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Title	Symmetry of the Wave Functions (エルゴード理論とその周辺)
Author(s)	HIDA, TAKEYUKI
Citation	数理解析研究所講究録 (1974), 204: 130-139
Issue Date	1974-03
URL	http://hdl.handle.net/2433/105131
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

Kyoto University

数理解析研究所講究録
204巻 1974年 130-139

Symmetry of the Wave functions

Takeyuki HIDA

§0. Introduction

The infinite dimensional unitary group has been introduced in connection with the complex white noise (see [1]). Roughly speaking, it is isomorphic to the group of all linear transformations of the space of generalized functions such that the measure ν of the complex white noise is kept invariant under them. In my previous paper [2], I have found an interesting subgroup, call it now G_0, which is generated by the shift and the Fourier-Mehler transform (the Fourier transform of fractional order) ; this group G_0 itself is a six-dimensional Lie group. Then, I have been able to see that the group G_0 describes several important properties of the complex white noise or those of the complex Brownian motion (see [3]). While, I have enjoyed the discussion with W. Miller Jr. to have found that the group G_0 is isomorphic to the symmetry group of the heat equation and to give an interpretation telling why they are so. The report of this result will be published soon.

In this short report, I shall discuss the symmetry of the solution space $\mathfrak{S}$ of the Schrödinger equation

(1) $$i\hbar \frac{\partial \psi}{\partial t} = H\psi , \qquad H : \text{Hamiltonian operator},$$

where the collection of the initial states are taken to be the (complex) Schwartz space $\mathcal{S}_c$. If a tranformation in G_0 is applied to an initial state, then we are given the associated transformation on $\mathcal{G}$ which carries a wave function to another wave function. In this manner, we can form a transformation group $\tilde{G}_0$ acting on $\mathcal{G}$ which is isomorphic to the group G_0. Indeed, $\tilde{G}_0$ turns out to be the symmetry group of the wave functions in some particular cases where the Hamiltonian is simple (and is typical as well).

§1. Complex white noise and unitary group $U(\mathcal{S}_c)$

Let E be a nuclear space, which is densely included in $L^2(R^1)$, such that

$$E \subset L^2(R^1) \subset E^*, \qquad E^* \text{ the dual space of } E.$$

The real white noise is the probability measure (E^*, μ_{σ^2}) where the characteristic functional $C(\xi)$ of μ_{σ^2} is given by

$$\text{(2)} \qquad C(\xi) = \int_{E^*} e^{i\langle x,\xi\rangle} d\mu_{\sigma^2}(x) = e^{-\frac{1}{2}\sigma^2\|\xi\|^2}, \quad \xi \in E.$$

In the above expression $\| \ \|$ stands for the $L^2(R^1)$-norm and $\langle x,\xi\rangle$, $x \in E^*$, $\xi \in E$, is the canonical bilinear form which connects E^* and E.

Complexifications of E and E^* can be done in a usual manner, call them E_c and E_c^*. Members in E_c and E_c^* are denoted by $\zeta = \xi + i\eta$, $\xi, \eta \in E$, and $z = x + iy$, $x, y \in E^*$, respectively. We extend $\langle x,\xi\rangle$ to

$$\langle z,\zeta\rangle = (\langle x,\xi\rangle + \langle y,\eta\rangle) + i(-\langle x,\eta\rangle + \langle y,\xi\rangle),$$

through which E_c^* becomes the dual space of E_c.

We are now ready to introduce the measure of the complex white noise :

$$\nu = \mu_{\frac{1}{2}} \times \mu_{\frac{1}{2}}.$$

The measure space (E_c^*, ν) is called the complex white noise.

Let $U(E_c)$ be the collection of all linear transformations g on E_c satisfying the following two conditions :

i) g is a homeomorphism of E_c,

ii) $\|g\zeta\| = \|\zeta\|$ for every $\zeta \in E_c$.

With the usual product $(g_1g_2)\zeta = g_1(g_2\zeta)$, $U(E_c)$ becomes a group. We can also topologize $U(E_c)$, for instance, by the ν-topology or by the compact-open topology, so that $U(E_c)$ becomes a topological group. This topological group $U(E_c)$ is called the infinite dimensional unitary group or simply the unitary group.

Theorem 1. For any $g \in U(E_c)$, the adjoint g^* is a measure-preserving transformation on (E_x^*,ν).

§2. Symmetries of the heat equation.

