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Some Remarks on Isolated Singularity and their Application to Algebraic Manifolds

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0. Introduction. In 1954 F. Hirzebruch [8] obtained an interesting formula which makes it possible to determine the alternating sum $\Sigma_q(-1)^q \dim H^q(V,\Omega^p(L^k))$ for any complete intersection V of hypersurfaces in a complex projective space and for any $k \in \mathbf{Z}$ where L is the analytic line bundle over V induced by hyperplanesection. He further determined $\dim H^q(V,\Omega^p)$ by using some vanishing theorem. In the author's knowledge, however, the general $\dim H^q(V,\Omega^p(L^k))$ seem not to have been determined yet. In this note we shall give a formula which determines directly $\dim H^q(V,\Omega^p(L^k))$ in case $0<q<\dim V$, by using the theory of isolated singularity. (See Theorem 2.3.1, Corollary 2.3.1.)

Part I is concerned with the general theory of isolated singularity and is of preparatory nature. The readers who have known the standard of the theory (e.g. Greuel [4]) may bypass it after they become familiar with the terminology and notations. Part II begins with the study of $\mathbf{C}^*$-actions over isolated singularities. The main theme there is to compute the characters of the representations of $\mathbb{C}^*$ over various cohomology groups attached to the singularities, We apply it to the cones associated with algebraic manifolds and prove the required formula finally.

The almost all results obtained in this paper have already been announced in [11], [12].

Part I: General Theory

1.1. Preliminaries. We shall often denote by (X,x) the pair of an analytic space X with a point $x \in X$ such that $X \setminus x$ is smooth and pure dimensional. We call such a pair an isolated singularity (even in case X is smooth). For an analytic space X, Ω_X^p denotes the sheaf of analytic p-forms on X; but we write often $\mathcal{O}_X$ for Ω_X^0. Suppose (X,x) is given and let ι be the inclusion $X \setminus x \hookrightarrow X$. Then, for a sheaf G over X, the sheaves $R^q\iota_*\iota^*G$ $(q>0)$, $\mathcal{H}_x^q(G)$ are concentrated into the point x, so we shall often identify them with their stalks over x. Whether these notations mean sheaves or stalks should be understood from the context. Now let us begin with the Serre type duality for (X,x).

Lemma 1.1.1. *Let (X,x), ι be as above and set $n = \dim X$. Then $R^q\iota_*\iota^*\Omega_X^p$, $R^{n-q-1}\iota_*\iota^*\Omega_X^{n-p}$ are finite dimensional (over $\mathbb{C}$) and are naturally dual each other provided* $0 < q < n-1$.

We can prove this easily by using Serre [14] and Andreotti-Grauert [1]. For the explicit pairing which gives the duality, see Section 2.1. Note that it can also be proved that there is a natural pairing between $(\iota_*\iota^*\Omega_X^p)_x$ and $R^{n-1}\iota_*\iota^*\Omega_X^{n-p}$ which is compatible with the structures of the complexes $\iota_*\iota^*\Omega_X^{\bullet}$, $R^{n-1}\iota_*\iota^*\Omega_X^{\bullet}$ and induces the duality between $H^p(\iota_*\iota^*\Omega_X^{\bullet})$ and $R^{n-p}(R^{n-1}\iota_*\iota^*\Omega_X^{\bullet})$.

The next lemma is concerning the coherency of local cohomology.

Lemma 1.1.2. Let G be a coherent $\mathcal{O}_X$-Module such that $S|_{X\setminus x}$ is locally free. Then $\mathcal{H}_x^q(G)$ is coherent for $q<\dim X$.

Proof. Siu [15].

We shall now introduce a condition for an isolated singularity which will turn out to be convenient later.

Definition. We say (X,x) satisfies condition (L) if $\mathcal{H}_x^q(\Omega_X^p) = 0$ for p,q such that $p+q < \dim X$.

Lemma 1.1.3. (Partial Poincare Lemma) If (X,x) satisfies the condition (L), then $\mathcal{H}_x^p(\mathbb{C})=0$, $H^p(\Omega_{X,x}^{\bullet})=0$ for $0<p<\dim X$, where $\Omega_X^{\bullet}$ denotes the Poincare complex of X, and $\Omega_{X,x}^{\bullet}$ its stalk over x.

This can be proved as follows: Consider the E_2-term of the spectral sequence $'E_2^{p,q}=\mathcal{H}_x^p(\mathcal{H}^q(\Omega_X^{\bullet}))$. These are zero except $'E_2^{p,0}=\mathcal{H}_x^p(\mathbb{C})$, $'E_2^{0,q}=H^0(\Omega_{X,x}^{\bullet})$ $(q>0)$. But, since there is a complete neighborhood system at x consisting of contractible ones only, it can be shown by Bloom-Herrera [2] that $H^{r-1}(\Omega_{X,x}^{\bullet})='E_r^{0,r-1} \longrightarrow$

$\xrightarrow{d_r} {}'E_r^{r,0} = \mathcal{H}_x^r(\mathbb{C})$ is zero-map for every $r>0$ (See the proof of Lemma 1.2.2 appearing later). Comparing this with E_1-terms ${}''E_1^{p,q} = \mathcal{H}_x^q(\Omega_X^p)$ of the other spectral sequence having the same limit, we obtain the conclusion of the lemma.

Definition. Let f be an analytic function on X such that $f(x)=0$, and that df_z, which is not the germ but the value at x of the differential form df, is not zero for any $z \in X \setminus x$. Then $(f^{-1}(0),x)$ is a new isolated singularity, and is called the hypersurfacesection of (X,x) defined by f.

The hypersurfacesection is a usefull device for the study of complete intersections since they are obtained from non-singular ones by iterated hypersurfacesections. See Hamm [5].

Now the method to prove Lemma 1.1.3 shows also

Lemma 1.1.4. Let (X,x) satisfy the condition (L) and $f \in \Gamma(X, \mathcal{O}_X)$ be such that $df_z \neq 0$ for every $z \in X \setminus x$. Then the sequence $0 \longrightarrow \Omega_X^0 \xrightarrow{df} \Omega_X^1 \xrightarrow{df} \cdots \xrightarrow{df} \Omega_X^{\dim X}$ is exact, where $\Omega_X^p \xrightarrow{df} \Omega_X^{p+1}$ denotes the exterior muptiplication by df.

Another usefull way to formulate this lemma is the exactness of the sequences

$$(1.1.1) \qquad 0 \longrightarrow \Omega_f^{p-1} \xrightarrow{df} \Omega_X^p \longrightarrow \Omega_f^p \longrightarrow 0 \qquad p \leqq \dim X$$

where we have put as in Brieskorn [3]

$$\Omega_f^p = \Omega_X^p \,/\, df \wedge \Omega_X^{p-1} \quad .$$

Lemma 1.1.5. Let (X,x), f be as in Lemma 1.1.4. Then, (X,x) satisfies the condition (L) if and only if $\mathcal{H}_x^q(\Omega_f^p)=0$ for any p,q such that $p+q<\dim X$.

Proof. Suppose (X,x) satisfies (L). Using the long exact sequence of the local cohomology associated with (1.1.1) we obtain monomorphisms $\mathcal{H}_x^{q-1}(\Omega_f^{p+1}) \hookrightarrow \mathcal{H}_x^q(\Omega_f^p)$ for p,q such that $p+q < \dim X$. Combining these, we have monomorphisms $\mathcal{H}_x^q(\Omega_f^p) \hookrightarrow \mathcal{H}_x^{p+q+1}(\Omega_f^{-1})=0$ when $p+q < \dim X$. This proves the "only if" part. Next, we note that $\mathcal{H}_x^0(\Omega_f^p)=0$ ($p< \dim X$) implies the exactness of (1.1.1). Thus the long exact sequence used above, again proves the "if" part.

We end this section by indicating briefly the topological meaning of the cohomology groups $H^p(\Omega^{\bullet}_{f,x})$. (Note the exterior differentiation d naturally induces the maps $\Omega_f^p \xrightarrow{d} \Omega_f^{p+1}$ by which the complex $\Omega_f^{\bullet}$ is defined.) Let (X,x), f be as in Lemma 1.1.4 and let $f(x)=0$. Let further (Y,y) be the hypersurfacesection by f, that is, $Y=f^{-1}(0)$, $y=x$. Then we can always assume by Milnor [10]

a) (X,x) is a closed analytic set in some open ball with center $x=0$ in $\mathbf{C}^N$: $(z_1, z_2, \cdots, z_N)$.

b) The restricted functions $r|_{X \setminus x}$, $r|_{Y \setminus y}$ have no critical point, where $r(z) = \Sigma_{i=1}^N |z_i|^2$.

Theorem A. Under the above assumptions, there is a neighbor-

hood S *of* 0 *in* $\mathbb{C}$ *such that* (i) $f : f^{-1}(S) \setminus Y \longrightarrow S \setminus 0$ *is a* C^{∞} *fibre bundle* (ii) $\mathbb{R}^p f_*(\Omega_f^{\bullet})|S$ *are coherent* $\mathcal{O}_S$*-Modules* (iii) *there is a natural isomorphism*

$$H^p(\Omega_{f,x}^{\bullet}) \cong \mathbb{R}^p f_*(\Omega_f^{\bullet})_0.$$

To Milnor [10], Hamm [5], the (i) is due. The assertion (ii) can be proved as follows: Take a smaller open ball B' and set $X' = X \cap B'$. Then the argument of Brieskorn [3] shows the restriction map $\Gamma(f^{-1}(T),\Omega_f^{\bullet}) \longrightarrow \Gamma(f^{-1}(T) \cap X',\Omega_f^{\bullet})$ is quasi-isomorphic for any open subset of T provided S is sufficiently small. But this map is also quasi-nuclear $\Gamma(T,\mathcal{O}_S)$-homomorphism in the sense of Kiehl-Verdier [9]. From the fundamental theorem of [9] it follows immediately (ii). The quasi-isomorphicity above proves (iii) also.

Note that this method can prove the finite-dimensionality of $H^p(\Omega_{X,x}^{\bullet})$ of Bloom-Herrera [2]. (To prove this, one may only replace f by the map $(X,x) \longrightarrow$ (point,point) in the above argument.)

1.2. *Conservation of* (L) *under hypersurfacesection and some consequences of* (L). Throughout this section we fix (X,x) and $f \in \Gamma(X,\mathcal{O}_X)$ such that $(Y,y) = (f^{-1}(0),x)$ is a hypersurface-section of (X,x).

Lemma **1.2.1.** *If* (X,x) *satisfies the condition* (L), *then* (Y,y) *also satisfies the condition* (L).

Proof. By Lemma 1.1.5, $\mathcal{H}_x^q(\Omega_f^p)=0$ when $p+q < \dim X$. But $\mathcal{H}_x^0(\Omega_X^p)=0$ $(p<\dim X)$ implies the exactness of the sequence

$$(1.2.1) \quad 0 \longrightarrow \Omega_f^p \xrightarrow{f} \Omega_f^p \longrightarrow \Omega_Y^p \longrightarrow 0 \qquad p < \dim X$$

where Ω_Y^p should be regarded as sheaves over X. Thus we obtain the long exact sequence

$$\cdots\cdot \longrightarrow \mathcal{H}_x^q(\Omega_f^p) \longrightarrow \mathcal{H}_x^q(\Omega_f^p) \longrightarrow \mathcal{H}_y^q(\Omega_Y^p) \longrightarrow \cdots\cdot$$

from which it follows $\mathcal{H}_y^q(\Omega_Y^p) = 0$ when $p+q < \dim Y$.

Remark. Using Lemma 1.1.1, Lemma 1.1.2 and Nakayama's lemma, we can supply the argument above to prove the stronger statement : (X,x) satisfies (L) if and only if (Y,y) satisfies (L) and $\dim \mathcal{H}_y^0(\Omega_Y^n)=\dim \mathcal{H}_y^1(\Omega_Y^n)$ $(n=\dim Y)$

Corollary 1.2.1. *If (X,x) is a complete intersection, then it satisfies the condition (L).*

As indicated before, this follows from Lemma 1.2.1 and Hamm [5].

