Is the Quantum State Real in the Hilbert Space Formulation?

Mani L. Bhaumik

Department of Physics and Astronomy, University of California, Los Angeles, USA. E-mail: bhaumik@physics.ucla.edu

Editors: ***Zvi Bern*** **&** ***Danko Georgiev***

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The persistent debate about the reality of a quantum state has recently come under limelight because of its importance to quantum information and the quantum computing community. Almost all of the deliberations are taking place using the elegant and powerful but abstract Hilbert space formalism of quantum mechanics developed with seminal contributions from John von Neumann. Since it is rather difficult to get a direct perception of the events in an abstract vector space, it is hard to trace the progress of a phenomenon. Among the multitude of recent attempts to show the reality of the quantum state in Hilbert space, the Pusey–Barrett–Rudolph theory gets most recognition for their proof. But some of its assumptions have been criticized, which are still not considered to be entirely loophole free. A straightforward proof of the reality of the wave packet function of a single particle has been presented earlier based on the currently recognized fundamental reality of the universal quantum fields. Quantum states like the atomic energy levels comprising the wave packets have been shown to be just as real. Here we show that an unambiguous proof of reality of the quantum states gleaned from the reality of quantum fields can also provide an explicit substantiation of the reality of quantum states in Hilbert space.
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1 Introduction

The debate about the reality of quantum states is as old as quantum physics itself. The objective reality underlying the manifestly bizarre behavior of quantum objects is conspicuously at odds with our daily classical physical reality. The scientific outlook of objective reality commenced with the precepts of classical physics that cemented our notion of reality for centuries. Discoveries starting in the last decade of the nineteenth century revealing the uncanny quantum world in the microscopic domain shook that perception.

With the particular exception of Einstein, who was the lone supporter of his postulated wave-particle duality for photons for about two decades, physicists continued to think of the microscopic quantum world within the confines of the ingrained classical physics. Finally, with de Broglie's proposed extension of the wave–particle duality to matter particles like electrons and its experimental verification, irrevocably opened the door ushering in the bizarre new world of quantum physics.

With some talented younger physicists like Schrödinger, Heisenberg, Born, Dirac and others, the development of the quantum physics proceeded in a break neck speed starting in early 1926. Although these efforts led to the most successful description of events in the atomic domain, the revelations of quantum physics were so weird that immediately a debate started about the significance of it all.

Soon the dispute came to a climax at the 1927 Solvay Conference in Belgium with the famed Bohr–Einstein debate. While Bohr insisting that there was no reality before a quantum state is measured, Einstein maintained there must be a reality even before a quantum state is observed. Almost a century later the debate is surprisingly still thriving. A conspicuous example is the substantial account presented in the recent review article by Matthew Leifer [1]. All of these contemporary considerations are conducted using the abstract Hilbert space formulation of quantum mechanics initiated by John von Neumann.

After John Bell's epochal paper [2] presented Bell's inequality and its numerous experimental substantiations, the reality of the quantum state is now more acceptable in contrast to the conclusion of earlier Bohr–Einstein debate. The most prominent recent theory of reality is presented by Matthew Pusey, Jonathan Barrett, and Terry Rudolph [3]. Other theories are also advanced by Lucien Hardy [4] as well as Roger Colbeck and Renato Renner [5]. However, none of these latest advances is considered to be entirely loophole free. Leifer's considerably extensive review [1] provides a distinct example of the difficulties of reaching a definitive conclusion using the circuitous way of deliberations in the abstract Hilbert space. Here we present a rather straightforward way to prove the reality of the quantum state.

In order to avoid any possible confusion, it would be prudent to agree upon the definition of reality. In this regard, we rely upon the generally acknowledged connotation of reality. We consider something to be physically real if it is independently observed by several people and they agree with each other that the result of their observations is the same. Accordingly, one could rely on the following notions of the distinguished contemporary physicists for our understanding of reality.

