proof is given in Appendix.

Theorem 5.3. *Assume that n_2 is small.*

(1) *The full cooperation equilibrium is uniquely stochastically stable if*

$$\min_{i \in N_2} \frac{\eta_i}{1+\eta_i} > \min_{i \in N_2} \frac{1}{1+\eta_i}.$$

(2) *The partial cooperation equilibrium is uniquely stochastically stable if*

$$\min_{i \in N_2} \frac{\eta_i}{1+\eta_i} < \min_{i \in N_2} \frac{1}{1+\eta_i}.$$

The risk dominance relation can be rewritten in a simpler form. Let η_M (η_m, resp.) be the highest (lowest, resp.) incentive ratio among all potential free riders in N_2. Then, the risk dominance condition (5.1) is equivalent to

$$\frac{\eta_m}{1+\eta_m} > \frac{1}{1+\eta_M},$$

which can be reduced to $\eta_m \eta_M > 1$, or

$$(a_m - d_m)(a_M - d_M) > (f_m - d_m)(f_M - d_M).$$

The last inequality makes it clear that the risk dominance here is a variant of the original version of Harsanyi and Selten (1988). In particular, the two coincide each other when $n_2 = 2$. Moreover, when the full cooperation equilibrium is strictly Pareto efficient, that is, $a_i > f_i$ for every $i \in N_2$, then the former risk dominates the latter, and thus the full cooperation equilibrium is stochastically stable.

It is now instructive to consider the following game. In the group formation game, fix the action of every player in N_1 at the participation. The resulting game is a coordination game played by potential free riders, in which there are exactly two strict equilibria, the partial cooperation and the full cooperation. The intuition behind Theorem 5.3 is that, when n_2 is small, the stochastically stable outcome of the whole game is the same as that of the restricted coordination game. Therefore the outcome is determined by the risk dominance relation. Contrary to the original version of Harsanyi and Selten (1988), however, the relevant risk dominance relation involves only the maximum incentive ratio and the minimum incentive ratio. In stochastic stability analysis, only the minimum number of mistakes to upset a given equilibrium matters. As a result, the outcome is insensitive to incentive ratios of "intermediate" players.

6 Concluding Remarks

We have investigated the problem of group formation in collective action in a game theoretic model. Our analysis has been focused on how heterogeneity of preferences affects the formation and dynamic stability of voluntary groups. In the model, heterogeneous preferences are described by the threshold of cooperation and the incentive ratio of cooperation and free-riding. We have shown that the heterogeneous preferences yield a genuine multiplicity of strict Nash equilibria, in that, in addition to the global defection equilibrium, there are in general many types of cooperative equilibria. By applying the stochastic stability theory, we have considered which equilibrium is more likely to prevail. The equilibrium selection problem has been analyzed in the group formation game with two types of individuals. We have shown that when the number of individuals less motivated to cooperate is larger than a critical level, the partial cooperation is uniquely stochastically stable. Otherwise, the stochastic stability selects a risk dominant equilibrium. The risk dominance relation is determined by the highest and the lowest incentive ratios among those of potential free riders. The full cooperation equilibrium is uniquely stochastically stable if there exists at least one individual whose incentive ratio is relatively high, or if there exists no individual whose incentive ratio is considerably low.

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Appendix

Proof of Lemma 5.1.

By definition,

$$r(e) = \min \left\{ \min_{i \in N_1} r_i(e, N_1),\ \min_{i \in N_2} r_i(e, N_2) \right\}.$$

Therefore Lemma 5.1 follows directly from Lemma A.1 below. Thus it suffices to prove Lemma A.1. Set

$$\alpha_i = \frac{f_i - c_i}{a_i - d_i + f_i - c_i} \quad \text{and} \quad \beta_i = \frac{a_i - d_i}{a_i - d_i + f_i - c_i}.$$

Lemma A.1. *Let $i \in N_2$. For sufficiently small ε_i, $r_i(e, N_l)$ $(l = 1, 2)$ is given as follows.*

(1) $r_i(e^1, N_1) = n_1 - 1$.

(2) $r_i(e^1, N_2) = n - 1$.

(3) $r_i(e^2, N_1) = k$.

(4) *If n_2 is large to exit from e^2 for player i, then $r_i(e^2, N_2) > k$.*

(5) *If n_2 is small to exit from e^2 for player i, then for sufficiently large k,*

$$(n_2 - 1)\alpha_i k \le r_i(e^2, N_2) \le (n_2 - 1)\lceil \alpha_i k \rceil < k.$$

(6) $r_i(e^3, N_1) = k$.

(7) *If* n_2 *is large to exit from* e^3 *for player* i, *then* $r_i(e^3, N_2) = k$.

(8) *If* n_2 *is small to exit from* e^3 *for player* i, *then for sufficiently large* k,

$$(n_2 - 1)\beta_i k \leq r_i(e^3, N_2) \leq (n_2 - 1)\lceil \beta_i k \rceil < k.$$

Proof of Lemma A.1.

We are going to evaluate each $r_i(e^j, N_l)$ by setting up the relevant integer program, and then evaluating its optimal value. In such a program, t and s are nonnegative integer variables that satisfy $t + s \leq k$, where k is the sample size of the adaptive play. Note that for sufficiently small ε_i,

$$\left\lceil \frac{k\varepsilon_i}{a_i - d_i + \varepsilon_i} \right\rceil = 1 \quad \text{and} \quad \left\lceil \frac{k(a_i - d_i)}{a_i - d_i + \varepsilon_i} \right\rceil = k.$$

Lemma A.1.(1). $r_i(e^1, N_1) = n_1 - 1$.

Proof. Recall that a player $i \in N_1$ optimally plays action 1 either against the action profile e^2_{-i} or against e^3_{-i}. For any other action profile, 0 is the unique best response. Thus any sample against which i can optimally choose 1 must contain a sufficient number of e^2_{-i} or e^3_{-i}. Consider first samples that do not contain e^3_{-i}. Specifically, consider a sample taken by player $i \in N_1$ that contains e^2_{-i} for t times, e^1_{-i} for $k - t - s$ times, and s others (excluding e^3_{-i}). If i is a first exitor, each 1 in e^2_{-i} (there are $n_1 - 1$ of them) is a mistake, and each of the "other s" profiles contains at least one mistake. Thus one obtains the minimum number of mistakes by solving the following integer program. Note that 1 is a best response against the sample if and only if the constraint of the program is satisfied:

$$\min\, (n_1 - 1)t + s, \quad \text{subject to} \quad ta_i + (k - t)c_i \geq kd_i. \tag{A.1}$$

The constraint in (A.1) is equivalent to

$$t \geq \frac{k(d_i - c_i)}{a_i - c_i} = \frac{k\varepsilon_i}{a_i - d_i + \varepsilon_i}.$$

For sufficiently small ε_i, $t = 1$ and $s = 0$ is a feasible solution of (A.1). Thus, for samples without e^3_{-i}, the minimum number of mistakes is $n_1 - 1$. It is clear that for samples with e^3_{-i}, the number of mistakes exceeds $n_1 - 1$. Thus $r_i(e^1, N_1) = n_1 - 1$. ∎

Lemma A.1.(2). $r_i(e^1, N_2) = n - 1$.