Consider the complex $\mathcal{H}_x^0(\Omega_X^\bullet)$ which is the torsion part of $\Omega_X^\bullet$ and set

$$"\Omega_X^\bullet = \Omega_X^\bullet / \mathcal{H}_x^0(\Omega_X^\bullet) \ .$$

We shall now prove the following sharper version of the Poincaré lemma:

<u>Lemma 1.2.2</u>. <u>If</u> (X,x) <u>satisfies the condition</u> (L), <u>then</u> $H^p(''\Omega_X^\bullet)=0$ <u>for</u> $0<p<\dim X$.

To prove this, [2] seems to be not adequate. We have therefore to rely on the earlier works of Herrera [6], [7]. When a semi-analytic set M is given in a real analytic manifold $\tilde{M}$, we define the sheaf ε_M^p as the quotient of the sheaf of C^∞ p-forms over $\tilde{M}$ by the subsheaf of p-forms inducing the null form on the non-singular part of M. There is a natural <u>onto</u> homomorphism $H^p(\Gamma(M, \varepsilon_m^\bullet)) \longrightarrow H^p(M, \mathbb{C})$.

<u>Proof of Lemma</u> 1.2.2. Take a contractible neighborhood U of x and consider the commutative diagram

$$\begin{array}{ccc} H^p(\Gamma(U, \varepsilon_U^\bullet)) & \longrightarrow & H^p(\Gamma(U\setminus x, \varepsilon_U^\bullet)) \\ \downarrow & & \downarrow \\ H^p(U, \mathbb{C}) & \longrightarrow & H^p(U\setminus x, \mathbb{C}) \end{array} .$$

This proves the composition $H^p(\Gamma(U, \varepsilon_U^\bullet)) \longrightarrow H^p(\Gamma(U\setminus x, \varepsilon_U^\bullet)) \longrightarrow H^p(U\setminus x, \mathbb{C})$ is zero. Note that there is a natural map $''\Omega_U^\bullet \longrightarrow \varepsilon_U^\bullet$ which induces the map $H^p(\Gamma(U, ''\Omega_U^\bullet)) \longrightarrow H^p(\Gamma(U, \varepsilon_U^\bullet))$. Composing this with the map above, we obtain the natural map $H^p(\Gamma(U, ''\Omega_U^\bullet)) \longrightarrow H^p(U\setminus x, \mathbb{C})$, or passing to the limit, the natural map $H^p(''\Omega_{X,x}^\bullet) \longrightarrow R^p i_* \mathbb{C}$ which is zero.

But this zero map can be factorized as follows: $H^p({}''\Omega^\bullet_{X,x}) \xrightarrow{\ \eta\ }$ $H^p(\iota_*\iota^*\Omega^\bullet_X) \xrightarrow{\ \varepsilon\ } R^p\iota_*\mathbb{C}$ $(\iota: X\setminus x \hookrightarrow X)$, where ε is injective for $p<\dim X$ as the edge homomorphism of spectral sequence $R^q\iota_*\iota^*\Omega^p_X \Longrightarrow R^{p+q}\iota_*\mathbb{C}$ because of the condition (L), and further η is also injective for $p<\dim X$ since ${}''\Omega^p_X \longrightarrow \iota_*\iota^*\Omega^p_X$ is isomorphism when $p<\dim X-1$ and monomorphism when $p=\dim X-1$. This proves $H^p({}''\Omega^\bullet_X)=0$ for $0<p<\dim X$.

From now on we suppose that (X,x) satisfies the condition (L), and we put $n=\dim X-1$, that is, $n=\dim Y$. By Lemma 1.1.5 $\mathcal{H}^0_x(\Omega^p_f)=\mathcal{H}^1_x(\Omega^p_f)=0$ for $p<n$. We thus have the isomorphisms

$$(1.2.2) \qquad \Omega^p_f \simeq \iota_*\iota^*\Omega^p_f \qquad p<n\ .$$

When $p=n$, $\mathcal{H}^0_x(\Omega^n_f)=0$, but $\mathcal{H}^1_x(\Omega^n_f)\neq 0$; so we have only the exact sequence

$$(1.2.3) \qquad 0 \longrightarrow \Omega^n_f \longrightarrow \iota_*\iota^*\Omega^n_f \longrightarrow \mathcal{H}^1_x(\Omega^n_f) \longrightarrow 0\ .$$

Now we denote by ${}'\Omega^\bullet_f$ the complex

$$0 \longrightarrow \Omega^0_f \xrightarrow{\ d\ } \Omega^1_f \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } \Omega^n_f \longrightarrow 0\ .$$

(In general Ω^p_f $(p>n)$ are not zero, so ${}'\Omega^\bullet_f$ does not coincide with $\Omega^\bullet_f$.) Then (1.2.2), (1.2.3) imply that

$$(1.2.4) \qquad H^p((\iota_*\iota^*\Omega^\bullet_f)_x) \cong H^p({}'\Omega^\bullet_{f,x}) \qquad p<n$$

and further that the sequence

(1.2.5) $0 \longrightarrow H^n('\Omega^\bullet_{f,x}) \longrightarrow H^n((\iota_*\iota^*\Omega^\bullet_f)_x) \longrightarrow \mathcal{H}^1_x(\Omega^n_f) \longrightarrow 0$

is exact. Since $\Omega^p_{f,x}$ $(p>n)$ are finite-dimensional, the cokernel of $H^p(\Omega^\bullet_{f,x}) \hookrightarrow H^p('\Omega^\bullet_{f,x})$ is always finite-dimensional, which, in view of Theorem A, shows that each $H^p('\Omega^p_{f,x})$ is finitely generated $\mathcal{O}_{\mathbb{C},0}$-module. Thus (1.2.3), (1.2.4) and Lemma 1.1.2 imply

Lemma 1.2.3. For all p, $H^p((\iota_*\iota^*\Omega^\bullet_f)_x)$ are finitely generated $\mathcal{O}_{\mathbb{C},0}$-modules.

Let G/f denote for short the quotient sheaf G/fG for an $\mathcal{O}_X$-Module G, and let us compute $\iota_*\iota^*\Omega^p_f/f$. First note there are isomorphisms $\iota_*\iota^*\Omega^p_f/f \cong \Omega^p_f/f \cong \Omega^p_Y$ $(p<n)$ by (1.2.2). Next consider the exact sequence $0 \longrightarrow \iota_*\iota^*\Omega^n_f \xrightarrow{f} \iota_*\iota^*\Omega^n_f \longrightarrow \iota_*\iota^*\Omega^n_Y \longrightarrow R^1\iota_*\iota^*\Omega^n_f$, where the last tern is isomorphic to $R^1\iota_*\iota^*\Omega^{n+1}_X$ since $\iota^*\Omega^n_f \cong \iota^*\Omega^{n+1}_X$. But this last space is, according to Lemma 1.1.1, the dual of $R^{n-1}\iota_*\iota^*\Omega^0_X = \mathcal{H}^n_x(\Omega^0_X)$ which is zero because (X,x) satisfies (L). We have thus proved

(1.2.6) $$\iota_*\iota^*\Omega^\bullet_f/f \cong \{0 \longrightarrow \Omega^0_Y \xrightarrow{d} \Omega^1_Y \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}_Y \xrightarrow{d} \iota_*\iota^*\Omega^n_Y \longrightarrow 0\} .$$

This gives now rise to the exact sequence of complexes

$$0 \longrightarrow {}''\Omega_Y^{\bullet} \longrightarrow \iota_*\iota^*\Omega_f^{\bullet}/f \longrightarrow \mathcal{H}_y^1(\Omega_Y^n) \longrightarrow 0$$

where the last term should be considered to be a complex concentrated in the degree n place. From this sequence and Lemma 1.2.2 it follows that

$$(1.2.7) \qquad H^p((\iota_*\iota^*\Omega_f^{\bullet}/f)_x) = 0 \qquad 0 < p < n .$$

and that the sequence

$$(1.2.8)\ 0 \longrightarrow H^n({}''\Omega_{Y,y}^{\bullet}) \longrightarrow H^n((\iota_*\iota^*\Omega_f^{\bullet}/f)_x) \longrightarrow \mathcal{H}_y^1(\Omega_Y^n) \longrightarrow 0$$

is exact.

Now consider the long exact sequence

$$\cdots \longrightarrow H^p((\iota_*\iota^*\Omega_f^{\bullet})_x) \xrightarrow{f} H^p((\iota_*\iota^*\Omega_f^{\bullet})_x) \longrightarrow H^p((\iota_*\iota^*\Omega_f^{\bullet}/f)_x) \longrightarrow \cdots$$

By (1.2.7), Lemma 1.2.3 and Nakayama's lemma, we have that $H^p((\iota_*\iota^*\Omega_f^{\bullet})_x) = 0$ for $p<n$, $H^n((\iota_*\iota^*\Omega_f^{\bullet}/f)_x) \cong H^n((\iota_*\iota^*\Omega_f^{\bullet})_x)/f$, and that $H^n((\iota_*\iota^*\Omega_f^{\bullet})_x)$ is <u>torsion free</u> $\mathcal{O}_{\mathbb{C},0}$-moudle. Consider the exact sequence $0 \longrightarrow \iota_*\iota^*\Omega_f^{\bullet}/f \longrightarrow \iota_*\iota^*\Omega_Y^{\bullet} \longrightarrow \mathcal{H}_y^1(\Omega_Y^{n-1}) \longrightarrow 0$ where the last complex should be considered to concentrate in the degree $(n-1)$ place. (Note that (Y,y) also satisfies (L) according to Lemma 1.2.1.) By the long exact sequence associated with this, we obtain the exact sequence

$$(1.2.9) \quad 0 \longrightarrow H^{n-1}((\iota_*\iota^*\Omega_Y^{\bullet})_y) \longrightarrow \mathcal{H}_y^1(\Omega_Y^{n-1})$$

$$\longrightarrow H^n((\iota_*\iota^*\Omega_f^{\bullet})_x)/f \longrightarrow H^n((\iota_*\iota^*\Omega_Y^{\bullet})_y) \longrightarrow 0 .$$

From the ecaxt sequence $0 \longrightarrow {''\Omega}^{\bullet}_{Y} \longrightarrow \iota_{*}\iota^{*}\Omega^{\bullet}_{Y,y} \longrightarrow \mathcal{H}^{1}_{y}(\Omega^{\bullet}_{Y}) \longrightarrow$ 0 it follows also the exact sequence

$$(1.2.10) \quad 0 \longrightarrow H^{n-1}((\iota_{*}\iota^{*}\Omega^{\bullet}_{Y})_{y}) \longrightarrow H^{n-1}(\mathcal{H}^{1}_{y}(\Omega^{\bullet}_{Y})) \longrightarrow H^{n}({''\Omega}^{\bullet}_{Y,y})$$

$$\longrightarrow H^{n}((\iota_{*}\iota^{*}\Omega^{\bullet}_{Y})_{y}) \longrightarrow H^{n}(\mathcal{H}^{1}_{y}(\Omega^{\bullet}_{Y})) \longrightarrow 0$$

To sum up all proved so far, we obtain

Theorem 1.2.1. *Let* (X,x), f, (Y,y), ${'\Omega}^{\bullet}_{f}$, ${''\Omega}^{\bullet}_{Y}$ *be as above and assume that* (X,x) *satisfies the condition* (L). *Then* $H^{p}({'\Omega}^{\bullet}_{f,x})=0=H^{p}((\iota_{*}\iota^{*}\Omega^{\bullet}_{f})_{x})$ $(p<n=\dim Y)$, *and* $H^{n}((\iota_{*}\iota^{*}\Omega^{\bullet}_{f})x)$ *is torsion free* $\mathcal{O}_{\mathbb{C},0}$*-module; moreover the exact sequences* (1.2.5), (1.2.9), (1.2.10) *are valid*.

Remark. The exact sequence (1.2.10) is always valid if (Y,y) satisfies the condition (L), even in case there is not an (X,x) of which (Y,y) is a hypersurfacesection.