Referring to the outstanding developments in the cutting-edge quantum field theory or QFT in short, the distinguished Physics Nobel Laureate Frank Wilczek asserts

> the standard model is very successful in describing reality—the reality we find ourselves inhabiting. [6, p. 96]

Wilczek additionally enumerates

> The primary ingredient of physical reality, from which all else is formed, fills all space and time. Every fragment, each space-time element, has the same basic properties as every other fragment. The primary ingredient of reality is alive with quantum activity. Quantum activity has special characteristics. It is spontaneous and unpredictable. [6, p. 74]

Another esteemed Physics Nobel Laureate Steven Weinberg confirms

> the Standard Model provides a remarkably unified view of all types of matter and forces (except for gravitation) that we encounter in our laboratories, in a set of equations that can fit on a single sheet of paper. We can be certain that the Standard Model will appear as at least an approximate feature of any better future theory. [7]

Thus, it would be cogent to consider the space filling universal Quantum Fields as the primary ingredients of physical reality uncovered by us so far. An abundant proof of this can be encountered all around us in several different ways. The most direct convincing evidence comes from the fact that elementary particles like an electron has exactly the same properties, such as mass-energy, charge, spin etc., irrespective of when or where in the universe it comes into existence—in the big bang, in astrophysical processes throughout the eons or anywhere in a lab in the world.

A manifestation of the fluctuations of the quantum fields in a phenomenon like the electron anomalous g-factor agrees up to an unprecedented twelve decimal places when the experimental results are compared to the theoretical computation. Observed phenomena like Lamb shift, Casimir effect further assert the existence of the fluctuations of the quantum fields. A very dramatic confirmation of the indispensable effects of the quantum field fluctuations comes from the mass of the composite particles like protons and neutrons. The mass of the three valence quarks in a proton provided by the Higgs boson is only about 9 Mev while the total proton mass is a whopping 938 Mev. This magical "mass without mass" ascends from the endowment of quantum fluctuations.

Perhaps the most spectacular graphic evidence is provided by the observed anisotropy in the cosmic microwave background radiation with their presumed origin in the cosmic inflation in the early universe when the quantum fluctuations of the reputed inflaton field enormously expanded from the very microscopic to macroscopic dimensions providing seeds for galaxy formation afterwards. Any reasonable concept of physical reality should then owe its eventual origin to the fundamental reality of quantum fields and their characteristic attributes.

The elementary particles like electrons, one of the members of the initial act of material formation from the abstract but physical quantum fields, are quanta of the fields. Each of them can be rendered as a wave packet consisting of an admixture of the various fields. Accordingly, the wave packet function of the elementary particle ought to be considered as real as the primary quantum fields. More generally the fields whose quantization produces the 24 other observed elementary particles in Nature, such as the photons, W^+, W^-, Z^0, quarks and gluons, embodied in the Standard Model of particle physics should be just as real. Consequently so would be the composite particles like protons and neutrons. We can then consider the quantum states of atoms and molecules comprising the elementary and composite particles real as well.

This would empower us with confidence to extend the reality of the quantum states to those in Hilbert space formalism including all its operations to be as real as the wave packet of a single quantum particle and the wave functions of all quantum states of the Schrödinger and Heisenberg formulation.

2 The wave function of an electron

A particle like an electron arises as a quantized excitation of the underlying electron quantum field. Such an energy-momentum eigenstate of the field can be expressed as a specific Lorentz covariant superposition of field shapes of the electron field along with all the other quantum fields of the Standard Model of particle physics.

Superposition of field shapes in a one-particle state evolves in a simple wavelike manner with time dependence $e^{-\imath\omega t}$. The individual field shapes, each with their own computable dynamic time evolution, are actually the vacuum fluctuations comprising the very structure of the energy-momentum eigenstate. The quantum fluctuations are evanescent in the sense that they pass away soon after coming into being. But new ones are constantly boiling up to establish an equilibrium distribution so stable that their contribution to the electron g-factor, as mentioned earlier, results in a measurement accuracy of one part in a trillion [8].

