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On the fixed point set of a unipotent  
transformation on generalized flag varieties

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Introduction

Let  $G = GL_n$  be the general linear group defined over a field  $K$ . Let  $P$  be a parabolic subgroup of  $G$ . For a unipotent element  $u$  of  $G$ , put

$$(G/P)_u = \{gP \in G/P \mid u \cdot gP = gP\},$$

the fixed point subvariety of  $u$  in a generalized flag variety  $G/P$ . The author [2] obtained a locally closed partition of  $(G/P)_u$  into affine spaces. This is a generalization of a result of N. Spaltenstein [3]. The purpose of this report is to give an alternate proof to the result of [2]. The proof in this report is simpler than that of [2] and, it seems, applicable for other groups. Some applications (in particular, on the character theory of the finite general linear groups) of this paper are described in [1] with other results on the Springer representations of Weyl groups for reductive groups.

Notations. Let  $V$  be a vector space over a field  $K$ . If  $\{x_\nu \mid \nu \in I\}$  is a subset of  $V$ , then we denote by  $\langle x_\nu \mid \nu \in I \rangle$  the subspace spanned by  $\{x_\nu\}$ . We denote by  $\mathbb{N}$  the set of all natural numbers. For  $n \in \mathbb{N}$ , let  $A^n$  be the  $n$ -dimensional affine space over  $K$ . If  $\{X_\nu\}$  is a family of subsets of a set  $X$ , then  $X = \coprod_\nu X_\nu$  means the direct sum decomposition of  $X$ . A partition  $\lambda$  of  $n$  means a sequence  $\lambda = (n_1, n_2, \dots, n_r)$  such that  $n_i \in \mathbb{N}$  ( $i=1, \dots, r$ ),  $n_1 + n_2 + \dots + n_r = n$  and  $n_1 \geq n_2 \geq \dots \geq n_r > 0$ .

### §1. Preliminaries

Let  $G, P$  and  $u$  be as in the introduction. There exists  $\mu = (\mu_1, \dots, \mu_r)$  (resp.  $\lambda = (\lambda_1, \dots, \lambda_s)$ ), a partition of  $n$ , such that  $P$  (resp.  $u$ ) is conjugate to  $P_\mu$  (resp.  $u_\lambda$ ), where  $P_\mu$  is a parabolic subgroup of  $G$  whose Levi subgroup is isomorphic to  $\prod_{i=1}^r GL_{\mu_i}$  (resp. the unipotent element of Jordan type diag  $(J_1, \dots, J_s)$ ,  $J_i = \left( \begin{array}{cccc} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & \\ & & & 1 \end{array} \right)_{\lambda_i}$ ). Then  $(G/P)_u$  is isomorphic to  $(G/P_\mu)_{u_\lambda}$ .

For  $\lambda$ , a partition of  $n$ , we can associate the Young diagram of type  $\lambda$ , in the usual way.

Definition 1. Let  $\lambda$  and  $\mu$  be partitions of  $n$ . Put  $\mu = (\mu_1, \dots, \mu_r)$

(1) A  $\mu$ -tableau of type  $\lambda$  is a Young diagram of type  $\lambda$  whose nodes are numbered with the figures from 1 to  $r$  such that the cardinality of the nodes with figure  $i$  is  $\mu_i$ .

(2) A  $\mu$ -tableau is said to be semi-standard if, in each row,

the sequence of the figures on the nodes increases (may be stationary).

Example. If  $\lambda = (3, 2, 1)$  and  $\mu = (2, 2, 1, 1)$ , then

$$(1) \begin{array}{|c|c|c|} \hline 1 & 3 & 1 \\ \hline 4 & 2 & \\ \hline 2 & & \\ \hline \end{array} \quad (2) \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 & \\ \hline 4 & & \\ \hline \end{array}$$

are  $\mu$ -tableaus of type  $\lambda$  ((2) is semi-standard). If  $\mu = (\mu_1, \mu_2)$ , then, for simplicity, we may write  $\boxed{1}$  as  $\begin{array}{|c|} \hline \text{diagonal lines} \\ \hline \end{array}$  and  $\boxed{2}$  as  $\square$ .