Remark. By Theorem A and (1.2.5), the Milnor number μ of (Y,y) is equal to the rank of $H^{n}((\iota_{*}\iota^{*}\Omega^{\bullet}_{f})_{x})$ over $\mathcal{O}_{\mathbb{C},0}$ provided $n=\dim Y \geqq 2$. Since this module is torsion free, it follows from (1.2.9)

$$\mu = \dim \mathcal{H}^{1}_{y}(\Omega^{n-1}_{Y}) + \dim H^{n}((\iota_{*}\iota^{*}\Omega^{\bullet}_{Y})_{y}) - \dim H^{n-1}((\iota_{*}\iota^{*}\Omega^{\bullet}_{Y})_{y})$$

Further, in case $n \geqq 3$, $0 \longrightarrow \mathcal{H}^{1}_{y}(\Omega^{n-1}_{Y}) \longrightarrow \mathcal{H}^{2}_{y}(\Omega^{n-1}_{f}) \longrightarrow \mathcal{H}^{2}_{y}(\Omega^{n-1}_{f}) \longrightarrow \mathcal{H}^{2}_{y}(\Omega^{n-1}_{Y}) \longrightarrow 0$ is exact, so $\dim \mathcal{H}^{1}_{y}(\Omega^{n-1}_{Y})$ may be replaced $\dim R^{1}\iota_{*}\iota^{*}\Omega^{n-1}_{Y}$. Thus the formula $(*)$ of [11] is valid for any isolated singularity which is a complete intersection.

Remark. Let (X,x), (Y,y) be as in Theorem 2.2.1 and assume X is smooth. In this case there are isomorphisms which are, in a way, canonical:

$$\mathcal{H}^0_y(\Omega^{n+1}_Y) \cong \mathcal{H}^1_y(\Omega^n_Y) \cong \cdots \cong \mathcal{H}^{n-1}_y(\Omega^2_Y)$$

$$\mathcal{H}^0_y(\Omega^n_Y) \cong \mathcal{H}^1_y(\Omega^{n-1}_Y) \cong \cdots \cong \mathcal{H}^{n-1}_y(\Omega^1_Y) .$$

Furtheremore the following conditions are equivalent: (i) $H^n(\Omega^{\bullet}_{Y,y})=0$ (ii) $H^n(''\Omega^{\bullet}_{Y,y})=0$ (iii) $\dim H^{n-1}((\iota_*\iota^*\Omega^{\bullet}_Y)_y)=\dim H^n((\iota_*\iota^*\Omega^{\bullet}_Y)_y)$; according to Saito [13] these are equivalent to the quasi-homogenith of (Y,y).

Part II. $\mathbb{C}^*$-actions over isolated singularity

2.1. Gysin sequence. Let $\mathbb{C}^* = \mathbb{C} \setminus 0$. A $\mathbb{C}^*$-action over an isolated singularity (X,x) is a family $T(c)$, $c \in \mathbb{C}^*$ of analytic homemorphisms of X onto itself satisfying that $T(c)x = x$, $T(c)T(c') = T(cc')$ $(c, c' \in \mathbb{C}^*)$, and that $T : X \times \mathbb{C}^* \ni (z,c) \to T(c)z \in X$ is analytic. Throughout Part II we will requir the following to be satisfied:

Assumption The constants are the only invariant elements of $\mathcal{O}_{X,x}$ under the action T.

The meaning of this is the following: Let ξ be the generating vector field of this action and L_ξ its Lie derivative. Then the assumption implies that L_ξ induces automorphsims on $\Omega^p_{X,x}$ for $p > 0$. (More precisely, if $0 \neq \omega \in \Omega^p_{X,x}$ $(p > 0)$ and if $T(c)^*\omega = c^m\omega$ $(c \in \mathbb{C}^*)$, then such an m is either always positive, or always negative. Note that in the positive case T can be extended to $\mathbb{C}$-action.) If we denote the interior multiplication of ξ by $i(\xi)$, then $L_\xi = i(\xi)d + di(\xi)$. From this identity we obtain

Lemma 2.1.1. Under the above assumption, the complexes $\Omega^{\bullet}_X$, $\mathcal{H}^0_x(\Omega^{\bullet}_X)$, $''\Omega^{\bullet}_X$ are all acyclic, and the sequence

$$\cdots \to \Omega^p_X \xrightarrow{i(\xi)} \Omega^{p-1}_X \xrightarrow{i(\xi)} \cdots \xrightarrow{i(\xi)} \Omega^0_X \xrightarrow{\alpha} (\iota_x)_*\mathbb{C} \longrightarrow 0$$

is exact, where ι_x denotes the inclusion $x \hookrightarrow X$, and α the average map $\Omega^0_{X,x} \ni f \to \int_0^1 T(\exp(2\pi i\theta))^*f \, d\theta \in \mathbb{C} = (\iota_x)_*\mathbb{C}_x$.

Consider the two spectral sequences $E_r^{p,q}(X,x)$ $'E_r^{p,q}(X,x)$ whose E_1-terms are $R^q\iota_x\iota^*(\Omega_X^p), \mathcal{H}_x^{q+1}(\Omega_X^p)$ respectively, where $\iota : X \setminus x \hookrightarrow X$. The limit of $E^{p,q}(X,x)$ is $R^{p+q}\iota_*\mathbb{C}$, while that of $'E_r^{p,q}(X,x)$ is $\mathcal{H}_x^{p+q+1}(\mathbb{C})$. It is evident that the natural maps $E_1^{p,q}(X,x) \to {}'E_1^{p,q}(X,x)$ are isomorphisms when $q > 0$. Since $H^p({}''\Omega_X^{\bullet}) = 0$ $(p > 0)$ by Lemma 2.1.1, it holds also $E_2^{p,0}(X,x) \overset{\sim}{\to} {}'E_2^{p,0}(X,x)$ by the exact sequence (1.2.10) and the remark following Theorem 1.2.1 provided (X,x) satisfies the condition (L).

Corollary 2.1.1. If (X,x) admits a $\mathbb{C}^*$-action and if it satisfies the condition (L), then the natural maps $E_2^{p,q}(X,x) \to {}'E_2^{p,q}(X,x)$ are isomorphisms except for the case $p = q = 0$.

If we set $\Omega_\xi^p = i(\xi)\Omega_X^{p-1}$ $(p > 0)$ and $\Omega_\xi^{-1} = (\iota_x)_*\mathbb{C}$, then we have the short exact sequences $0 \to \Omega_\xi^p \to \Omega_X^p \to \Omega_\xi^{p-1} \to 0$ $(p \geq 0)$. From these, we obtain the long exact sequences of Gysin type

$$(2.1.1)\quad \begin{array}{l} 0 \to \mathcal{H}_x^0(\Omega_\xi^p) \to \mathcal{H}_x^0(\Omega_X^p) \to \mathcal{H}_x^0(\Omega_\xi^{p-1}) \overset{\delta}{\to} \cdots \\ \cdots \overset{\delta}{\to} \mathcal{H}_x^q(\Omega_\xi^p) \to \mathcal{H}_x^q(\Omega_X^p) \to \mathcal{H}_x^q(\Omega_\xi^{p-1}) \overset{\delta}{\to} \cdots . \end{array}$$

These play key role in the study of this section. Before proceeding, we have to give a more explicit description to the connecting homomorphism δ. Since $i(\xi)$ is an anti-derivation, $\Omega_\xi^{\bullet} = \sum_p \Omega_\xi^p$ is the sheaf of graded algebra (by exterior multiplication), so $A = \sum_{p,q} A^{p,q}$, where $A^{p,q} = R^q\iota_*\iota^*\Omega_\xi^p$, is a bigraded algebra by the cup-multiplication. Thus $1 \in A^{0,0}$, $\delta 1 \in A^{1,1}$. We set $ch(\xi) = \delta\cdot 1$ and call it the Chern class of ξ (or of the action T). Note that $B = \sum_{p,q} B^{p,q}$, where $B^{p,q} = \mathcal{H}_x^{q+1}(\Omega_\xi^p)$, is a bigraded A-module such that the natural map $A \to B$ is a A-homomorphism. Now it is easy to prove

Lemma 2.1.2. The connecting homomorphism $\delta: B^{p,q} \to B^{p+1,q+1}$ is nothing but the multiplication of $ch(\xi)$ (up to sign).

Here we shall give a precise formulation of the Serre duality given in Lemma 1.1.1. Regard the canonical generator γ of $R^{2n-1}\iota_*\mathbb{Z}$ ($\cong \mathbb{Z}$) as an element of $R^{2n-1}\iota_*\mathbb{C}$ and define ε' : $R^{n-1}\iota_*\iota^*\Omega_X^n \to \mathbb{C}$ by $\varepsilon'(\alpha)\gamma = \varepsilon(\alpha)$ where $n = \dim X$, ε is the edgehomomorphism of $R^{n-1}\iota_*\iota^*\Omega_X^n$ onto $R^{2n-1}\iota_*\mathbb{C}$. Let $\alpha \in R^q\iota_*\iota^*\Omega_X^p$ $\beta \in R^{n-1-q}\iota_*\iota^*\Omega_X^{n-p}$ and set

$$\langle\alpha,\beta\rangle = \varepsilon'(\alpha \cup \beta)$$

where $\cup$ denote the cup-multiplication. This pairing gives the duality stated in Lemma 1.1.2.

Now note $\iota^*\Omega_\xi^n = 0$, so that $\iota^*\Omega_X^n \cong \iota^*\Omega_\xi^{n-1}$. Thus we obtain the isomorphism $\eta: A^{(n-1),(n-1)} = R^{n-1}\iota_*\iota^*\Omega_\xi^{n-1} \cong R^{n-1}\iota_*\iota^*\Omega_X^n$. We see then that the value $\varepsilon'\circ\eta(ch(\xi)^{n-1})$ is not zero. For, using a suitable embedding of X into its Zariski tangent at x, we can construct a real valued function ψ on $X\setminus x$ such that $i\partial\bar{\partial}\psi$ is positive definite, $L_\xi(\psi) = \sqrt{-1}$, and that $\psi^{-1}(c)$ is compact for $c \in \mathbb{R}$. It can be shown then

$$\varepsilon'\circ\eta(ch(\xi)^{n-1}) = \sqrt{-1}^n\int_{\psi^{-1}(c)} \partial\psi \wedge (\partial\bar{\partial}\psi)^{n-1},$$

where the right hand side is obviously not zero. In view of Lemma 2.1.2, this fact proves

Lemma 2.1.3. The iterated connecting homomorphism $\delta^{\ell} : \mathcal{H}_x^p(\Omega_\xi^{p-1}) \to \mathcal{H}_x^{p+\ell}(\Omega_\xi^{p+\ell-1})$ is not zero if $p+\ell \leq \dim X$.

From now on we assume that (X,x) satisfies the condition (L). Then $\mathcal{H}_x^q(\Omega_X^p) = 0$ if $p+q<n = \dim X$, and further, by Lemma 1.1.1, $\mathcal{H}_x^q(\Omega_X^p) = 0$ if $p+q>n+1$, $q<n$. In view of the sequence (2.1.1), these imply

Lemma 2.1.4. The connecting homomorphism $\delta : \mathcal{H}_x^q(\Omega_\xi^{p-1}) \to \mathcal{H}_x^{q+1}(\Omega_\xi^p)$ is (i) isomorphism in case $p+q < n-1$ or in case $p+q>n+1$, $q<n-1$, (ii) injective in case $p+q = n-1$, (iii) surjective in case $p+q = n+1$, $q<n-1$.

Trivially $\mathcal{H}_x^q(\Omega_\xi^{-1}) = 0$ if $q \neq 0$. We know also $\mathcal{H}_x^q(\Omega_\xi^{n-1}) \cong \mathcal{H}_x^q(\Omega_X^n) = 0$ for $2 \leq q \leq n-1$. Lemma 2.1.4, combined with these, proves

Lemma 2.1.5. $\mathcal{H}_x^q(\Omega_\xi^p) = 0$ if $p+q<n$, $q-p \neq -1$ or if $p+q>n$, $q<p+1$. Let $\mu = [n/2]-1$, $\nu = -[-n/2]$. Then there are isomorphisms

(2.1.2) $\mathbb{C} \cong \mathcal{H}_x^0(\Omega_\xi^{-1}) \cong \mathcal{H}_x^1(\Omega_\xi^0) \cong \ldots \cong \mathcal{H}_x^{\mu+1}(\Omega_\xi^\mu)$

(2.1.3) $\mathcal{H}_x^{\nu+1}(\Omega_\xi^\nu) \cong \mathcal{H}_x^{\nu+2}(\Omega_\xi^{\nu+1}) \cong \ldots \cong \mathcal{H}_x^{n-1}(\Omega_\xi^{n-2})$.