The Lorentz covariant superposition of fluctuations of all the quantum fields in the one-particle quantum state can be conveniently depicted leading to a well behaved smooth wave packet that is everywhere continuous and continuously differentiable.

However, the detailed quantitative calculations are quite involved. For simplicity, we can get the result using a heuristic perspective. For this purpose, let us consider an isolated single quantum of a non-interacting electron quantum field. Since no force is acting on such an electron, its momentum would be constant and therefore its position would be indefinite since a regular ripple from a free electron field with a very well-defined energy and momentum is represented by a delocalized periodic function.

Recalling that the electron in reality is an admixture of all the quantum fields of the standard model, it should be noted that in the non-relativistic regime, there would not be enough energy to create any new particle. Consequently, the contribution of the different quantum fields to the single particle would comprise of irregular disturbances of the fields with energy off the mass shell, resulting in the electron ripple to be very highly distorted.

It is well known that such a pulse, no matter how deformed, can be expressed by a Fourier integral with weighted linear combinations of simple periodic wave forms like trigonometric functions [9]. The result would be a wave packet function to represent an electron that would embody a fundamental reality of the universe since all the amplitudes of the wave packet would consist of contributions of irregular disturbances of the various real primary quantum fields.

The wave function $\psi(x)$, for simplicity in one dimension, will be given by the inverse Fourier transform

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{\psi}(k)\, e^{\imath kx} dk \tag{1}$$

where $\tilde{\psi}(k)$ is a function that quantifies the amount of each wave number component $k = \frac{2\pi}{\lambda}$ that gets added to the combination.

From Fourier analysis, we also know that the spatial wave function $\psi(x)$ and the wave number function $\tilde{\psi}(k)$ constitute a Fourier transform pair. Therefore, we can find the wave number function through the forward Fourier transform as

$$\tilde{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(x)\, e^{-\imath kx} dx \tag{2}$$

Thus, the Fourier transform relationship between $\psi(x)$ and $\tilde{\psi}(k)$, where x and k are known as conjugate variables, can help us determine the frequency or the wave number content of any spatial wave function.

So far, we have considered only a fixed momentum. Obviously in any dynamical system, momentum would be expected to change all the time. As the momentum increases, the shape of the curve representing equation (2) in the momentum space would be taller and thinner while, being a conjugate variable, the curve in the position space would be shorter and wider. This naturally leads us to the postulate of famed uncertainty principle, which is evidently a natural consequence of the position and momentum being conjugate variables in wave packets.

Fortunately, a fairly rigorous underpinning of the wave packet for a single particle QFT state in position space for a scalar quantum field has been provided by Robert Klauber [10, p. 275]. Since particles of all quantum fields are invariably an admixture of contributions from essentially all the fields of the Standard Model, the wave packet function of a single particle of a scalar quantum field can be considered to be qualitatively representative of those of the vector fields like the electron quantum field.

The mathematical form in position space of a wave packet function of a particle for a scalar field is [10]

$$|\phi\rangle = \left| \int (2\pi)^{-\frac{3}{2}} A(k')\, e^{-\imath k' x}\, d^3k' \right\rangle \tag{3}$$

Using the more common convention of $e^{\imath k' x}$ in position space for an inverse Fourier transform, k instead of k' for momentum, $\psi(x)$ for $|\phi\rangle$ and $\tilde{\psi}(k)$ for $A(k')$, equation (3) in one space dimension becomes exactly equation (1), thus confirming that the single particle wave packet function given by (1) can be rigorously derived from QFT.

In order to determine the time evolution of the wave packet function, we need to incorporate the time term to the spatial function. Accordingly,

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{\psi}(k)\, e^{i(kx-\omega t)} dk \tag{4}$$

where $\omega = 2\pi\nu$ is the angular frequency.

On a cursory glance, the wave packet in equation (4) looks very similar to the one that has always been used in quantum mechanics so far, for example, to solve Schrödinger equation for the hydrogen atom. But in reality the wave function given by (4) is substantially different in character since the amplitudes are fundamentally real physical entities owing their origin to the primary reality of different quantum fields.

