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A Generalization of a Prime Number Thorem of Rodosskii-
Tatsuzawa for an algebraic Field

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1. Introduction

The Prime Number Theorem in an Arithmetic Progressions is stated as follows ;

Theorem A . If (q , l) = 1 and $q \leqq \exp (c \sqrt{\log x})$ then

$$\pi(x, q, l) = \frac{1}{\varphi(q)} \int_2^x \frac{dt}{\log t} + O\left(\frac{x^{\beta_1}}{\varphi(q) \log x}\right) + O\left(x e^{-c' \sqrt{\log x}}\right)$$

where β_1 is a so-called Siegel zero.

For q , which does not satisfy the above restriction , we have the following theorem ;

Theorem B (Rodosskii - Tatsuzawa) (See , Prachar (8) Chap . 9 Satz 2 . 2)

There exist constant c_1 and c_2 , such that , if

$$c_1 \log q \log\log q \leqq \log x \leqq c_2 (\log q)^2$$

$$\pi(x, q, l) = \frac{1}{\varphi(q)} \int_2^x \frac{dt}{\log t} + O\left(\frac{x^{\beta_1}}{\varphi(q) \log x}\right) + O\left(\frac{x}{\varphi(q) \log x} e^{-c \frac{\log x}{\log q}}\right)$$

The prime ideal theorem in an ideal classes of an algebraic field , which is a generalization of theorem A , has been already obtained under the similar restriction. (see, Mitsui (5))

The purpose of this paper is to offer a generalization of theorem B for an algebraic field under the similar condition. We now write it down ;

Theorem K *is an algebraic field of degree* n *and* $\mathfrak{f}$ *is an integral ideal in* K*. We denote a product of* $\mathfrak{f}$ *and some infinite prime ideals by* $\tilde{\mathfrak{f}}$ $h(\tilde{\mathfrak{f}})$ *is the number of ideal classes mod* $\tilde{\mathfrak{f}}$ *and* $\mathfrak{l}$ *is one of the ideal classes . Then we have*

for all $c > 0$ *, there exist* c_1 *and* c_2 *such that, under the condition*

$$c_1 \log N(\mathfrak{f}) \log\log N(\mathfrak{f}) \leqq \log x \leqq c\,(\log N(\mathfrak{f}))^2$$

$$\pi(x,\mathfrak{l}) = \sum_{\substack{N(\mathfrak{z})<x \\ \mathfrak{z}\in\mathfrak{l}}} 1 = \frac{1}{h(\tilde{\mathfrak{f}})}\int_2^x \frac{dt}{\log t} + O\left(\frac{x^{\beta_1}}{h(\tilde{\mathfrak{f}})\log x}\right) + O\left(\frac{x}{h(\tilde{\mathfrak{f}})\log x} e^{-c_2\frac{\log x}{\log N(\mathfrak{f})}}\right)$$

(The constant in O *depend only on* K *and* c *.)*

The auther has already given another proof of theorem B in (2) , using Gallagher's mean value theorem (see, Gallagher (1) Theorem 2) In this paper, we use its generalization for an algebraic field , again . The functions $\mu(\mathfrak{a})$, $d(\mathfrak{a})$, $\Lambda(\mathfrak{a})$ *and* $\varphi(\mathfrak{a})$ *are that of generalization for an algebraic field of Möbius , divisor , von Mangoldt and Euler function , respectively . All constants of all estimations depend only on* K *and* c *.*

The auther wishes to express his thanks to Prof. T . Tatszawa , who taught him Theorem 2 and 3 in this paper , and encouraged him during preparing this paper .

2. The number of ideals of an ideal class mod $\tilde{f}$ whose norm lie in a given interval

Let K be an algebraic field of degree n with r_1 real conjugates and 2 r_2 complex conjugates and f an integral ideal of K . And we put $\tilde{f} = f \mathfrak{z}_\infty^{(1)} \cdots \mathfrak{z}_\infty^{(8)}$ and h $(\tilde{f})$ is the number of ideal classes mod $\tilde{f}$ .Then ,

Theorem 1

$$h(\tilde{f}) = \frac{h\,\varphi(\tilde{f})}{e(\tilde{f})} \qquad (1)$$

where h is the ideal class number of K , e $(\tilde{f})$ the number of residue classes mod $\tilde{f}$ of units of K , and $\varphi(\tilde{f}) = 2^{8}\varphi(f)$