Observe that the iterated connecting homomorphism $\delta^{\nu-\mu} : \mathcal{H}_x^{\mu+1}(\Omega_\xi^\mu) \to \mathcal{H}_x^{\nu+1}(\Omega_\xi^\nu)$ is injective since $\dim \mathcal{H}_x^{\mu+1}(\Omega_\xi^\mu) = 1$ by (2.1.2) and since this $\delta^{\nu-\mu}$ is not zero by Lemma 2.1.3. We shall discuss the consequence from this fact and the vanishing of

$\mathcal{H}^q_x(\Omega^p_\xi)$ in Lemma 2.1.5. For this purpose, we separate the case $n = 2m$ and the case $n = 2m+1$. a) Case $n = 2m$. By the first satement of Lemma 2.1.5 and the sequence (2.1.1) we obtain isomorphisms $\mathcal{H}^q_x(\Omega^{p+1}_X) \xrightarrow{\sim} \mathcal{H}^q_x(\Omega^p_\xi)$ in case $p+q = n$, $0 < q \leqq m$ and $\mathcal{H}^q_x(\Omega^p_\xi) \xrightarrow{\sim} \mathcal{H}^q_x(\Omega^p_X)$ in case $p+q = n$, $0 < q < m$. Since $\mathcal{H}^m_x(\Omega^{m-1}_\xi) \xrightarrow{\delta} \mathcal{H}^{m+1}_x(\Omega^m_\xi)$ is injective as observed above, we have also by (2.1.1) the isomorphism $\mathcal{H}^m_x(\Omega^m_\xi) \simeq \mathcal{H}^m_x(\Omega^m_X)$. Combining these, we obtain isomorphism $\mathcal{H}^q_x(\Omega^{p+1}_X) \simeq \mathcal{H}^q_x(\Omega^p_X)$ induced by $i(\xi):\Omega^{p+1}_X \to \Omega^p_X$ for p,q such that $p+q = n$, $0 < q \leqq m$. But the restriction $0 < q \leqq m$ can be replaced by $0 < q < n$ according to the Serre duality of Lemma 1.1.1. b) Case $n = 2m + 1$. The isomorphisms $\mathcal{H}^q_x(\Omega^{p+1}_X) \xrightarrow{\sim} \mathcal{H}^q_x(\Omega^p_X)$, for p,q such that $p+q = n$, $0<q<m+1$, can immedialely be obtained as in the previous case. To prove $\mathcal{H}^{m+1}_x(\Omega^{m+1}_X) \xrightarrow{\sim} \mathcal{H}^{m+1}_x(\Omega^m_X)$, consider the following two exact sequences which are some parts of (2.1.1) :

$$0 \to \mathcal{H}^{m+1}_x(\Omega^{m+1}_X) \to \mathcal{H}^{m+1}_x(\Omega^m_\xi) \xrightarrow{\delta} \mathcal{H}^{m+2}_x(\Omega^{m+1}_\xi)$$
$$\Vert$$
$$0 \to \mathcal{H}^m_x(\Omega^{m-1}_\xi) \xrightarrow{\delta} \mathcal{H}^{m+1}_x(\Omega^m_\xi) \to \mathcal{H}^{m+1}_x(\Omega^m_X).$$

Since $\delta^2 : \mathcal{H}^m_x(\Omega^{m-1}_\xi) \to \mathcal{H}^{m+2}_x(\Omega^{m+1}_\xi)$ is injective as was observed above, the composed map $\mathcal{H}^{m+1}_x(\Omega^{m+1}_X) \to \mathcal{H}^{m+1}_x(\Omega^m_\xi) \to \mathcal{H}^{m+1}_x(\Omega^m_X)$ is also injective. But by the duality of Lemma 1.1.1 $\mathcal{H}^{m+1}_x(\Omega^{m+1}_X)$, $\mathcal{H}^{m+1}_x(\Omega^m_X)$ have the same dimension. Thus this map $\mathcal{H}^{m+1}_x(\Omega^{m+1}_X) \to \mathcal{H}^{m+1}_x(\Omega^m_X)$ is also isomorphism. Again by the Serre duality we have $\mathcal{H}^q_x(\Omega^{p+1}_X) \xrightarrow{\sim} \mathcal{H}^q_x(\Omega^p_X)$ for $p+q = n$, $0 < q < n$.

To sum up,

<u>Theorem 2.1.1</u>. <u>Assume that</u> (X,x) <u>satisfies the condition</u> (L). <u>Then the interior multiplication</u> $i(\xi):\Omega_X^{p+1} \to \Omega_X^p$ <u>induces the isomorphism</u> $\mathcal{H}_x^q(\Omega_X^{p+1}) \to \mathcal{H}_x^q(\Omega_X^p)$ for $0 < p < \dim X$ <u>where</u> $q = \dim X - p$. <u>Further</u> $\mathcal{H}_x^q(\Omega_\xi^p) = 0$ <u>if</u> $p+q \neq \dim X$, $q-p \neq 1$, $0 < q < \dim X$, <u>and</u> $\delta^{\nu-\mu}:\mathcal{H}_x^{\mu+1}(\Omega_\xi^\nu) \to \mathcal{H}_x^{\nu+1}(\Omega_\xi^\nu)$ <u>is isomorphic</u>, <u>where</u> μ,ν <u>are as in Lemma</u> 2.1.5.

Finally we remark this theorem provides us a clear insight into the structure of $E_2^{p,q}(X,x)$. Because of the condition (L) we have $E_2^{p,q}(X,x) = 0$ if $p+q \neq n-1$, $p+q \neq n$, $p+q \neq 0$, $p+q \neq 2n-1$ where $n = \dim X$. (For the vanishing of $E_2^{p,q}(X,x)$ when $q = 0$ or $q = n-1$, see Lemma 1.1.3 and the remark following Lemma 1.1.1.) Obviously $E_2^{0,0}(X,x) \cong \mathbb{C}$ $E_2^{n,n-1}(X,x) \cong \mathbb{C}$. If $p+q = n-1$, Corollary 2.1.1 implies then, $E_2^{p,q}(X,x) \cong \mathrm{Ker}(\mathcal{H}_x^{q+1}(\Omega_X^p) \xrightarrow{d} \mathcal{H}_x^{q+1}(\Omega_X^{p+1}))$, $E_2^{p+1,q}(X,x) \cong \mathrm{Cok}(\mathcal{H}_x^{q+1}(\Omega_X^p) \xrightarrow{d} \mathcal{H}_x^{q+1}(\Omega_X^{p+1}))$. In view of the identity $L_\xi = i(\xi)d+di(\xi)$, Theorem 2.1.1 now proves

$$(2.1.4)\quad E_2^{p,q}(X,x) \cong \mathrm{Ker}(\mathcal{H}_x^{q+1}(\Omega_X^p) \xrightarrow{L_\xi} \mathcal{H}_x^{q+1}(\Omega_X^p))$$

$$(2.1.5)\quad E_2^{p+1,q}(X,x) \cong \mathrm{Cok}(\mathcal{H}_x^{q+1}(\Omega_X^{p+1}) \xrightarrow{L_\xi} \mathcal{H}_x^{q+1}(\Omega_X^{p+1}))$$

$$(\cong \mathrm{Ker}(\mathcal{H}_x^{q+1}(\Omega_X^{p+1}) \xrightarrow{L_\xi} \mathcal{H}_x^{q+1}(\Omega_X^{p+1}))).$$

where $p+q = n-1$. Note also these two groups are isomorphic under the map induced by $i(\xi):\Omega_X^{p+1} \to \Omega_X^p$.

2.2. Characteristic functions. In this section we also suppose that (X,x), T are as in Section 2.1, that (X,x) satisfies the condition (L), and that the action T fulfills the assumption mentioned at the beginning of Section 2.1. If V is a certain cohomology group attached to (X,x), we denote by $T(c)^*|V$ the automorphism of V induced by the map $T(c)$. (But, in case it is obviously understood from the context what this V is, we simply write $T(c)^*$ for $T(c)^*|V$.) According to this convention we set

$$\chi_X^q(t) = \mathrm{Trace}\,(T(t)^*|\mathcal{H}_x^q(\Omega_X^{n-q}))$$

where $0\leqq q<n = \dim X$. That is, $\chi_X^q(t)$, $0\leqq q<n$ are the characters of the representation of $\mathbb{C}^*$ over $\mathcal{H}_x^q(\Omega_X^{n-q})$. When regarded as functions in t, they are rational and have poles only at $t = 0$. In view of the duality of Lemma 1.1.1, we have

$$\chi_X^q(t) = \chi_X^{n-q+1}(t^{-1}) \qquad 2\leqq q\leqq n-1,$$

so it will be reasonable to set

$$\chi_X^n(t) = \chi_X^1(t^{-1}), \quad \chi_X^{n+1}(t) = \chi_X^0(t^{-1}).$$

We set now

$$\chi_X(s,t) = \sum_{q=0}^{n+1} \chi_X^q(t)\, s^q$$

and call it the characteristic function of the action T.

Note the isomorphism $\mathcal{H}^q_x(\Omega^{p+1}_X) \xrightarrow{\sim} \mathcal{H}^q_x(\Omega^p_X)$ (p+q = n, 0<q<n) of Theorem 2.1.1 is $\mathbb{C}^*$-equivariant. Thus

$$\chi^q_X(t) = \text{Trace } (T(t)^*|\mathcal{H}^q_x(\Omega^{n-q+1}_X)) \qquad 0<q<n.$$

This identity will be frequently used in the following discussion.

Now we shall study how the characteristic function changes when one makes a hypersurfacesection which is compatible with $\mathbb{C}^*$-action. Let f be analytic function on X such that $df_z \neq 0$ for $z \in X \setminus x$, $T(c)^*f = c^d f$ ($c \in \mathbb{C}^*$) where d is a positive integer. (The assumption $d>0$ implies that, if $0 \neq \omega \in \Omega^p_X$ ($p>0$) and if $T(c)^*\omega = c^m\omega$, then $m>0$, as was remarked at the beginning of Section 2.1. Thus, in particular, $\chi^0_X(t)$ is a polynomial in t without constant term. This kind of remarks will often be applied below.) As in Part I, we denote by (Y,y) the hypersurfacesection defined by f; that is, $Y = f^{-1}(0)$, $y=x$. Since T induces naturally a $\mathbb{C}^*$-action on (Y,y), we can define the characters $\chi^0_Y(t)$, $\chi^1_Y(t),\ldots,$ $\chi^n_Y(t)$, and the characteristic function $\chi_Y(s,t)$ of the induced action, which we shall denote also by T. The sheaves Ω^p_f being defined as in Part I, we set further

$$\chi^q_f(t) = \text{Trace } (T(t)^*|\mathcal{H}^q_x(\Omega^{n-q}_f)) \qquad 0 \leq q<n$$

$$\chi^n_f(t) = t^{-d}\,\chi^1_f(t^{-1})$$

$$\chi^{n+1}_f(t) = t^{-d}\chi^0_f(t^{-1}).$$

Note that, just as we deduced $\mathcal{H}^q_x(\Omega^p_f) = 0$ $(p+q<n)$ from $\mathcal{H}^q_x(\Omega^p_X) = 0$ $(p+q<n)$ in the proof of Lemma 1.1.5, so we can deduce $\mathcal{H}^q_x(\Omega^p_f) = 0$ $(p+q>n,\ q<n)$ from $\mathcal{H}^q_x(\Omega^p_X) = 0$ $(p+q>n+1,\ q<n)$. Thus, from the short exact sequence $0 \to \Omega^{n-q-1}_f \xrightarrow{df} \Omega^{n-q}_X \to \Omega^{n-q}_f \to 0$, it follows the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{H}^q_x(\Omega^{n-q}_X) & \longrightarrow & \mathcal{H}^q_x(\Omega^{n-q}_f) & \longrightarrow & \mathcal{H}^{q+1}_x(\Omega^{n-q-1}_f) & \longrightarrow & \mathcal{H}^{q+1}_x(\Omega^{n-q}_X) & \longrightarrow 0 \\
 & \downarrow T(c)^* & & \downarrow T(c)^* & & \downarrow c^d\ T(c)^* & & \downarrow T(c)^* & \\
0 \longrightarrow & * & \longrightarrow & * & \longrightarrow & * & \longrightarrow & * & \longrightarrow 0
\end{array}$$

where the bottom row is identical with the top row. From this and the Serre duality we obtain for $0 \leqq q \leqq n$

$$(2.2.1) \qquad t^d \chi^{q+1}_f(t) - \chi^q_f(t) = \chi^{q+1}_X(t) - \chi^q_X(t).$$