When quantum mechanics, with its essential requirement of a wave packet function to represent a localized particle, emerged in the atomic domain of physics and commenced explaining its mysterious accomplishments with uncanny consistency, it appeared totally contrary to our intuition developed from classical physics. Naturally, the wave packet of a particle was considered to be just a fictitious mathematical construct merely necessary for carrying out calculations.

At best, the amplitudes of a wave packet are considered to be real probability amplitudes to find the particle during a measurement. This is consistent with the fact that the amplitudes of the wave packet represent distinctly different properties of various quantum fields except energy. The energy represented by any amplitude of the wave packet has the same characteristics even though the other properties of the amplitude may have different attributes corresponding to their origin in the various fields.

Although the amplitude of the wave function is complex-valued, its squared modulus is real-valued. Following Einstein, Born [11] had interpreted the square of the modulus of the wave amplitude as probability for the occurrence of the particle. Thus probability $p(x)$ of finding the particle at the position x in the interval x and $x + dx$ is

$$p(x) = \psi^*(x,t)\, \psi(x,t)\, dx \tag{5}$$

Even a century later, quantum mechanics still perplexes most people including many scientists. This is consistent with the fact that on an average at least one popular book on the riddles of quantum mechanics is still being published every year by notable authors. However, as has been explained in detail [12], there appear to be plausible answers to the enigmas of quantum mechanics derived from the discoveries of the quantum field theory of the standard model.

For example, a wave packet function is localized and therefore can represent a quantum particle, but just holistically, since only the totality of the wave packet represents all the conserved quantities of the energy-momentum eigenstate of a particle such as mass, charge, and spin. This particularly important fact requires the total wave function to collapse during measurement, which has been one of the most puzzling aspects of quantum mechanics. It can now be evident in fact as a requirement because of the particular factual nature of the wave packet.

Most of the topics in quantum mechanics, extensively used in many diverse fields like chemistry, biology, material science, quantum information science, involve nonrelativistic quantum mechanics since the particle speeds are lower than the speed of light. The very extensively studied quantum objects are the hydrogen atom and the quantum harmonic oscillator.

In the hydrogen atom, the electron revolves in the central field of the proton producing discrete energy levels that correspond to characteristics standing wave patterns called orbitals. These are calculated by using the time-independent Schrödinger equation using the real electron wave function in equation (1). The normalized hydrogen wave functions, using polar coordinates for convenience, can be found in textbooks on quantum mechanics for example in [13, § 4.2].

The foremost aspect to underscore here is that the standing wave patterns representing the quantum states of the hydrogen atom are embodiments of the wave packet function in equation (1) that is based on the primary reality of the quantum fields. These standing wave patterns of the orbitals representing the quantum states are thus objectively as real as the primary quantum fields.

In fact, reality of some of these orbitals has been established by experimental measurements [14]. Consequently it would be compelling to infer that all quantum states of quantum mechanics are based on reality, which can be extended to the quantum states described in the abstract Hilbert space formalism, the brain child of David Hilbert and John von Neumann.

3 Emergence of Hilbert space

Although von Neumann was an exceptionally talented polymath, his contribution to the development of the Hilbert space formalism is one of his outstanding legacies to science. Following Hilbert's preliminary thoughts, he not only initiated the formulation but also coined the name Hilbert space providing an advanced comprehensive mathematical foundation of quantum theory.

3.1 Von Neumann's contribution

Shortly after receiving his PhD in Physics from the University of Budapest in 1926, von Neumann decided to work under David Hilbert, likely the foremost mathematician of the time, for completing his habilitation program. At the University of Göttingen, considered to be the premiere center for mathematics in the world for quite a while, Hilbert became interested in the indispensable application of intricate mathematics in the emerging topics like relativity and quantum physics. Perhaps with that perception, he was delighted to attract this mathematical genius to his group and secured for him a six-month fellowship from the Rockefeller financed International Education Board with additional letters of recommendation from two of his former students, Richard Courant and Hermann Weyl.