The proof of this theorem, for example , appears in Suetsuna (9) Chap . 2 Th . 4 (p . 55)

Cor .

$$h(\tilde{f}) \ll N(f) \qquad (2)$$

Theorem 2

$$\sum_{\substack{N(\mathfrak{a}) < x \\ \mathfrak{a} \in l}} 1 = \frac{1}{h(\tilde{f})}\frac{\varphi(f)}{N(f)} c_K x + O\left(\frac{1}{h(\tilde{f})}\frac{\varphi(f)}{N(f)} N(f)^{\frac{1}{n}} x^{1-\frac{1}{n}}\right) \qquad (3)$$

where c_K is the constant only determined by K .

proof) see, Tatsuzawa (10)

Theorem 3 If $y \ll x$, then

$$\sum_{\substack{x \leqq N(\mathfrak{a}) \leqq x+y \\ \mathfrak{a} \in \mathfrak{l}}} 1 \ll \frac{y}{h(\tilde{\mathfrak{f}})} + \frac{N(\mathfrak{f})^{\frac{1}{n}}}{h(\tilde{\mathfrak{f}})} x^{1-\frac{1}{n}} + 1 \quad (4)$$

We now choose a ideal class $\mathfrak{l}$ mod $\mathfrak{f}$, and fix it .

3 . A Generalization of Gallagher's Mean Value Theorem

Lemma If $z \ll y$, and $a(\mathfrak{a})$ are complex numbers , then

$$\sum_{\chi \bmod \tilde{\mathfrak{f}}} \Big| \sum_{y \leqq N(\mathfrak{a}) \leqq y+z} a(\mathfrak{a}) \chi(\mathfrak{a}) \Big|^2 \ll \left(z + N(\mathfrak{f})^{\frac{1}{n}} y^{1-\frac{1}{n}} + h(\tilde{\mathfrak{f}})\right) \sum_{y \leqq N(\mathfrak{a}) \leqq y+z} |a(\mathfrak{a})|^2 \quad (5)$$

proof) By the orthogonal relation , Schwarz inequality and theorem 3 ,

we have

$$\sum_{\chi \bmod \tilde{\mathfrak{f}}} \Big| \sum_{y \leqq N(\mathfrak{a}) \leqq y+z} a(\mathfrak{a}) \chi(\mathfrak{a}) \Big|^2 \ll \sum_{\mathfrak{l} \bmod \tilde{\mathfrak{f}}} h(\tilde{\mathfrak{f}}) \Big| \sum_{\substack{y \leqq N(\mathfrak{a}) \leqq y+z \\ \mathfrak{a} \in \mathfrak{l}}} a(\mathfrak{a}) \Big|^2$$

$$\ll \sum_{\mathfrak{l} \bmod \tilde{\mathfrak{f}}} h(\tilde{\mathfrak{f}}) \Big(\sum_{\substack{y \leqq N(\mathfrak{a}) \leqq y+z \\ \mathfrak{a} \in \mathfrak{l}}} 1 \Big) \sum_{\substack{y \leqq N(\mathfrak{a}) \leqq y+z \\ \mathfrak{a} \in \mathfrak{l}}} |a(\mathfrak{a})|^2$$

$$\ll \left(z + N(\mathfrak{f})^{\frac{1}{n}} x^{1-\frac{1}{n}} + h(\tilde{\mathfrak{f}})\right) \sum_{y \leqq N(\mathfrak{a}) \leqq y+z} |a(\mathfrak{a})|^2$$

Theorem 4 . χ is a character mod $\tilde{\mathfrak{f}}$, and

$$S(\chi, t) = \sum a(\mathfrak{a}) \chi(\mathfrak{a}) N(\mathfrak{a})^{-it}$$

is an absolute convergent Dirichlet series. Then, if $T \geqq 1$, we have

$$\sum_{\chi \bmod \tilde{\mathfrak{f}}} \int_{-T}^{T} |S(\chi,t)|^2 dt \ll \sum \left(h(\tilde{\mathfrak{f}}) T + N(\mathfrak{f})^{\frac{1}{n}} T N(\mathfrak{a})^{1-\frac{1}{n}} + N(\mathfrak{a})\right) |a(\mathfrak{a})|^2 \quad (6)$$

proof) We first refer the following inequality .