Here we have used the isomorphism in Theorem 2.1.1 of course. Using the exact sequence $0 \to \Omega^{n-q}_f \to \Omega^{n-q}_f \to \Omega^{n-q}_Y \to 0$ and reasoning similarly, we obtain also

$$(2.2.2) \qquad (t^d-1)\ \chi^q_f(t) = \chi^{q-1}_Y(t) - \chi^q_Y(t)$$

for $0<q\leqq n$. In view of the identities $\chi^{n+1}_f(t) = t^{-d}\chi^0_f(t^{-1})$, $\chi^{n+1}_X(t) = \chi^0_X(t^{-1})$, $\chi^n_Y(t) = \chi^0_Y(t^{-1})$, we can reformulate (2.2.1), (2.2.2) as follows

$$(2.2.1)' \qquad t^d(\chi_f(s,t) - \chi^0_f(t)) - s(\chi_f(s,t) - s^{n+1}\, t^{-d}\chi^0_f(t^{-1}))$$
$$= (\chi_X(s,t) - \chi^0_X(t)) - s(\chi_X(s,t) - s^{n+1}\chi^0_X(t^{-1}))$$

$$(2.2.2)' \qquad (t^d-1)(\chi_f(s,t) - \chi_f^0(t) - s^{n+1}\, t^{-d}\chi_f^0(t^{-1}))$$

$$= s(\chi_Y(s,t) - s^n\chi_Y^0(t^{-1})) - (\chi_Y(s,t) - \chi_Y^0(t))$$

where we have set $\chi_f(s,t) = \sum_{q=0}^{n+1} \chi_f^q(t)\, s^q$. From these, we can easily obtain

<u>Theorem 2.2.1</u> <u>The notation and the assumption being as above, it holds</u>

$$(2.2.3) \qquad \chi_f^0(t) = \chi_X^0(t) + \chi_Y^0(t)$$

$$(2.2.4) \qquad s(\chi_Y(s,t) - s^n\chi_Y^0(t^{-1})) - t^d(\chi_Y(s,t) - \chi_Y^0(t))$$

$$= (t^d-1)(\chi_X(s,t) - \chi_X^0(t) - s^{n+1}\chi_X^0(t^{-1})).$$

<u>Proof</u>. Setting $s=1$ in (2.2.1)', (2.2.2)' we have

$$\chi_f^0(t) - \chi_X^0(t) - \chi_Y^0(t) = \chi_f^0(t^{-1}) - \chi_X^0(t^{-1}) - \chi_Y^0(t^{-1}).$$

But $\chi_f^0(t)$, $\chi_Y^0(t)$ are polynomials without constant term just as $\chi_X^0(t)$ is a polynomial without constant term. Thus (2.2.3) is proved. The formula (2.2.4) follows easily from (2.2.1)', (2.2.2)' and (2.2.3).

Since $\chi_X^0(t) = \chi_X(0,t)$, by (2.2.4) one can know $\chi_Y(s,t)$ when he knows $\chi_Y^0(t)$, $\chi_X(s,t)$; further to know $\chi_Y^0(t)$ it suffices to know $\chi_f^0(t)$ in view of (2.2.3). This will be of particular importance when one wants to compute the characteristic functions for quasi-homogeneous complete intersections.

Now we shall study some interesting consequences from Theorem 2.2.1. Let ζ be a d-th root of unity. Setting $t = \zeta$ in (2.2.4), we obtain

$$\chi_Y(s,\zeta) = (\chi_Y^0(\zeta) - s^{n+1}\chi_Y^0(\zeta^{-1}))/(1-s).$$

Since the left hand side is a polynomial in s, we have

(2.2.5) $$\chi_Y^0(\zeta) = \chi_Y^1(\zeta) = \cdots = \chi_Y^n(\zeta) = \chi_Y^0(\zeta^{-1}) \qquad (\zeta^d = 1).$$

This means that the automorphisms of $\mathcal{H}_y^q(\Omega_Y^{n-q-1})$, $0 \leq q < n-1$ induced by $\varphi = T(\exp(2\pi i/d))$ have all the same characteristic polynomial. On the other hand, the exact sequence (1.2.9) (with n replaced by n-1) shows that φ induces the automorphisms having the same characteristic polynomial over $\mathcal{H}_y^1(\Omega_Y^{n-2})$ and $H^n((\iota_*\iota^*\Omega_f^{\bullet})_x)/f$. Since $H^n((\iota_*\iota^*\Omega_f^{\bullet})_x)$ is a torsion free $\mathcal{O}_{\mathbb{C},0}$-module and since the cokernels of the inclusions $H^n(\Omega_{f,x}^{\bullet}) \hookrightarrow H^n(\Omega_{f,x}^{\bullet}) \hookrightarrow H^n((\iota_*\iota^*\Omega_f^{\bullet})_x)$ are all finite-dimensional, this polynomial coincides with the characteristic polynomial of the monodromy of the Milnor fibering given in Theorem A of §1.1. Thus we have proved

Lemma 2.2.1. The characteristic polynomials of the automorphisms of $\mathcal{H}_y^q(\Omega_Y^{n-q-1})$ $(0 \leq q < n-1)$ induced by $T(\exp(2\pi i/d))$ are identical with that of the monodromy of the Milnor fibering defined by f. In particular

(2.2.6) $\dim H^{n-1}(X\setminus Y, \mathbb{C})$ $(=\dim H^n(X\setminus Y, \mathbb{C}))$

$$= d^{-1} \sum_{m=0}^{d-1} \chi_Y^0 (\exp(2\pi i m/d)).$$

The last formula can be proved by the Wang sequence applied to the fibering. (See Milnor [10].)

We can now prove the degeneracy of $E_2^{p,q}(Y,y)$ $E_2^{p,q}(X,x)$. By Lemma 1.1.1 together with the remark following that and by (2.1.4), (2.1.5), the sums $\sum_{p+q=n-1} \dim E_2^{p,q}(X,x)$, $\sum_{p+q=n-2} \dim E_2^{p,q}(Y,y)$ are equal to the constant terms $\chi_X(1,t) - \chi_X^0(t) - \chi_X^{n+1}(t)$, $\chi_Y(1,t) - \chi_Y^0(t) - \chi_Y^n(t)$ in their laurent expansions at $t=0$ respectively. Set $s=1$ in (2.2.4) and observe the resulting identity:

$$\chi_Y^0(t) + (t^d/1-t^d)\ \chi^0(t^{-1})$$

$$= (\chi_X(1,t)-\chi_X^0(t)-\chi_X^0(t^{-1}))-(\chi_Y(1,t)-\chi_Y^0(t)-\chi_Y^0(t^{-1})).$$

Since $\chi_Y^0(t)$ is a polynomial without constant term, we obtain by comparing the constant terms of both sides

$$\sum_{p+q=n-1} \dim E_2^{p,q}(X,x) + \sum_{p+q=n-2} \dim E_2^{p,q}(Y,y)$$

$$= d^{-1} \sum_{m=0}^{d-1} \chi_Y^0(\exp(2\pi i\ m/d))$$

$$= \dim H^{n-1}(X\setminus Y, \mathbb{C}).$$

From this it follows the inequality $\dim H^{n-1}(X\setminus x, \mathbb{C}) + \dim H^{n-2}(Y\setminus y, \mathbb{C}) \leq \dim H^{n-1}(X\setminus Y, \mathbb{C})$. But the opposite inequality is obvious from the standard exact sequence

(2.2.7) $$0 \to H^{n-1}(X\setminus x, \mathbb{C}) \to H^{n-1}(X\setminus Y, \mathbb{C}) \to H^{n-2}(Y\setminus y, \mathbb{C})$$

$$\to H^{n}(X\setminus x, \mathbb{C}) \to H^{n}(X\setminus Y, \mathbb{C}) \to H^{n-1}(Y\setminus y, \mathbb{C}) \to 0.$$

Thus $\sum_{p+q=n-1} \dim E_2^{p,q}(X,x) = \dim H^{n-1}(X\setminus x, \mathbb{C})$, $\sum_{p+q=n-2} \dim E_2^{p,q}(Y,y) = \dim H^{n-2}(Y\setminus y, \mathbb{C})$. These prove

Theorem 2.2.1. *The E_2-terms $E_2^{p,q}(X,x)$, $E_2^{p,q}(Y,y)$ are degnerate. The exact sequence (2.2.7) splits into two exact sequences*

$$0 \to H^{n-1}(X\setminus x, \mathbb{C}) \to H^{n-1}(X\setminus Y, \mathbb{C}) \to H^{n-2}(Y\setminus y, \mathbb{C}) \to 0$$

$$0 \to H^{n}(X\setminus x, \mathbb{C}) \to H^{n}(X\setminus Y, \mathbb{C}) \to H^{n-1}(Y\setminus y, \mathbb{C}) \to 0$$

2.3. Explicit calculation of characteristic function.

In this section we shall briefly discuss the isolated singularities admitting $\mathbb{C}^*$-action and being complete intersections, and compute some of the characteristic functions defined for them. First we shall fix a $\mathbb{C}^*$-action T on $\mathbb{C}^N:(z_1,z_2,\cdots,z_N)$. Let $\alpha_1, \alpha_2,\cdots, \alpha_N$ be positive integers and set for $z = (z_1,z_2,\cdots,z_N)$ and $c \in \mathbb{C}^*$

$$T(c)z = (c^{\alpha_1}z_1, c^{\alpha_2}z_2,\cdots, c^{\alpha_N}z_N).$$

A polynomial f in z is said to be quasi-homogeneous (with respect to T) if $T(c)^*f = c^d f$ $(c \in \mathbb{C}^*)$ for some integer $d > 0$. The integer d is called the quasi-degree of f. If sufficiently general $(f_1,f_2,\cdots,f_r)$ $(r \leq N)$ are given, then $X_{(f_1,f_2,\cdots,f_r)} = \{z \in \mathbb{C}^N; f_i(z)=0, 1\leq i\leq r\}$ is the complete intersection of the hypersurfaces $f_i(z) = 0$ and has singularity at most at $z = 0$. More precisely, given a system $\delta = (d_1,d_2,\cdots,d_r)$, we define inductively the set $V(\delta)$ of r-tuples of quasi-homogeneous polynomials $(f_1,f_2,\cdots,f_r)$ of quasi-degree $d_1, d_2, \cdots, d_r$ respectively, by the requirement that $(f_1,f_2,\cdots,f_r) \in V(\delta)$ if and only if $(f_2,\cdots,f_r) \in V(d_2,\cdots,d_r)$ and $X_{(f_1,f_2,\cdots,f_r)}$ is a hypersurfacesection of $X_{(f_2,f_3,\cdots,f_r)}$ by f_1. Ordering the coefficients of $f_i, 1 \leq i \leq r$ in some fixed manner, we can regard $V(\delta)$ as a Zariski open subset of a complex euclidean space. Now we shall also fix $\delta = (d_1,d_2,\cdots,d_r)$ and let τ denote a general element of $V(\delta)$. Thus X_τ denotes the set $X_{(f_1,f_2,\cdots,f_r)}$ if $\tau = (f_1,f_2,\cdots,f_r)$. By Lemma 1.2.1 each $(X_\tau, 0)$ satisfies the condition (L). By some elementary argument,[1] it can be shown that the Milnor number of $(X_\tau, 0)$ is a constant $\mu(\delta)$. We thus have by Lemma 2.2.1

$$\dim \mathcal{H}^0_0(\Omega^n_{X_\tau}) = \mu(\delta)$$

where $n = N - r$. Setting $t = 1$ in (2.2.3) we obtain

<u>Lemma 2.3.1.</u> <u>Let</u> $\tau = (f_1, f_2, \cdots, f_r) \in V(\delta)$ <u>and set</u> $n = N-r$. <u>Then the dimension of</u> $\Omega^{n+1}_{f_1,0} = \Omega^{n+1}/(\sum_{i=2}^r f_i \Omega^{n+1} + \sum_{i=1}^r df_i \wedge \Omega^n)$ <u>does not depend on</u> τ, <u>where</u> Ω^p <u>is the stalk</u> $\Omega^p_{\mathbf{C}^N,0}$ <u>over</u> 0 <u>of the</u> p-<u>forms on</u> $\mathbf{C}^N$.