With all the exhilaration of the emergence of quantum physics in 1926, Hilbert opted to devote his long-standing winter lectures on mathematics that year to the novel topic and convinced von Neumann to join the effort under his guidance. These impressive lectures covered almost all the contemporary developments including Heisenberg's matrix mechanics, Schrödinger's wave mechanics, Born's probability interpretation, and finally Jordan's and Dirac's transformation theory to unify them into a single formalism. These lectures were published [15], which was the last publication by Hilbert on the subject possibly because of his failing health he could not keep up with the fast moving subject. However, he was thrilled by the marvelous coincidence that the mathematical formalism developed by him before the advent of quantum mechanics ideally harmonized with the formulation of the mathematical apparatus necessary for an elegant and rigorous treatment of the emerging new subject.

Von Neumann, as the most outstanding of Hilbert's heirs, continued to carry the torch and in 1927 published three brilliant papers [16–18] that placed quantum mechanics on a meticulous mathematical foundation including a rigorous proof of the equivalence of matrix and wave mechanics. He also coined the name Hilbert space that soon took off with every one starting to use it for the new formalism. The significance of the concept of a Hilbert space was underlined with the realization that it offers one of the best mathematical formulations of quantum mechanics.

His habilitation (qualification to conduct independent university teaching) was completed in December of 1927, and he began lecturing as a Privatdozent at the University of Berlin in 1928 at the age of 25, the youngest Privatdozent ever elected in the university's history in any subject. In 1929, he briefly became a Privatdozent at the University of Hamburg, where the prospects of becoming a tenured professor were better, but in October of that year a better offer presented itself when he was invited to Princeton University. In 1933, he was offered a lifetime professorship at the Institute for Advanced Study in New Jersey. He remained a mathematics professor there until his death, although he had announced his intention to resign and become a professor at large at the University of California, Los Angeles.

As a true polymath, he made significant contributions to various diverse fields throughout his momentous career. It is fascinating to note that he himself considered his pioneering contributions to the foundations of quantum mechanics especially the Hilbert state formalism as his most important scientific contribution.

In a recent article, Klaas Landsman [19] summarizes von Neumann's main accomplishments in his pioneering contribution to the development of the Hilbert state formalism of quantum mechanics, with the admiring comment that any one of these would have been a significant achievement for a 23 year old. These are:

> 1. Axiomatization of the notion of a Hilbert space (previously known only in examples).
>
> 2. Establishment of a spectral theorem for (possibly unbounded) self-adjoint operators.
>
> 3. Axiomatization of quantum mechanics in terms of Hilbert spaces (and operators):
>
> (a) Identification of observables with (possibly unbounded) self-adjoint operators.
> (b) Identification of pure states with one-dimensional projections (or rays).
> (c) Identification of transition amplitudes with inner products.
> (d) A formula for the Born rule stating the probability of measurement outcomes.
> (e) Identification of general states with density operators.
> (f) Identification of propositions with closed subspaces (or the projections thereon). [19]

3.2 Dirac's contribution

In the meantime, another star shined in the rising quantum firmament. Paul Dirac received his PhD in Physics under Ralph Fowler at Cambridge in 1926 just about the same time did von Neumann. It is quite interesting to note that both of them acquired a PhD in engineering before veering off to theoretical physics. Dirac, however, was the first physicist to ever get a PhD in quantum mechanics, more specifically in matrix mechanics, which he improved by formally characterizing it using non-commutative operators and Poisson's brackets.

Schrödinger initially revealed the similarity between the two seemingly different formulations of quantum mechanics, his own wave mechanics and Heisenberg's matrix mechanics. In early 1927, Dirac [20] and Jordan [21, 22], independently of one another, published their versions of a general formalism tying the various forms of the new quantum theory together in full generality. This formalism has come to be known as the Dirac–Jordan transformation theory.