$S(t) = \sum a_n n^{it}$ is an absolute convergent Dirichlet series, and $\tau = e^{\frac{1}{T}}$. Then

$$\int_{-T}^{T} |S(t)|^2 dt \ll T^2 \int_0^\infty \Big| \sum_{y}^{y\tau} a_n \Big|^2 \frac{dy}{y}$$

(See Gallagher (1) Theorem 1) Then we have

$$\sum_{\chi \bmod \tilde{f}} \int_{-T}^{T} |S(\chi,t)|^2 dt \ll T^2 \int_0^\infty \sum_{\chi \bmod \tilde{f}} \Big| \sum_{y \le N(\mathfrak{a}) \le y\tau} a_{\mathfrak{a}} \chi(\mathfrak{a}) \Big|^2 \frac{dy}{y}$$

By the assumption $T \geqq 1$, we can apply the above lemma to the inner sum .

$$\ll T^2 \int_0^\infty \left(y(\tau-1) + N(f)^{\frac{1}{n}} y^{1-\frac{1}{n}} + h(\tilde{f}) \right) \sum_{y \le N(\mathfrak{a}) \le y\tau} |a_{\mathfrak{a}}|^2 \frac{dy}{y}$$

The coefficient of $|a_{\mathfrak{a}}|^2$ is ($n>1$)

$$T^2 \int_{N(\mathfrak{a})/\tau}^{N(\mathfrak{a})} \left(y(\tau-1) + N(f)^{\frac{1}{n}} y^{1-\frac{1}{n}} + h(\tilde{f}) \right) \frac{dy}{y}$$

$$= h(\tilde{f})T + T^2 N(\mathfrak{a})(\tau-1)(1-\tau^{-1})$$
$$+ \frac{1}{1-\frac{1}{n}} N(f)^{\frac{1}{n}} N(\mathfrak{a})^{1-\frac{1}{n}} (1-\tau^{-1+\frac{1}{n}}) T^2$$

$$\ll h(\tilde{f})T + N(\mathfrak{a}) + N(f)^{\frac{1}{n}} T \, N(\mathfrak{a})^{1-\frac{1}{n}}$$

In the case $n = 1$, the proof is similarly done . Then we get the theorem .

4 . Some properties on $L(s, \chi)$

In this paragraph we pick up some properties of $L(s, \chi)$. These proofs can be found in Mitsui (5) or be similarly done as the case of Dirichlet L-functions , using theorem 2 and 3 .

Theorem 5 . The following estimations are true uniformly on $\sigma \geqq 1-\frac{1}{2n}$

$$L(s,\chi) = \sum_{N(\mathfrak{a})<M} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^{s}} + E(\chi)\,\frac{N(\mathfrak{f})}{h(\mathfrak{f})\,\varphi(\mathfrak{f})}\,c_K\,\frac{M^{1-s}}{s-1}$$
$$+\,O\left(|s|\,\frac{N(\mathfrak{f})^{\frac{1}{n}}}{M^{\sigma+\frac{1}{n}-1}}\right) \qquad (7)$$

where $E(\chi)$ *is 1 if* χ *is principal , 0 if not .*

Theorem 6

$$L(s,\chi)\,(s-1)^{E(\chi)} \ll N(\mathfrak{f})^{2}\,(|t|+2)^{\frac{2}{3}n(1-\sigma)+E(\chi)}\,\log((|t|+2)N(\mathfrak{f}))$$

(*uniformly* $-\frac{1}{2}\leq\sigma\leq 4$) (8)

Theorem 7

$$\frac{L'}{L}(s,\chi) = \sum_{|\gamma_0-t|\leq 1}\frac{1}{s-\rho_0} - \frac{E(\chi)}{s-1} + O(\log(N(\mathfrak{f})(|t|+2)))$$

where $\rho_0=\beta_0+i\gamma_0$ *is a zero of* $L(s,\chi)$ (9)

Theorem 8 *There exists a constant* $A>0$ *, such that , in the region*

$$\sigma \geqq 1-\frac{A}{\log(N(\mathfrak{f})(|t|+2))}, \quad |t|>0 \qquad (10)$$

all $L(s,\chi)$ *whose character are defined mod* $\mathfrak{f}$ *, are zero-free .*

For $t=0$ *, all* $L(s,\chi)$ *except one* $L(s,\chi_1)$ *where* χ_1 *is a real character mod* $\mathfrak{f}$*, are zero-free in the interval* $1\geqq\sigma\geqq 1-\frac{A}{\log N(\mathfrak{f})}$

The number of the exceptional zeros is at most one with multiplicity .