Consider the heat equation

$$\frac{\partial}{\partial t} v(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2} v(x,t), \qquad x \in R^1, \quad t \in [0,\infty). \tag{3}$$

To obtain the symmetries of the equation (3) we proceed as follows. Set

$$Q = \frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2}, \tag{4}$$

and let $\tilde{\mathfrak{g}}$ be the vector space of differential operators L of the form

$$L = X\frac{\partial}{\partial X} + T\frac{\partial}{\partial t} + U \tag{5}$$

satisfying the following two conditions :

i) X, T and U are analytic in both x and t,

ii) $QLv = 0$ whenever $Qv = 0$.

Now, three propositions are in order.

Proposition 1. An operator L belongs to $\tilde{\mathfrak{g}}$ if and only if L

satisfies the commutation relation

(6) $$[L, Q] = R(x,t)Q ,$$

where R is an analytic function of x and t depending only on L.

Proposition 2. The vector space $\tilde{g}$ forms a Lie algebra with the product [,].

Proposition 3. The collection

(7) $$\tilde{G} = \{\exp \alpha_1 L_1 \cdots \exp \alpha_n L_n \; ; \; L_j \in \tilde{g}, \; \alpha_j \in R, \; j = 1, 2, \cdots, n\}$$

forms a local Lie group.

Definition. The group $\tilde{G}$ given by (7) is called the symmetry group of Q.

Theorem 2. (Miller [4]). The Lie algebra $\tilde{g}$ is six-dimensional, and a basis of $\tilde{g}$ is given by

(8) $$I \text{ (identity)}, \quad L_{-2} = \frac{\partial}{\partial t}, \quad L_{-1} = \frac{\partial}{\partial x},$$
$$L_0 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \quad L_1 = t\frac{\partial}{\partial x} + x, \quad L_2 = tx\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + (x^2+t)/2.$$

We now come to a description illustrating how our unitary group contributes to the symmetry group of the heat equation.

Let $\mathfrak{G}$ be the collection of the functions v given by

(9) $$v \equiv v(x,t;\zeta) = \int_{-\infty}^{\infty} \zeta(u) g(t,x-u)\,du, \qquad \zeta \in \mathscr{S}_c,$$

where $g(t,x)$ is the Gauss kernel. Obviously, $\mathfrak{G}$ is a subspace of the solution space. In addition, since the mapping T of the space $\mathscr{S}_c$:

$$T : \zeta(u) \longrightarrow v(x,t;\zeta), \qquad \zeta \in \mathscr{S}_c,$$

is a bijection, $\mathfrak{G}$ can be topologized so as to be isomorphic to $\mathscr{S}_c$.

For a one-parameter subgroup $\{g_t\}$ of $U(\mathscr{S}_c)$ we are given the associated one-parameter group $\{\tilde{g}_t\}$ acting on $\mathfrak{G}$ in such a way that

(10) $$\tilde{g}_t v(x,t;\zeta) = v(x,t;g_t\zeta).$$

In what follows we shall restrict our attention to the six one-parameter subgroups of G_0 which generate respective one-parameter subgroups listed below.

1) gauge transform I_t :

$$I_t\zeta(u) = e^{it}\zeta(u).$$

2) shift S_t :

$$S_t\zeta(u) = \zeta(u - t).$$

3) multiplication π_t :

$$\pi_t\zeta(u) = e^{itu}\zeta(u).$$

4) dilation (tension) τ_t :

$$\tau_t\zeta(u) = \zeta(ue^t)e^{\frac{1}{2}t}.$$

5) Fourier-Mehler transform $\mathcal{F}_\theta$:

$$\mathcal{F}_\theta\zeta(u) = \int_{-\infty}^{\infty} K_\theta(u,v)\zeta(v)\,dv,$$

where

$$K_\theta(u,v) = \{\pi(1 - e^{2i\theta})\}^{-\frac{1}{2}} \exp[-\frac{i(u^2+v^2)}{2\tan\theta} + \frac{iuv}{\sin\theta}].$$

Their infinitesimal generators can easily be obtained :

	one-parameter subgroup	generator
	I_t	iI
	S_t	$-d/du \equiv \mathcal{s}$
(11)	π_t	$iu \equiv i\pi$
	τ_t	$u\frac{d}{du} + \frac{1}{2} \equiv \tau$
	$\mathcal{F}_\theta$	$-\frac{1}{2}i(d^2/du^2 - u^2 + 1) \equiv i\mathcal{f}$

There can naturally be introduced a generator $\sigma = \frac{1}{2}[\tau,\mathcal{f}]$ so that we are given a Lie algebra.

