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A New Coincident Index of Business Cycles Based on Monthly and Quarterly Series

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Abstract

Maximum likelihood factor analysis of time series is possible even when some series are quarterly and others are monthly. Treating quarterly series as monthly series with missing observations and replacing them with artificial observations independent of the model parameters, one can apply the Kalman filter to a state-space representation of a factor model and evaluate the likelihood function. An application to quarterly real GDP and monthly coincident business cycle indicators gives a new coincident index of business cycles. The new index is essentially the smoothed estimate of latent monthly real GDP and should improve upon the Stock–Watson index.

KEY WORDS: Factor analysis; Time series; Missing observation; State-space model; Kalman filter; Stock–Watson index.

1 Introduction

There is no doubt that, as a measure of the aggregate state of an economy, real GDP is one of the most important coincident business cycle indicators (BCI). Popular U.S. coincident indices of business cycles, however, do not use real GDP, e.g., the composite index (CI), currently published by the Conference Board, and the Experimental Coincident Index (XCI) by Stock and Watson (1989). This is presumably because real GDP is quarterly. Without a statistically rigorous method to construct a monthly index from monthly and quarterly series, they ignore quarterly BCIs. The Japanese coincident CI uses a quarterly BCI (operating profits), but they simply transform it into a monthly series by linear interpolation.

This paper proposes a new coincident index of business cycles based on monthly and quarterly BCIs. Applying maximum likelihood factor analysis (ML-FA) to a one-factor model for the four monthly coincident BCIs that make up the CI, Stock and Watson (1991) obtain an index known as the Stock–Watson index (SWI). We extend the SWI by including quarterly real GDP.

Technically, we consider ML-FA of time series when some series are quarterly and others are monthly. Treating quarterly series as monthly series with missing observations, we obtain a state-space representation of a factor model with missing observations. Following Brockwell and Davis (1991, sec. 12.3) and Brockwell, Davis, and Salehi (1991), we replace missing observations with artificial observations from the standard normal distribution independent of the model parameters and rewrite the state-space model accordingly, so that we can apply the standard Kalman filter (KF) to evaluate the likelihood function. Numerical maximization of the likelihood function is straightforward. Shumway and Stoffer (1982) apply the EM algorithm; see also Shumway and Stoffer (2000, sec. 4.4). The resulting index should improve upon the SWI because it uses the most important coincident BCI that the SWI does not use, namely, real GDP.

The SWI is essentially the updated estimate of the common factor in the BCIs. We use the smoothed estimate instead, not only for more accurate estimation but also for the following reason. Let y_t be a vector of BCIs (usually the first differences of their logs) and f_t be the common factor in y_t. Let for $t \geq 1$, $Y_t := (y_1, \ldots, y_t)$. To be precise, the SWI is the updated estimate of the *cumulative* common factor. Notice that for $t > 1$,

$$\mathrm{E}\left(\sum_{j=1}^{t} f_j \middle| Y_t \right) \neq \sum_{j=1}^{t} \mathrm{E}(f_j | Y_j),$$

i.e., the updated estimate of the cumulative common factor is not equal to the sum of the updated estimates of the common factor. To obtain the former, Stock and Watson (1991) include the cumulative common factor in the state vector. Among recent extensions of the SWI that introduce regime-switching, Kim and Yoo (1995) and Chauvet (1998) obtain the former in the same way, but Kim and Nelson (1998) obtain only the latter. Obviously, this problem does not occur to the smoothed estimate.

Another benefit of including real GDP in one-factor models for coincident BCIs is a new interpretation of the common factor as the monthly growth rate (to be precise, the first difference of the log) of latent monthly real GDP. This interpretation leads to natural identification of the mean and the variance of the common factor; we identify the mean of the common factor as the mean monthly growth rate of quarterly real GDP, and assume that the factor loading of latent monthly real GDP is 1. Stock and Watson (1991), on the other hand, identify the mean of the common factor as a weighted average of the mean growth rates of the monthly BCIs, and normalize the variance of the common factor to be 1. As a result, the economic (not statistical) meaning of the common factor, and hence of the SWI, is unclear.

The plan of the paper is as follows. Section 2 sets up a static one-factor model for monthly series, including latent series underlying quarterly series, and derives a state-space model for monthly and quarterly series. Section 3 explains estimation of state-space models with missing observations and fixed-interval smoothing given the model parameters. Section 4 applies the method to the U.S. quarterly real GDP and monthly coincident BCIs to obtain a new coincident index of business cycles. Section 5 contains some concluding remarks.

2 The Model

2.1 One-Factor Model

Let $\{Y_{1,t}\}_{t=-\infty}^{\infty}$ be an $N_1 \times 1$ random sequence of quarterly BCIs observable every third month and $\{Y_{2,t}\}_{t=-\infty}^{\infty}$ be an $N_2 \times 1$ random sequence of monthly BCIs. Let $N := N_1 + N_2$. Let $\{Y_{1,t}^*\}_{t=-\infty}^{\infty}$ be an $N_1 \times 1$ latent random sequence such that for all t,

$$\ln Y_{1,t} = \frac{1}{3}\left(\ln Y_{1,t}^* + \ln Y_{1,t-1}^* + \ln Y_{1,t-2}^*\right), \tag{1}$$

i.e., $Y_{1,t}$ is the geometric mean of $Y_{1,t}^*$, $Y_{1,t-1}^*$, and $Y_{1,t-2}^*$. Taking the three-period differences, for all t,

