Title	Full-Harmonic Structures on a Green Space : Abstract (倉持型理想境界と解析学)
Author(s)	MAEDA, FUMI-YUKI
Citation	数理解析研究所講究録 (1967), 26: 75-86
Issue Date	1967-05
URL	http://hdl.handle.net/2433/107512
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
26巻 1967年 75-86

[D] Full-harmonic Structures on a Green Space (Abstract)

Fumi-Yuki MAEDA, Dept. of Math., Hiroshima Univ.

Chap. I. Full-harmonic structures and associated kernels.

§ 1. Full-harmonic structure.

Let Ω be a non-compact space of type $\mathcal{E}$ in the sense of Brelot-Choquet [1] or an open Riemann surface. For an open set $G \subset \Omega$, let $H(G)$ be the linear space of all harmonic functions on G. Let $\mathcal{D}$ be the family of all domains D in Ω such that D is not relatively compact and its relative boundary ∂D is compact. Let $\mathcal{G}$ be the family of all open sets G in Ω such that ∂G is compact. Then every component D of $G \in \mathcal{G}$ is either relatively compact or $D \in \mathcal{D}$.

Suppose there is given a family $\mathcal{F} = \{\widetilde{H}(D)\}_{D \in \mathcal{D}}$ satisfying the following three conditions:

(i) Each $\widetilde{H}(D)$ is a linear subspace of $H(D)$;

(ii) If $D, D' \in \mathcal{D}$, $D' \subset D$ and $u \in \widetilde{H}(D)$, then $u|_{D'} \in \widetilde{H}(D')$;

(iii) If $u \in H(D)$ and if there exists a compact set K in Ω such that $u|_{D'} \in \widetilde{H}(D')$ for every component D' of $D - K$, then $u \in \widetilde{H}(D)$.

A domain $D \in \mathcal{D}$ is called $\mathcal{F}$-regular if, for any non-negative continuous function f on ∂D, there exists a unique non-negative function u_f on $\overline{D}$ such that $u_f|_D \in \widetilde{H}(D)$ and $u_f|_{\partial D} = f$. An open set $G \in \mathcal{G}$ is called $\mathcal{F}$-regular if each relatively compact component of G is regular with respect to the Dirichlet problem and each component D of G such that $D \in \mathcal{D}$ is $\mathcal{F}$-regular.

$\mathcal{F}$ is called a full-harmonic structure if it satisfies the following fourth condition (cf. [6]):

(iv) For any compact set K_o in Ω, there exists another compact set K in Ω such that K^i(the interior of K)$\supset K_o$ and $\Omega - K$ is $\mathcal{F}$-regular.

In this note, we shall assume the following condition (iv)' instead of (iv):

(iv)' If $D \in \mathcal{D}$ has smooth boundary, then it is $\mathcal{F}$-regular.

Let D be an $\mathcal{F}$-regular domain. For $f \in C(\partial D)$, we define $u_f = u_{f^+} - u_{f^-}$. Then $u_f \in \widetilde{H}(D)$ and the mapping $f \to u_f$ is linear. For each $x \in D$, the mapping $f \to u_f(x)$ is positive linear, so that there exists a measure $\mu_x^D \equiv \mu_{\mathcal{F},x}^D$ on ∂D such that $u_f(x) = \int f d\mu_x^D$ for all $f \in C(\partial D)$. If $G \in \mathcal{G}$ is $\mathcal{F}$-regular, then we define $\mu_x^G \equiv \mu_{\mathcal{F},x}^G$ as follows: Let D be the component containing x. If $D \in \mathcal{D}$, then define $\mu_x^G = \mu_x^D$. If D is relatively compact, then let μ_x^G be the usual harmonic measure of D with respect to x.

§ 2. Full-superharmonic functions.

Let $\mathcal{F}$ be a full-harmonic structure and let $D \in \mathcal{D}$. A superharmonic function s on D is called $\mathcal{F}$-full-superharmonic if there exists a sequence $\{K_n\}$ of compact sets such that $\partial D \subset K_1^i$, $K_n \subset K_{n+1}^i$, $\bigcup_{n=1}^{\infty} K_n = \Omega$, each $\Omega - K_n$ is $\mathcal{F}$-regular and

$$s(x) \geq \int s d\mu_{\mathcal{F},x}^{D-K_n}$$

for all n and for all $x \in D - K_n$.