A few months later, in response to these publications by Dirac and Jordan, John von Neumann published his Hilbert space formalism for quantum mechanics. Jordan's work in time went into oblivion while Dirac firmly embraced the Hilbert space formalism advanced by von Neumann. Additionally Dirac introduced the elegant bra-ket notation as well as delta functions. Von Neumann did not considered use of the delta functions to be rigorous. However, a later version oddly dubbed the rigged Hilbert space was constructed, which restored rigorousness to Dirac's approach.

The contributions of von Neumann and Dirac to the foundations of quantum theory using Hilbert space are considered to be equal as reflected in the embraced phrase the Dirac–von Neumann axioms in mathematical formulation of quantum mechanics. Compilations of their innovation were published as books by Paul Dirac [23] in 1930 and John von Neumann [24] in 1932. In many ways their contributions are mutually complementary. For example, while von Neumann's contributions often emphasized mathematical rigor, Dirac emphasized pragmatic concerns such as utility and intuitiveness.

4 Fundamentals of Hilbert Space Formalism

The Hilbert space, acknowledged as the most appropriate for mathematical formulations of quantum mechanics, is a square integrable, complex, linear, abstract space of vectors possessing a positive definite inner product assured to be a number. The states of a quantum mechanical system are vectors in a multidimensional Hilbert space containing an orthonormal basis set of eigenfunctions. The observables are Hermitian operators on that space, and measurements are orthogonal projections. Unitary operators are used for changing a vector from one basis set to another. For elegance, Dirac's bra-ket notation is used to characterize the vectors and delta functions are used to express orthogonality of vectors.

No physical property of a quantum system changes by going from wave or matrix mechanics rendering to the Hilbert state formalism. A very simple example illustrates the essence. The normalization relation for a single particle wave packet function in position space is presented in the wave mechanical description as

$$\int_{-\infty}^{+\infty} \psi^*(x)\,\psi(x)\,dx = 1 \tag{6}$$

Using Dirac's bra-ket notation equation (6) becomes

$$\int_{-\infty}^{+\infty} \psi^*(x)\,\psi(x)\,dx = \langle\psi|\psi\rangle = 1 \tag{7}$$

where the bra vector $\langle\psi|$ is complex conjugate transpose of the ket $|\psi\rangle$.

The orthogonality relation is given by

$$\int_{-\infty}^{+\infty} \psi_m^*(x)\,\psi_n(x)\,dx = \langle\psi_m|\psi_n\rangle = \delta_{mn} \tag{8}$$

where δ_{mn} is the Kronecker delta

$$\delta_{mn} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases} \tag{9}$$

The quantum wave functions, for example, the solutions of the Schrödinger equation describing physical states in wave mechanics are considered as the set of components $\psi(x)$ of the abstract vector Ψ called the state vector. However, the state vector does not depend upon any particular choice of coordinates. The same state vector can be described in terms of the wave function in position or momentum state or write as an expansion in wave functions $\psi_n(x)$ of definite energy

$$\Psi = \sum_n c_n \psi_n(x) \tag{10}$$

suggesting that every linear combination of vectors in a Hilbert space is again a vector in the Hilbert space. The normalized square moduli $|c_n|^2$ of the complex coefficients are then interpreted as the probability for the system to be in the state ψ_n analogous to Born's initial proposal where $|\psi(x)|^2$ is interpreted as the probability density for the particle to be at x.

4.1 Projective measurement

Every vector in the Hilbert space as a linear combination of the basis vectors ψ_n with complex coefficients c_n can be expressed in Dirac's notation as

$$|\psi\rangle = \sum_n c_n|\psi_n\rangle. \tag{11}$$

Multiplying both sides of equation (11), by $\langle\psi_m|$ gives

$$\langle\psi_m|\psi\rangle = \sum_n c_n\langle\psi_m|\psi_n\rangle. \tag{12}$$

Since $\langle\psi_m|\psi_n\rangle = \delta_{mn}$, $\langle\psi_n|\psi_n\rangle = 1$

$$c_n = \langle\psi_n|\psi\rangle \tag{13}$$

which is the transition amplitude of state $|\psi\rangle$ to state $|\psi_n\rangle$.