$$\begin{aligned}\ln Y_{1,t} - \ln Y_{1,t-3} \quad &= \quad \frac{1}{3}\left(\ln Y_{1,t}^* - \ln Y_{1,t-3}^*\right) + \frac{1}{3}\left(\ln Y_{1,t-1}^* - \ln Y_{1,t-4}^*\right) \\ &\quad + \frac{1}{3}\left(\ln Y_{1,t-2}^* - \ln Y_{1,t-5}^*\right),\end{aligned}$$

or

$$\begin{aligned} y_{1,t} &= \frac{1}{3}\left(y_{1,t}^* + y_{1,t-1}^* + y_{1,t-2}^*\right) + \frac{1}{3}\left(y_{1,t-1}^* + y_{1,t-2}^* + y_{1,t-3}^*\right) \\ &\quad + \frac{1}{3}\left(y_{1,t-2}^* + y_{1,t-3}^* + y_{1,t-4}^*\right) \\ &= \frac{1}{3}y_{1,t}^* + \frac{2}{3}y_{1,t-1}^* + y_{1,t-2}^* + \frac{2}{3}y_{1,t-3}^* + \frac{1}{3}y_{1,t-4}^*, \end{aligned} \tag{2}$$

where $y_{1,t} := \Delta_3 \ln Y_{1,t}$ and $y_{1,t}^* := \Delta \ln Y_{1,t}^*$. We observe $y_{1,t}$ every third month, and never observe $y_{1,t}^*$.

Let for all t,

$$y_t := \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix}, \quad y_t^* := \begin{pmatrix} y_{1,t}^* \\ y_{2,t} \end{pmatrix},$$

where $y_{2,t} := \Delta \ln Y_{2,t}$. Assume a static one-factor structure for $\{y_t^*\}_{t=-\infty}^{\infty}$ such that for all t,

$$\begin{aligned}
\begin{pmatrix} y_{1,t}^* \\ y_{2,t} \end{pmatrix} &= \begin{pmatrix} \mu_1^* \\ \mu_2 \end{pmatrix} + \beta f_t + u_t, \qquad (3) \\
\phi_f(L) f_t &= v_{1,t}, \\
\Phi_u(L) u_t &= v_{2,t}, \\
\begin{pmatrix} v_{1,t} \\ v_{2,t} \end{pmatrix} &\sim \text{NID}\left(0, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \Sigma_{22} \end{bmatrix}\right),
\end{aligned}$$

where $\beta \in \Re^N$ is a factor loading vector, $\{f_t\}_{t=-\infty}^{\infty}$ is a scalar stationary sequence of common factors, $\{u_t\}_{t=-\infty}^{\infty}$ is an $N \times 1$ stationary sequence of specific factors, L is the lag operator, $\phi_f(.)$ is a pth-order polynomial on $\Re$, and $\Phi_u(.)$ is a qth-order polynomial on $\Re^{N\times N}$. For identification, assume that (i) the first element of β is 1 and (ii) $\Phi_u(.)$ and Σ_{22} are diagonal.

Since we never observe $y_{1,t}^*$, we consider the associated dynamic one-factor model for $\{y_t\}_{t=-\infty}^{\infty}$ such that for all t,

$$\begin{aligned}
\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} &= \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} \beta_1 \left(\frac{1}{3} f_t + \frac{2}{3} f_{t-1} + f_{t-2} + \frac{2}{3} f_{t-3} + \frac{1}{3} f_{t-4}\right) \\ \beta_2 f_t \end{pmatrix} \\
&\quad + \begin{pmatrix} \frac{1}{3} u_{1,t} + \frac{2}{3} u_{1,t-1} + u_{1,t-2} + \frac{2}{3} u_{1,t-3} + \frac{1}{3} u_{1,t-4} \\ u_{2,t} \end{pmatrix}, \qquad (4)
\end{aligned}$$

where $\mu_1 = 3\mu_1^*$.

2.2 A State-Space Representation

Assuming that $p, q \leq 4$, a state-space representation of (4) is

$$s_t = F s_{t-1} + G v_t, \qquad (5)$$

$$y_t = \mu + H s_t, \qquad (6)$$

where

$$s_t := \begin{pmatrix} f_t \\ \vdots \\ f_{t-4} \\ u_t \\ \vdots \\ u_{t-4} \end{pmatrix},$$

$$
\begin{aligned}
v_t &:= \begin{pmatrix} v_{1,t} \\ v_{2,t} \end{pmatrix}, \\
F &:= \left[\begin{array}{cccccc}
\begin{matrix} \phi_{f,1} & \dots & \phi_{f,p} & o'_{5-p} \\ 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{matrix} & O_{5\times 5N} \\
O_{5N\times 5} & \begin{matrix} \Phi_{u,1} & \dots & \Phi_{u,q} & O_{N\times(5-q)N} \\ I_N & & 0 & O_{N\times N} \\ & \ddots & & \vdots \\ 0 & & I_N & O_{N\times N} \end{matrix}
\end{array}\right], \\
G &:= \begin{bmatrix} 1 & o'_N \\ 0 & o'_N \\ \vdots & \vdots \\ o_N & I_N \\ o_N & O_{N\times N} \\ \vdots & \vdots \end{bmatrix}, \\
H &:= \begin{bmatrix} \frac{\beta_1}{3} & \frac{2\beta_1}{3} & \beta_1 & \frac{2\beta_1}{3} & \frac{\beta_1}{3} & \frac{1}{3}I_{N_1} & O_{N_1\times N_2} & \frac{2}{3}I_{N_1} & O_{N_1\times N_2} & \dots \\ \beta_2 & & O_{N_2\times 4} & & & O_{N_2\times N_1} & I_{N_2} & & O_{N_2\times N} & \dots \end{bmatrix},
\end{aligned}
$$

where o_n is the $n \times 1$ zero vector and $O_{m\times n}$ is the $m \times n$ zero matrix.