It can be seen that if s is $\mathcal{F}$-full-superharmonic on D, then for any compact set K such that $K^i \supset \partial D$ and $\Omega - K$ is $\mathcal{F}$-regular,

$s(x) \geq \int s d\mu^{D-K}_{\mathcal{F},x}$ for all $x \in D - K$.

For $D \in \mathcal{D}$, $u \in \widetilde{H}(D)$ if and only if u and $-u$ are both $\mathcal{F}$-full-superharmonic. Thus, $u \in \widetilde{H}(D)$ is said to be $\mathcal{F}$-full-harmonic on D.

Full-superharmonic functions have many properties analogous to those of superharmonic functions (cf. [6]). For example, we have:

Minimum Principle: Let $D \in \mathcal{D}$ be such that $\overline{D} \neq \Omega$. If u is a full-superharmonic function on D and if $\underline{\lim}_{x \to \xi, x \in D} u(x) \geq 0$ for all $\xi \in \partial D$, then $u \geq 0$.

§ 3. Full-superharmonic functions of potential type.

Let a full-harmonic structure $\mathcal{F}$ be given. We take a domain $\Omega_0 \in \mathcal{D}$ such that $\overline{\Omega}_0 \neq \Omega$ and fix it in the sequel.

It is shown that any non-negative full-superharmonic function on Ω_0 has the greatest full-harmonic minorant (cf. [6]).

Definition. A non-negative $\mathcal{F}$-full-superharmonic function on Ω_0 is called of potential type if its greatest $\mathcal{F}$-full-harmonic minorant on Ω_0 is zero. We denote by $\mathcal{P} \equiv \mathcal{P}(\Omega_0; \mathcal{F})$ the set of all $\mathcal{F}$-full-superharmonic functions of potential type.

Any non-negative $\mathcal{F}$-full-superharmonic function u on Ω_0 is decomposed into $u = v + h$ with $v \in \mathcal{P}$ and h $\mathcal{F}$-full-harmonic.

Let $\mathcal{P}_b \equiv \mathcal{P}_b(\Omega_0; \mathcal{F})$ be the set $\mathcal{P}(\Omega_0; \mathcal{F}) \cap H(\Omega_0)$.

§ 4. $\mathcal{F}$-Green kernel.

The domain Ω_0 has the Green function $G_x(y) \equiv G_x^{\Omega_0}(y)$ (cf. [1]). For a full-harmonic structure $\mathcal{F}$, an $\mathcal{F}$-Green kernel is a non-negative extended real valued function $M_x(y) \equiv M_{\mathcal{F},x}(y) \equiv M^{\Omega_0}_{\mathcal{F},x}(y)$ on $\Omega_0 \times \Omega_0$ such that

(i) for each $x \in \Omega_o$, $M_x - G_x$ is harmonic on Ω_o;

(ii) for each $x \in \Omega_o$, $M_x \in \wp(\Omega_o;\mathcal{F})$ and M_x is $\mathcal{F}$-full-harmonic on $\Omega_o - \{x\}$.

<u>Theorem.</u> The $\mathcal{F}$-Green kernel $M_x \equiv M^{\Omega_o}_{\mathcal{F},x}$ always exists and is unique. It is continuous on $\Omega_o \times \Omega_o$. Furthermore, $M_x = G_x + U_x$ with $U_x \in \wp_b(\Omega_o;\mathcal{F})$.

<u>Sketch of the proof:</u> For each $x \in \Omega_o$, let $\mathcal{U}_x = \{u \in \wp;\ G_x + u \in \wp\}$.

(i) $\mathcal{U}_x$ is non-empty. To show it, choose a closed sphere K_o such that $K_o \cap \bar{\Omega}_o = \emptyset$ and let $h(y) = \int d\mu^{\Omega - K_o}_{\mathcal{F},y}$ and h_o be the Dirichlet solution on $\Omega - K_o$ with boundary values 1 on ∂K_o, 0 on the ideal boundary. $h - h_o$ is $\mathcal{F}$-full-superharmonic and non-negative on Ω_o. Let $h - h_o = v + h_1$, where $v \in \wp(\Omega_o;\mathcal{F})$ and h_1 is $\mathcal{F}$-full-harmonic on Ω_o. Choose a compact set K with smooth boundary such that $K^i \supset \partial\Omega_o \cup \{x\}$ and let

$\lambda = \sup_{y \in \partial K \cap \Omega_o} [G_x(y)/h_o(y)]$. Then $0 < \lambda < +\infty$. We can show that

$G_x + \lambda v \in \wp$, so that $\lambda v \in \mathcal{U}_x$.