Example. If  $\lambda = (3, 2, 1)$  and  $\mu = (4, 2)$ , then

$$\begin{array}{|c|c|c|} \hline 2 & 1 & 1 \\ \hline 1 & 2 & \\ \hline 1 & & \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline \text{diagonal lines} & \text{diagonal lines} & \text{diagonal lines} \\ \hline \text{diagonal lines} & \square & \\ \hline \text{diagonal lines} & & \\ \hline \end{array}$$

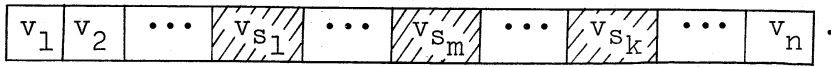
Let  $\widetilde{L}_\mu(\lambda)$  (resp.  $L_\mu(\lambda)$ ) be the set of all  $\mu$ -tableaus of type  $\lambda$  (resp. the set of all semi-standard  $\mu$ -tableaus of type  $\lambda$ ).

§2. The Grassmann manifold

Let  $V = \langle v_1, \dots, v_n \rangle$  be an  $n$ -dimensional vector space over a field  $K$  with basis  $\{v_1, \dots, v_n\}$ . We denote by  $G_k(V)$  the Grassmann manifold defined by the set of all  $k$ -dimensional subspaces of  $V$ . Put  $L_k = \{(s_1, \dots, s_k) \in \mathbb{N}^k \mid 1 \leq s_1 < s_2 < \dots < s_k \leq n\}$ , a set of increasing sequence of natural numbers. For  $s = (s_1, \dots, s_k) \in L_k$ , let  $S_s$  be the set of vector subspaces defined by

$$\{ \langle v_{s_m} + \sum_{i>s_m} a_{mi} v_i \mid 1 \leq m \leq k \rangle \mid a_{mi} \in K \}.$$

We remark that we can associate to  $S_s$  the following tableau :



The next lemma gives a well-known cellular decomposition of the Grassmann manifold.

Lemma 1. (1)  $G_k(V) = \bigsqcup_{s \in L_k} S_s,$

(2)  $\langle v_{s_m} + \sum_{i > s_m} a_{mi} v_i \mid 1 \leq m \leq k \rangle = \langle v_{s_m} + \sum_{I_m} a'_{mi} v_i \mid 1 \leq m \leq k \rangle,$

where  $I_m$  is a condition:  $i > s_m, i \neq s_{m+1}, \dots, s_k,$

(3) put  $e(s) = \sum_{m=1}^k \{(n-s_m) - (k-m)\},$  then by (2), we have an

isomorphism  $A^{e(s)} \xrightarrow{\sim} S_s$  under a mapping:  $(\dots, a_{mi}, \dots) \mapsto \langle v_{s_m} + \sum_{I_m} a_{mi} v_i \mid 1 \leq m \leq k \rangle,$

(4)  $S_s$  is a locally closed subset of  $G_k(V)$  in the Zariski topology.

Let  $N$  be a nilpotent transformation of  $V$ . We take a Jordan basis  $\{w_{ij_i} \mid 1 \leq j_i \leq l_i\}$  of  $V$  satisfying the following requirement:

$$l_1 \leq l_2 \leq \dots \leq l_d, \quad Nw_{ij} = w_{i+1j} \quad \text{and} \quad Nw_{dj} = 0.$$

We remark that this basis forms a Young diagram of degree  $n$  and of type  $\lambda = \lambda(N) = (\underbrace{d, \dots, d}_{l_1}, \dots, \underbrace{1, \dots, 1}_{l_d - l_{d-1}}).$

Example. Let  $\dim V = 8$ . If  $N$  has two Jordan blocks of dimension 3 and one Jordan block of dimension 2, then

$w_{31}$	$w_{21}$	$w_{11}$
$w_{32}$	$w_{22}$	$w_{12}$
$w_{33}$	$w_{23}$	

Put  $u_\lambda = 1_n + N$ ,  $1_n$  is the identity matrix of size  $n$ , then  $u_\lambda$  is a unipotent element of  $GL_n = GL(V)$  of Jordan type  $\lambda$ . We place  $w_{ij}$  in the following way :

$$v_1 = w_{1\ell_1}, \dots, v_{\ell_1} = w_{11}, v_{\ell_1+1} = w_{2\ell_2}, \dots, v_{\ell_1+\ell_2} = w_{21}, \dots, v_n = w_{d1}.$$

For  $\bar{k} = (k, n-k)$  and  $\lambda = \lambda(N)$ , put

$$\widetilde{L_{\bar{k}}}(\lambda) = \{ \bar{k}\text{-tableaus of type } \lambda \} = \left\{ \begin{array}{c} \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} & \left| \begin{array}{l} \text{the number of} \\ \square \text{ is } k \end{array} \right. \end{array} \right\}.$$

We have a bijective correspondence between  $\widetilde{L_{\bar{k}}}(\lambda)$  and  $L_k$  by making a sequence  $(s_1, \dots, s_k) \in L_k$  if  $v_{s_i}$  is in a node  $\square$ .

