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Complete Integrability of Nonlinear Differential-Difference Equations

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Abstract Two classes of nonlinear differential-difference equations of evolution which have been solved by the inverse scattering method are shown to describe completely integrable Hamiltonian systems. One is associated with a linear eigenvalue equation

$$\sqrt{a_{n+1}}\,u_{n+1} + \sqrt{a_n}\,u_{n-1} = \lambda\, u_n,$$

and the other with coupled linear eigenvalue equations

$$v_{1,n+1} = z\, v_{1,n} + Q_n v_{2,n} + S_n v_{2,n+1},$$

$$v_{2,n+1} = z^{-1} v_{2,n} + R_n v_{1,n} + T_n v_{1,n+1}.$$

§ 1. Introduction

In recent years a large class of nonlinear differential-difference equations have been solved by the method of inverse scattering. The variety of such equations now becomes comparable with that of nonlinear differential equations, describing continuous systems, which are solvable by the inverse scattering method. For the continuous cases it has been well established that the system governed by an equation of evolution to which the inverse scattering method can be applied is a completely integrable Hamiltonian system.[1)-3)] The discrete counterpart of this statement is believed to be correct, but so far only two relevant systems have been known to us ; Flaschka and McLaughlin[4),5)] have shown that the Toda lattice is a completely integrable Hamiltonian system and the present authors have proved in a previous paper[6)] that the generalized Volterra system is also completely integrable.

In this paper we want to put into this category two classes of nonlinear differential-difference equations solvable by the method of inverse scattering. The first part deals with the complete integrability of a class of equations generated by a linear eigenvalue equation

$$\sqrt{a_{n+1}}\, u_{n+1} + \sqrt{a_n}\, u_{n-1} = \lambda\, u_n \tag{1.1}$$

and the time evolution of eigenfunctions

$$du_n/dt = \lambda \sqrt{a_{n+1}} A_n\, u_{n+1} + [B_n + \frac{1}{2} \sum_{k=-\infty}^{n} d(\ln a_k)/dt]\, u_n. \tag{1.2}$$

Here A_n and B_n depend in general on the 'potential' a_n. This class contains among others the equation for Volterra system

$$da_n/dt = a_n(a_{n+1} - a_{n-1}). \tag{1.3}$$

Manakov solved this equation by the inverse scattering method.[7)] At this point we remark that Eq. (1.3) is a special case of the equations for a *generalized* Volterra system given by

$$\begin{aligned} dc_n/dt &= 2\, z_o\, c_n(d_n - d_{n+1}), \\ dd_n/dt &= 2\, z_o^{-1}\, d_n(c_{n-1} - c_n)\, ; \end{aligned} \tag{1.4}$$

indeed we obtain (1.3) by setting $z_o = 1$, $c_n = -a_{2n+1}/2$ and $d_n = -a_{2n}/2$ in (1.4). The latter equations were solved by the method of inverse scattering[8] and proved to describe a completely integrable Hamiltonian system[6] by Fujii and the present authors.

Another member of the first class is a discrete version of the KdV equation

$$da_n/dt = 3a_n(a_{n-1} - a_{n+1}) + \frac{1}{2}a_n[a_{n+1}(a_{n+2} + a_{n+1}) + a_n(a_{n+1} - a_{n-1}) - a_{n-1}(a_{n-1} + a_{n-2})], \qquad (1.5)$$

which reduces to

$$w_\tau - 6ww_x + w_{xxx} = 0$$

if we put $\ln a_n = -(\Delta x)^2 w(x, \tau)$ with $x = n\Delta x$ and $\tau = (\Delta x)^3 t$ and let Δx tend to zero.

We will show that these equations and other members of the first class all together describe completely integrable Hamiltonian systems. As usual action-angle variables are defined in terms of scattering data and it is verified that the inverse scattering method is a canonical transformation. Characteristic are the facts that the Poisson brackets defined at the outset by the potentials and the subsequent canonical transformations are common to all the equations of the class and that the Hamiltonian of each equation is given by a certain linear combination of conserved quantities of a nonlinear differential-difference equation derived from (1.1) and (1.2).