5 . Notations and Remarks

Now we put the restriction

$$C_1 \log N(\mathfrak{f}) \log\log N(\mathfrak{f}) \leqq \log x \leqq C_2 (\log N(\mathfrak{f}))^2 \qquad (11)$$

and N ($\mathfrak{f}$) is sufficiently large . We define the following functions .

$$\psi(x,\chi) = \sum_{N(\mathfrak{a})<x} \chi(\mathfrak{a})\,\Lambda(\mathfrak{a}) \qquad (12)$$

$$\psi_1(x,\chi) = \sum_{N(\mathfrak{a})<x} \chi(\mathfrak{a})\,\Lambda(\mathfrak{a}) \log\frac{x}{N(\mathfrak{a})} \qquad (13)$$

$$Q(s,\chi) = \sum_{N(\mathfrak{a})<N(\mathfrak{f})T^n} \frac{\mu(\mathfrak{a})\,\chi(\mathfrak{a})}{N(\mathfrak{a})^s} \qquad (14)$$

Furthermore , we assume that $\log T \ll \log N(\mathfrak{f}) \qquad (15)$

$$\alpha = 1+\frac{1}{\log x}\ , \quad \beta = 1-\frac{1}{2n}+\frac{1}{\log x} \qquad (16)$$

$$\psi(x,\mathfrak{l}) = \sum_{\substack{N(\mathfrak{a})<x\\ \mathfrak{a}\in\mathfrak{l}}} \Lambda(\mathfrak{a}) \qquad (17)$$

and

$$\psi_1(x,\mathfrak{l}) = \sum_{\substack{N(\mathfrak{a})<x\\ \mathfrak{a}\in\mathfrak{l}}} \Lambda(\mathfrak{a}) \log\frac{x}{N(\mathfrak{a})} \qquad (18)$$

Then ,

$$\psi_1(x,\mathfrak{l}) = \frac{1}{h(\tilde{\mathfrak{f}})} \sum_{\chi \bmod \mathfrak{f}} \overline{\chi(\mathfrak{l})}\,\psi_1(x,\chi) \qquad (19)$$

We begin with the estimation of $\psi_1(x,\mathfrak{l})$

6 . The estimation of $\psi_1(x,\mathfrak{l})$

6 . 1 Transformation of $\psi_1(x,\mathfrak{l})$

<u>Lemma</u> $a > 0$, then

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{x^s}{s^2}\,ds = \begin{cases} 0 & 0<x\leqq 1 \\ \log x & x \geqq 1 \end{cases} \qquad (20)$$

The proof of this lemma is immediately done .

By the above lemma and the definition of $L(s, \chi)$, we have

$$\psi_1(x, \chi) = -\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{L'}{L}(s,\chi)\frac{x^s}{s^2}ds$$

$$= -\frac{1}{2\pi i}\left(\int_{\alpha-iT}^{\alpha+iT} + \int_{\alpha-i\infty}^{\alpha-iT} + \int_{\alpha+iT}^{\alpha+i\infty}\right)\frac{L'}{L}(s,\chi)\frac{x^s}{s^2}ds$$

$$= -\frac{1}{2\pi i}\left(I_1(x) + I_2(x) + I_3(x)\right) \qquad (21)$$

(*Say !*)

By theorem 8 and the condition (15) , we may assume the following conditions .

1) All $L(s, \chi)$ except one real zero of $L(s, \chi_1)$ are zero- free in the region

$$\sigma \geqq 1 - \frac{A}{\log N(f)}, \quad |t| \leqq T \qquad (A>0) \qquad (22)$$

2) All zeros and poles of all $L(s, \chi)$ mod f are far from the line

$$\sigma = 1 - \frac{A}{\log N(f)}, \quad |t| \leqq T \qquad (23)$$

by $\gg \dfrac{1}{\log N(f)}$.