Now we can easily prove the stability of the characteristic function $\chi_{X_\tau}(s,t)$ (defined for the given action T). For this purpose we set for $\tau = (f_1, f_2, \cdots, f_r) \in V(\delta)$

$$Q_\tau = \sum_{i=2}^r f_i \Omega^{n+1} + \sum_{i=1}^r df_i \wedge \Omega^n$$

and we set further for all integers m

$$\Omega(m) = \{\omega \in \Omega^{n+1};\ T(c)^*\omega = c^m \omega (c \in \mathbf{C}^*)\}$$

$$Q_\tau(m) = \{\omega \in Q_\tau;\ T(c)^*\omega = c^m \omega (c \in \mathbf{C}^*)\}.$$

$\Omega(m)$, $Q_\tau(m)$ are all finite-dimensional and $Q_\tau(m)$ depends continuously on τ. Thus $\dim(\Omega(m)/Q_\tau(m))$ is upper semi-continuous. But $\sum_m \dim(\Omega(m)/Q_\tau(m))$ is constant by Lemma 2.3.1. Thus each $\dim(\Omega(m)/Q_\tau(m))$ itself is constant. In other words, the character of $\mathbf{C}^*$ over $\Omega^{n+1}_{f_1,0}$ does not depend on τ. In view of (2.2.3) and (2.2.4) this proves through the induction on r,

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[1] This is supplied in the last half of the appendix.

<u>Lemma 2.3.2. The characteristic function of</u> $(X_\tau, 0)$ <u>with respect to the action</u> T <u>does not depend on special choice of</u> $\tau \in V(\delta)$.

Here we shall denote this characteristic function by $\chi_\delta(s,t)$ though it is determined not only by δ but also by $(\alpha_1,\alpha_2,\cdots,\alpha_N)$. Let us now determine $\chi_\delta(s,t)$ under the following assumption:

<u>Assumption. Each</u> d_i <u>is divisible by</u> $\alpha_1, \alpha_2,\cdots, \alpha_N$.

This restriction means that there is in the family $(X_\tau,0)$, $\tau \in V(\delta)$ a complete intersection of the Brieskorn varieties: Let (a_{ij}) be a given (r, N)-matrix whose all r-minors are not zero. Set

$$f_i^0(z) = \sum_j a_{ij} z_j^{d_j/\alpha_j} \qquad i = 1,2,\cdots,r.$$

Then certainly $\tau^0 = (f_1^0, f_2^0,\cdots,f_r^0) \in V(\delta)$. According to the formulas (2.2.3) and (2.2.4), in order to compute $\chi_\delta(s,t)$ it is sufficient to determine the character of $\mathbf{C}^*$ over the space

$$\Omega^{n+1}_{f_1^0,0} = \Omega^{n+1}/(\textstyle\sum_{n=2}^r f_j^0 \Omega^{n+1} + \sum_{j=1}^r df_j^0 \wedge \Omega^n),$$

so we have first to make the structure of this space as clear as possible. For this purpose we denote by F_i the analytic set where the values of the forms $df_1^0, df_2^0,\cdots, df_i^0$ are not linearly independent. (That is, the set where the coordinate functions of the form $df_1^0 \wedge \cdots \wedge df_i^0$ (with respect to an arbitrary trivialization of $\Omega^i_{\mathbb{C}^N}$) vanish. This is obviously the union of the (i-1)-dimensional coordinate linear varieties of $\mathbb{C}^N$.) Set now

$$\Omega^p(i) = \Omega^p_{\mathbb{C}^N}/\textstyle\sum_{j=1}^i df_j^0 \wedge \Omega^{p-1}_{\mathbb{C}^N}.$$

Then $\Omega^p(i)$ is locally free outside F_i. Note that $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_r$ and $F_i \setminus F_{i-1}$ is non-singular. Thus we have $\mathcal{H}^q_{F_i \setminus F_{i-1}}(\Omega^p(i-1)|_{\mathbb{C}^N \setminus F_{i-1}}) = 0$ for $q \neq N-i+1$. In particular, the natural map $\mathcal{H}^q_{F_{i-1}}(\Omega^p(i-1)) \to \mathcal{H}^q_{F_i}(\Omega^p(i-1))$ is isomorphic for $q < N-i+1$. Using this fact we shall prove by the induction on i that $\mathcal{H}^q_{F_i}(\Omega^p(i)) = 0$ when $p+q < N-i+1$. Suppose $\mathcal{H}^q_{F_{i-1}}(\Omega^p(i-1)) = 0$ when $p+q < N-i+2$. Then the isomorphism proved above shows $\mathcal{H}^q_{F_i}(\Omega^p(i-1)) = 0$ in case $p+q < N-i+1$. Note there is a natural exact sequence $\Omega^{p-1}(i) \to \Omega^p(i-1) \to \Omega^p(i) \to 0$ where the first map is monomorphic outside F_i. Hence we obtain the long exact sequence

$$0 \to \mathrm{Ker}(\Omega^{p-1}(i) \to \Omega^p(i-1)) \to \mathcal{H}^0_{F_i}(\Omega^{p-1}(i))$$

$$\to \mathcal{H}^0_{F_i}(\Omega^p(i-1)) \to \mathcal{H}^0_{F_i}(\Omega^p(i)) \to \mathcal{H}^1_{F_i}(\Omega^{p-1}(i)) \to \cdots .$$

Thus we obtain monomorphisms $\mathcal{H}^q_{F_i}(\Omega^p(i)) \hookrightarrow \mathcal{H}^{q+1}_{F_i}(\Omega^{p-1}(i))$ when $p+q < N-i$. Combining these, we have completed the induction. In particualr the map $\Omega^{N-i}(i) \xrightarrow{df_i^0} \Omega^{N-i+1}(i-1)$ is always monomorphic, and thus we obtain monomorphism

$$\Omega^{n+1}/(\sum_{j=1}^{r-1} df_j^0 \wedge \Omega^n) \xhookrightarrow{df_1^0 \wedge df_2^0 \wedge \cdots \wedge df_{r-1}^0} \Omega^N$$

the image of which obviously coincides with that of $\Omega^{n+1} \xrightarrow{df_1^0 \wedge \cdots \wedge df_{r-1}^0} \Omega^N$. Through the isomorphism $\mathcal{O}_{\mathbb{C}^N,0} \xrightarrow{dz_1 \wedge \cdots \wedge dz_N} \Omega^N$, we obtain the isomorphism $\Omega^{n+1}/(\sum_{j=1}^{r-1} df_j^0 \wedge \Omega^n) \overset{\sim}{\to} A$ where A is the ideal of $\mathcal{O}_{\mathbb{C}^N,0}$ generated by the elements $\Pi_{j\in I} z_j^{(d_j/\alpha_j - 1)}$ where I ranges over all sets consisting of $r - 1$ elements of $\{1,2,\cdots,N\}$. We thus obtain finally the isomorphism

$$\eta: \Omega^{n+1}_{f_1^0,0} \overset{\sim}{\to} A/(\sum_{j=2}^{r} f_j^0 A + A')$$

where A' is the ideal of $\mathcal{O}_{\mathbb{C}^N,0}$ generated by $\Pi_{j\in J} z_j^{(d_j/\alpha_j - 1)}$ with J ranging over all sets of r-elements of $\{1,2,\cdots,N\}$. Although η is not $\mathbb{C}^*$-equivariant, there is the relation $\eta \circ T(c)^* = c^{n'} T(c)^* \circ \eta$ which, in view of (2.2.3), implies that $\chi_\delta(0,t) + \chi_{\delta'}(0,t)$, where $\delta' = (d_2,d_3,\cdots,d_r)$, is equal to the character of $\mathbb{C}^*$ over $A/(\sum_{j=2}^{r} f_j^0 A + A')$ times $t^{n'}$ $\left(n' = \sum_{i=1}^{N} \alpha_i - \sum_{i=2}^{r} d_i\right)$. But this last group is explicit enough to accomplish the computation of the character over it. The result is, however, rather complicated, and for its formulation we still need the following notation: Let $u = (u_1,u_2,\cdots,u_N)$ be indeterminates and define inductively the polynomials $P^i(u;z_1, z_2,\cdots,z_i)$, with rational functions in u as coefficients, by the identities:

$$P^1(u;z_1) = \Pi_{i=1}^{N} (z_1 - u_i)/u_i$$

$$P^{i+1}(u;z_1,z_2,\cdots,z_{i+1})$$

$$= \frac{z_1P^i(u;z_2,z_3,\cdots,z_{i+1})-z_2P^i(u;z_1,z_3,\cdots,z_{i+1})}{z_2-z_1}$$

<u>Theorem 2.3.1.</u> <u>The notation being as above, it holds</u>

$$\chi_\delta(s,t) = \frac{1}{t^{d_1}-s}\{t^{d_1}Q_r(t)-s^{n+2}Q_r(t^{-1})\} +$$
$$+ s\sum_{j=1}^{r-1}\frac{\Pi_{i=1}^{j}(1-t^{d_i})}{\Pi_{i=1}^{j+1}(t^{d_i}-s)}\{Q_{r-j}(t)-t^{d_{j+1}}s^{n+j}Q_{r-j}(t^{-1})\}$$

<u>where</u> $Q_i(t)$, $1 \leq i \leq r$ <u>are given by</u>

$$Q_i(t) = P^i(t^{\alpha_1}-1,\cdots t^{\alpha_N}-1;\ t^{d_1}-1,t^{d_2}-1,\cdots,t^{d_i}-1).$$

<u>Remark.</u> The argument used to obtain the isomorphism η is essentially due to Greuel [4]. See the proof of "De Rham Lemma" which is formulated in a much more general way.