The second class of nonlinear differential-difference equations we will prove their complete integrability is related to the scheme for four potentials Q_n, R_n, S_n and T_n which was analyzed by Ablowitz and

Ladik[9),10)]

$$v_{1,n+1} = z\, v_{1,n} + Q_n v_{2,n} + S_n v_{2,n+1},$$
$$v_{2,n+1} = z^{-1} v_{2,n} + R_n v_{1,n} + T_n v_{1,n+1}, \tag{1.6}$$

where z is the eigenvalue. The time dependence of eigenfunctions for Eq. (1.6) is assumed to be determined by the differential equations

$$dv_{1,n}/dt = A_n v_{1,n} + B_n v_{2,n},$$
$$dv_{2,n}/dt = C_n v_{1,n} + D_n v_{2,n}. \tag{1.7}$$

It would be possible to give a through discussion as in the first part on the complete integrability of nonlinear differential-difference equations generated by Eqs. (1.6) and (1.7), but we have not yet succeeded in finding a satisfactory way to put all the equations of the second class into canonical forms. What we can here say and will give a full account in a near future is that in the special case for which $S_n = T_n = 0$ the generalized Wronskian technique, discussed by Calogero and Degasperis[11)] and employed by Chiu and Ladik to generate exactly soluble nonlinear discrete evolution equations,[12)] enables us to write down equations in canonical form once for all. In the second part, therefore, we will be content with the study of the simplest of the equations generated by Eqs. (1.6) and (1.7), namely, a set of equations[9)]

$$dQ_n/dt = (1 - R_nQ_n)(S_n - S_{n-1}),$$
$$dR_n/dt = (1 - R_nQ_n)(T_n - T_{n-1}), \tag{1.8}$$
$$dS_n/dt = (1 - S_nT_n)(Q_{n+1} - Q_n),$$

$$dT_n/dt = (1 - S_nT_n)(R_{n+1} - R_n),$$

and prove that the system governed by these equations is a completely integrable Hamiltonian system. We note that on setting $Q_n = -R_n = V_n$ and $S_n = -T_n = I_n$ Eqs (1.8) reduce to Hirota's self-dual network equations[13)]

$$dV_n/dt = (1 + V_n^2)(I_n - I_{n-1}),$$
$$dI_n/dt = (1 + I_n^2)(V_{n+1} - V_n). \tag{1.9}$$

It should be added that in spite of our restriction the Poisson brackets for potentials we shall set up later are available for any set of nonlinear differential-difference equations generated by Eqs. (1.7) and (1.8) when they are put into canonical form, and moreover that all of these sets of equation have the same conserved quantities as (1.8) ; the Hamiltonian for any set of equations should be provided by some linear combination of the conserved quantities.

We think the crucial point in the problem of the complete integrability of nonlinear evolution equations is how to find a systematic way of expressing their Hamiltonian in terms of potentials, so in what follows we will make a supreme effort in disclosing this point. Since we are at present familiar with scattering problem, conservation laws and other subjects of the inverse scattering method, we will attempt to avoid as far as possible the repetition of similar stories ; indeed the scheme used for the generalized Volterra system in the previous paper[6)] can be applied *mutatis mutandis* to the inverse scattering problem in the present cases.

Part I

§ 2. Scattering problem and conservation laws

In this section we briefly summarize the results of the scattering problem of Eq. (1.1) and derive the conservation laws for nonlinear differential-difference equations generated by Eqs. (1.1) and (1.2).

By the reqirement that Eqs. (1.1) and (1.2) are integrable and the eigenvalue λ is time-invariant, one can obtain after eliminating B_n a nonlinear differential-difference equation

$$d(\ln a_n)/dt + \lambda^2(A_n - A_{n-1}) + (a_{n-1}A_{n-2} - a_nA_n + a_nA_{n-1} - a_{n+1}A_{n+1}) = 0. \tag{2.1}$$

To make the problem definite we determine Eq. (2.1) explicitly by supposing that A_n is expressed as a polynomial of degreee M in λ^2:

$$A_n = \lambda^{2M} \sum_{j=0}^{M} A_n^{(j)} \lambda^{-2j}. \tag{2.2}$$

Then Eq. (2.1) yields

$$A_n^{(j+1)} - A_{n-1}^{(j+1)} + (a_{n-1}A_{n-2}^{(j)} - a_nA_n^{(j)} + a_nA_{n-1}^{(j)} - a_{n+1}A_{n+1}^{(j)}) = 0, \quad 0 \leqq j \leqq M-1 \tag{2.3}$$

and

$$\frac{d}{dt}\ln a_n = a_{n+1}A_{n+1}^{(M)} + a_nA_n^{(M)} - a_nA_{n-1}^{(M)} - a_{n-1}A_{n-2}^{(M)}. \tag{2.4}$$