Proof. Recall that a player $i \in N_2$ optimally plays action 1 only against e^3_{-i}. For any other action profile, 0 is the unique best response. Thus any sample against which i can optimally choose 1 must contain at least one e^3_{-i}. If i is a first exitor, each 1 in e^3_{-i} (there are $n-1$ of them) is a mistake. Thus $r_i(e^1, N_2) \geq n - 1$. To show the reverse inequality, consider a sample taken by player $i \in N_2$ that contains the action profile e^1_{-i} for $k-1$ times and an e^3_{-i}. 1 is a best response against the sample if and only if

$$a_i + (k-1)c_i \geq kd_i, \tag{A.2}$$

which is equivalent to

$$1 \geq \frac{k(d_i - c_i)}{a_i - c_i} = \frac{k\varepsilon_i}{a_i - d_i + \varepsilon_i}.$$

This inequality holds for sufficiently small ε_i. Therefore $r_i(e^1, N_2) \leq n - 1$. □

Lemma A.1.(3). $r_i(e^2, N_1) = k$.

Proof. Recall that a player $i \in N_1$ optimally plays action 1 either against the action profile e^2_{-i} or against e^3_{-i}. For any other action profile, 0 is the unique best response. Consider a sample taken by player $i \in N_1$ that contains the action profile e^3_{-i} for t times, e^2_{-i} for $k - t - s$ times, and s others. For i to optimally play 0 against this sample, there must be sufficient number of "other" profiles. In fact, let us show that $s = k$. 0 is a best response against the sample if and only if

$$kd_i \geq (k - t - s)a_i + ta_i' + sc_i,$$

where $a_i' = u_i(C, n_1 + n_2 - 1)$. Since $a_i' \geq a_i$, it is necessary that

$$kd_i \geq (k - s)a_i + sc_i,$$

which is equivalent to

$$s \geq \frac{k(a_i - d_i)}{a_i - c_i} = \frac{k(a_i - d_i)}{a_i - d_i + \varepsilon_i}.$$

When ε_i is sufficiently small, this implies that $s \geq k$. Thus in order to optimally choose 0, $i \in N_1$ has to have a sample that consists entirely of "other" profiles. If $i \in N_1$ is a first exitor, any profile that is neither e^2_{-i} nor e^3_{-i} must contain at least one mistake. Therefore $r_i(e^2, N_1) = k$. □

Lemma A.1.(4). *If n_2 is large to exit from e^2 for player i, then $r_i(e^2, N_2) > k$ for any sufficiently small ε_i.*

Proof. Recall that a player $i \in N_2$ optimally plays action 1 only against e^3_{-i}. For any other action profile, 0 is the unique best response. Thus any sample against which i can optimally choose 1 must contain a sufficient number of e^3_{-i}. If i is a first exitor, each 1 in e^3_{-i} played by a member in N_2 is a mistake (there are $n_2 - 1$ of them), and each of the "other s" profiles contains at least one mistake. Consider a sample taken by player $i \in N_2$ that contains the action profile e^3_{-i} for t times, e^2_{-i} for $k - t - s$ times, and s others. One can evaluate the minimum number of mistakes by solving the following integer program. Note that 1 is a best response against this sample if and only if the constraint of the following program is satisfied:

$$\min\,(n_2 - 1)t + s, \quad \text{subject to} \quad ta_i + (k - t)c_i \geq (k - t - s)f_i + (t + s)d_i. \tag{A.3}$$

The exact value of $r_i(e^2, N_2)$ is given by program (A.3) with integer constraint. Ignoring integer constraint, the optimal value of (A.3) is less than or equal to $r_i(e^2, N_2)$. Thus it suffices to show that the optimal value of (A.3) exceeds k. Note that when the objective function passes through an optimal solution of (A.3), its intercept on s axis gives the optimum value of (A.3).

The constraint in (A.3) is equivalent to

$$s \geq \frac{k(f_i - c_i)}{f_i - d_i} - \left(\frac{a_i - d_i + f_i - c_i}{f_i - d_i}\right) t.$$

Draw a horizontal t axis and a vertical s axis. In this coordinate, the boundary of constraint (A.3) is a line that has a negative slope steeper than -1 and its intercept on s axis is above k. See Figure 5. On the other hand, the slope of the objective function is $-(n_2 - 1)$. Now assume that n_2 is large to exit from e^2 for player i. There are two cases to consider.

(Figure 5 appears about here.)

Case 1. $n_2 - 2 > \eta_i$.

In this case,

$$n_2 - 1 > \frac{a_i - d_i + f_i - d_i}{f_i - d_i}.$$

Thus for any sufficiently small ε_i,

$$n_2 - 1 \geq \frac{a_i - d_i + f_i - c_i}{f_i - d_i},$$

where $c_i = d_i - \varepsilon_i$. Therefore the objective function is (weakly) steeper than the constraint boundary. Thus $(\underline{t}, \underline{s})$ in Figure 5 is an optimum solution. Clearly, when the objective function passes through $(\underline{t}, \underline{s})$, the intercept is above $\overline{s}$, which in turn strictly exceeds k. Therefore $r_i(e^2, N_2) > k$.

Case 2. $n_2 - 2 = \eta_i$.

In this case,

$$n_2 - 1 < \frac{a_i - d_i + f_i - c_i}{f_i - d_i}$$

for any $\varepsilon_i > 0$. Therefore the objective function is flatter than the constraint boundary. Thus $(\alpha_i k, 0)$ in Figure 5 is the unique solution of (A.3), where

$$\alpha_i k = \frac{k(f_i - c_i)}{a_i - d_i + f_i - c_i}.$$

When the objective function passes through $(\alpha_i k, 0)$, its equation is given by

$$s = -(n_2 - 1)(t - \alpha_i k) = -(n_2 - 1)t + (n_2 - 1)\alpha_i k.$$

Therefore it suffices to show that $(n_2 - 1)\alpha_i > 1$. Since $n_2 - 2 = \eta_i$, we have

$$\frac{(n_2 - 1)(f_i - d_i)}{a_i - d_i + f_i - d_i} = 1.$$

On the other hand, it follows from $a_i - d_i > 0$ that

$$\alpha_i = \frac{f_i - c_i}{a_i - d_i + f_i - c_i} > \frac{f_i - d_i}{a_i - d_i + f_i - d_i}$$

for any $\varepsilon_i > 0$, where $c_i = d_i - \varepsilon_i$. Hence $(n_2 - 1)\alpha_i > 1$. □

Lemma A.1.(5). *If n_2 is small to exit from e^2 for player i, then for sufficiently large k,*

$$(n_2 - 1)\alpha_i k \leq r_i(e^2, N_2) \leq (n_2 - 1)\lceil \alpha_i k \rceil < k.$$

Proof. Assume that n_2 is small to exit from e^2 for player i. Then,

$$n_2 - 1 < \frac{a_i - d_i + f_i - c_i}{f_i - d_i}$$

for every $\varepsilon_i > 0$. Thus the slope of the objective function in (A.3) is flatter than that of the constraint boundary. Similarly to Lemma A.1.(4), it suffices to evaluate program (A.3). Ignoring integer constraint, the optimum solution is the intersection of the best response constraint and

t (horizontal) axis. In Figure 5, it is denoted by $(\alpha_i k, 0)$. Thus the optimal value with integer constraint is at least $(n_2 - 1)\alpha_i k$. Rounding $\alpha_i k$ gives $\lceil \alpha_i k \rceil$. Since $\lceil \alpha_i k \rceil$ is an integer, the optimal value with integer constraint is at most $(n_2 - 1) \lceil \alpha_i k \rceil$.