Let us now discuss the case $\alpha_1 = \alpha_2 = \cdots = \alpha_\gamma = 1$. In this case the divisibility assumption given above is trivially satisfied and the quasi-homogeneity means the usual homogeneity, so each $\tau = (f_1,f_2,\cdots,f_r) \in V(\delta)$ defines the algebraic manifold V_τ which is the complete interection of hypersurfaces $f_i = 0$, regarding $(z_1,z_2,\cdots,z_N)$ as the homogeneous coordinates of $P^{N-1}(\mathbb{C})$. X_τ is then the cone $C(V_\tau)$ over V_τ; in other words, $X_\tau\backslash 0$ is identified with L^{-1} minus the zero section where L is the line bundle over V_τ induced by the hyperplanesection of $P^{N-1}(\mathbb{C})$. Thus there is the canonical projection $\pi : X_\tau \setminus 0 \to V_\tau$. Evidently we have natural isomorphism

$$\mathcal{O}_{X\setminus 0} \underset{\pi^*\mathcal{O}_{V_\tau}}{\otimes} \pi^*\Omega^p_{V_\tau} \cong \Omega^p_\xi|_{X\setminus 0}.$$

(The sheaves Ω^p_ξ are to be defined as in Section 2.1.) Since the fiber of $\pi : X_\tau \setminus 0 \to V_\tau$ is Stein, it follows

$$\begin{aligned} R^q\iota_*\iota^*\Omega^p_\xi &\cong H^q(X\setminus 0,\ \Omega^p_\xi) \\ &\cong \hat{\sum}_{k\in\mathbf{Z}} H^q(V_\tau,\ \Omega^p(L^k)) \end{aligned}$$

where $\iota : X_\tau \setminus 0 \hookrightarrow X_\tau$ and the last sum is infinite sum converging with respect to some suitable topology when $q = 0$ or $q = n-1$. (Note the sum is finite if $0 < q < n-1$ according to the vanishing theorem of Kodaira.) Using the Gysin sequence (2.1.1) and Theorem 2.1.1, we have for $0 < q < n-1$

$$\chi^{q+1}_{X_\tau}(t) = -\delta_{q,n-q-1} + \sum_{k\in\mathbf{Z}} t^k \dim H^q(V_\tau, \Omega^p(L^k)).$$

Moreover, by the exact sequence $\Omega^p_\xi \to \iota_*\iota^*\Omega^p_\xi \to \mathcal{H}^1_0(\Omega^p_\xi) \to 0$ and by the fact that, if $0 \neq \omega \in \Omega^p_{\xi,0}$ and if $T(c)^*\omega = c^k\omega$ $(c \in \mathbf{C}^*)$, then $k \geq p+1$ (Recall $\Omega^p_\xi = i(\xi)\Omega^{p+1}_X$), we can show that

$$\chi^1_{X_\tau}(t) - \sum_{k<n} t^k \dim H^0(V_\tau, \Omega^{n-1}(L^k))$$

is a polynomial divisible by t^n. We can as well prove the vanishings $H^q(V_\tau, \Omega^p(L^k)) = 0$ $(p+q \neq n-1,\ k \neq 0)$ and $H^q(V_\tau,\ \Omega^p) = 0$ $(p+q \neq n-1,\ p \neq q)$ by (2.1.1) and Theorem 2.1.1. Now let the polynomials $R^i(z_1, z_2, \cdots, z_i)$ $i=1,2,\cdots$ be defined inductively by

$$R^1(z_1) = (z_1-1)^{n+r}$$

$$R^{i+1}(z_1,z_2,\cdots,z_{i+1})$$
$$= \frac{z_1R^i(z_2,z_3,\cdots,z_{i+1}) - z_2R^i(z_1,z_3,\cdots,z_{i+1})}{z_2 - z_1}.$$

Then it follows from Theorem 2.3.1

Corollary 2.3.1. Let V be the complete intersection of r-hypersurfaces of degree $d_1, d_2, \cdots, d_r$ in $\mathbb{P}^{n+r-1}(\mathbb{C})$ and L the line bundle over V induced by the hyperplane section. Then $H^q(V,\Omega^p(L^k)) = 0$ if $p+q \neq n-1,\ k \neq 0$ or if $p+q \neq n-1,\ p \neq q$. Further it holds the congruence

$$\chi^0_{C(V)}(t) + s\chi^1_{C(V)}(t)$$
$$+ \sum_{q=1}^{n-2} s^{q+1}(-\delta_{q,n-q-1}+\sum_{k\in\mathbb{Z}} t^k \dim H^q(V,\Omega^{n-q-1}(L^k)))$$
$$\equiv \frac{t^{d_1}}{t^{d_1}-s} R^r(\frac{t^{d_1}-1}{t-1},\dots\frac{t^{d_r}-1}{t-1})$$
$$+ s\sum_{j=1}^{r-1} \frac{\Pi_{i=1}^{j}(1-t^{d_i})}{\Pi_{i=1}^{j+1}(t^{d_i}-s)} R^{r-j}(\frac{t^{d_{j+1}}-1}{t-1},\dots,\frac{t^{d_r}-1}{t-1})$$
$$\mod s^n$$

where the right hand should be interpreted as power series in s whose coefficients are rational functions in t. More over $\chi^1_{\mathbb{C}(V)}(t) - \sum_{k<n} t^k \dim H^0(V,\Omega^{n-1}(L^k))$ is a polynomial divisible by t^n.

This corollary, combined with Hirzebruch [8], determines all of the dimensions of $H^q(V,\Omega^p(L^k))$.

Appendix

The purpose of this appendix is to prove the duality stated in Lemma 1.1.1 and the statement mentioned at the top of p.28. Let M be a complex manifold of dimension n, V an analytic vector bundle over M and V^* its dual. We denote by $A^{(p,q)}(V)$ (resp. $\mathcal{D}_c^{(p,q)}(V^*)$) the space of V-valued C^∞ (p,q)-forms on M (resp. the space of V^*-valued distribution (p,q)-forms on M with compact support). Between $A^{(p,q)}(V)$ and $\mathcal{D}_c^{(n-p,n-q)}(V^*)$ there is a natural pairing which, through the Dolbeault isomorphism, gives rise to a pairing

$$H^q(M,\Omega^p(V)) \times H_c^{n-q}(M,\Omega^{n-p}(V^*)) \ni (\alpha,\beta) \to \langle\alpha,\beta\rangle \in \mathbf{C}$$

where we have denoted by $\Omega^s(V)$, $\Omega^s(V^*)$ the sheaves of analytic s-forms on X with values in V, V^* respectively. The problem is to examine whether this $\langle\ ,\ \rangle$ define the actual duality or not. Our object is the following

Theorem A.1. Assume that there is a smooth proper map φ of M onto an open interval (a,b) (possibly $a = -\infty$ or $b = \infty$) such that $d\varphi$ vanishes nowhere in M and that the complex Hessian $i\partial\bar{\partial}\varphi$ is positive definite everywhere. Then the pairing $\langle\ ,\ \rangle$ defines the duality between $H^q(M, \Omega^p(V))$ and $H_c^{n-q}(M,\Omega^{n-p}(V^*))$ for any $q \leq n-2$.

Proof, Step I. By Andreotti-Grauert [1], $H^q(M,\Omega^p(V))$ is finite dimensional for $0 < q < n-1$, so $A^{(p,q)}(V) \overset{\bar{\partial}}{\to} A^{(p,q+1)}(V)$ has closed image for $q < n-2$. (This is also true $q > n-2$ by Malgrange, Bull. Soc. Math. France 85 (1957), p.236.) Hence, by Serre [14], the pairing $\langle\ ,\ \rangle$ defines the duality between $H^q(M,\Omega^p(V))$ and $H_c^{n-q}(M,\Omega^{n-p}(V^*))$ for $q \leq n-3$. Unfortunately the general theory of [14] seems to be not adequate to prove the duality for $q = n-2$. But we can at least prove the following statement: <u>It holds always the inequality</u> $\dim H^{n-2}(M,\Omega^p(V)) \leq \dim H_c^2(M,\Omega^{n-p}(V^*))$ <u>where the equality holds if and only if the duality for</u> $q = n-2$ <u>holds</u>. For, let $\omega \in A^{(p,n-2)}(V)$ be a $\bar\partial$-closed form such that $\langle\omega,\psi\rangle = 0$ for every $\bar\partial$-closed $\psi \in \mathcal{D}_c^{(n-p,2)}(V^*)$. Then, by Hahn-Banach Theorem, ω lies in the closure of the image of $A^{(p,n-3)}(V) \xrightarrow{\bar\partial} A^{(p,n-2)}(V)$. But this image is certainely closed as indicated above. This shows that $\langle\alpha,\beta\rangle = 0$ for all $\beta \in H_c^2(M,\Omega^{n-p}(V^*))$ implies $\alpha = 0$. We have thus proved the required assertion.

<u>Step</u> II. It remains to prove the opposite inequality $\dim H^{n-2}(M,\Omega^p(V)) \geq \dim H_c^2(M,\Omega^{n-p}(V^*))$. To show this, it suffices to prove the isomorphisms $H^q(M,\mathcal{G}) \cong H_c^{q+1}(M,\mathcal{G})$, $0 < q < n-1$ for any locally free $\mathcal{O}_M$-Module $\mathcal{G}$. For, if these isomorphisms are true, then $H^{n-2}(M,\Omega^p(V)) \cong H_c^{n-1}(M,\Omega^p(V))$, $H_c^2(M,\Omega^{n-p}(V^*)) \cong H^1(M,\Omega^{n-p}(V^*))$. Therefore, it suffices to prove $\dim H^1(M,\Omega^{n-p}(V^*)) \leq \dim H_c^{n-1}(M,\Omega^p(V))$. But this is just the inequality obtained in Step I in case $n = 3$ when p, V are replaced by $n-p$, V^*. In case $n \geq 4$, $1 \leq n-3$, so, again by what was proved in Step I, we obtain the equality $\dim H^1(M,\Omega^{n-p}(V^*)) = \dim H^{n-1}(M,\Omega^p(V))$.

We have thus reduced the roof of Theorem A.1 to the isomorphisms $H^q(M,\mathcal{G}) \cong H_c^{q+1}(M,\mathcal{G})$, $0 < q < n-1$, which, combined again with Theorem A.1, prove Lemma 1.1.1. We start with the following consequence of Andreotti-Grauert [1].

Lemma A.1. Let $\varphi: M \to (a,b)$ be as in Theorem A.1. Let further $\mathcal{G}$ be a locally free $\mathcal{O}_M$-Module. Then, for $c \in (a,b)$, the restriction maps

$$H^q(M,\mathcal{G}) \to H^q(\varphi^{-1}((a,c)),\mathcal{G}) \qquad (q \neq 0)$$

$$H^q(M,\mathcal{G}) \to H^q(\varphi^{-1}((c,b)),\mathcal{G}) \qquad (q \neq n-1)$$

are isomorphisms.

Let now $\mathcal{G}$ be as in this lemma. Take a fine resolution $\mathcal{G}^\bullet$ of $\mathcal{G}$ and set $\Gamma_+(M,\mathcal{G}^\bullet) = \varinjlim_{c\nearrow b} \Gamma(\varphi^{-1}((c,b)),\mathcal{G}^\bullet)$, $\Gamma_-(M,\mathcal{G}^\bullet) = \varinjlim_{c\searrow a} \Gamma(\varphi^{-1}((a,c)),\mathcal{G}^\bullet)$. Then the restriction maps $\Gamma(M,\mathcal{G}^\bullet) \to \Gamma_\pm(M,\mathcal{G}^\bullet)$ give rise to the exact sequence $0 \to \Gamma_c(M,\mathcal{G}^\bullet) \to \Gamma(M,\mathcal{G}^\bullet) \to \Gamma_+(M,\mathcal{G}^\bullet) \oplus \Gamma_-(M,\mathcal{G}^\bullet) \to 0$ where $\Gamma_c(M,\mathcal{G}^\bullet)$ denotes the complex of sections with compact support. From this it follows the long exact sequence

$$\text{(a.1)} \qquad \cdots \to H_c^q(M,\mathcal{G}) \to H^q(M,\mathcal{G}) \to H_+^q(M,\mathcal{G}) \oplus H_-^q(M,\mathcal{G}) \to \cdots$$

where we have put $H_+^q(M,\mathcal{G}) = \varinjlim_{c\nearrow b} H^q(\varphi^{-1}((c,b)),\mathcal{G})$, $H_-^q(M,\mathcal{G}) = \varinjlim_{c\searrow a} H^q(\varphi^{-1}((a,c)),\mathcal{G})$. Combined with Lemma A.1, this implies

Lemma A.2. $H_+^q(M,\mathcal{G}) \cong H_c^{q+1}(M,\mathcal{G})$ if $q \neq 0$, and $H_-^q(M,\mathcal{G}) \cong H_c^{q+1}(M,\mathcal{G})$ if $q \neq n-1$. In particular $H^q(M,\mathcal{G}) \cong$

$H_c^{q+1}(M,\mathcal{G})$ if $0 < q < n-1$.

Let now (X,x) be an isolated singularity and suppose X is imbedded into $\mathbb{C}^N:(z_1,z_2,\cdots,z_N)$ so that $x = 0$ and $r|_{X\setminus x}$ has no critical point where $r(z) = \sum_{i=1}^N |z_i|^2$. Then, setting $M = X\ x$, $\varphi = r|_{X\setminus x}$ and applying Theorem A.1 and Lemmas A.1 and A.2, we obtain Lemma 1.1.1.