Equation (2.1) or, more specifically, Eq. (2.4) for M fixed arbitrarily is an equation which can be solved by the inverse scattering method for Eq. (1.1). To compute the right hand side of (2.4) we define a sequence of functions $\{C_n^{(\ell)}\}_{\ell=0}^{\infty}$ recurrently by the relations of the same form as (2.3)

$$C_n^{(\ell+1)} - C_{n-1}^{(\ell+1)} + (a_{n-1}C_{n-2}^{(\ell)} - a_n C_n^{(\ell)} + a_n C_{n-1}^{(\ell)} - a_{n+1}C_{n+1}^{(\ell)}) = 0, \qquad (2.5)$$

accompanied by the condition that $C_n^{(0)} = 1$ for all n and $C_n^{(\ell)}$, $\ell \geqq 1$, approaches zero as $|n| \to \infty$. If A_n has an asymptotic form

$$A_n \sim \lambda^{2M} \sum_{j=0}^{M} A_-^{(j)} \lambda^{-2j} \qquad \text{as } n \to -\infty, \qquad (2.6)$$

then $A_n^{(j)}$ can be represented as

$$A_n^{(j)} = \sum_{i=0}^{j} A_-^{(i)} C_n^{(j-i)}$$

and Eq. (2.4) becomes

$$\frac{d}{dt} \ln a_n = \sum_{j=0}^{M} A_-^{(j)} \{C_n^{(M+1-j)} - C_{n-1}^{(M+1-j)}\}. \qquad (2.7)$$

The early members of $\{C_n^{(\ell)}\}$ are found to be

$$C_n^{(0)} = 1, \qquad C_n^{(1)} = a_{n+1} + a_n - 2,$$

$$C_n^{(2)} = a_{n+1}^2 + a_n^2 + a_{n+2}a_{n+1} + 2a_{n+1}a_n + a_n a_{n-1}$$
$$- 2(a_{n+1} + a_n) - 2.$$

The choice $A_-^{(0)} = 1$ for $M = 0$ reduces Eq. (2.7) to the Volterra equation (1.3), and the choice $A_-^{(0)} = 1/2$ and $A_-^{(1)} = -2$ for $M = 1$ to the discretized KdV equation (1.5).

In the next section we shall show that functions $\{C_n^{(\ell)}\}$ are simply related to the conserved quantities for Eq. (2.4).

For the scattering problem of Eq. (1.1) it is more convenient to write it in the form

$$\exp(b_{n+1}/2)u_{n+1} + \exp(b_n/2)u_{n-1} = \lambda u_n, \tag{2.8}$$

where we have set $a_n = \exp(b_n)$. In the sequel we shall often use z instead of λ by putting $\lambda = z + z^{-1}$.

Let u_n and u'_n be two solutions of (2.8) with eigenvalues λ and λ', respectively. Then it is easy to see the relations

$$(\lambda \pm \lambda')u_n u'_n = W^{\pm}_{n+1}(u, u') \pm W^{\pm}_n(u, u'), \tag{2.9}$$

where

$$W^{\pm}_n(u, u') = \exp(b_n/2)(u_n u'_{n-1} \pm u_{n-1}u'_n). \tag{2.10}$$

The boundary condition we assume for the potential b_n is that $b_n \to 0$ sufficiently rapidly as $|n| \to \infty$. The Jost functions $\phi_n(z)$ and $\psi_n(z)$ are defined as usual as solutions of (2.8) subject to the boundary conditions

$$\begin{aligned} \phi_n(z) &\sim z^n \quad \text{as } n \to -\infty, \\ \psi_n(z) &\sim z^{-n} \quad \text{as } n \to \infty. \end{aligned} \tag{2.11}$$

The scattering data $\{\alpha(z), \beta(z), z_k, \beta_k\}$ are defined as follows: Through the expansion

$$\phi_n(z) = \alpha(z)\psi_n(z^{-1}) + \beta(z)\psi_n(z) \tag{2.12}$$

for $z \neq \pm 1$, the functions $\alpha(z)$ and $\beta(z)$ are given by

$$\alpha(z) = W_n^-(\phi, \psi)/(z - z^{-1}),$$
$$\beta(z) = - W_n^-(\phi, \overline{\psi})/(z - z^{-1}), \tag{2.13}$$

where $\overline{\psi}_n(z)$ stands for $\psi_n(z^{-1})$. It is not hard to verify that $\alpha(z)$ and $\beta(z)$ are both even functions of z. Under the hypothesis of the potentials $\{b_n\}$ the function $\alpha(z)$ is analytic outside the unit circle $|z| = 1$. The points z_k for which $\alpha(z_k) = 0$, $k = \pm 1, \pm 2, \ldots, \pm N$, on the real axis outside the unit circle correspond to the bound states of (2.8). All zeros are assumed to be simple. They appear in pairs and are numbered so that $z_{-k} = - z_k$. At $z = z_k$ we have from (2.8)

$$\phi_{n,k} = \beta_k \psi_{n,k}, \tag{2.14}$$

where $\beta_k = \beta(z_k)$, $\phi_{n,k} = \phi_n(z_k)$ and $\psi_{n,k} = \psi_n(z_k)$. From (2.9) follows the relation

$$\sum_{n=-\infty}^{\infty} \phi_{n,k} \psi_{n,k} = z_k \dot{\alpha}(z_k), \tag{2.15}$$

where the dot signifies the differentiation with respect to z.