Then by Lemma 1, (1), we can write  $G_k(V) = \bigsqcup_{\ell \in \widetilde{L_{\bar{k}}}(\lambda)} S_\ell$ . Put

$$G_k(V)^N = \{ W \in G_k(V) \mid N(W) \subset W \},$$

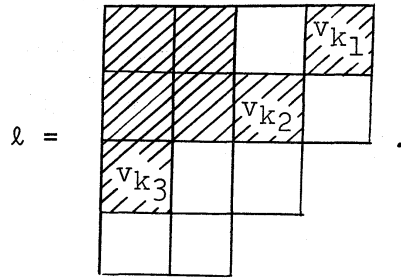
$$S_\ell^N = S_\ell \cap G_k(V)^N.$$

Let  $L_{\bar{k}}(\lambda)$  be the set of all semi-standard  $\bar{k}$ -tableau of type

$$\lambda = \lambda(N), \text{ e.g. } L_{\bar{k}}(\lambda) = \left\{ \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \right\}.$$

Lemma 2. Let  $\ell \in \widetilde{L_{\bar{k}}}(\lambda)$ . In order to have  $S_\ell^N \neq \emptyset$ , it is necessary and sufficient that  $\ell \in L_{\bar{k}}(\lambda)$

Proof. We assume  $S_\ell^N \neq \emptyset$ . For this  $\ell \in \widetilde{L_{\bar{k}}}(\lambda)$ , let  $v_{k_m} = w_{i_m j_m}$  ( $m = 1, 2, \dots$ ;  $k_1 < k_2 < \dots$ ) be the  $w_{ij}$  which is in the rightest node in the  $m$ -th row from the top in the tableau obtained by extracting the nodes  $\square$  from  $\ell$ . For example



For  $W \in S_{\lambda}^N$ , there exists  $a_{mj} \in K$  such that

$$v_{k_m} + \sum_{j>k_m} a_{mj} v_j \in W \quad (m=1,2,\dots).$$

By  $N(W) \subset W$ , we have

$$N^{h_m}(v_{k_m} + \sum_{j>k_m} a_{mj} v_j) \in W \quad (0 \leq h_m \leq d-i_m).$$

The set  $\{N^{h_m}(v_{k_m} + \sum_{j>k_m} a_{mj} v_j) \mid \substack{m=1,2,\dots \\ 0 \leq h_m \leq d-i_m}\}$  is linearly

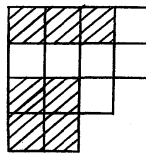
independent and the number of its elements is greater than  $k = \dim W$ . Hence the set

$$\left\{ N^{h_m}(v_{k_m} + \sum_{j>k_m} a_{mj} v_j) \mid \substack{m=1,2,\dots \\ 0 \leq h_m \leq d-i_m} \right\}$$

must be a basis of  $W$ , which implies, by definition,

$$\lambda \in L_{\overline{K}}(\lambda).$$

Conversely, for a semi-standard  $\lambda \in L_{\overline{K}}(\lambda)$ , put



$W = \langle w_{ij} \text{ in } \square \rangle$ . Then  $W \in S_{\lambda}^N$ . This means that  $S_{\lambda}^N \neq \emptyset$ . The proof of the lemma is thus completed.

Let  $\lambda \in L_{\overline{K}}(\lambda)$ . In the tableau  $\lambda$ , let  $v_{k_m} = w_{i_m j_m}$  ( $m=1,2,\dots$ ;  $k_1 < k_2 < \dots$ ) be as in the proof of Lemma 2. Put

$$M_\ell = \left\{ N^{h_m} v_{k_m} \mid \begin{array}{l} m=1,2,\dots, \\ 0 \leq h_m \leq d-i_m \end{array} \right\} = \{w_{ij} \text{ in } \square\}.$$

Lemma 3. For  $\ell \in L_{\overline{k}}(\lambda)$ , we have

$$S_\ell^N = \left\{ \left\langle N^{h_m} (v_{k_m} + \sum_{j>k_m} a_{mj} v_j) \mid \begin{array}{l} m=1,2,\dots, \\ 0 \leq h_m \leq d-i_m \end{array} \right\rangle \mid \begin{array}{l} a_{mi} \in K, \\ a_{mi}=0 \text{ if } v_i \in M_\ell \end{array} \right\}.$$