Now we can speak of the generators of $\{\tilde{g}_t\}$ which come from the group G_0 via the expression (10). Simple computations prove that for $\{g_t\} \subset G_0$ the infinitesimal generator

$$\frac{d}{dt} \tilde{g}_t \Big|_{t=0} = T \frac{d}{dt} g_t \Big|_{t=0}$$

exists and we have

$$(12) \quad \begin{aligned} & \quad T \\ I & \longrightarrow I \text{ (acting on } \mathfrak{G}) \\ \mathcal{s} & \longrightarrow -\frac{\partial}{\partial x} \\ \pi & \longrightarrow t\frac{\partial}{\partial x} + x \\ \tau & \longrightarrow 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \frac{1}{2} \\ f & \longrightarrow (t^2 - 1)\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + \frac{1}{2}(x^2 + t - 1) \\ \sigma & \longrightarrow (t^2 + 1)\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + \frac{1}{2}(x^2 + t) \end{aligned}$$

Proposition 4. The algebra generated by the differential operators in (12) is the same as $\tilde{\underline{g}}$ in Theorem 2.

§3. Heat equation in higher dimensional spaces.

In order to generalize the results in the last section we shall first present somewhat different observation of the algebra $\tilde{\underline{g}}$.

Since the space $\mathfrak{G}$ is topologized so as to be isomorphic to $\mathcal{S}_c$ via T, there is a one-to-one correspondence between a curve $\{v_s\}$ in $\mathfrak{G}$ and a curve $\{\zeta_s\}$ in $\mathcal{S}_c$. Given a smooth curve $\{\zeta_s\}$ in $\mathcal{S}_c$ with $\zeta_0 = \zeta$ we have the infinitesimal transformation α in such a way that $\alpha\zeta = \frac{d}{ds}\zeta_s\Big|_{s=0}$. For such an α there corresponds an infinitesimal transformation L :

$$(13) \qquad L = T \circ \alpha.$$

We are now able to state our requirements for the set $\mathbb{A}$ of the α's.

a) each $L = T\circ\alpha$, $\alpha \in A$, is of the form (5) satisfying i) and ii),

b) the generator f of $\{\mathcal{J}_\theta\}$ is a member of $\mathbb{A}$,

c) $\mathbb{A}$ forms a Lie algebra.

With these assumptions we can prove

Proposition 5. The maximal algebra generated by the $L = T\circ\alpha$, $\alpha \in \mathbb{A}$ satisfying the requirements a), b) and c) coincides with the Lie algebra $\tilde{\underline{g}}$.

Having had such an illustration we are now in a position to discuss the symmetry group of the heat equation in the n-dimensional Euclidean space R^n. The only thing we should note in such a generalization is that the generator f should be replaced by

(14) $$-\frac{1}{2}\sum_{j=1}^{n}\left(\frac{\partial^2}{\partial u_j^2} - u_j^2 + 1\right).$$

Consider the heat equation

(15) $$\frac{\partial}{\partial t} v(x,t) = \frac{1}{2}\Delta v(x,t), \quad \Delta = \sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}, \quad x = (x_1, x_2, \cdots, x_n).$$

Again we restrict our attention to the subspace $\mathcal{G}$ of the solution space to the equation (15) which is obtained in a similar manner to T using (9) :

(16) $$\zeta \longrightarrow v(x,t;\zeta) = (2\pi t)^{-\frac{n}{2}}\int_{R^n}\zeta(u)\exp\left[-\frac{1}{2t}\|x-u\|^2\right] du^n$$

where ζ is in the complex Schwartz space $\mathcal{S}_c(R^n)$.

Let L be defined as in (5) :

(17) $$L = \sum_{k=1}^{n} X_k\frac{\partial}{\partial x_k} + T\frac{\partial}{\partial t} + U$$

satisfying

i) X_k, T and U are analytic in both x and t,

ii) $QLv = 0$ whenever $Qv = 0$,

where Q is the operator given by

$$Q = \frac{\partial}{\partial t} - \frac{1}{2}\Delta.$$

The Lie algebra formed by such L's will be denoted by the same symbol $\tilde{\underline{g}}$.

Theorem 3. i) The algebra $\tilde{\underline{g}}$ is isomorphic to the algebra $\underline{g}$ formed by the α with the restrictions (20).

ii) The dimension of the symmetry group of Q is $n(n+3)/2 + 4$.

§4. Symmetries of wave functions for free particle.

In this section we shall discuss the symmetry group of the wave functions for a free particle. The Schrödinger equation is now of the form :

(18) $$\frac{1}{i}\frac{\partial\psi}{\partial t} = \frac{1}{2}\Delta\psi,$$

and the Green's function K is given by

(19) $$K(t, x - u) = (2\pi i t)^{-\frac{n}{2}} \exp[\frac{i}{2t}\| x - u\|^2].$$

Here we are, as usual, assuming that $m = 1$, and $h = 1$. In what follows we shall deal only with the one-dimensional case. The idea together with the technique is exactly the same as in the case of the heat equation. Far from that, the space $\mathcal{S}_c$, being a subspace of the complex $L^2(R^1)$, is fitting to express the initial states for a particle.