3 Estimation

3.1 Likelihood Function

Let θ be the parameter vector. Let $\{y^+_{1,t}\}^{\infty}_{t=-\infty}$ be such that for all t,

$$
y^+_{1,t} := \begin{cases} y_{1,t} & \text{if } y_{1,t} \text{ is observable} \\ z_t & \text{otherwise} \end{cases},
$$

where $z_t \sim \mathrm{NID}(0, I_{N_1})$ does not depend on θ. Let for $t \geq 1$, $Y_t := (y_1, \dots, y_t)$ and $Y^+_t := \left(y^+_1, \dots, y^+_t\right)$. Then the maximum likelihood (ML) estimator of θ given Y_T and that given Y^+_T are equivalent. Indeed, by the prediction error decomposition of a joint pdf of Y^+_T,

$$
\begin{aligned}
f\left(Y^+_T; \theta\right) &= \prod_{t=1}^{T} f\left(y^+_t | Y^+_{t-1}; \theta\right) \\
&= \prod_{t=1}^{T} f\left(y^+_t | Y_{t-1}; \theta\right) \\
&= \prod_{t\in A} f(y_t | Y_{t-1}; \theta) \prod_{t\notin A} f(z_t, y_{2,t} | Y_{t-1}; \theta) \\
&= \prod_{t\in A} f(y_t | Y_{t-1}; \theta) \prod_{t\notin A} f(z_t) f(y_{2,t} | Y_{t-1}; \theta)
\end{aligned}
$$

$$
\begin{aligned}
&= \prod_{t\in A} f(y_t|Y_{t-1};\theta) \prod_{t\notin A} f(y_{2,t}|Y_{t-1};\theta) \prod_{t\notin A} f(z_t) \\
&= \prod_{t=1}^{T} f(y_t|Y_{t-1};\theta) \prod_{t\notin A} f(z_t) \\
&= f(Y_T;\theta) \prod_{t\notin A} f(z_t),
\end{aligned}
$$

where $y_{1,t}$ is observable for $t \in A \subset \{1,\dots,T\}$. Thus the log-likelihood function of θ given Y_T and that given Y_T^+ are different only by a constant. Since the ML estimator of θ does not depend on z_t, we can set $z_t = 0$ for its realization without loss of generality. Now that we observe y_t^+ every month, we can apply the standard KF to evaluate the likelihood function of θ given Y_T^+.

Write (6) as

$$
\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} s_t.
$$

Then we have for all t,

$$
\begin{pmatrix} y_{1,t}^+ \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} \mu_{1,t} \\ \mu_2 \end{pmatrix} + \begin{bmatrix} H_{1,t} \\ H_2 \end{bmatrix} s_t + \begin{pmatrix} w_{1,t} \\ 0 \end{pmatrix},
$$

where

$$
\begin{aligned}
\mu_{1,t} &:= \begin{cases} \mu_1 & \text{if } y_{1,t} \text{ is observable} \\ 0 & \text{otherwise} \end{cases}, \\
H_{1,t} &:= \begin{cases} H_1 & \text{if } y_{1,t} \text{ is observable} \\ 0 & \text{otherwise} \end{cases}, \\
w_{1,t} &:= \begin{cases} 0 & \text{if } y_{1,t} \text{ is observable} \\ z_t & \text{otherwise} \end{cases}.
\end{aligned}
$$

We consider a state-space model for $\{y_t^+\}_{t=-\infty}^{\infty}$ such that for all t,

$$
s_t = F s_{t-1} + G v_t, \tag{7}
$$

$$
y_t^+ = \mu_t + H_t s_t + w_t, \tag{8}
$$

where

$$
\mu_t := \begin{pmatrix} \mu_{1,t} \\ \mu_2 \end{pmatrix}, \quad H_t := \begin{bmatrix} H_{1,t} \\ H_2 \end{bmatrix}, \quad w_t := \begin{pmatrix} w_{1,t} \\ 0 \end{pmatrix}.
$$

Let for $t \geq 1$,

$$
\begin{aligned}
\mu_{t|t-1}(\theta) &:= \mathrm{E}\left(y_t^+|Y_{t-1}^+;\theta\right), \\
\Sigma_{t|t-1}(\theta) &:= \mathrm{var}\left(y_t^+|Y_{t-1}^+;\theta\right),
\end{aligned}
$$

where $Y_0^+ = \emptyset$. Then for $t \geq 1$,

$$
\begin{aligned}
f\left(y_t^+ | Y_{t-1}^+; \theta\right) &= (2\pi)^{-N/2} \det\left(\Sigma_{t|t-1}(\theta)\right)^{-1/2} \\
& \quad \exp\left(-\frac{1}{2}\left(y_t^+ - \mu_{t|t-1}(\theta)\right)' \Sigma_{t|t-1}(\theta)^{-1} \left(y_t^+ - \mu_{t|t-1}(\theta)\right)\right).
\end{aligned}
$$

