数理解析研究所講究録
222巻 1974年 51-55

ON THE COSINE PROBLEM OF S. CHOWLA

Saburô UCHIYAMA

Department of Mathematics, Okayama University

Let A be a set of $n \geqq 1$ distinct integers a_1, $a_2, \ldots, a_n$. There will be no loss in generality in assuming that the a_j are non-negative, or positive, integers. Define for real x

$$E(x) = \sum_{j=1}^{n} e(a_j x) \qquad [e(x) = e^{2\pi i x}],$$

$$C(x) = \frac{1}{2}\{E(x) + E(-x)\} = \sum_{j=1}^{n} \cos 2\pi a_j x ,$$

and write

$$I(A) = \int_0^1 |E(t)|dt ,$$

$$M(A) = -\min_{0 \leqq x < 1} C(x) .$$

We have

$$M(A) \geqq \frac{1}{2}\int_0^1 |C(t)|dt ;$$

indeed,

$$\int_0^1 |C(t)|dt = \int_0^1 \{|C(t)| - C(t)\}dt \leqq 2M(A) .$$

Also, since

$$|C(x)| = \frac{1}{2} |\{E(-x) + E(x)\}e(ax)|$$

for any integer a, we find

$$M(A) \geqq \frac{1}{4} I(A_1) ,$$

where A_1 is a set of $2n$ distinct (positive, or non-negative) integers.

The Cosine Problem of Ankeny-Chowla (cf. [2]) is to prove that to an arbitrary positive integer K there corresponds an integer $n_0 = n_0(K)$ such that

$$M(A) > K$$

for all $n > n_0$, where $A = \{a_1, \ldots, a_n\}$ is *any* set of n different integers.

This problem is, as we have seen above, closely related to the

Problem of J. E. Littlewood (cf. [9]): Is it true or not that

$$I(A) > B \log n$$

for all sets A of n distinct positive integers $a_1, \ldots, a_n$? Here, and in what follows, B stands for an (unspecified) absolute constant > 0. Littlewood's conjecture $I(A) > B \log n$ is, if true, substantially the best possible.

In 1960 P. J. Cohen [5] and H. Davenport [6] succeeded in proving that for some $B > 0$ and all $n \geqq 3$

$$I(A) > B\left(\frac{\log n}{\log\log n}\right)^{\frac{1}{4}},$$

so that

$$M(A) > B\left(\frac{\log n}{\log\log n}\right)^{\frac{1}{4}},$$

which gives a solution to the Cosine Problem of Ankeny-Chowla.

In 1963 S. Chowla (cf. [3], [4]) revised the Cosine Problem and stated the conjecture: For any set A of n distinct integers one has

$$M(A) > B\, n^{\frac{1}{2}} .$$

This is, if true, essentially the best possible.

The Cosine Problem of S. Chowla (thus revised) seems to be still unsolved, and the best result known so far is the one due to K. F. Roth [17] who proved that

$$M(A) > B\left(\frac{\log n}{\log\log n}\right)^{\frac{1}{2}}.$$

The main purpose of this article is to provide a list of papers published up to 1973, on the Cosine Problem and Littlewood's problem, as well as some allied ones. The list given below is, of course, not exhaustive.

References

[1] J. Chidambaraswamy: On the mean modulus of trigonometric polynomials and a conjecture of S. Chowla. Proc. Amer. Math. Soc. 36(1972), 195-200.

[2] S. Chowla: The Riemann zeta and allied functions. Bull. Amer. Math. Soc. 58(1952), 287-305.

[3] S. Chowla: Some applications of a method of A. Selberg. Det Kongelige Norske Vid. Selsk. Forh. 36(1963), 194-200.

[4] S. Chowla: Some applications of a method of A. Selberg. J. Reine Angew. Math. 217(1965), 128-132.

[5] P. J. Cohen: On a conjecture of Littlewood and idempotent measures. Amer. J. of Math. 82(1960), 191-212.

[6] H. Davenport: On a theorem of P. J. Cohen. Mathematika 7(1960), 93-97.

[7] P. Erdös: Some unsolved problems. MTA Mat. Kut. Intéz. Közl. 6A(1961), 221-254.

[8] P. Erdös: On trigonometric sums with gaps. MTA Mat. Kut. Intéz. Közl. 7A(1962), 37-42.

[9] G. H. Hardy and J. E. Littlewood: A new proof of a theorem on rearrangements. J. London Math. Soc. 23(1948), 163-168.

[10] S. Izumi: Some trigonometrical series XX. Proc. Japan Acad. 32(1956), 93-96.

[11] M. Izumi and S. Izumi: On some trigonometrical polynomials. Math. Scand. 21(1967), 38-44.

[12] L. C. Kurtz and S. M. Shah: On the L^1 norm and the mean value of a trigonometric series. Proc. Amer. Math. Soc. 19(1969), 1023-1028.

[13] D. J. Newman: Norms of polynomials. Amer. Math. Monthly 67(1960), 778-779.

[14] А. М. Олевский: Ряды Фурье непрерывных функций по ограниченным ортонормальным системам. Изв. Акад. наук СССР Сер. мат. 30(1966), 387-432.

[15] А. М. Олевский: О порядке роста функций Лебега ограниченных

ортонормальных систем. Доклады Акад. наук СССР 176 (1967), 1247-1250.

[16] А. М. Олевский: Ряды Фурье и функции Лебега. Успехи мат. наук 22(1967), 237-239.

[17] K. F. Roth: On cosine polynomials corresponding to sets of integers. Acta Arith. 24(1973), 87-98.

[18] R. Salem: On a problem of Littlewood. Amer. J. of Math. 77(1955), 535-540.

[19] M. Uchiyama and S. Uchiyama: On the cosine problem. Proc. Japan Acad. 36(1960), 475-479.

[20] S. Uchiyama: A propos d'un problème de M. J. E. Littlewood. C. R. Acad. Sci. Paris 260(1965), 2675-2678.

[21] S. Uchiyama: On the mean modulus of trigonometric polynomials whose coefficients have random signs. Proc. Amer. Math. Soc. 16(1965), 1185-1190.