By assumption,

$$(n_2 - 1)\left(\frac{f_i - d_i}{a_i - d_i + f_i - d_i}\right) < 1.$$

Therefore for sufficiently small ε_i and sufficiently large k,

$$(n_2 - 1)\left(\frac{f_i - c_i}{a_i - d_i + f_i - c_i}\right) + \frac{n_2 - 1}{k} = (n_2 - 1)\alpha_i + \frac{n_2 - 1}{k} < 1,$$

from which it follows that $(n_2 - 1) \lceil \alpha_i k \rceil < k$. □

Lemma A.1.(6). $r_i(e^3, N_1) = k$.

Proof. Consider a sample taken by player $i \in N_1$ that contains the action profile e^2_{-i} for t times, e^3_{-i} for $k - t - s$ times, and s others. Analogously to Lemma A.1.(3), one can show that in order for an $i \in N_1$ to optimally choose 0 against this sample it must consist entirely of "other" profiles. □

Lemma A.1.(7). *If n_2 is large to exit from e^3 for player i, then $r_i(e^3, N_2) = k$.*

Proof. Recall that a player $i \in N_2$ optimally plays action 1 only against e^3_{-i}. For any other action profile, 0 is the unique best response. Consider a sample taken by player $i \in N_2$ that contains the action profile e^2_{-i} for t times, e^3_{-i} for $k - t - s$ times, and s others. If i is a first exitor, each 0 in e^2_{-i} played by a member in N_2 is a mistake (there are $n_2 - 1$ of them), and each of the "other s" profiles contains at least one mistake. One can evaluate the minimum number of mistakes by solving the following integer program. Note that 0 is a best response against this sample if and only if the constraint of the following program is satisfied:

$$\min\,(n_2 - 1)t + s, \quad \text{subject to} \quad tf_i + (k - t)d_i \geq (k - t - s)a_i + (t + s)c_i. \tag{A.4}$$

The constraint in (A.4) is equivalent to

$$s \geq \frac{k(a_i - d_i)}{a_i - c_i} - \left(\frac{a_i - d_i + f_i - c_i}{a_i - c_i}\right) t.$$

(Figure 6 appears about here.)

In the t-s coordinate, the boundary of constraint (A.4) is a line that has a negative slope steeper than -1 and its intercept on s-axis is below k. See Figure 6. On the other hand, the slope of the objective function is $-(n_2 - 1)$. Now assume that n_2 is large to exit from e^3. Then we have

$$n_2 - 1 \geq \frac{a_i - d_i + f_i - d_i}{a_i - d_i}.$$

Noting that $f_i - d_i > 0$, it follows that

$$n_2 - 1 > \frac{a_i - d_i + f_i - c_i}{a_i - c_i}$$

for any $\varepsilon_i > 0$. Thus the objective function is steeper than the constraint boundary. Ignoring integer constraint, the optimum solution is the intersection of the best response constraint and the s (vertical) axis. Its s coordinate $\overline{s}$ is given by

$$\overline{s} = \frac{k(a_i - d_i)}{a_i - c_i} = \frac{k(a_i - d_i)}{a_i - d_i + \varepsilon_i}.$$

For sufficiently small ε_i, rounding $\overline{s}$ gives $r_i(e^3, N_2) = k$. □

Lemma A.1.(8). *If n_2 is small to exit from e^3 for player i, then for sufficiently large k,*

$$(n_2 - 1)\beta_i k \leq r_i(e^3, N_2) \leq (n_2 - 1)\lceil \beta_i k \rceil < k.$$

Proof. Similarly to Lemma A.1.(7), it suffices to evaluate program (A.4). Assume that n_2 is small to exit from e^3 for $i \in N_2$. Then

$$n_2 - 1 < \frac{a_i - d_i + f_i - d_i}{a_i - d_i}.$$

Therefore the objective function is flatter than the constraint boundary for sufficiently small ε_i. Ignoring integer constraint, the optimum solution is the intersection of the best response constraint and t (horizontal) axis. Its t coordinate is given by

$$\beta_i k = \frac{k(a_i - d_i)}{a_i - d_i + f_i - c_i}.$$

Thus the optimal value with integer constraint is at least $(n_2 - 1)\beta_i k$. Rounding $\beta_i k$ gives $\lceil \beta_i k \rceil$. Since $\lceil \beta_i k \rceil$ is an integer, the optimal value with integer constraint is at most $(n_2 - 1)\lceil \beta_i k \rceil$.

By assumption,

$$(n_2 - 1)\left(\frac{a_i - d_i}{a_i - d_i + f_i - d_i}\right) < 1.$$

Therefore for sufficiently large k,

$$(n_2 - 1)\left(\frac{a_i - d_i}{a_i - d_i + f_i - c_i}\right) + \frac{n_2 - 1}{k} < 1,$$

from which it follows that $(n_2 - 1)\lceil \beta_i k \rceil < k$. □

Proof of Lemma 5.2

Lemma 5.2.(1). *$r(e^1, e^2) \le n_1$ and $r(e^1, e^2) < r(e^1, e^3)$.*

Proof. One can construct a path from e^1 to e^2 that has exactly n_1 mistakes as follows. See Figure 3. In the path depicted in the figure, players $i = 1, \ldots, n_1$ simultaneously choose action 1 by mistake on, say, date 1. On date 1 players $i = n_1 + 1, \ldots, n_1 + n_2$ optimally choose 0. From date 2 on, every player samples the most recent k profiles, and plays optimally against it. Clearly, the path moves into $h(e^2)$. Thus $r(e^1, e^2) \le n_1$.

(Figure 7 appears about here.)

It remains to show that $r(e^1, e^3) > r(e^1, e^2)$. If $n_1 = 2$, Figure 7 shows that $r(e^1, e^2) = 1$. In this case, it is clear that $r(e^1, e^3) > 1$. Thus we can assume that $n_1 \ge 3$. For the remaining cases, it suffices to show that $r(e^1, e^3) > n_1$, since we know by the preceding paragraph that $n_1 \ge r(e^1, e^2)$. Assume first that $n_2 \ge 3$, and consider a path from e^1 to e^3. If there is a first exitor in N_2, then by Lemma A.1.(2) the path contains at least $n_1 + n_2 - 1$ mistakes. Thus we can assume that $j \in N_1$ is a first exitor of the path, and that there is no first exitor in N_2. Assume that player $i \in N_2$ optimally chooses 1 for the first time during the path on date τ, and that any other $i' \in N_2$ ($i' \ne i$) optimally chooses 1 for the first time during the path no earlier than date τ. Prior to date τ, there must be a date τ^* on which e^3_{-i} is played. Every 1 chosen by $i' \in N_2$ ($i' \ne i$) on date τ^* is a mistake. There are $n_2 - 1$ of them. On the other hand, since $j \in N_1$ is a first exitor, prior to date τ there is a date τ^{**} on which e^2_{-j} or e^3_{-j} is played. On date τ^{**}, every 1 chosen by $j' \in N_1$ ($j' \ne j$) is a mistake. There are at least $n_1 - 1$ of them. Thus, prior to date τ the path contains at least $n_1 + n_2 - 2$ mistakes. This shows that $r(e^1, e^3) > n_1$ when $n_2 \ge 3$. Thus it remains to consider the case that $n_1 \ge 3$ and $n_2 = 2$. To deal with this case, we show the following.