(ii) Let $U_x = \inf \mathcal{U}_x$. It is easy to see that $U_x \in \wp_b$ and $G_x + U_x \in \wp$. By Theorem 3 of [6], we also see that $G_x + U_x$ is full-harmonic on $\Omega_o - \{x\}$. Hence $M_x = G_x + U_x$ satisfies (i) and (ii).

(iii) The continuity follows from the fact that, given $\varepsilon > 0$, $U_{x'} + \varepsilon v \in \mathcal{U}_x$ if x' is sufficiently close to x.

§ 5. Integral representation of $\wp$-functions.

Let $u \in \wp(\Omega_o;\mathcal{F})$. There exists a unique positive measure μ on Ω_o such that

$$u = h + \int_{\Omega_0} M^{\Omega_0}_{\mathcal{F},x} d\mu(x)$$

with $h \in \mathcal{P}_b(\Omega_0;\mathcal{F})$ (cf. [5]).

In order to consider the integral representation of $h \in \mathcal{P}_b(\Omega_0;\mathcal{F})$, we construct an ideal boundary associated with $\mathcal{F}$: Fix $x_0 \in \Omega_0$ and let

$$K^{\Omega_0}_{\mathcal{F},x}(y) = \begin{cases} \dfrac{M^{\Omega_0}_{\mathcal{F},x}(y)}{M^{\Omega_0}_{\mathcal{F},x}(x_0)} & \text{if } x \neq y \text{ or } x \neq x_0 \\ 1 & \text{if } y = x = x_0. \end{cases}$$

For each $y \in \Omega_0$, $x \to K_x(y) \equiv K^{\Omega_0}_{\mathcal{F},x}(y)$ is continuous on Ω_0. We can define ideal boundary points with respect to K_x just as the Martin boundary points. The ideal boundary thus obtained is denoted by $\Delta_{\mathcal{F}}(\Omega_0)$. The corresponding metric defines a topology on $\Omega_0 \cup \Delta_{\mathcal{F}}(\Omega_0)$ and $\Delta_{\mathcal{F}}(\Omega_0)$ is compact with respect to this topology. It is shown that $\Delta_{\mathcal{F}}(\Omega_0)$ does not depend on $x_0 \in \Omega_0$. For each $\xi \in \Delta_{\mathcal{F}}(\Omega_0)$ let $K_\xi(y) = \lim_{x \to \xi, x \in \Omega_0} K_x(y)\ (y \in \Omega_0)$.

On $\Delta_{\mathcal{F}}(\Omega_0)$, there appear $\mathcal{F}$-minimal points and $\mathcal{F}$-non-minimal points. The set of $\mathcal{F}$-minimal points on $\Delta_{\mathcal{F}}(\Omega_0)$ is a G_δ-set and is denoted by $\Delta^1_{\mathcal{F}}(\Omega_0)$. As in the case of the Martin boundary, we obtain the following integral representation theorem (cf. [2], [5] and [6]):

__Theorem.__ Given $u \in \mathcal{P}_b(\Omega_0;\mathcal{F})$, there exists a unique positive measure μ on $\Delta_{\mathcal{F}}(\Omega_0)$ such that $\mu(\Delta_{\mathcal{F}} - \Delta^1_{\mathcal{F}}) = 0$ and

$$u = \int_{\Delta_{\mathcal{F}}} K_\xi \, d\mu(\xi).$$

Thus, for any $u \in \mathcal{S}(\Omega_0;\mathcal{F})$, there exists a unique positive measure ν on $\Omega_0 \cup \Delta_{\mathcal{F}}(\Omega_0)$ such that $\nu(\Delta_{\mathcal{F}} - \Delta_{\mathcal{F}}^1) = 0$ and

$$u = \int_{\Omega_0 \cup \Delta_{\mathcal{F}}} K_x d\nu(x) .$$

§ 6. Ordering among full-harmonic structures.

Let $\mathcal{F}_1$, and $\mathcal{F}_2$ be two full-harmonic structures. If, for any $D \in \mathcal{D}$ with smooth boundary and for any $x \in D$, $\mu^D_{\mathcal{F}_1,x} \le \mu^D_{\mathcal{F}_2,x}$ (i.e., $\int f d\mu^D_{\mathcal{F}_1,x} \le \int f d\mu^D_{\mathcal{F}_2,x}$ for all $f \in C(\partial D)$ such that $f \ge 0$), then we write $\mathcal{F}_1 \le \mathcal{F}_2$. This is an order relation in the class of full-harmonic structures on Ω.