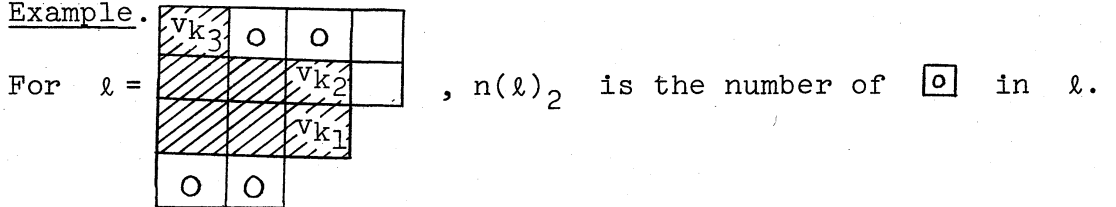
Proof. It is obvious that  $S_\ell^N$  contains the right-hand side. Apply Lemma 1 (2) to elements of  $S_\ell$ . Then the proof of this lemma is similar to that of Lemma 2. Thus the lemma.

Definition 2. Let  $\ell \in L_{\overline{k}}(\lambda)$ . For  $v_{k_m} = w_{i_m j_m}$  ( $m=1,2,\dots$ ), let  $n(\ell)_m$  be the number of  $\square$  in  $\ell$  which lies in the left-hand side of the column on which  $v_{k_m}$  lies, or in the upper position than that of  $\square_{v_{k_m}}$  in the column on which  $v_{k_m}$  lies. Put

$$n(\ell) = n(\ell)_1 + n(\ell)_2 + \dots.$$

We remark that  $n(\ell) \leq e(\ell)$ , where  $e(\ell)$  is defined in Lemma 1 (3).

Example.



In this case  $n(\ell)_2 = n(\ell)_1 = 4$ ,  $n(\ell)_3 = 0$  and

$$n(\ell) = n(\ell)_1 + n(\ell)_2 + n(\ell)_3 = 4 + 4 + 0 = 8.$$



In view of Lemma 1, (3), we have :

Corollary. For  $\ell \in L_{\overline{k}}(\lambda)$ , we have

$$S_{\ell}^N \simeq \mathbb{A}^{n(\ell)}.$$

Put  $T_{\ell} = S_{\ell}^N$ . By Lemma 3,  $T_{\ell}$  is a closed subset (linear subvariety) of  $S_{\ell}$ . Summing up the above statements, we have :

Theorem 1. Let the notations be as above. We have

$$G_k(V)^N = \bigsqcup_{\ell \in L_{\overline{k}}(\lambda)} T_{\ell},$$

where  $T_{\ell}$  is a locally closed subset of  $G_k(V)^N$  and isomorphic to an  $n(\ell)$ -dimensional affine space  $\mathbb{A}^{n(\ell)}$ .

### §3. The flag manifold

Let  $\mu = (\mu_1, \dots, \mu_p, \mu_{p+1})$  be a partition of  $n$ . Put  $k_j = \mu_1 + \dots + \mu_j$  ( $j=1, 2, \dots, p, p+1$ ). Then  $1 \leq k_1 < k_2 < \dots < k_p < k_{p+1} = n$ . For  $j=1, 2, \dots, p$ , we denote by  $\mathcal{F}_j$  the flag manifold of type  $(k_1, \dots, k_j)$  defined by

$$\{(W_1, \dots, W_j) \in G_{k_1}(V) \times \dots \times G_{k_j}(V) \mid W_i \subset W_{i+1} \ (1 \leq i \leq j-1)\}.$$

Then,  $\mathcal{F}_j$  is isomorphic to  $GL_{k_{j+1}}/P(\mu_1, \dots, \mu_{j+1})$ , where

$P(\mu_1, \dots, \mu_{j+1})$  is a parabolic subgroup of  $GL_{k_{j+1}}$  whose Levi subgroup is isomorphic to  $\prod_{i=1}^{j+1} GL_{\mu_i}$ . In particular, if  $j=p$ , then

$\mathcal{F}_p \simeq GL_n/P_{\mu}$ . For a nilpotent transformation  $N$  of  $V$ , put

$$\mathcal{Y}_j^N = \{(W_i) \in \mathcal{Y}_j \mid N(W_i) \subset W_i \quad (1 \leq i \leq j)\}.$$

If  $u_\lambda = 1_n + N$  is the corresponding unipotent element of  $GL_n$ , then  $\mathcal{Y}_p^N \simeq (GL_n/P_\mu)_{u_\lambda}$ .