To prove the fact remarked after Lemma 1.1.1, we first note that the refriction map $H^q(M,\Omega^\bullet) \to H^q_\pm(M,\Omega^\bullet)$ is quasi-isomorphism. For, by Lemma A.1, $H^q(M,\Omega^\bullet) \to H^q_+(M,\Omega^\bullet)$ $(q \neq 0)$, $H^q(M,\Omega^\bullet) \to H^q_-(M,\Omega^\bullet)$ $(q \neq n-1)$ are actually isomorphisms. Morover, $E_2^{p,q} = H^p(H^q(M,\Omega^\bullet))$, ${}_\pm E_2^{p,q} = H^p(H^q_\pm(M,\Omega^\bullet))$ are regarded E_2-terms of the three spectral sequences which converge to the same limit. (The facts that $\varphi: M \to (a,b)$ is proper and that $d\varphi$ vanish nowhere, imply that the inclusions $\varphi^{-1}((a,c)), \varphi^{-1}((c,b)) \hookrightarrow M$ are homotopy equivalences). Therefore, $H^p(H^q(M,\Omega^\bullet)) \cong H^p(H^q_\pm(M,\Omega^\bullet))$. Next, set $M = X\setminus x$, $\varphi = r|_{X\setminus x}$ and observe that $H^0_-(X\setminus x,\Omega^\bullet) = (\iota_*\iota^*\Omega^\bullet_X)_x$ and that $H^p((\iota_*\iota^*\Omega^p_X)_x)$ are finite dimensional when $n \geqq 2$. (Note $H^p(\Omega^\bullet_{X,x})$ are finite-dimensional, and the kernel and the cokernel of $\Omega^\bullet_{X,x} \to (\iota_*\iota^*\Omega^\bullet_X)_x$ are finite dimensional) Note that $H^0(M,\Omega^p)$ are Fréchet, that the complex $H^0(M,\Omega^\bullet)$ has finite-dimensional cohomology and that $H^0(M,\Omega^p)$ and $H^n_c(M,\Omega^{n-p})$ are dual each other. Thus, argueing as in [14], we conclude that $H^p(H^0(M,\Omega^\bullet)) \cong H^p((\iota_*\iota^*\Omega^\bullet_X)_x)$ and $H^{n-p}(H^n_c(M,\Omega^\bullet)) \cong H^{n-p}(H^{n-1}_-(M,\Omega^\bullet)) \cong H^{n-p}((R^{n-1}\iota_*\iota^*\Omega^\bullet)_X)$ are mutually dual, which was to be proved.

In the rest of this appendix we shall prove the statement mentioned at the top of p.28, that is, that the Milnor number of $(X_\tau, 0)$, $\tau \in V(\delta)$ does not depend on τ. Let the action T over $\mathbb{C}^N:(z_1,\dots,z_N)$ be as in Section 2.3 and recall that $V(\delta)$ (δ is a system $(d_1,d_2,\dots,d_r)$ of quasi-degrees with respect to T) is a Zariski open subset of some complex euclidean space. In this space we take a linear system of coordinotes $\tau = (\tau_1, \tau_2,\dots,\tau_\rho)$. We shall further define the polynomials F_i, $i = 1, 2,\dots,r$ of $\tau_1,\tau_2,\dots,\tau_\rho$, $z_1,z_2,\dots,z_N$ as follows: Recalling that each $\tau \in V(\delta)$ is a system of quasi-homogeneous polynomials $f_1(z),f_2(z),\dots,f_r(z)$ of quasi-degree $d_1,d_2,\dots,d_r$ respectively, we set $F_i(\tau,z) = f_i(z)$, $1 \leq i \leq r$. Setting $X_\tau^i = \{z \in \mathbb{C}^N;\ F_i(\tau,z) = F_{i+1}(\tau,z) = \dots = F_r(\tau,z)\}$ for $\tau \in V(\delta)$, we obtain a series of isolated singularities $(X_\tau,0) = (X_\tau^1,0) \hookrightarrow (X_\tau^2,0) \hookrightarrow \dots (X_\tau^r,0) \hookrightarrow (\mathbb{C}^N,0)$, where each $(X_\tau^i,0)$ is a hypersurfacesection of $(X_\tau^{i+1},0)$. Let $\alpha_1,\alpha_2,\dots,\alpha_N$ be such that $T(c)(z_1,z_2,\dots,z_N) = (c^{\alpha_1}z_1, c^{\alpha_2}z_2,\dots,c^{\alpha_N}z_N)$ and define $\beta_1,\beta_2,\dots,\beta_N$ so that $\alpha_i\beta_i$ are equal to the smallest common multiple m of $\alpha_1,\alpha_2,\dots,\alpha_N$. If we set $r(z) = \sum_{i=1}^N |z_i|^{2\beta_i}$, we know by Milnor [10] (Corollary 2.8), for any $\tau \in V(\delta)$, there is $\varepsilon>0$ such that $r(z)|_{X_\tau^i \cap \{z\in\mathbb{C}^N;r(z)<\varepsilon\}\setminus 0}$, $1 \leq i \leq r$ have no critical points. But the formula $r(T(c)z) = |c|^{2m}r(z)$ implies that $r(z)|_{X_\tau^i\setminus 0}$ have no critical points. (Note that for $z \in X_\tau^i$ we can find $c \in \mathbb{C}^*$ such that $T(c)z \in X_\tau^i \cap \{z;r(z)<\varepsilon\}$.). This fact means that, for (τ,z) such that $z \in X_\tau^i\setminus 0$, the following $2(r-i+1)+1$ real linear forms in $t = (t_1,t_2,\dots,t_N) \in \mathbb{C}^n$ are linearly independent over $\mathbb{R}$:

$$A_k^{(\tau,z)}(t) = \mathrm{Re}[\textstyle\sum_{i=1}^N (t_i\partial F_k(\tau,z)/\partial z_i + \bar{t}_i\partial F_k(\tau,z)/\partial \bar{z}_i)],\quad i \leq k \leq r$$

$$B_k^{(\tau,z)}(t) = \mathrm{Im}[\textstyle\sum_{i=1}^N (t_i \partial F_k(\tau,z)/\partial z_i + \bar{t}_i \partial F_k(\tau,z)/\partial \bar{z}_i)],\ i \leq k \leq r$$

$$C^{(\tau,z)}(t) = \textstyle\sum_{i=1}^N (t_i \partial r(z)/\partial z_i + \bar{t}_i \partial r(z)/\partial \bar{z}_i)$$

(The linear independence of the first $2(r-i+1)$ forms in the restatement of that $(X_\tau^i, 0)$ is an isolated singularity.) From this it follows

<u>Lemma A.3</u>. <u>One can find</u> (<u>real</u>) C^∞ <u>vector fields</u> Z_i $(i=1,2,\ldots,\rho)$ over $V(\delta) \times (\mathbb{C}^N \setminus 0)$ such that $Z_i \tau_j = \delta_{ij}$, $Z_i r(z) = 0$, $Z_i F_j \equiv 0$ mod. $F_j, F_{j+1}, \ldots, F_r, \bar{F}_j, \bar{F}_{j+1}, \ldots, \bar{F}_r$, <u>where we have let</u> $g \equiv 0$ mod. $h_1, h_2, \ldots, h_s$ <u>mean that</u> g <u>lies in the ideal generated by</u> $h_1, h_2, \ldots, h_s$ <u>in the ring of</u> C^∞ <u>functions</u>.

<u>Proof</u>. We express the required Z_i in the form $\mathrm{Re}[\partial/\partial \tau_i - \sum_{s=1}^N t_{is}(\tau,z)\, \partial/\partial z_s]$ where $t_{is}(\tau,z)$ are C^∞ functions in $V(\delta) \times (\mathbb{C}^N \setminus 0)$. Then $Z_i F_k = 0$ is equivalent to $A_k^{(\tau,z)}(t_{i1}(\tau,z), t_{i2}(\tau,z), \ldots, t_{iN}(\tau,z)) = \mathrm{Re}[\partial F_k(\tau,z)/\partial \tau_i + \partial F_k(\tau,z)/\partial \bar{\tau}_i]$, $B_k^{(\tau,z)}(t_{i1}(\tau,z), t_{i2}(\tau,z), \ldots, t_{iN}(\tau,z)) = \mathrm{Im}[\partial F_k(\tau,z)/\partial \tau_i + \partial F_k(\tau,z)/\partial \bar{\tau}_i]$, and further $Z_i r(z) = 0$ is equivalent to $C^{(\tau,z)}(t_{i1}(\tau,z), t_{i2}(\tau,z), \ldots, t_{iN}(\tau,z)) = 0$. Since $2(r-i+1)+1 \leq 2N$ ($\because r \leq N$), by what was remarked before the lemma, in some neighborhood of (τ_0, z_0) such that $z_0 \in X_{\tau_0}^j \setminus 0$, we can find the solution $t_{is}(\tau,z)$, $1 \leq i \leq \rho$, $1 \leq s \leq N$ of these equations for $j \leq k \leq r$. Suppose now $z \in X_\tau^{j+1} \setminus X_\tau^j$.

Then we can find vector fields Z_i' in a neighborhood of (τ,z) such that $Z_i' \tau_k = \delta_{ik}$, $Z_i' r(z) = 0$, $Z_i' F_k$ vanish identically for $j < k$. By shrinking the neighborhood if necessary, we also see that the vector fields Z_i' satisfy $Z_i'F_k \equiv 0$ mod. $F_k, \dots, F_r$, $\overline{F}_k, \dots, \overline{F}_r$ for $k \leq j$, since F_j is among $F_k, \dots, F_r$ and $z \notin X_\tau^j$ implies $F_j(\tau, z) \neq 0$. We have thus shown the local existence of the required vector fields. The global existence is proved now by using the partition of unity.

Using this lemma, we shall show

Theorem A.2. Set $S_\varepsilon = \{z \in \mathbf{C}^N;\ r(z) = \varepsilon\}$ and $M_\tau^i(\varepsilon) = X_\tau^i \cap S_\varepsilon$ for $1 \leq i \leq r$, $\tau \in V(\delta)$, and $\varepsilon > 0$. For any $\tau', \tau'' \in V(\delta)$ there is a diffeomorphism of S_ε onto itself which maps $M_{\tau'}^i(\varepsilon)$ onto $M^i_{\tau''}(\varepsilon)$.

Proof It suffices to prove when τ', τ'' are sufficiently near. Let U be an open subset of $V(\delta)$ which is convex in the linear space containing $V(\delta)$ and let $\tau' = (\tau_1', \tau_2', \dots, \tau_\rho')$, $\tau'' = (\tau_1'', \tau_2'', \dots, \tau_\rho'') \in U$. We set $Z = \sum_{i=1}^{\rho} (\tau_i'' - \tau_i') Z_i$. Since $Zr(z) = 0$, Z is tangent to the surfaces $U \times S_\varepsilon$. Since the projection $U \times S_\varepsilon \to U$ is proper, the definition domain of $\exp tZ$ is just the product of S_ε and the definition domain of $\exp t(\sum_{i=1}^{\rho} (\tau_i'' - \tau_i') \partial / \partial \tau_i)$, for every $t \in \mathbf{R}$. Thus $\exp Z$ induces a diffeomorphism of $\{\tau'\} \times S_\varepsilon$ onto $\{\tau''\} \times S_\varepsilon$. We regard this as a diffeomorphism φ of S_ε onto S_ε through the projection $U \times S_\varepsilon \to S_\varepsilon$. Then the conditon $ZF_j \equiv 0$ mod. $F_j, \dots, F_r, \overline{F}_j, \dots, \overline{F}_r$ implies φ maps each $M_{\tau'}^j(\varepsilon)$ onto $M_{\tau''}^j(\varepsilon)$.

Recall that the Milnor number of $(X_\tau^i, 0)$ is a topological invariant of $M_\tau^{i+1}(\varepsilon) \setminus M_\tau^i(\varepsilon)$ (Milnor [10](Remark 8.6) combined with Hamm [5]). Hence Theorem A.2 proves that the Milnor number of $(X_\tau, 0)$ is a constant, which was to be proved.

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