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ON A PROBLEM IN DIOPHANTINE APPROXIMATION

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We are concerned in this article with a property of badly approximable real numbers. Professor W. M. Schmidt has made among many other things a number of interesting and important contributions in the study of such numbers.

An  $n$ -tuple  $(\alpha_1, \dots, \alpha_n)$  of real numbers is, by definition, badly approximable if

$$(*) \quad |x|^{1/n} \cdot \max(\|\alpha_1 x\|, \dots, \|\alpha_n x\|) \geq \gamma$$

for some constant  $\gamma > 0$ , whenever  $x \neq 0$  is an integer.

Here,  $\|t\|$  denotes as usual the distance from the real number  $t$  to the nearest integer, so that we always have  $0 < \|t\| \leq 1/2$ .

We shall prove the following

**THEOREM.** Let  $(\alpha_1, \dots, \alpha_n)$  be a badly approximable  $n$ -tuple of real numbers, i.e. an  $n$ -tuple satisfying (\*). Let  $\beta_1, \dots, \beta_n$  be  $n$  arbitrary real numbers, and  $X \geq 2$  an arbitrary integer. Put

$$D = \left[ \frac{(n+1)n^{n-1} X^{4n+2}}{2\gamma} \right] + 1.$$

Then, for any integer  $d \geq 0$  there is an integer  $x$  in the interval

$$d < x \leq d + D$$

such that

$$\|\alpha_i x - \beta_i\| < \frac{1}{x} \quad (i = 1, 2, \dots, n).$$

By a transference theorem (cf. e.g. [2; Chap. V, Corollary to Theorem II]), an  $n$ -tuple  $(\alpha_1, \dots, \alpha_n)$  is badly approximable if, and only if for every integral  $n$ -vector  $\underline{x} \neq \underline{0}$

$$\|\alpha_1 x_1 + \dots + \alpha_n x_n\| \geq \frac{\gamma'}{|\underline{x}|^n}$$

for some constant  $\gamma' > 0$ . Here

$$|\underline{x}| = \max(|x_1|, \dots, |x_n|)$$

if  $\underline{x} = (x_1, \dots, x_n)$ . We may take  $\gamma' = n^{n+1}\gamma$  in case the  $n$ -tuple  $(\alpha_1, \dots, \alpha_n)$  satisfies (\*). Thus, in particular, if  $(\alpha_1, \dots, \alpha_n)$  is badly approximable, then  $1, \alpha_1, \dots, \alpha_n$  are linearly independent over the rationals.

Now, following H. Bohr and B. Jessen [1], we define

$$F(t) = 1 + e^{2\pi i(\alpha_1 t - \beta_1)} + \dots + e^{2\pi i(\alpha_n t - \beta_n)}$$

and, with Fejér's kernel

$$K_N(t) = \sum_{v=-N}^N \left(1 - \frac{|v|}{N}\right) e^{2\pi i vt} = \frac{1}{N} \left(\frac{\sin \pi N t}{\sin \pi t}\right)^2,$$

$$\mathbb{K}_N(t) = K_N(\alpha_1 t - \beta_1) \cdots K_N(\alpha_n t - \beta_n)$$

$$= 1 + \left(1 - \frac{1}{N}\right) (e^{-2\pi i(\alpha_1 t - \beta_1)} + \dots \\ + e^{-2\pi i(\alpha_n t - \beta_n)}) + R(t);$$

here,  $R(t)$  is a trigonometric polynomial whose exponents, divided by  $2\pi$ , are all different from the numbers  $0, -\alpha_1, \dots, -\alpha_n$ .

We have

$$F(t) K_N(t) = 1 + \left(1 - \frac{1}{N}\right) n + S(t),$$

where  $S(t)$  is a trigonometric polynomial whose exponents are all different from zero modulo  $2\pi$ . Hence

$$\frac{1}{D} \sum_{d < x \leq d+D} F(x) K_N(x) = 1 + \left(1 - \frac{1}{N}\right) n \\ + \frac{1}{D} \sum_{d < x \leq d+D} S(x),$$

where we find easily

$$\left| \frac{1}{D} \sum_{d < x \leq d+D} S(x) \right| \leq \frac{N^n}{D} \frac{N^n}{2\gamma'}.$$

(Note that the sum of the coefficients of  $K_N(t)$  equals  $N$ .)

By the positivity of the kernel  $K_N(t)$  we have, since

$$\frac{1}{D} \sum_{d < x \leq d+D} K_N(x) \leq 1 + \frac{N^n}{D} \frac{N^n}{2\gamma'},$$

$$\max_{d < x \leq d+D} |F(x)| \geq 1 + \left(1 - \frac{1}{N}\right) n - \frac{N^{2n}}{2\gamma' D}.$$

$$\cdot \left( 1 + \frac{N^{2n}}{2\gamma'D} \right)^{-1}.$$

Taking

$$N = nX^2,$$

$$D = \left[ \frac{(n+1)n^{2n}X^{4n+2}}{2\gamma'} \right] + 1,$$

we get

$$\max_{d < x \leq d+D} |F(x)| \geq n+1 - \frac{3}{X^2}.$$

Let  $\alpha, \beta$  be any one of the pairs  $\alpha_i, \beta_i$  ( $1 \leq i \leq n$ ).

Then, since

$$|F(x)| \leq n-1 + |1 + e^{2\pi i(\alpha x - \beta)}|,$$

we have

$$|1 + e^{2\pi i(\alpha x - \beta)}| \geq 2 - \frac{3}{X^2}.$$

Noticing that  $|1 + e^{2\pi it}| = 2|\cos \pi t|$  and  $|\sin \pi t| \geq 2\|t\|$ , we deduce from the above inequalities for  $|F(x)|$  that

$$\|\alpha x - \beta\| \leq \frac{\sqrt{3}}{2} \frac{1}{X} < \frac{1}{X}$$

for some integer  $x$ , independent of the particular  $\alpha, \beta$ , in the interval  $d < x \leq d+D$ . This completes the proof of our theorem.

It should be observed that our method can be applied to any  $n$ -tuple of real numbers  $\alpha_1, \dots, \alpha_n$  such that 1 and the  $\alpha_i$ 's are linearly independent over the field of rational

numbers, obtaining a result similar to the theorem above with a suitably defined  $D$  in terms of  $M_n$ , where

$$\frac{1}{M_n} = \min_{0 < \underline{x} \leq n} \|\alpha_1 x_1 + \dots + \alpha_n x_n\|.$$

We thus have a sort of quantitative formulation of the (small) approximation theorem of Kronecker's.

#### References

- [1] H. Bohr and B. Jessen: One more proof of Kronecker's theorem. J. London Math. Soc., 7(1932), 274-275.
- [2] J. W. S. Cassels: An Introduction to Diophantine Approximation. Cambridge Tracts in Math. & Math. Phys. No. 45. Cambridge Univ. Press, 1957.