We put $\gamma = 1 - \dfrac{A}{\log N(f)}$ (24)

We change the way of integration ,and have

$$I_1(x) = -2\pi i\, E(x)\, x + 2\pi i\, E_1(x)\frac{x^{\beta_1}}{\beta_1^2} + \int_{\gamma-iT}^{\gamma+iT}\frac{L'}{L}\frac{x^s}{s^2}ds$$
$$+ \left(\int_{\gamma+iT}^{\alpha+iT} - \int_{\gamma-iT}^{\alpha-iT}\right)\frac{L'}{L}(s,\chi)\frac{x^s}{s^2}ds$$

$$= -2\pi i\, E(x)\, x + 2\pi i\, E_1(x)\frac{x^{\beta_1}}{\beta_1^2} + I_4(x) + I_5(x) \qquad (25)$$

(*Say!*)

$$E_1(\chi) = \begin{cases} 1 & \text{if there is a zero in the region} \quad . \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

We put $E_1^* = \sum_{\chi \bmod f} E_1(\chi)\overline{\chi(l)}$ (27)

Using (14) , we transform $I_4(\chi)$ as follows ;

$$\begin{aligned} I_4(\chi) &= \int_{\gamma-iT}^{\gamma+iT} \frac{L'}{L}(s,\chi)\left(1-L(s,\chi)Q(s,\chi)\right)^2 \frac{x^s}{s^2}\,ds \\ &\quad + \left(\int_{\beta-iT}^{\beta+iT} + \int_{\beta+iT}^{\gamma+iT} - \int_{\beta-iT}^{\gamma-iT}\right)(2L'Q - L'LQ^2)\frac{x^s}{s^2}\,ds \\ &= I_7(\chi) + I_8(\chi) + I_9(\chi) + I_{10}(\chi) \quad (\text{say}) \end{aligned} \tag{28}$$

6 . 2 Estimation of $I_2(\chi)$ and $I_3(\chi)$

By theorem 7 , we have the following estimations on the line $\sigma = \alpha$

$$\frac{L'}{L}(s,\chi) \ll \log(N(f)(|t|+2))\log x \tag{29}$$

Hence ,

$$I_3(\chi) + I_4(\chi) \ll x \log x \int_T^\infty \frac{\log N(f)(|t|+2)}{\alpha^2+T^2}\,dt \ll \frac{x\log^2 x}{T} \tag{30}$$

6 . 3 Estimation of $I_5(\chi)$ and $I_6(\chi)$

As the above paragraph , we have

$$I_5(\chi) + I_6(\chi) \ll \frac{\log^2 x}{T^2}\int_\gamma^\alpha x^\sigma\,d\sigma \ll \frac{x\log^2 x}{T^2} \tag{31}$$

6 . 4 Estimation of $I_9(\chi)$ and $I_{10}(\chi)$

By using Cauchy's integration formula , we have

$$L'(s,\chi) \ll N(\mathfrak{f})^2 (|t|+2)^{\frac{2}{3}n(1-\sigma)} \log^3 N(\mathfrak{f})(|t|+2) \tag{32}$$

On the other hand , the following estimations are obtained by using convexity argument .

$$Q(s,\chi) \ll N(\mathfrak{f}) T^{n(1-\sigma)} \log x \quad \text{for} \quad \beta \leq \sigma \leq \alpha . \tag{33}$$

Therefore we have

$$I_9(x) + I_{10}(x) \ll N(\mathfrak{f})^6 \frac{x}{T^2} \log^5 x \int_\beta^\gamma \left(\frac{T^{\frac{10}{3}n}}{x}\right)^{1-\sigma} d\sigma$$

$$\ll N(\mathfrak{f})^6 \frac{x^\gamma}{T^2} \log^5 x \tag{34}$$

$$(\text{if} \quad T^{\frac{10}{3}n} \leq x \quad)$$

6 . 5 Estimation of $I_8(\chi)$

By theorem 6 , (32) ,and (33) , we have

$$I_8(x) \ll N(\mathfrak{f})^6 T^{\frac{10}{3}n(1-\beta)} x^\beta \log^6 x$$

$$= N(\mathfrak{f})^6 T^{5/3} x^\beta \log^6 x \tag{35}$$

6 . 6 Estimations of $I_7(\chi)$

Putting $M = N(\mathfrak{f})T^n$ in theorem 5 , we have on the line $\sigma = \alpha$