The log-likelihood function of θ given Y_T^+ is

$$
\begin{aligned}
\ln L\left(\theta; Y_T^+\right) &= -\frac{NT}{2} \ln 2\pi - \frac{1}{2} \sum_{t=1}^{T} \ln \det\left(\Sigma_{t|t-1}(\theta)\right) \\
& \quad -\frac{1}{2} \sum_{t=1}^{T} \left(y_t^+ - \mu_{t|t-1}(\theta)\right)' \Sigma_{t|t-1}(\theta)^{-1} \left(y_t^+ - \mu_{t|t-1}(\theta)\right).
\end{aligned}
$$

To evaluate this, we must evaluate $\left\{\mu_{t|t-1}(\theta), \Sigma_{t|t-1}(\theta)\right\}_{t=1}^{T}$. Let for $t, s \geq 0$,

$$
\begin{aligned}
\hat{s}_{t|s} &:= \mathrm{E}\left(s_t | Y_s^+; \theta\right), \\
P_{t|s} &:= \operatorname{var}\left(s_t | Y_s^+; \theta\right).
\end{aligned}
$$

From (8), for $t \geq 1$,

$$
\begin{aligned}
\mu_{t|t-1}(\theta) &= \mu_t + H_t \hat{s}_{t|t-1}, \\
\Sigma_{t|t-1}(\theta) &= H_t P_{t|t-1} H_t' + \Sigma_{ww,t},
\end{aligned}
$$

where

$$
\Sigma_{ww,t} := \begin{cases} O_{N \times N} & \text{if } y_{1,t} \text{ is observable} \\ \begin{bmatrix} I_{N_1} & O_{N_1 \times N_1} \\ O_{N_1 \times N_2} & O_{N_2 \times N_2} \end{bmatrix} & \text{otherwise} \end{cases}.
$$

Given θ, we can evaluate $\left\{\hat{s}_{t|t-1}, P_{t|t-1}\right\}_{t=1}^{T}$ using the KF.

3.2 Kalman Filter

3.2.1 Initial State

To start the KF, we must specify $\hat{s}_{1|0}$ and $P_{1|0}$. For the exact ML estimator, we set

$$
\begin{aligned}
\hat{s}_{1|0} &= \mu_s, \\
P_{1|0} &= \Gamma_{ss}(0),
\end{aligned}
$$

where $\mu_s := \mathrm{E}(s_1)$ and $\Gamma_{ss}(0) := \operatorname{var}(s_1)$. Since $\{s_t\}_{t=-\infty}^{\infty}$ is stationary, taking expectations on both sides of (7),

$$
\mu_s = F \mu_s.
$$

Assuming that $I_{5+5N} - F$ is nonsingular,

$$\mu_s = 0.$$

From (7), we also get

$$\begin{aligned} \Gamma_{ss}(0) &= F\Gamma_{ss}(1)' + G\Sigma_{vv}G', \\ \Gamma_{ss}(1) &= F\Gamma_{ss}(0), \end{aligned}$$

where $\Gamma_{ss}(1) := \operatorname{cov}(s_1, s_0)$. Eliminating $\Gamma_{ss}(1)$,

$$\Gamma_{ss}(0) = F\Gamma_{ss}(0)F' + G\Sigma_{vv}G',$$

or

$$\begin{aligned} \operatorname{vec}(\Gamma_{ss}(0)) &= \operatorname{vec}(F\Gamma_{ss}(0)T') + \operatorname{vec}(G\Sigma_{vv}G') \\ &= (F \otimes F)\operatorname{vec}(\Gamma_{ss}(0)) + \operatorname{vec}(G\Sigma_{vv}G') \\ &= \left(I_{(5+5N)^2} - F \otimes F\right)^{-1} \operatorname{vec}(G\Sigma_{vv}G'). \end{aligned}$$

Alternatively, we can simply set

$$\begin{aligned} \hat{s}_{0|0} &= 0, \\ P_{0|0} &= 0, \end{aligned}$$

which implies that

$$\begin{aligned} \hat{s}_{1|0} &= 0, \\ P_{1|0} &= G\Sigma_{vv}G'. \end{aligned}$$

The resulting estimator is asymptotically equivalent to the ML estimator.

3.2.2 Updating

Notice that for $t \geq 1$,

$$s_t | Y_{t-1}^{+} \sim \mathrm{N}\left(\hat{s}_{t|t-1}, P_{t|t-1}\right).$$

We have for $t \geq 1$,

$$y_t^+ - \hat{y}_{t|t-1}^+ = H_t \left(s_t - \hat{s}_{t|t-1}\right) + w_t,$$

where $\hat{y}_{t|t-1}^+ := \mathrm{E}\left(y_t^+ | Y_{t-1}^+\right)$. Thus for $t \geq 1$,

$$\begin{pmatrix} s_t \\ y_t^+ \end{pmatrix} | Y_{t-1}^+ \sim \mathrm{N}\left(\begin{pmatrix} \hat{s}_{t|t-1} \\ \hat{y}_{t|t-1}^+ \end{pmatrix}, \begin{bmatrix} P_{t|t-1} & P_{t|t-1}H_t' \\ H_t P_{t|t-1} & H_t P_{t|t-1} H_t' + \Sigma_{ww,t} \end{bmatrix}\right).$$