> **Claim.** Assume that $n_1 \ge 3$. Let σ be an action profile in which $\sigma_i = 1$ for every $i \in N_1$. Consider a path from e^1 to σ such that each 1 chosen by $i \in N_1$ in the "most recent" σ is not a mistake. This path contains at least n_1 mistakes by members of N_1.
>
> For each $i \in N_1$, let τ_i be the date on which i chooses 1 as a best response for the first time during the path. Every 1 chosen by i prior to date τ_i is a mistake.

Prior to date τ_i, there must be a date τ_i^* on which either e_{-i}^2 or e_{-i}^3 is played. If there are more than one such dates for i, let τ_i^* be the earliest date. Now there are two cases to consider. Assume first that there are two different players in N_1, say i_1 and i_2, such that dates $\tau_{i_1}^*$ and $\tau_{i_2}^*$ are actually the same date, τ^*. In this case, the play of date τ^* is either e^2 or e^3, thus the path contains at least n_1 mistakes. The remaining case is that dates $\tau_1^*, \ldots, \tau_{n_1}^*$ are all different. Assume that they appear as $\tau_1^* < \cdots < \tau_{n_1}^*$. On date τ_1^*, there are at least $n_1 - 1$ mistakes. On date τ_2^*, 1 by player 1 may be a best response, but 1 by player 3, who exists since $n_1 \geq 3$, must be a mistake. Thus the path contains at least n_1 mistakes. This concludes the proof of the claim.

Assume that $n_1 \geq 3$ and $n_2 = 2$. We show that $r(e^1, e^3) > n_1$. Assume that player $n_1 + 1 \in N_2 = \{n_1 + 1, n_1 + 2\}$ optimally chooses 1 for the first time during the path on date τ, and that $n_1 + 2$ optimally chooses 1 for the first time during the path no earlier than date τ. Prior to date τ, there must be a date τ^* on which $e_{-(n_1+1)}^3$ is played. Then 1 chosen by $n_1 + 2$ on date τ^* is a mistake. Thus there is at least one mistake by members of N_2. On date τ^*, every $i \in N_1$ chooses 1. Let γ be the number of best responses among 1s chosen by $i \in N_1$ on date τ^*. There are three cases to consider. First, if $\gamma = 0$, then date τ^* contains n_1 mistakes by members of N_1. Together with the mistake by $n_1 + 2$, date τ^* contains $n_1 + 1$ mistakes. Second, let $n_1 > \gamma \geq 1$. Then date τ^* contains $n_1 - \gamma$ mistakes by members of N_1. Since $\gamma \geq 1$, prior to date τ^* there are at least $n_1 - 1$ mistakes by members of N_1. Hence, up to date τ^*, there are at least $n_1 - \gamma + n_1 - 1$ mistakes by members of N_1. Together with the mistake by $n_1 + 2$, the path contains at least $n_1 - \gamma + n_1 > n_1$ mistakes. Finally, let $\gamma = n_1$. In this case, Claim 1 implies that the path contains at least n_1 mistakes by members of N_1. Together with the mistake by $n_1 + 2$, the path contains at least $n_1 + 1$ mistakes. □

Lemma 5.2.(2). $r(e^2, e^1) = r(e^2)$.

Proof. Assume first that n_2 is large to exit from e^2. In this case, $r(e^2) = k$ by Lemma 5.1.(1). In addition, $n_2 \geq 3$. Consider the following path from e^2. The path consists of phases 1 and 2. Denote the date τ action of player i by σ_i^τ. See Figure 8.

(Figure 8 appears about here.)

Phase 1 ($\tau = 1, \ldots, k$): Every player samples e^2_{-i} k times from $h(e^2)$. Every player $i \neq n_1 + 1$ optimally responds to the sample. Let $\sigma^\tau_{n_1+1} = 1$ for $\tau = 1, \ldots, k$.

Phase 2 ($\tau = k + 1, \ldots, 2k$): Every player samples $(\sigma^1_{-i}, \ldots, \sigma^k_{-i})$, and optimally responds to it. Then $\sigma^\tau_i = 0$ for every $i \in N$. Note, in particular, that $\sigma^\tau_i = 0$ for every $i = n_1 + 2, \ldots, n_1 + n_2$ since $n_2 \geq 3$.

Clearly, the path moves into $h(e^1)$. During this path, mistakes are those 1s by player $n_1 + 1$ in phase 1. There are k of them. Thus $r(e^2, e^1) = k$.

Assume next that n_2 is small to exit from e^2. Then we have $r(e^2) = \min_{i \in N_2} r_i(e^2, N_2)$ by Lemma 5.1.(2). Assume that $r(e^2) = r_{n_1+1}(e^2, N_2)$. Recall from the proof of Lemma A.1.(5) that $r_{n_1+1}(e^2, N_2)$ is the optimal value of program (A.3) with integer constraints. Let (t^*, s^*) be an optimal solution of (A.3).[5] By the construction of (A.3), it follows that when player $n_1 + 1$ samples $e^3_{-n_1+1}$ for t^* times, $e^2_{-n_1+1}$ for $k - t^* - s^*$ times, and others for s^* times, her best response is 1. Assume for the moment that $n_2 \geq 3$, and consider the following path from e^2. See Figure 9.

(Figure 9 appears about here.)

Phase 1 ($\tau = 1, \ldots, k$): Every player samples e^2_{-i} k times from $h(e^2)$. From $\tau = 1$ to $k - t^* - s^*$, every player optimally responds to the sample. From $\tau = k - t^* - s^* + 1$ to $k - t^*$, every player $i \neq n_1 + 2$ continues to respond to the sample optimally. Let $\sigma^\tau_{n_1+2} = 1$. From $\tau = k - t^* + 1$ to k, every player $i = 1, \ldots, n_1, n_1 + 1$ continues to respond to the sample optimally. For player $i = n_1 + 2, \ldots, n_1 + n_2$, let $\sigma^\tau_i = 1$.

Phase 2 ($\tau = k + 1, \ldots, 2k$): Every player samples $(\sigma^1_{-i}, \ldots, \sigma^k_{-i})$, and optimally responds to it. Then $\sigma^\tau_i = 1$ for $i = 1, \ldots, n_1$. By construction, $\sigma^\tau_i = 0$ for $i = n_1 + 2, \ldots, n_1 + n_2$ and $\sigma^\tau_{n_1+1} = 1$.