$$L(s,\chi)Q(s,\chi) - 1 = \sum_{N(\mathfrak{f})T^n \leq N(\mathfrak{a}) \leq (N(\mathfrak{f})T^n)^2} b(\mathfrak{a})\chi(\mathfrak{a})N(\mathfrak{a})^{-s} + O\left(|Q(s,\chi)|\left(\frac{|s|}{T} + E(x)\log x\right)\right) \tag{36}$$

where
$$b(\mathfrak{a}) = \sum_{\mathfrak{z}|\mathfrak{a},\, N(\mathfrak{z})<N(\mathfrak{f})T^n} \mu(\mathfrak{z}) \tag{37}$$

Therefore ,

$$\sum_{\chi \bmod \tilde{f}} |I_\gamma(x)| \ll x^\gamma \log^2 x \left\{ \int_{-T}^{T} \frac{\sum_{\chi \bmod \tilde{f}} \left| \sum_{N(f)T^n \leq N(\mathfrak{r}) \leq (N(f)T^n)^2} b(\mathfrak{r})\chi(\mathfrak{r})N(\mathfrak{r})^{-\gamma - it} \right|^2}{|\gamma + it|^2} dt \right.$$

$$\left. + \frac{1}{T^2}\int_{-T}^{T} \sum_{\chi \bmod \tilde{f}} |Q(\gamma+it,\chi)|^2 dt + \log^2 x \int_{-T}^{T} \frac{|Q(\gamma+it,\chi_0)|^2}{|\gamma+it|^2} dt \right\} \qquad (38)$$

We now apply Gallagher's mean value theorem to the above integrals .

Then we have

$$\sum_{\chi \bmod \tilde{f}} \int_{-T}^{T} |Q(\gamma+it,\chi)|^2 dt \ll \sum_{N(\mathfrak{r}) < N(f)T^n} \left(h(\tilde{f})T + N(f)^{\frac{1}{n}} T N(\mathfrak{r})^{-\frac{1}{n}} + N(\mathfrak{r})\right) \frac{1}{N(\mathfrak{r})^{2\gamma}}$$

$$\ll N(f)T \qquad (39)$$

and

$$\sum_{\chi \bmod \tilde{f}} \int_{-T}^{T} \frac{\left| \sum_{N(f)T^n \leq N(\mathfrak{r}) \leq (N(\mathfrak{r})T^n)^2} b(\mathfrak{r})\chi(\mathfrak{r})N(\mathfrak{r})^{-\gamma - it} \right|^2}{|\gamma+it|^2} dt$$

$$\ll \sum_{N(f)T^n \leq N(\mathfrak{r}) \leq (N(f)T^n)^2} \left(h(\tilde{f})T + N(f)^{\frac{1}{n}} T N(\mathfrak{r})^{1-\frac{1}{n}} + N(\mathfrak{r})\right) \frac{|b(\mathfrak{r})|^2}{N(\mathfrak{r})^{2\gamma}}$$

$$\ll \log^4 x \qquad (40)$$

(We use $\sum_{N(\mathfrak{r}) < x} |b(\mathfrak{r})|^2 \ll \sum_{N(\mathfrak{r}) \leq x} |d(\mathfrak{r})|^2 \ll x \log^3 x$ *)*

The last integral is trivially estimated and we obtain

$$\sum_{\chi \bmod \tilde{f}} I_\gamma(x) \ll x^\gamma \left(\log^4 x + \frac{N(f)}{T}\right) \qquad (41)$$

6 . 7 Estimation of $\psi(x, l)$

Now we collect the results of preceding paragraphs , then we have

$$\psi_1(x, l) = \frac{x}{h(\tilde{f})} - E_1^* \frac{x^{\beta_1}}{h(\tilde{f})\beta_1^2} + O\left(\frac{x}{T} \log^2 x\right)$$

$$+ O\left(x^{1-\frac{1}{2n}} N(f)^6 T^{\frac{5}{3}} \log^6 x \right) + O\left(\frac{x^6}{h(\tilde{f})} \log^4 x \right) \quad (42)$$