Inserting equation (13) into equation (11) gives

$$|\psi\rangle = \sum_n |\psi_n\rangle\langle\psi_n|\psi\rangle. \tag{14}$$

Defining a projection operator $\hat{P}_n = |\psi_n\rangle\langle\psi_n|$, equation (14) becomes

$$|\psi\rangle = \sum_n \hat{P}_n|\psi\rangle. \tag{15}$$

leading to

$$\sum_n \hat{P}_n = \hat{I} \tag{16}$$

signifying that the sum of all the projection operators is unity.

The outer product $|\psi\rangle\langle\psi|$ is called the projection operator since it projects an input ket vector $|\phi\rangle$ into a ray defined by the ket $|\psi\rangle$, as follows

$$|\psi\rangle\langle\psi||\phi\rangle = (\langle\psi|\phi\rangle)\,|\psi\rangle \tag{17}$$

with a probability $|\langle\psi|\phi\rangle|^2$ as the inner product between two state vectors is a complex number known as probability amplitude. This is usually known as projective measurement and we will notice that it is important for the measurement of a mixed state consisting of an ensemble of pure states.

4.2 Operator Valued Observables

In a quantum system, what can be measured in an experiment are the eigenvalues of various observable physical quantities like position, momentum, energy, etc. These observables are represented by linear, self-adjoint Hermitian operators acting on Hilbert space.

Each eigenstate of an observable corresponds to eigenvector $|\psi_n\rangle$ of the operator $\hat{A}$, and the associated eigenvalue λ_n corresponds to the value of the observable in that eigenstate

$$\hat{A}|\psi_n\rangle = \lambda_n|\psi_n\rangle \tag{18}$$

For a self-adjoint Hermitian operator, quantum states associated with different eigenvalues of $\hat{A}$ are orthogonal to one another

$$\langle\psi_m|\psi_n\rangle = \delta_{mn} \tag{19}$$

The possible results of a measurement are the eigenvalues of the operator, which explains the choice of self-adjoint operators for all the eigenvalues to be real. The probability distribution of an observable in a given state can be found by computing the spectral decomposition of the corresponding operator. For a Hermitian operator $\hat{A}$ on an n-dimensional Hilbert space, this can be expressed in terms of its eigenvalues λ_n following equation (18) as

$$\hat{A} = \sum_n \lambda_n|\psi_n\rangle\langle\psi_n| \tag{20}$$

If the observable $\hat{A}$, with eigenstates $\{|\psi_n\rangle\}$ and spectrum $\{\lambda_n\}$ is measured on a system described by the state vector $|\psi\rangle$, the probability for the measurement to yield the value λ_n is given by

$$p(\lambda_n) = |\langle\psi_n|\psi\rangle|^2 \tag{21}$$

After the measurement the system is in the eigenstate $|\psi_n\rangle$ corresponding to the eigenvalue λ_n found in the measurement, which is called the reduction of state. This seemingly unphysical reduction of state is a shortcut for the description of the measurement process and the fact that the system becomes entangled with the state of the macroscopic measurement equipment. The entanglement leads to the necessary decoherence of the superposition of states of the measured system leading solely to the observed eigenvalue with its specific probability.

4.3 Unitary Operators

If the inverse of an operator $\hat{U}$ is the adjoint operator

$$\hat{U}^{-1} = \hat{U}^\dagger \tag{22}$$

then this operator is called a unitary operator and

$$\hat{U}^\dagger\hat{U} = \hat{U}\hat{U}^\dagger = \hat{I} \tag{23}$$

Unitary operators play a significant role in quantum mechanics representing transformation in the state space. Time evolution is just one example since the evolution of state vectors with time is unitary. This means the state vector changes smoothly preserving the total probability.