We preserve the notations in §2. For  $\ell \in L_{\bar{k}_p}(\lambda)$  ( $\bar{k}_p = (k_p, \mu_{p+1})$ ), put  $V_\ell = \langle w_{ij} \text{ in } \square \rangle$ . We remark that  $V_\ell$  is a element of  $T_\ell = S_\ell^N$ . If  $W \in T_\ell$ , then the projection  $f: V \rightarrow V_\ell$  induces an  $N$ -module isomorphism  $f_W: W \xrightarrow{\sim} V_\ell$ . By the projection

$$\pi_p: \mathcal{Y}_p \longrightarrow G_{k_p}(V) \quad ((W_1, \dots, W_p) \mapsto W_p),$$

we have the following trivialization:

$$\pi_p^{-1}(T_\ell) \xrightarrow{\sim} \mathcal{Y}_{p-1} \times T_\ell \quad ((W_i) \mapsto (f_{W_p}(W_1), \dots, f_{W_p}(W_{p-1})), W_p).$$

Under this trivialization, we have

$$\pi_p^{-1}(T_\ell) \cap \mathcal{Y}_p^N \xrightarrow{\sim} \mathcal{Y}_{p-1}^N \times T_\ell,$$

and therefore  $\mathcal{Y}_p^N \xrightarrow{\sim} \frac{1 \quad 1}{\ell \in L_{\bar{k}_p}(\lambda)} \mathcal{Y}_{p-1}^N \times T_\ell$ . By induction, we have

$$\mathcal{Y}_p^N \xrightarrow{\sim} \frac{1 \quad 1 \quad 1}{\substack{\ell_j \in L_{\bar{k}_j}(\lambda_j) \\ j=1, \dots, p}} T_{\ell_1} \times T_{\ell_2} \times \dots \times T_{\ell_p},$$

where  $\lambda_j$  is the Young tableau obtained by extracting the nodes with figure  $j+2, \dots, p+1$ . Therefore, we can write

$$\mathcal{Y}_p^N = \frac{1 \quad 1}{\ell \in L_\mu(\lambda)} T_\ell,$$

where  $L_\mu(\lambda)$  is the set of all semi-standard  $\mu$ -tableaus of type  $\lambda$  and  $T_\ell$  is isomorphic to some  $T_{\ell_1} \times \dots \times T_{\ell_p}$  ( $\ell_j \in L_{\bar{k}_j}(\lambda_j)$ ,  $j=1, \dots, p$ ).

Remark. Similarly, we can prove that

$$\mathcal{F}_p = \sum_{\ell \in \widetilde{L}_\mu(\lambda)} S_\ell,$$

where  $\widetilde{L}_\mu(\lambda)$  is the set of all  $\mu$ -tableaus of type  $\lambda$ . About this decomposition, we note that  $T_\ell = S_\ell^N$  and  $S_\ell^N \neq \emptyset$  if and only if  $\ell \in L_\mu(\lambda)$ .

Definition 3. For  $\ell \in L_\mu(\lambda)$ , let  $n(\ell)$  be a non-negative integer defined by the following recurrence rule :

(1) If  $\mu = (\mu_1, \mu_2)$  or  $(n)$ , then  $n(\ell)$  is defined in Definition 2.

(2) For  $\mu = (\mu_1, \dots, \mu_p, \mu_{p+1})$ , put  $\mu' = (\mu_1, \dots, \mu_p)$  and  $k_p = \mu_1 + \dots + \mu_p$ . Let  $\ell_1 \in L_{\bar{k}_p}(\lambda)$  ( $\bar{k}_p = (k_p, \mu_{p+1})$ ) be the semi-standard  $\bar{k}_p$ -tableau obtained from  $\ell$  by changing the figures  $p+1$  into 2 (or  $\square$ ) and figures  $i$  ( $1 \leq i \leq p$ ) into 1 (or  $\boxplus$ ). Let  $\ell_2$  be the  $\mu'$ -tableau obtained by extracting the nodes with figure  $p+1$  from  $\ell$  and by rearranging the rows in the appropriate order. Thus  $\ell_2 \in L_{\mu'}(\lambda_{p-1})$  for some partition  $\lambda_{p-1}$  of  $k_p$ . Then we defines

$$n(\ell) = n(\ell_1) + n(\ell_2).$$

Theorem 2. Let  $\lambda$  and  $\mu$  be a partition of  $n$ . The variety  $(GL_n/P_\mu)_{u_\lambda}$  has a partition

$$(GL_n/P_\mu)_{u_\lambda} = \sum_{\ell \in L_\mu(\lambda)} T_\ell,$$

where  $T_\ell$  is a locally closed subset of  $(GL_n/P_\mu)_{u_\lambda}$  and isomorphic to an  $n(\ell)$ -dimensional affine space  $\mathbb{A}^{n(\ell)}$  and this

partition is defined over  $K$ .

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