The Kalman gain matrix is for $t \geq 1$,

$$B_t := P_{t|t-1} H_t' \left(H_t P_{t|t-1} H_t' + \Sigma_{ww,t}\right)^{-1}. \tag{9}$$

The updating equations for the state vector and its variance–covariance matrix are for $t \geq 1$,

$$\hat{s}_{t|t} = \hat{s}_{t|t-1} + B_t \left(y_t^+ - \mu_t - H_t \hat{s}_{t|t-1}\right), \tag{10}$$

$$P_{t|t} = P_{t|t-1} - B_t H_t P_{t|t-1}. \tag{11}$$

3.2.3 Prediction

From (7), the prediction equations for the state vector and its variance–covariance matrix are for $t \geq 1$,

$$\hat{s}_{t|t-1} = F \hat{s}_{t-1|t-1}, \tag{12}$$

$$P_{t|t-1} = F P_{t-1|t-1} F' + G \Sigma_{vv} G'. \tag{13}$$

Combining the updating and prediction equations, we get $\left\{\hat{s}_{t|t-1}, P_{t|t-1}\right\}_{t=1}^{T}$.

3.3 Fixed-Interval Smoothing

Sometimes we want $\left\{\hat{s}_{t|T}\right\}_{t=1}^{T}$. Hamilton (1994, sec. 13.6) gives the following simple derivation of the smoothing equation for the state vector. We have for $t \geq 1$,

$$\begin{pmatrix} s_t \\ s_{t+1} \end{pmatrix} | Y_t^+ \sim \mathrm{N}\left(\begin{pmatrix} \hat{s}_{t|t} \\ \hat{s}_{t+1|t} \end{pmatrix}, \begin{bmatrix} P_{t|t} & P_{t|t}F' \\ F P_{t|t} & P_{t+1|t} \end{bmatrix}\right).$$

Hence for $t \geq 1$,

$$\mathrm{E}\left(s_t | s_{t+1}, Y_t^+\right) = \hat{s}_{t|t} + P_{t|t} F' P_{t+1|t}^{-1} \left(s_{t+1} - \hat{s}_{t+1|t}\right).$$

We can write for all t and for $j \geq 1$,

$$\begin{aligned}
y_{t+j}^{+} &= \mu_{t+j} + H_t s_{t+j} + w_{t+j} \\
&= \mu_{t+j} + H_t (F s_{t+j-1} + G v_{t+j}) + w_{t+j} \\
&= \dots \\
&= \mu_{t+j} + H_t \left(G v_{t+j} + \dots + G^{j-2} v_{t+2} + F^{j-1} s_{t+1} \right) + w_{t+j}.
\end{aligned}$$

Notice that for all t and for $j \geq 1$, y_{t+j} is independent of s_t given s_{t+1}. Hence for $t = 1, \dots, T$,

$$\begin{aligned}
\mathrm{E}\left(s_t | s_{t+1}, Y_T^{+}\right) &= \mathrm{E}\left(s_t | s_{t+1}, Y_t^{+}\right) \\
&= \hat{s}_{t|t} + P_{t|t} F' P_{t+1|t}^{-1} \left(s_{t+1} - \hat{s}_{t+1|t}\right).
\end{aligned}$$

Taking conditional expectations given Y_T^{+} on both sides and applying the law of iterated expectations (LIE), we obtain the smoothing equation for the state vector such that for $t = 1, \dots, T$,

$$\hat{s}_{t|T} = \hat{s}_{t|t} + P_{t|t} F' P_{t+1|t}^{-1} \left(\hat{s}_{t+1|T} - \hat{s}_{t+1|t}\right). \tag{14}$$

In practice, it may be difficult to take the inverse of $P_{t+1|t}$ when its dimension is large. The following algorithm by de Jong (1988, 1989) is useful in such cases; see also Koopman (1998). Let for $t = 1, \dots, T+1$,

$$r_t := P_{t|t-1}^{-1} \left(\hat{s}_{t|T} - \hat{s}_{t|t-1}\right),$$

so that

$$\hat{s}_{t|T} = \hat{s}_{t|t-1} + P_{t|t-1} r_t.$$

Plugging (10) into (14), for $t = 1, \dots, T$,

$$\hat{s}_{t|T} = \hat{s}_{t|t-1} + B_t \left(y_t^{+} - \mu_t - H_t \hat{s}_{t|t-1}\right) + P_{t|t} F' P_{t+1|t}^{-1} \left(\hat{s}_{t+1|T} - \hat{s}_{t+1|t}\right).$$

Comparing the previous two equations, for $t = 1, \dots, T$,

$$\begin{aligned}
P_{t|t-1} r_t &= B_t \left(y_t^{+} - \mu_t - H_t \hat{s}_{t|t-1}\right) + P_{t|t} F' P_{t+1|t}^{-1} \left(\hat{s}_{t+1|T} - \hat{s}_{t+1|t}\right) \\
&= B_t \left(y_t^{+} - \mu_t - H_t \hat{s}_{t|t-1}\right) + P_{t|t} F' r_{t+1},
\end{aligned}$$