Taking $T = N(f)^4$, *we have*

$$\frac{x}{T} \log^2 x \ll \frac{x}{h(\tilde{f})} e^{-c_3 \frac{\log x}{\log N(f)}} \quad (43)$$

$$x^{1-\frac{1}{2n}} N(f)^6 T^{\frac{5}{3}} \log^5 x \ll \frac{x}{h(\tilde{f})} N(f)^{-c_4 \log\log N(f)}$$

$$\ll \frac{x}{h(\tilde{f})} e^{-c_5 \frac{\log x}{\log N(f)}} \quad (44)$$

and

$$\frac{x^{\gamma}}{h(\tilde{f})} \log^4 x << \frac{x}{h(\tilde{f})} e^{-\frac{A}{2} \frac{\log x}{\log N(f)}} \quad (45)$$

Therefore we get

$$\psi_1(x, l) = \frac{x}{h(\tilde{f})} - E_1^* \frac{x^{\beta_1}}{h(\tilde{f}) \beta_1^2} + O\left(\frac{x}{h(\tilde{f})} e^{-c_6 \frac{\log x}{\log N(f)}} \right) \quad (46)$$

7 . Estimation of $\psi(x, l)$

Proposition 1 . For all $c^* > 0$ *, there exists a constant* c_1^* 0 *,*

such that , under the condition

$$c_1^* \log N(f) \log\log N(f) \leqq \log x \leqq c^* (\log N(f))^2$$

$$\psi(x, l) = \frac{x}{h(\tilde{f})} - E_1^* \frac{x^{\beta_1}}{h(\tilde{f}) \beta_1} + O\left(\frac{x}{h(\tilde{f})} e^{-c_7^* \frac{\log x}{\log N(f)}} \right) \quad (47)$$

proof) Let $c_2 = 2c^*$ *and take* θ *, such that* $0 \leqq \theta \leqq 1$ *,*

we have by (46) ,

$$\psi_1(x + \theta x, l) - \psi_1(x, l) = \frac{\theta x}{h(\tilde{f})} - E_1^* \frac{x^{\beta_1}}{h(\tilde{f}) \beta_1} (\theta + O(\theta^2))$$

$$+ O\left(\frac{x}{h(\tilde{f})} e^{-c_8 \frac{\log x}{\log N(f)}} \right) \quad (48)$$

And by the definition of $\psi_1(x, l)$ *, we get*

$$\psi_1(x+\theta x, \mathfrak{l}) - \psi_1(x, \mathfrak{l}) = \sum_{N(\mathfrak{a})<x,\ \mathfrak{a}\in\mathfrak{l}} \Lambda(\mathfrak{a}) \log(1+\theta) + \sum_{\substack{x\le N(\mathfrak{a})<x+\theta x \\ \mathfrak{a}\in\mathfrak{l}}} \Lambda(\mathfrak{a}) \log\frac{x}{N(\mathfrak{a})} \qquad (49)$$

Using theorem 3 , we obtain

$$\sum_{\substack{x\le N(\mathfrak{a})<x+\theta x \\ \mathfrak{a}\in\mathfrak{l}}} \Lambda(\mathfrak{a}) \log\frac{x}{N(\mathfrak{a})} = O\left(\theta \log x \left(\frac{\theta x}{h(\mathfrak{f})} + \frac{N(\mathfrak{f})^{\frac{1}{n}}}{h(\mathfrak{f})} x^{1-\frac{1}{n}} + 1\right)\right) \qquad (50)$$

and

$$\sum_{N(\mathfrak{a})\le x,\ \mathfrak{a}\in\mathfrak{l}} \Lambda(\mathfrak{a}) \log(1+\theta) = \theta\psi(x, \mathfrak{l}) + O\left(\frac{\theta^2 x \log x}{h(\mathfrak{f})}\right) \qquad (51)$$

The proposition is immediately deduced , taking

$$\theta = e^{-\frac{c_6}{2}\frac{\log x}{\log N(\mathfrak{f})}} \qquad (52)$$

The theorem is easily proved from this proposition .

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