Phase 3 ($\tau = 2k + 1, \ldots, 3k$): Every player samples $(\sigma^{k+1}_{-i}, \ldots, \sigma^{2k}_{-i})$, and optimally responds to it. Then $\sigma^\tau_i = 0$ for every $i = 1, \ldots, n_1 + 1$. Since $n_2 \geq 3$, $\sigma^\tau_i = 0$ for every $i = n_1 + 2, \ldots, n_1 + n_2$ as well.

Clearly, the path moves into $h(e^1)$. During this path, mistakes are those 1s by players $i = n_1 + 2, \ldots, n_1 + n_2$ in phase 1. There are $(n_2 - 1)t^* + s^*$ of them. Therefore $r(e^2, e^1) = (n_2 - 1)t^* + s^*$.

[5] That is, $(n_2 - 1)t^* + s^* = r_{n_1+1}(e^2, N_2) = r(e^2)$, and both t^* and s^* are nonnegative integers.

Finally, let $n_2 = 2$. Consider the path in Figure 9 again. In phase 3, if $n_2 = 2$ then $\sigma^\tau_{n_1+2} = 1$ and $\sigma^\tau_i = 0$ for every $i \neq n_2 + 2$. Add phase 4 in which everyone samples $(\sigma^{2k+1}_{-i}, \ldots, \sigma^{3k}_{-i})$, to which everyone optimally responds. In this way, the path moves into $h(e^1)$. □

Lemma 5.2.(3). $r(e^3, e^1) = r(e^3)$.

Proof. Assume first that n_2 is large to exit from e^3. Then $r(e^3) = k$ by Lemma 5.1.(3). In addition, $n_2 \geq 3$. In this case, the path in Figure 10 shows that $r(e^3, e^1) = k$. Thus assume that n_2 is small to exit from e^3. Then we have $r(e^3) = \min_{i \in N_2} r_i(e^3, N_2)$ by Lemma 5.1.(4). Assume that $r(e^3) = r_{n_1+1}(e^3, N_2)$. Recall from the proof of Lemma A.1.(8) that $r_{n_1+1}(e^3, N_2)$ is the optimal value of program (A.4) with integer constraints. Let (t^*, s^*) be an optimal solution of (A.4). By the construction of (A.4), it follows that when player $n_1 + 1$ samples $e^2_{-n_1+1}$ for t^* times, $e^3_{-n_1+1}$ for $k - t^* - s^*$ times, and others for s^* times, her best response is 0. Consider the following path that starts from e^3. See Figure 11.

(Figure 10 appears about here.)

(Figure 11 appears about here.)

Phase 1 ($\tau = 1, \ldots, k$): Every player samples e^3_{-i} k times from $h(e^3)$. From $\tau = 1$ to $k - t^* - s^*$, every player optimally responds to the sample. From $\tau = k - t^* - s^* + 1$ to $k - t^*$, every player $i \neq n_1 + 2$ continues to respond to the sample optimally. For player $i = n_1 + 2$, let $\sigma^\tau_i = 0$. From $\tau = k - t^* + 1$ to k, every player $i = 1, \ldots, n_1, n_1 + 1$ continues to respond to the sample optimally. For player $i = n_1 + 2, \ldots, n_1 + n_2$, let $\sigma^\tau_i = 0$.

Phase 2 ($\tau = k + 1, \ldots, 2k$): Every player samples $(\sigma^1_{-i}, \ldots, \sigma^k_{-i})$, and optimally responds to it. Then $\sigma^\tau_i = 1$ for every $i = 1, \ldots, n_1$. By construction, $\sigma^\tau_1 = 1$ for every $i = n_1 + 2, \ldots, n_1 + n_2$ and $\sigma^\tau_{n_1+1} = 0$.

Phase 3 ($\tau = 2k + 1, \ldots, 3k$): Every player samples $(\sigma^{k+1}_{-i}, \ldots, \sigma^{2k}_{-i})$, and optimally responds to it. Then $\sigma^\tau_i = 0$ for every $i \neq n_1 + 1$ and $\sigma^\tau_{n_1+1} = 1$.

Phase 4 ($\tau = 3k + 1, \ldots, 4k$): Every player samples $(\sigma^{2k+1}_{-i}, \ldots, \sigma^{3k}_{-i})$, and optimally responds to it. Then $\sigma^\tau_i = 0$ for every $i \in N$ (Phase 4 is not depicted in Figure 11).

Clearly, the path moves into $h(e^1)$. During this path, mistakes are those 0s by players $i = n_1 + 2, \ldots, n_1 + n_2$ in phase 1. There are $(n_2 - 1)t^* + s^*$ of them. Therefore $r(e^3, e^1) = (n_2 - 1)t^* + s^* = r_{n_1+1}(e^3, N_2) = r(e^3)$. Note that the path in Figure 11 does work even if $n_2 = 2$. □

Lemma 5.2.(4). *If n_2 is large to exit from e^2, then $r(e^2, e^3) > k$.*

Proof. Assume that n_2 is large to exit from e^2. Thus $n_2 \geq 3$. Take any exiting path from e^2 to e^3. If there is an $i \in N_2$ who is a first exitor of this path, then this path contains more than k mistakes by Lemma A.1.(4). Thus assume that there is no first exitor in N_2. Let $i^* \in N_2$ be a first player in N_2 who chooses 1 as a best response during the path. Prior to the date, say date τ, on which i^* optimally chooses 1 for the first time, at least one $e^3_{-i^*}$ appears in the path. Fix such an $e^3_{-i^*}$, denote the date on which this $e^3_{-i^*}$ occurs by τ'. By the choice of i^*, every 1 chosen by $i \in N_2$ ($i \neq i^*$) in this $e^3_{-i^*}$ is a mistake. There are $n_2 - 1$ of them. On the other hand, since i^* is not a first exitor, prior to date τ there is an $i^{**} \in N_1$ who chooses 0 as a best response. It follows from Lemma A.1.(3) (and its proof) that prior to date τ there are k dates on each of which at least one mistake is made. If date τ' is not one of these "k dates," then the path contains at least $k + n_2 - 1$ mistakes prior to date τ. If date τ' is one of these "k dates," one of the $n_2 - 1$ mistakes may be counted as one of k mistakes, but still at least $k + n_2 - 2$ mistakes has been made prior to date τ. In either case, the number of mistakes exceeds k since $n_2 \geq 3$. ☐

Proof of Theorem 5.3

To prove Theorem 5.3, we need the following lemma.

Lemma A.2.

(1) If n_2 is small to exit from e^2, then $k > (n_2 - 1)\lceil \alpha k \rceil + n_2$ for sufficiently large k.

(2) $r(e^1, e^3) \leq r(e^1, e^2) + n_2$.

(3) $r(e^3, e^1) \leq (n_2 - 1)\lceil \beta k \rceil$.

Proof. (1) By Lemma 5.1.(2), $1 > (n_2 - 1)\alpha$. Therefore for sufficiently large k,

$$1 > (n_2 - 1)\alpha + \frac{2n_2 - 1}{k},$$

or equivalently, $k > (n_2 - 1)(\alpha k + 1) + n_2$. The right hand side is larger than $(n_2 - 1)\lceil \alpha k \rceil + n_2$.