Table 1: U.S. Coincident Business Cycle Indicators

BCI	Description
	Quarterly
GDP	Real gross domestic product (billions of chained 1992 \$, SA, AR)
	Monthly
EMP	Employees on nonagricultural payrolls (thousands, SA)
INC	Personal income less transfer payments (billions of chained 1992 \$, SA, AR)
IIP	Index of industrial production (1992 = 100, SA)
SLS	Manufacturing and trade sales (millions of chained 1992 \$, SA)

NOTE: SA means “seasonally-adjusted,” and AR means “annual rate.”

or using (9) and (11),

$$
\begin{aligned}
r_t &= P_{t|t-1}^{-1} B_t \left(y_t^+ - \mu_t - H_t \hat{s}_{t|t-1} \right) + P_{t|t-1}^{-1} P_{t|t} F' r_{t+1} \\
&= H_t' \left(H_t P_{t|t-1} H_t' + \Sigma_{ww,t} \right)^{-1} \left(y_t^+ - \mu_t - H_t \hat{s}_{t|t-1} \right) + (I - H_t' B_t') F' r_{t+1}.
\end{aligned}
$$

The algorithm starts from $r_{T+1} := 0$ and iterates for $t = T, \ldots, 1$,

$$
\begin{aligned}
r_t &= H_t' \left(H_t P_{t|t-1} H_t' + \Sigma_{ww,t} \right)^{-1} \left(y_t^+ - \mu_t - H_t \hat{s}_{t|t-1} \right) + (I - H_t' B_t') F' r_{t+1}, \\
\hat{s}_{t|T} &= \hat{s}_{t|t-1} + P_{t|t-1} r_t.
\end{aligned}
$$

4 New Coincident Index

4.1 Data

We apply the method to U.S. coincident BCIs to obtain a new coincident index of business cycles. The BCIs are quarterly real GDP and the four monthly coincident BCIs that currently make up the CI; see Table 1. The data are from CITIBASE. The sample period is 1959:1–1998:12. To stationarize the series, we take the first difference of the log of each series and multiply it by 100, which is approximately equal to the monthly or quarterly percentage growth rate series.

Table 2 summarizes descriptive statistics of the series. We see that EMP has substantially lower monthly mean than the others including GDP, and that EMP and INC have smaller standard deviations (s.d.) than IIP and SLS. The low mean and the small s.d. of EMP strongly pulls the growth rate of the CI downward,

Table 2: Descriptive Statistics of the Business Cycle Indicators

BCI	Mean	S.D.	Min.	Max.
		Quarterly		
GDP	0.80	0.92	−2.43	3.73
		Monthly		
EMP	0.19	0.24	−0.86	1.23
INC	0.26	0.42	−1.27	1.68
IIP	0.28	0.89	−4.25	6.00
SLS	0.29	1.05	−3.27	3.55
CI	0.23	0.37	−1.47	1.89

because the CI weights the growth rates of the BCIs according to the inverses of their s.d.'s.

To reduce the number of parameters, we estimate the dynamic factor model (4) without the constant term for the demeaned series. We apply the approximate ML estimator instead of the exact one, because the two are asymptotically equivalent. We use Ox 2.20 for computation; see Doornik (1999).

4.2 Lag-Order Selection

Before estimation, we must determine p and q, the orders of autoregressive (AR) models for the common and specific factors respectively. We use a model selection criterion for that purpose; in particular, we check Akaike's information criterion (AIC) and Schwartz's Bayesian information criterion (SBIC). In our case,

$$\begin{aligned} \text{AIC} &:= \frac{1}{T}\left\{\ln L\left(\hat{\theta}\right) - [(N-1)+p+Nq+1+N]\right\}, \\ \text{SBIC} &:= \frac{1}{T}\left\{\ln L\left(\hat{\theta}\right) - \frac{\ln T}{2}[(N-1)+p+Nq+1+N]\right\}, \end{aligned}$$

where $\hat{\theta}$ is the ML estimator of θ.

Table 3 shows AIC and SBIC for various p and q. AIC selects $(p,q)=(1,3)$ and SBIC selects $(p,q)=(1,2)$. We follow SBIC here, preferring the simpler model.

Table 3: Lag-Order Selection for the Factor Model With Real GDP

(p, q)	Log-likelihood	AIC	SBIC
(0,0)	$-1,710.32$	-3.5915	-3.6350
(0,1)	$-1,661.58$	-3.5002	-3.5655
(0,2)	$-1,615.65$	-3.4147	-3.5018
(0,3)	$-1,601.91$	-3.3965	-3.5053
(0,4)	$-1,597.40$	-3.3975	-3.5281
(1,0)	$-1,641.27$	-3.4494	-3.4973
(1,1)	$-1,602.98$	-3.3799	-3.4496
(1,2)	$-1,561.66$	-3.3041	-3.3955
(1,3)	$-1,548.40$	-3.2868	-3.4001
(1,4)	$-1,544.13$	-3.2884	-3.4234
(2,0)	$-1,638.67$	-3.4461	-3.4983
(2,1)	$-1,600.97$	-3.3778	-3.4518
(2,2)	$-1,560.79$	-3.3044	-3.4002
(2,3)	$-1,547.85$	-3.2878	-3.4054
(2,4)	$-1,543.18$	-3.2885	-3.4278
(3,0)	$-1,638.36$	-3.4475	-3.5041
(3,1)	$-1,600.86$	-3.3797	-3.4580
(3,2)	$-1,560.17$	-3.3052	-3.4053
(3,3)	$-1,547.42$	-3.2890	-3.4109
(3,4)	$-1,543.06$	-3.2903	-3.4340