4.4 Quantum Entanglement

In all of the operations in the Hilbert space formalism, the reality of the quantum state is not altered. We now consider the landmark paper [25], of Einstein–Podolsky–Rosen (EPR) where they presented a thought experiment, which attempted to show that "the quantum-mechanical description of physical reality given by wave functions is not complete", indicating the possible existence of some hidden variables to explain the apparent violation of locality enshrined in Einstein's theory of relativity. However, in 1964, John Stewart Bell offered [2] his celebrated theoretical explanation known as Bell's inequality revealing that one of the key assumptions, the principle of locality, as applied to the kind of hidden variables interpretation hoped for by EPR, was mathematically inconsistent with the predictions of quantum theory.

Among an overabundance of experimental efforts performed under the generic topic of quantum entanglement, a loophole-free Bell inequality violation has presumably claimed to have been conclusively demonstrated [26]. These experiments demonstrate that although there are some possible non-local correlation in quantum systems, it does not violate causality since no information can be transferred faster than the speed of light consistent with the theory of relativity.

A detailed discussion has been provided by Bhaumik [27] illustrating how the reality of the quantum state is not violated by quantum entanglement. In brief, the expectation value of an overall pure quantum state of a composite system does not change although in an entanglement experiment, wave function of a constituent mixed subsystem can change violating locality.

This is due to the fact that the actions of an experimentalist on a subsystem of an entangled state can be described as applying a unitary operator to that subsystem. Although this produces a change on the wave function of the complete system, such a unitary operator cannot change the density matrix describing the rest of the system. In brief, if distant particles 1 and 2 are in an entangled state, nothing an experimentalist with access only to particle 1 can do that would change the density matrix of particle 2.

The density matrix $\hat{\rho}$ of an ensemble of states $|n\rangle$ with probabilities p_n is given by

$$\hat{\rho} = \sum_n p_n |n\rangle\langle n| \tag{24}$$

where $|n\rangle\langle n|$ are projection operators and the sum of the probabilities are $\sum_n p_n = 1$. Thus there can be various ensembles of states with each one having its own probability distribution that will give the same density matrix.

The expectation value of an observable $\langle \hat{A} \rangle$ is

$$\langle \hat{A} \rangle = \mathrm{Tr}\left(\hat{\rho}\hat{A}\right) \tag{25}$$

Furthermore, the time evolution of $\hat{\rho}$ only depends upon the commutator of $\hat{\rho}$ with the Hamiltonian $\hat{H}$ following the von Neumann equation

$$\imath\hbar \frac{d}{dt}\hat{\rho}(t) = [\hat{H}, \hat{\rho}(t)] \tag{26}$$

Thus as long as $\hat{\rho}$ remains the same, a change in the wave function of particle 2 does not affect any observable since all observable results can be predicted from the density matrix, without needing to know the ensemble used to construct it. Consequently no useful signal can be sent using entanglement and nonlocality between two observers separated by an arbitrary distance thereby no violation of the sanctified tenets of special theory of relativity ensues.

5 Conclusion

It would be cogent to acknowledge the space filling, ever immutable, universal quantum fields to constitute the primary ingredients of reality uncovered by us so far. Elementary particles like electrons are quanta of these fields and as such are as real as the fields themselves. And so is the wave packet functions depicting elementary particles constructed in terms of the attributes of the quantum fields.

The quantum states of, for example, a hydrogen atom portrayed in terms of the wave functions that are solutions of the Schrödinger equation using the wave packet function ought to be real as well. In fact, recent experimental observations confirm this reality. It is now well known that the same quantum states can also be described in terms of Heisenberg's matrix mechanics. Eventually it was recognized that the most elegant and efficient way to treat the quantum states is by utilizing the abstract Hilbert space formalism developed predominantly by John von Neumann and Paul Dirac.

There have been concerns about whether the quantum states described by the abstract Hilbert space have any reality. Using appropriate assumptions several recent theoretical treatments suggest that the quantum states in Hilbert space are real. However, the somewhat roundabout ways of going about the proof leave rooms for possible loopholes. We show here that the reality of the quantum states can be confirmed in a straightforward manner relying on the primary reality of quantum fields.

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