(2) Assume first that $n_1 \geq 3$. Then $r(e^1, e^2) = n_1$ by the Claim in the proof of Lemma 5.2.(1) and Figure 3. Figure 12 shows that $r(e^1, e^3) \leq n_1 + n_2$. Assume next that $n_1 = 2$. Then we know by Figure 7 that $r(e^1, e^2) = 1$. Now Figure 13 shows that $r(e^1, e^3) \leq 1 + n_2$.

(Figure 12 appears about here.)

(Figure 13 appears about here.)

(3) If n_2 is small to exit from e^3, the conclusion follows from Lemma 5.1.(4) and 5.2.(3). If n_2 is large to exit from e^3, $r(e^3, e^1) = k$ by Lemma 5.1.(3) and 5.2.(3). Thus it suffices to show that $(n_2 - 1)\lceil \beta k \rceil \geq k$. Let $\beta_i = \beta$. If n_2 is large to exit from e^3, then, in the program (A.4) without integer constraint, $(0, \overline{s})$ is an optimal solution and $(\beta_i k, 0)$ is a feasible solution (see Figure 6). Therefore $(n_2 - 1)\beta_i k \geq k(a_i - d_i)/(a_i - c_i)$. Since $\lceil \beta_i k \rceil \geq \beta_i k$, $(n_2 - 1)\lceil \beta_i k \rceil \geq k(a_i - d_i)/(a_i - c_i)$. By monotonicity of $\lceil \cdot \rceil$,

$$\lceil (n_2 - 1)\lceil \beta_i k \rceil \rceil \geq \left\lceil \frac{k(a_i - d_i)}{a_i - c_i} \right\rceil.$$

For sufficiently small ε_i, the right hand side is equal to k. It is clear that the left hand side is equal to $(n_2 - 1)\lceil \beta_i k \rceil$. □

Proof of Theorem 5.3. For (1), assume that $\min_i \eta_i/(1 + \eta_i) > \min_i 1/(1 + \eta_i)$, or equivalently

$$\min_{i \in N_2} \frac{a_i - d_i}{a_i - d_i + f_i - d_i} > \min_{i \in N_2} \frac{f_i - d_i}{a_i - d_i + f_i - d_i}.$$

For sufficiently large k and sufficiently small ε_i, we have

$$\min_{i \in N_2} \frac{a_i - d_i}{a_i - d_i + f_i - c_i} > \min_{i \in N_2} \frac{f_i - c_i}{a_i - d_i + f_i - c_i} + \frac{1}{k}\left(1 + \frac{n_2}{n_2 - 1}\right),$$

where $c_i = d_i - \varepsilon_i$. This implies $\beta > \alpha + (1 + \frac{n_2}{n_2 - 1})/k$, and thus $\beta k > \lceil \alpha k \rceil + n_2/(n_2 - 1)$. We are going to show that this inequality and Lemma A.2.(1) constitute a sufficient condition for e^3 to be stochastically stable.

Since n_2 is small to exit from e^2, $r(e^2, e^1) \leq (n_2 - 1)\lceil \alpha k \rceil$ by Lemma 5.1.(2) and 5.2.(2). On the other hand, $r(e^1, e^3) \leq r(e^1, e^2) + n_2$ by Lemma A.2.(2). Therefore

$$\rho(T_8) \leq r(e^1, e^2) + (n_2 - 1)\lceil \alpha k \rceil + n_2.$$

By Lemma 5.4, it suffices to show that T_8 dominates T_5. Assume first that n_2 is small to exit from e^3. Then by Lemma 5.1.(4), $r(e^3, e^1) \geq (n_2 - 1)\beta k$. Thus

$$\rho(T_5) \geq r(e^1, e^2) + (n_2 - 1)\beta k.$$

Therefore e^3 is uniquely stochastically stable if

$$\rho(T_5) - \rho(T_8) \geq r(e^1, e^2) + (n_2 - 1)\beta k - (r(e^1, e^2) + (n_2 - 1)\lceil \alpha k \rceil + n_2)$$

$$= (n_2 - 1)\beta k - (n_2 - 1) \lceil \alpha k \rceil - n_2 > 0.$$

The last inequality is equivalent to $\beta k > \lceil \alpha k \rceil + n_2/(n_2 - 1)$. Assume next that n_2 is large to exit from e^3. Then by Lemma 5.1.(3), $r(e^3, e^1) \geq k$. Similarly to the above, e^3 is uniquely stochastically stable if $k - (n_2 - 1) \lceil \alpha k \rceil - n_2 > 0$, which is equivalent to Lemma A.2.(1).

For (2), assume that $\min_i 1/(1 + \eta_i) > \min_i \eta_i/(1 + \eta_i)$. Then, similarly to (1), $\alpha k > \lceil \beta k \rceil$ for small ε_i and large k. Thus it suffices to show that the last inequality is a sufficient condition for e^2 to be stochastically stable.

By Lemma A.2.(3), $r(e^3, e^1) \leq (n_2 - 1) \lceil \beta k \rceil$. Thus

$$\rho(T_5) \leq r(e^1, e^2) + (n_2 - 1) \lceil \beta k \rceil .$$

By Lemma 5.4, it suffices to show that T_5 dominates T_7 and T_8. Since n_2 is small to exit from e^2, Lemma 5.1.(2) implies that $r(e^2, e^j) \geq (n_2 - 1)\alpha k$ for $j = 1, 3$. On the other hand, $r(e^1, e^3) > r(e^1, e^2)$ by Lemma 5.2.(1). Thus

$$\min\{\rho(T_7), \rho(T_8)\} \geq r(e^1, e^2) + (n_2 - 1)\alpha k.$$

Therefore e^2 is uniquely stochastically stable if

$$\begin{aligned}\min\{\rho(T_7), \rho(T_8)\} - \rho(T_5) &\geq r(e^1, e^2) + (n_2 - 1)\alpha k - r(e^1, e^2) - (n_2 - 1) \lceil \beta k \rceil \\ &= (n_2 - 1)(\alpha k - \lceil \beta k \rceil) > 0.\end{aligned}$$

□

Figures

		m			$k-t^*$			t^*			
σ_1	1	$\cdots$	1	1	$\cdots$	1	1	$\cdots$	1	1	$\cdots$
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$
σ_{n_1}	1	$\cdots$	1	1	$\cdots$	1	1	$\cdots$	1	1	$\cdots$
σ_{n_1+1}	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	1	$\cdots$
σ_{n_1+2}	0	$\cdots$	0	0	$\cdots$	0	1^*	$\cdots$	1^*	0	$\cdots$
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	
$\sigma_{n_1+n_2}$	0	$\cdots$	0	0	$\cdots$	0	1^*	$\cdots$	1^*	0	$\cdots$