Table 4: Estimation Result for the Factor Model With Real GDP

Parameter	GDP	EMP	INC	IIP	SLS
β	1.00	0.48	0.83	2.10	1.71
		(0.04)	(0.06)	(0.13)	(0.11)
ϕ_f			0.56		
			(0.05)		
σ_1^2			0.08		
			(0.01)		
$\phi_{u,1}$	-0.02	0.11	-0.04	-0.03	-0.44
	(0.12)	(0.05)	(0.05)	(0.07)	(0.05)
$\phi_{u,2}$	-0.78	0.45	0.02	-0.06	-0.22
	(0.12)	(0.05)	(0.05)	(0.06)	(0.05)
Σ_{22}	0.19	0.02	0.10	0.26	0.60
	(0.05)	(0.00)	(0.01)	(0.03)	(0.04)

NOTE: Numbers in parentheses are asymptotic standard errors (s.e.).

4.3 Estimation Result

Table 4 summarizes the estimation result. Since the BCIs have different s.d.'s, we should compare factor loadings for the standardized BCIs, i.e., factor loadings divided by the s.d.'s of the corresponding BCIs. After this normalization, IIP has the largest factor loading, while SLS has the smallest. Since we cannot estimate the monthly s.d. of GDP, we cannot compare its factor loading with others. The common factor has substantial positive autocorrelation. The specific factors have different time series properties: those of GDP and SLS have negative autocorrelation, that of EMP has positive autocorrelation, and those of INC and IIP have almost no autocorrelation.

Fixed-interval smoothing gives a sequence of the smoothed estimates of the common factor associated with the ML estimator of the model parameters. From this, we construct our new coincident index of business cycles as follows:

1. Add the monthly mean of GDP to the smoothed estimates of the common factor and divide them by 100. This gives the first difference series of the log of the new index, or latent monthly GDP.

2. Construct the level series by taking the partial sums and their exponentials.

Figure 1 plots the new index. We see that it captures the NBER business cycle reference dates very well.

4.4 Comparison with Other Indices

We compare our new index with the CI and the SWI, both of which do not use GDP. First, we construct the CI and the SWI from our data.

In the U.S., the Conference Board calculates the coincident CI in the following five steps:

1. Construct the monthly symmetric growth rate series of each BCI.

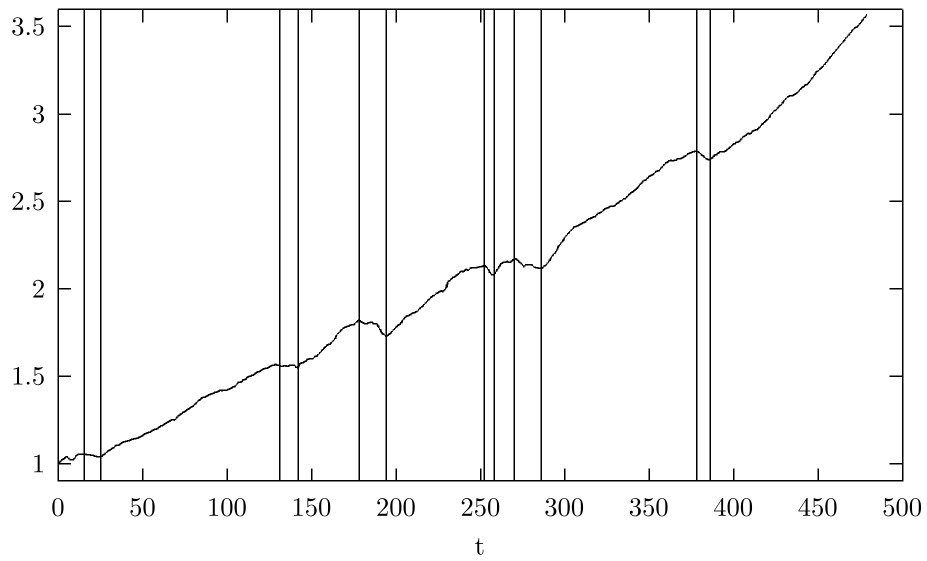

Figure 1: New Coincident Index of Business Cycles

2. Compute the s.d. of each series, excluding outliers.

3. Take a weighted cross-section average of the series using weights proportional to the inverses of their s.d.'s. This gives the monthly symmetric growth rate series of the CI.

4. Construct the level series from the symmetric growth rate series.

5. Rebase the level series to average 100 in the base year.

See the December 1996 issue of *Business Cycle Indicators* for details. For comparison, we take the difference in log instead of the symmetric growth rate, and do not exclude outliers when computing the s.d.'s.