<u>Theorem.</u> If $\mathcal{F}_1 \le \mathcal{F}_2$ then $M^{\Omega_0}_{\mathcal{F}_1,x} \le M^{\Omega_0}_{\mathcal{F}_2,x}$ for all $x \in \Omega_0$.

Let $G \in \mathcal{G}$ be regular with respect to the Dirichlet problem. For $f \in C(\partial G)$, the Dirichlet solution with boundary values f on ∂G, 0 on the ideal boundary is denoted by $h^G_{0,f}$. Let

$$\widetilde{H}_0(D) = \left\{ \begin{array}{l} u \in H(D);\ \text{there exists a compact set } K \text{ such that} \\ K^i \supset \partial D \text{ and } \Omega - K \text{ is regular and } h^{D-K}_{0,u} = u \text{ on } D - K \end{array} \right\}.$$

Then, $\mathcal{F}_0 = \{\widetilde{H}_0(D)\}_{D \in \mathcal{D}}$ is a full-harmonic structure and the $\mathcal{F}_0$-Green kernel on Ω_0 is the Green function $G^{\Omega_0}_x(y)$. $\widetilde{\mathcal{F}}_0$ is the smallest full-harmonic structure. The corresponding ideal boundary $\Delta_{\widetilde{\mathcal{F}}_0}(\Omega - K_0)$ is the Martin boundary.

Chap. II. <u>Boundary value problems and examples of full-harmonic structures.</u>

Let Ω be a Green space or a hyperbolic Riemann surface, and let K be a compact set in Ω with smooth boundary $\partial K (\neq \emptyset)$. Let $C^1(\partial K)$ be the set of functions which are C^1 in a neighborhood of ∂K. We consider a compactification Ω^* of Ω and we shall treat the problem to find a harmonic function u on $\Omega - K$ satisfying a preassigned boundary condition on $\Delta = \Omega^* - \Omega$ and $u = f$ on ∂K.

§ 1. Dirichlet problem. (cf. [2] and [4])

For an extended real valued function g on Δ, we consider the classes

$$\overline{\mathcal{S}}_g = \left\{ \begin{array}{l} s;\ \text{superharmonic, bounded below on } \Omega, \\ \underline{\lim}_{x \to \xi, x \in \Omega} s(x) \geq g(\xi) \text{ for any } \xi \in \Delta \end{array} \right\} \cup \{\infty\}$$

$$\underline{\mathcal{S}}_g = \{-s;\ s \in \overline{\mathcal{S}}_{-g}\},$$

and let $\overline{H}_g(x) = \inf\{s(x);\ s \in \overline{\mathcal{S}}_g\}$ and $\underline{H}_g(x) = \sup\{s(x);\ s \in \underline{\mathcal{S}}_g\}$. If $\overline{H}_g = \underline{H}_g$ and is finite, then we say that g is resolutive and we write $H_g = \overline{H}_g = \underline{H}_g$. H_g is a harmonic function on Ω. If any $g \in C(\Delta)$ is resolutive, then Ω^* is called a resolutive compactification.

Assumption 1. Ω^* is a resolutive compactification.

The harmonic measure $\mu_x (x \in \Omega)$ is a positive Radon measure on Δ such that $H_g(x) = \int g d\mu_x$ for any $g \in C(\Delta)$.

Let g be a resolutive function on Δ and let $f \in C^1(\partial K)$. Then

$$g_1 = \left\{ \begin{array}{l} g \text{ on } \Delta \\ \\ f \text{ on } \partial K \end{array} \right.$$

is resolutive with respect to the compactification $\Delta \cup (\Omega - K) \cup \partial K$ of $\Omega - K$. The solution will be denoted by $H_{g,f}^{\Delta,\partial K}$. We shall write

$$u_{0,f}^{K} \equiv H_{0,f}^{\Delta,\partial K} \quad \text{and} \quad v_{g}^{K} \equiv H_{g,0}^{\Delta,\partial K}.$$

§ 2. HD-space.

For an open set G and two harmonic functions u and v on G, let $\langle u, v\rangle_G$ be the mixed Dirichlet integral over G and let $\|u\|_G^2 = \langle u, u\rangle_G$. Let $HD(G)$ be the set of all harmonic functions u on G such that $\|u\|_G < +\infty$.