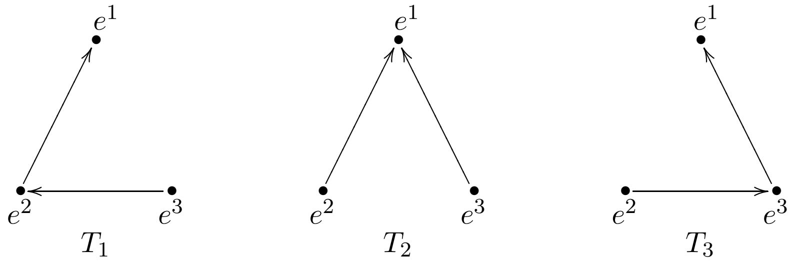
Figure 1: An exit from e^2 via direct transition.

		m				$k-1$			
σ_1	1	$\cdots$	1	1	1	$\cdots$	1	0	$\cdots$
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$
σ_{n_1-1}	1	$\cdots$	1	1	1	$\cdots$	1	0	$\cdots$
σ_{n_1}	1	$\cdots$	1	1	0^*	$\cdots$	0^*	0	$\cdots$
σ_{n_1+1}	0	$\cdots$	0	0	0	$\cdots$	0	1	$\cdots$
σ_{n_1+2}	0	$\cdots$	0	1^*	0	$\cdots$	0	0	$\cdots$
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$
$\sigma_{n_1+n_2}$	0	$\cdots$	0	1^*	0	$\cdots$	0	0	$\cdots$

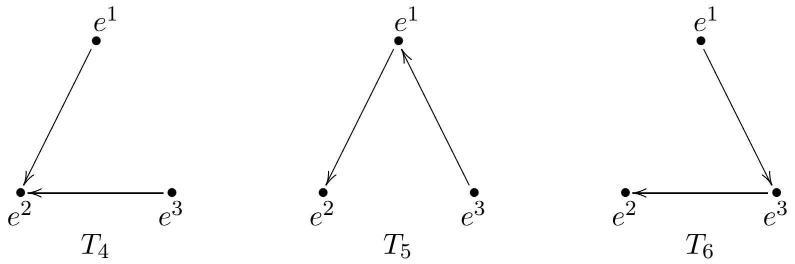
Figure 2: An exit from e^2 via indirect transition.

	m				k		
σ_1	0	$\cdots$	0	1^*	1	$\cdots$	1
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
σ_{n_1}	0	$\cdots$	0	1^*	1	$\cdots$	1
σ_{n_1+1}	0	$\cdots$	0	0	0	$\cdots$	0
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\sigma_{n_1+n_2}$	0	$\cdots$	0	0	0	$\cdots$	0

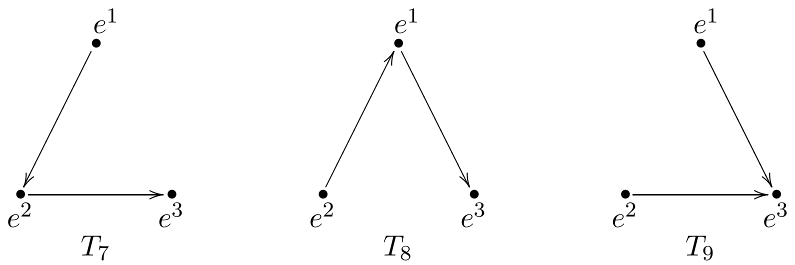
Figure 3: A path from e^1 to e^2 with n_1 mistakes.



e^1-trees.



e^2-trees.



e^3-trees.

Figure 4: Trees in the group formation game.

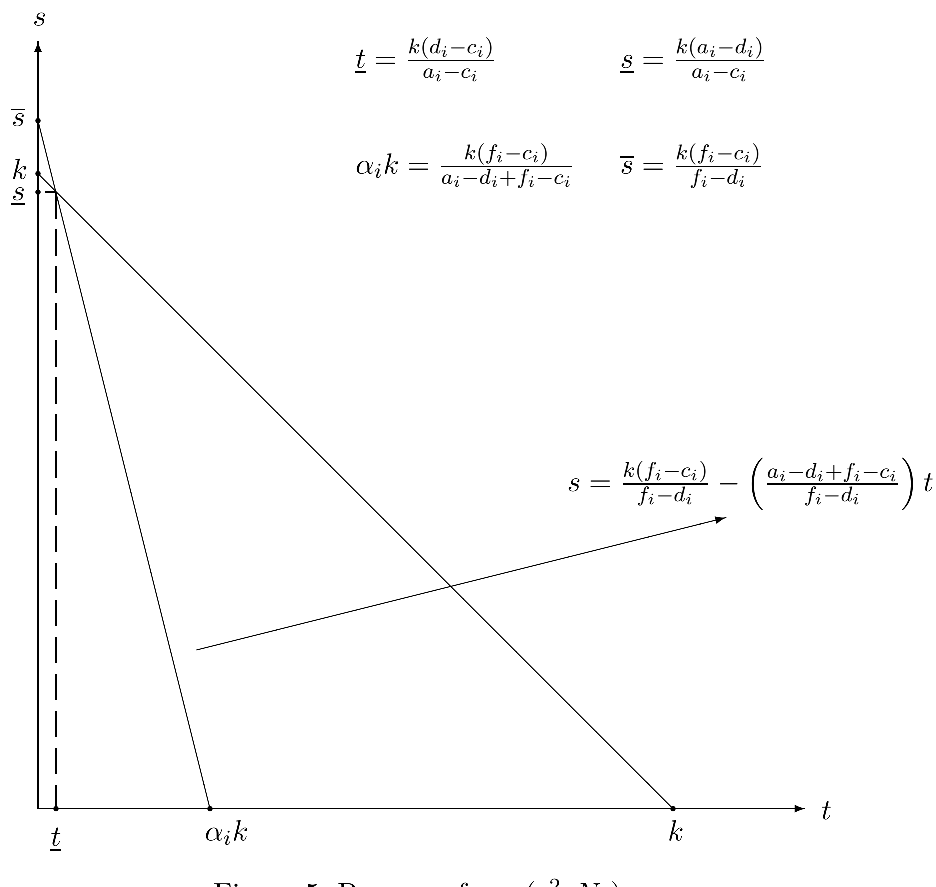


Figure 5: Program for $r_i(e^2, N_2)$.