ML-FA of the four monthly coincident BCIs gives the SWI. For comparison, we estimate the factor model (3) without the constant term for the demeaned series. For identification, we follow Stock and Watson (1991) and normalize the variance of the common factor to be 1 instead of restricting the factor loading vector. Before estimation, we must determine p and q. Table 5 shows AIC and SBIC for various

Table 5: Lag-Order Selection for the Factor Model Without Real GDP

(p,q)	Log-likelihood	AIC	SBIC
(0,0)	$-1,279.44$	-2.6878	-2.7226
(0,1)	$-1,246.57$	-2.6275	-2.6798
(0,2)	$-1,201.01$	-2.5407	-2.6104
(0,3)	$-1,190.17$	-2.5264	-2.6135
(0,4)	$-1,186.34$	-2.5268	-2.6313
(1,0)	$-1,206.08$	-2.5367	-2.5759
(1,1)	$-1,183.11$	-2.4971	-2.5537
(1,2)	$-1,145.17$	-2.4262	-2.5003
(1,3)	$-1,134.24$	-2.4118	-2.5032
(1,4)	$-1,130.60$	-2.4125	-2.5214
(2,0)	$-1,201.40$	-2.5290	-2.5726
(2,1)	$-1,180.18$	-2.4931	-2.5540
(2,2)	$-1,144.24$	-2.4264	-2.5048
(2,3)	$-1,133.76$	-2.4129	-2.5087
(2,4)	$-1,129.98$	-2.4133	-2.5265
(3,0)	$-1,200.91$	-2.5301	-2.5780
(3,1)	$-1,179.72$	-2.4942	-2.5595
(3,2)	$-1,143.68$	-2.4273	-2.5101
(3,3)	$-1,133.42$	-2.4142	-2.5144
(3,4)	$-1,129.63$	-2.4147	-2.5323

Table 6: Estimation Result for the Factor Model Without Real GDP

Parameter	EMP	INC	IIP	SLS
β	0.14	0.23	0.60	0.48
	(0.01)	(0.02)	(0.03)	(0.03)
ϕ_f		0.57		
		(0.05)		
$\phi_{u,1}$	0.10	-0.02	-0.08	-0.42
	(0.05)	(0.05)	(0.07)	(0.05)
$\phi_{u,2}$	0.45	0.04	-0.09	-0.21
	(0.05)	(0.05)	(0.07)	(0.05)
σ_v^2	0.02	0.10	0.25	0.61
	(0.00)	(0.01)	(0.03)	(0.05)

NOTE: Numbers in parentheses are asymptotic s.e.'s.

Table 7: Correlations Between Alternative Indices

	CI	SWI	New
CI	1.000		
SWI	0.970	1.000	
New	0.960	0.984	1.000

p and q. Following SBIC again, we select $(p, q) = (1, 2)$. Table 6 summarizes the estimation result, which is essentially the same as that in Table 4 except that we do not have GDP here and that we estimate different sets of parameters because of different identification restrictions.

As a by-product, we obtain a sequence of the updated estimates of the common factor associated with the ML estimator of the model parameters. For comparison, we construct our version of the SWI simply by adding the mean of the common factor defined below to this sequence and converting it to the level series. In fact, Kim and Nelson (1999, sec. 3.5) define the SWI in this way. Note that the SWI here is not the updated estimate of the cumulative common factor.

Stock and Watson (1991) identify the mean of the common factor as follows. Combining the updating equation (10) and the prediction equation (12),

$$\begin{aligned}\hat{s}_{t|t} &= F\hat{s}_{t-1|t-1} + B_t(y_t - \mu - HF\hat{s}_{t-1|t-1}) \\ &= (I - B_tH)FL\hat{s}_{t|t} + B_t(y_t - \mu) \\ &= [I - (I - B_tH)FL]^{-1}B_t(y_t - \mu),\end{aligned}$$

where I is the identity matrix and L is the lag operator. Without quarterly series, μ and H are time-independent. The first element of $\hat{s}_{t|t}$, i.e., the updated estimate of the common factor, is a linear combination of the current and past y_t's. Thus it is natural to define the mean of the common factor as the first row of $[I - (I - BH)F]^{-1}B$ times $\mathrm{E}(y_t)$, where B is the steady-state Kalman gain matrix.

Table 7 shows correlations between alternative indices. The SWI and the new index have the highest correlation, while the CI and the new index have the lowest.

Table 8: Business Cycle Turning Points Determined by Alternative Indices

NBER	CI	SWI	New
Peaks			
1960/4	0	−2	−2
1969/12	−2	−2	−2
1973/11	0	0	0
1980/1	0	0	0
1981/7	0	0	0
1990/7	−1	−1	−1
Troughs			
1961/2	0	0	−2
1970/11	0	0	0
1975/3	0	0	0
1980/7	0	0	0
1982/11	+1	+1	−1
1991/3	0	0	0

Table 8 compares business cycle turning points determined by alternative indices with the NBER business cycle reference dates. The CI captures the NBER reference dates best among the three. We do not conclude that the CI is the best index, however, because the NBER reference dates may not be "correct." Indeed, the result suggests that the NBER peak in December 1969 may be two months late, and the peak in July 1990 may be a month late. The new index does not agree with the NBER reference dates at three peaks and two troughs. Interestingly, the NBER peaks and troughs are always late at these turning points.

5 Concluding Remarks

One cannot claim that turning points determined by any procedure is better than the NBER business cycle reference dates without knowing the details of how the NBER Business Cycle Dating Committee determines the reference dates. On one hand, their procedure seems to lack rigorous statistical foundations; on the other hand, they look at more BCIs, both monthly and quarterly, than those used in this paper. The line of research initiated by Hamilton (1989) and Stock and Watson (1989, 1991) propose several objective procedures for determining business cycle turning points. To the best of our knowledge, however, none of them use monthly and quarterly BCIs together. To mimic or defeat the NBER procedure, it seems crucial to use quarterly BCIs as well as monthly BCIs. This paper makes the first step in that direction.

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