We denote by $R_D(\Delta)$ the set of all resolutive functions g on Δ such that $H_g \in HD(\Omega)$. We know that if $g \in R_D(\Delta)$ and $f \in C^1(\partial K)$, then $H_{g,f}^{\Delta,\partial K} \in HD(\Omega - K)$; in particular, $u_{0,f}^{K}$, $v_g^K \in HD(\Omega - K)$.

Doob's lemma (cf. [3] and [4]). There exists $M > 0$, depending only on K and x, such that, for any $g \in R_D(\Delta)$,

$$\int g^2 d\mu_x \leq M \|v_g^K\|^2.$$

Corollary. $R_D(\Delta) \subseteq L^2(\mu)$.

§ 3. Normal derivative on an ideal boundary.

Let $x_0 \in \Omega$ be fixed and let $\mu \equiv \mu_{x_0}$. Let A be a μ-measurable set on Δ. Given $u \in HD(\Omega - K)$ and a μ-measurable function φ on A, we say that u has a normal derivative φ on A, or φ is a normal derivative of u on A, if, for any $g \in R_D(\Delta)$ such that $g = 0$ μ-a.e. on $\Delta - A$, φg is μ-summable on A and

$$\langle u, v_g^K\rangle_{\Omega-K} = -\int_A \varphi g d\mu.$$

φ may not be uniquely (even μ-a.e.) determined by u. But, under the following assumption, if $u, v \in HD(\Omega)$ have the same normal derivative on Δ, then $u = v + \text{const}$.

Assumption 2. For any $u \in HD(\Omega)$, there exists $u^* \in R_D(\Delta)$ such that $u = H_{u^*}$.

Such u^* is uniquely determined μ-a.e. We hereafter assume Assumptions 1 and 2. The Wiener, Royden, Kuramochi and Martin compactifications are examples.

We can show that, for any $u \in HD(\Omega - K)$ such that $u = f \in C^1(\partial K)$ on ∂K, there exists $u^* \in R_D(\Delta)$ (uniquely determined μ-a.e.) such that $u = H^{\Delta, \partial K}_{u^*, u}$ on $\Omega - K$.

§ 4. Boundary value problems.

Let A_0 be a μ-measurable subset of Δ and let $\beta(\xi)$ be a non-negative μ-measurable function on $\Delta - A_0$, finite μ-a.e. We consider the following boundary condition (A_0, β):

$$\begin{cases} u^* = 0 & \text{on } A_0 \quad \mu\text{-a.e.} \\ \varphi = \beta u^* & \text{on } \Delta - A_0 \quad \mu\text{-a.e. for a normal derivative } \varphi \\ & \text{of } u \text{ on } \Delta - A_0. \end{cases}$$

If we write

$$\alpha = \begin{cases} 0 & \text{on } A_0 \\ \dfrac{1}{1+\beta} & \text{on } \Delta - A_0, \end{cases}$$

then α is a μ-measurable function on Δ, $0 \leq \alpha \leq 1$ and the condition (A_0, β) is equivalent to

(α): $(1 - \alpha)u^* = \alpha\varphi$ μ-a.e. for a normal derivative φ of u on $\Delta - A_0$.

Given $f \in C^1(\partial K)$, if there exists $u \in HD(\Omega - K)$ such that $u = f$ on ∂K and u satisfies the consition (α) or (A_0, β), then u will be denoted by $u^K_{\alpha,f}$. If $\alpha \equiv 0$, then our problem reduces to the Dirichlet problem, so that $u^K_{0,f}$ coincides with the one defined in § 1.

Theorem 1. $u^K_{\alpha,f}$ is unique if it exists. If $f \geq 0$ and $\alpha_1 \leq \alpha_2$, then $u^K_{\alpha_1,f} \leq u^K_{\alpha_2,f}$; in particular, for any α, $0 \leq u^K_{0,f} \leq u^K_{\alpha,f} \leq u^K_{1,f}$.

§ 5. Existence theorem.

Theorem 2. If β is bounded, then $u^K_{\alpha,f}$ exists for any K and f.

Sketch of the proof (cf. [3] and [4]): The case $\alpha \equiv 0$ μ-a.e. (i.e., $A_0 = \Delta$) is already known. Thus, we assume $\mu(\Delta - A_0) > 0$.