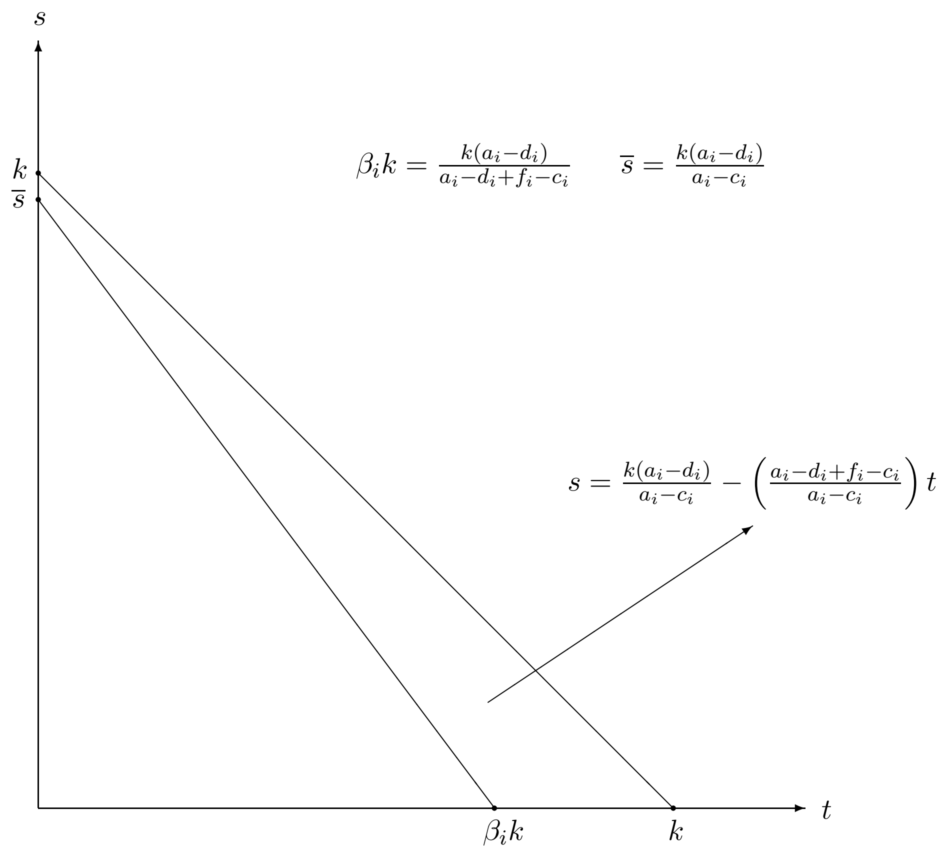


Figure 6: Program $r_i(e^3, N_2)$.

	m						k		
σ_1	0	$\cdots$	0	1^*	0	1	$\cdots$	1
σ_2	0	$\cdots$	0	0	1	1	$\cdots$	1
σ_{2+1}	0	$\cdots$	0	0	0	0	$\cdots$	0
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
σ_{2+n_2}	0	$\cdots$	0	0	0	0	$\cdots$	0

Figure 7: A path from e^1 to e^2 when $n_1 = 2$.

	$h(e^2)$			Phase 1			Phase 2		
	m			k			k		
σ_1	1	$\cdots$	1	1	$\cdots$	1	0	$\cdots$	0
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
σ_{n_1}	1	$\cdots$	1	1	$\cdots$	1	0	$\cdots$	0
σ_{n_1+1}	0	$\cdots$	0	1^*	$\cdots$	1^*	0	$\cdots$	0
σ_{n_1+2}	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\sigma_{n_1+n_2}$	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0

Figure 8: A path from e^2 to e^1 when n_2 is large to exit from e^2.

	$h(e^2)$			Phase 1									Phase 2			Phase 3		
	m			$k-t^*-s^*$			s^*			t^*			k			k		
σ_1	1	$\cdots$	1	1	$\cdots$	1	1	$\cdots$	1	1	$\cdots$	1	1	$\cdots$	1	0	$\cdots$	0
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
σ_{n_1}	1	$\cdots$	1	1	$\cdots$	1	1	$\cdots$	1	1	$\cdots$	1	1	$\cdots$	1	0	$\cdots$	0
σ_{n_1+1}	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	1	$\cdots$	1	0	$\cdots$	0
σ_{n_1+2}	0	$\cdots$	0	0	$\cdots$	0	1^*	$\cdots$	1^*	1^*	$\cdots$	1^*	0	$\cdots$	0	0	$\cdots$	0
σ_{n_1+3}	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	1^*	$\cdots$	1^*	0	$\cdots$	0	0	$\cdots$	0
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\sigma_{n_1+n_2}$	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	1^*	$\cdots$	1^*	0	$\cdots$	0	0	$\cdots$	0

Figure 9: A path from e^2 to e^1 when n_2 is small to exit from e^2.

	$h(e^3)$			Phase 1			Phase 2			Phase 3		
	m			k			k			k		
σ_1	1	$\cdots$	1	1	$\cdots$	1	0	$\cdots$	0	0	$\cdots$	0
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
σ_{n_1}	1	$\cdots$	1	1	$\cdots$	1	0	$\cdots$	0	0	$\cdots$	0
σ_{n_1+1}	1	$\cdots$	1	0^*	$\cdots$	0^*	1	$\cdots$	1	0	$\cdots$	0
σ_{n_1+2}	1	$\cdots$	1	1	$\cdots$	1	0	$\cdots$	0	0	$\cdots$	0
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\sigma_{n_1+n_2}$	1	$\cdots$	1	1	$\cdots$	1	0	$\cdots$	0	0	$\cdots$	0

Figure 10: A path from e^3 to e^1 when n_2 is large to exit from e^3.

	$h(e^3)$			Phase 1									Phase 2			Phase 3		
	m			$k-t^*-s^*$			s^*			t^*			k			k		
σ_1	1	$\cdots$	1	1	$\cdots$	1	1	$\cdots$	1	1	$\cdots$	1	1	$\cdots$	1	0	$\cdots$	0
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
σ_{n_1}	1	$\cdots$	1	1	$\cdots$	1	1	$\cdots$	1	1	$\cdots$	1	1	$\cdots$	1	0	$\cdots$	0
σ_{n_1+1}	1	$\cdots$	1	1	$\cdots$	1	1	$\cdots$	1	1	$\cdots$	1	0	$\cdots$	0	1	$\cdots$	1
σ_{n_1+2}	1	$\cdots$	1	1	$\cdots$	1	0^*	$\cdots$	0^*	0^*	$\cdots$	0^*	1	$\cdots$	1	0	$\cdots$	0
σ_{n_1+3}	1	$\cdots$	1	1	$\cdots$	1	1	$\cdots$	1	0^*	$\cdots$	0^*	1	$\cdots$	1	0	$\cdots$	0
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\sigma_{n_1+n_2}$	1	$\cdots$	1	1	$\cdots$	1	1	$\cdots$	1	0^*	$\cdots$	0^*	1	$\cdots$	1	0	$\cdots$	0

Figure 11: A path from e^3 to e^1 when n_2 is small to exit from e^3.

	$\overbrace{\qquad}^{m}$				$\overbrace{\qquad}^{k}$		
σ_1	0	$\cdots$	0	1^*	1	$\cdots$	1
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
σ_{n_1}	0	$\cdots$	0	1^*	1	$\cdots$	1
σ_{n_1+1}	0	$\cdots$	0	1^*	1	$\cdots$	1
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\sigma_{n_1+n_2}$	0	$\cdots$	0	1^*	1	$\cdots$	1

Figure 12: A path from e^1 to e^3 with $n_1 + n_2$ mistakes.

	$\overbrace{\qquad}^{m}$						$\overbrace{\qquad}^{k}$		
σ_1	0	$\cdots$	0	1^*	0	1	1	$\cdots$	1
σ_2	0	$\cdots$	0	0	1	1	1	$\cdots$	1
σ_{2+1}	0	$\cdots$	0	0	0	1^*	1	$\cdots$	1
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
σ_{2+n_2}	0	$\cdots$	0	0	0	1^*	1	$\cdots$	1

Figure 13: A path from e^1 to e^3 when $n_1 = 2$.