We still keep up the algebra $\underline{g}$ and use the expression

$$\psi(x,t;\zeta) = \int_{R^1} \zeta(u)K(t, x - u)du, \quad \zeta \in \mathcal{S}_c,$$

similar to (9). The collection of such ψ's is denoted by $\mathfrak{G}$. The operator $\alpha \in \underline{g}$ turns into an operator acting on $\mathfrak{G}$ via the bijection

$$T : \mathcal{S}_c \ni \zeta(u) \longrightarrow \psi(x,t;\zeta) \in \mathfrak{G}.$$

We list the correspondence below (see (12)).

$$
\begin{array}{lll}
 & & T \\
 & I & \longrightarrow I \\
 & \mathcal{S} & \longrightarrow -\dfrac{\partial}{\partial x} \\
 & \pi & \longrightarrow it\dfrac{\partial}{\partial x} + x \\
(20) & \tau & \longrightarrow 2t\dfrac{\partial}{\partial t} + x\dfrac{\partial}{\partial x} + \dfrac{1}{2} \\
 & f & \longrightarrow i(t^2 + 1)\dfrac{\partial}{\partial t} + itx\dfrac{\partial}{\partial x} + (x^2 + it - 1)/2. \\
 & \sigma & \longrightarrow i(t^2 - 1)\dfrac{\partial}{\partial t} + itx\dfrac{\partial}{\partial x} + (x^2 + it)/2.
\end{array}
$$

Theorem 4. The symmetry group of $\mathfrak{G}$ for a one-dimensional free particle is a six-dimensional Lie group and a basis of the associated Lie algebra is given by (20).

We should like to note that slight modification enables us to discuss the symmetry of a free particle in a constant external field F.

§5. Simple harmonic oscillator.

Finally we shall discuss the symmetry of a simple harmonic oscillator which is governed by the equation

$$(21)\qquad \frac{1}{i}\frac{\partial}{\partial t} \quad = \frac{1}{2}\frac{\partial^2}{\partial x^2}\psi - \frac{\omega^2}{2}x^2\psi.$$

The Green's function K can be expressed as

$$(22)\qquad K(t,x,u) = (\omega/2\pi i \sin \omega t)^{\frac{1}{2}} \exp[\frac{i\omega}{2\sin \omega t}\{(x^2+u^2)\cos \omega t - 2xu\}].$$

As before we obtain the following correspondence.

$$
\begin{array}{ll}
 & T \\
I & \longrightarrow I \\
\mathcal{S} & \longrightarrow L_{\mathcal{S}} \equiv -(i\omega x \sin \omega t + \cos \omega t \dfrac{\partial}{\partial x}) \\
\pi & \longrightarrow L_{\pi} \equiv x \cos \omega t + i\dfrac{\sin \omega t}{\omega}\dfrac{\partial}{\partial x} \\
\tau & \longrightarrow L_{\tau} = i\omega x^2 \sin 2\omega t + \cos^2 \omega t - \dfrac{1}{2} \\
 & \qquad + x \cos 2\omega t \dfrac{\partial}{\partial x} + \dfrac{\sin 2\omega t}{\partial}\dfrac{\partial}{\partial t}
\end{array}
$$

$$f \longrightarrow L \equiv \frac{i}{2}\frac{\partial}{\partial t}$$

$$\sigma \longrightarrow L_\sigma = x^2 \cos 2\omega t + \frac{i}{2\omega} \sin 2\omega t - \frac{i \cos 2\omega t}{\omega^2}\frac{\partial}{\partial t} + \frac{ix \sin 2\omega t}{\omega}\frac{\partial}{\partial x} .$$

Set

$$Q = i \frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{\omega^2}{2} x^2 .$$

Then we have, as is expected (see the commutation relation (6)),

$$[Q, L] = 0, \qquad [Q, L_\pi] = 0,$$

$$[Q. L_\tau] = 2 \cos 2\omega t \cdot Q,$$

$$[Q. L] = 0,$$

$$[Q. L_\sigma] = \frac{2i \sin 2\omega t}{\omega} \cdot Q .$$

We also note that we can generalize the result not only to higher dimensional cases but also to a case where a harmonic oscillator is driven by a constant external field. Furthermore, if one observes the Green's function of the Schrödinger equation for a charged particle in a constant external magnetic field in R^3, one might have hopes to get the symmetry group in a similar but somewhat complicated manner.

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