Case I. $\beta \equiv 0$. Consider the family

$$\Phi_f = \{u \in HD(\Omega - K);\ u = f \text{ on } \partial K,\ u^* = 0 \ \ \mu\text{-a.e. on } A_0\}.$$

Φ_f is non-empty, convex and complete with respect to the norm $\| \ \|_{\Omega-K}$. Hence there exists $u_0 \in \Phi_f$ such that $\|u_0\|_{\Omega-K} = \min_{u \in \Phi_f} \|u\|_{\Omega-K}$ and we see that 0 is a normal derivative of u_0 on $\Delta - A_0$.

Case II. $\beta \not\equiv 0$ on $\Delta - A_0$ and β is bounded. Let $A_1 = \{\xi \in \Delta - A_0;\ \beta(\xi) = 0\}$ and $A' = \{\xi \in \Delta - A_0;\ \beta(\xi) > 0\}$. We may assume $\mu(A') > 0$. Using the Doob's lemma, we can prove

Lemma. Given $\varphi \in L^2(\mu;\ \Delta - A_0)$, there exists a unique $u \in \Phi_0$ such that φ is a normal derivative of u on $\Delta - A_0$.

Next, consider the space $\mathcal{L} = \{\varphi \in L^2(\mu;\ A');\ \int_{A'} \frac{\varphi^2}{\beta} d\mu < \infty\}$. $\mathcal{L}$ is a Hilbert space with respect to $(\varphi, \psi)_\beta = \int_{A'} \frac{\varphi\psi}{\beta} d\mu$. By the above lemma, for each $\varphi \in \mathcal{L}$, there exists a unique $u_\varphi \in \Phi_0$ such that u has a normal derivative φ on A', 0 on A_1. The mapping $T : \varphi \to \beta(u^*_\varphi|_{A'})$ is a symmetric, negative definite operator on $\mathcal{L}$, so that $T - I$ is invertible. Let $v \in \mathfrak{E}_f$ be such that 0 is a normal derivative of v on $A_1 \cup A'$ and $v = f$ on ∂K, determined in Case I. Then $\beta(v^*|_{A'}) \in \mathcal{L}$ and there exists $\psi \in \mathcal{L}$ such that $\beta u^*_\psi - \psi = -\beta v^*$ on A'. Then we see that $u = v + u_\psi$ is our solution.

§ 6. α-full-harmonic structure $\mathcal{F}_\alpha$.

Let α be a function as above for which $u^K_{\alpha,f}$ always exists. For $D \in \mathcal{D}$, we define

$$\tilde{H}_\alpha(D) = \left\{ u \in H(D);\ \text{there exists a compact set } K \text{ with smooth boundary such that } K^i \supset \partial D \text{ and } u = u^{\partial K \cap D}_{\alpha,u} \text{ on } D - K \right\}.$$

Then, $\mathcal{F}_\alpha = \{\tilde{H}_\alpha(D)\}_{D \in \mathcal{D}}$ is a full-harmonic structure. By Theorem 1, if $\alpha_1 \le \alpha_2$, then $\mathcal{F}_{\alpha_1} \le \mathcal{F}_{\alpha_2}$.

Let K_0 be a closed sphere in Ω. Let us denote the $\mathcal{F}_\alpha$-Green kernel on $\Omega - K_0$ by $M_{\alpha,x}(y)$. Remark that $M_{0,x} = G_x$ and $M_{1,x}$ is the N-Green function N_x of Kuramochi. Thus, if $\alpha_1 \le \alpha_2$, then

$$0 \le G_x \le M_{\alpha_1,x} \le M_{\alpha_2,x} \le N_x.$$

The ideal boundary $\Delta_{\mathcal{F}_1}(\Omega - K_0)$ is the Kuramochi boundary.

<u>Theorem</u> 3. For any α, $M_{\alpha,x}(y) = M_{\alpha,y}(x)$ ($\forall x, y \in \Omega - K_0$).

References.

[1] M. Brelot and G. Choquet, Espaces et lignes de Green, Ann. Inst. Fourier, 3(1952), 199-263.

[2] C. Constantinescu and A. Cornea, Ideale Ränder Riemannscher Flächen, Berlin, 1963.

[3] J. L. Doob, Boundary properties of functions with finite Dirichlet integrals, Ann. Inst. Fourier, 12(1962), 573-621.

[4] F-Y. Maeda, Normal derivatives on an ideal boundary, J. Sci. Hiroshima Univ., 28(1964), 113-131.

[5] F-Y. Maeda, On full-superharmonic functions, Proceedings of Symposium on Kuramochi boundary, 1965, 24 pp.

[6] F-Y. Maeda, Axiomatic treatment of full-superharmonic functions, J. Sci. Hiroshima Univ., 30(1966), 197-215.