It is an important consequence of the inverse scattering method that $\alpha(z)$ is time-invariant, the fact which enables us to get an infinite number of conserved quantities of Eq. (2.7) by the asymptotic expansion of $\ln \alpha(z)$ for large z:

$$\ln \alpha(z) = \sum_{\ell=0}^{\infty} z^{-2\ell} I_\ell. \tag{2.16}$$

The first several of the conserved quantities are found to be

$$I_o = -\frac{1}{2} \sum_{n=-\infty}^{\infty} b_n, \qquad I_1 = - \sum_{n=-\infty}^{\infty} [\exp(b_n) - 1],$$

$$I_2 = - \sum_{n=-\infty}^{\infty} [\frac{1}{2}\exp(2b_n) + \exp(b_{n+1}+b_n) - 2\exp(b_n) + \frac{1}{2}]. \tag{2.17}$$

§ 3. Hamiltonians, Poisson brackets and canonical transformations

Let us calculate the variation of the scattering data as the potential changes. First we consider the case when $\alpha(z)$ does not have zeros outside the unit circle. By solving Eq. (2.8) for the Jost function ϕ_n (or ψ_n) iteratively towards $n \to -\infty$ (or $n \to \infty$) and taking derivatives of the solutions with respect to b_n we have

$$\begin{aligned} \partial\phi_n/\partial b_n &= -\phi_n/2, \qquad \partial\phi_{n-1}/\partial b_n = 0, \\ \partial\psi_{n-1}/\partial b_n &= -\psi_{n-1}/2, \qquad \partial\psi_n/\partial b_n = 0. \end{aligned} \tag{3.1}$$

Then, from (2.13) follow the gradients of $\alpha(z)$ and $\beta(z)$:

$$\begin{aligned} \frac{\partial}{\partial b_n}\alpha(z) &= -\frac{1}{2(z-z^{-1})} W_n^+(\phi, \psi), \\ \frac{\partial}{\partial b_n}\beta(z) &= \frac{1}{2(z-z^{-1})} W_n^+(\phi, \bar{\psi}). \end{aligned} \tag{3.2}$$

The asymptotic behaviour of the Jost function as $|z| \to \infty$ can also be seen from Eq. (2.8). It is not difficult to observe that

$$\begin{aligned} \phi_n(z) &\sim z^n \exp\{-\frac{1}{2}\sum_{k=-\infty}^{n} b_k\}, \\ \psi_n(z) &\sim z^{-n} \exp\{-\frac{1}{2}\sum_{k=n+1}^{\infty} b_k\} \end{aligned} \tag{3.3}$$

for large $|z|$. Accordingly we obtain from (2.10)

$$W_n(\phi, \psi) \sim z \exp\{-\frac{1}{2} \sum_{k=-\infty}^{\infty} b_k\} \qquad \text{as} \quad |z| \to \infty. \tag{3.4}$$

Now let us define a function Φ_n by

$$\Phi_n = W_n^+(\phi, \psi)/[\lambda\alpha(z)]. \tag{3.5}$$

It satisfies the equation

$$\exp(b_{n+1})(\Phi_{n+2} + \Phi_{n+1}) - \exp(b_n)(\Phi_n + \Phi_{n-1}) = \lambda^2(\Phi_{n+1} - \Phi_n) \tag{3.6}$$

and is normalized so that $\Phi_n \to 1$ as $|z| \to \infty$ and hence as $\lambda \to \infty$. It is to be noted that the denominator $\lambda\alpha(z)$ in (3.5) is responsible only for the normalization. We solve Eq. (3.6) by expanding Φ_n into inverse power series of λ^2 as

$$\Phi_n = \Phi_n^{(0)} + \lambda^{-2} \Phi_n^{(1)} + \lambda^{-4} \Phi_n^{(2)} + \cdots$$

with $\Phi_n^{(0)} = 1$. The substitution of the series into Eq. (3.6) enables us to determine $\Phi_n^{(\ell)}$ in the following compact form :

$$\Phi_n^{(\ell)} = L_n[\Phi_n^{(\ell-1)}] = (L_n)^\ell[1], \tag{3.7}$$

where $L_n[\cdot]$ is a linear operator defined by

$$L_n[f] = \sum_{k=-\infty}^{n} [\exp(b_k)(f_{k+1} + f_k) - \exp(b_{k-1})(f_{k-1} + f_{k-2})]. \tag{3.8}$$

By comparing (3.7) and (2.5) we easily find that there holds the relation

$$2C_n^{(\ell)} = \Phi_{n+1}^{(\ell)} + \Phi_n^{(\ell)}. \tag{3.9}$$

On the other hand Eq. (3.2) implies

$$\frac{\partial}{\partial b_n} \ln \alpha(z) = \frac{z + z^{-1}}{2(z - z^{-1})} \Phi_n . \tag{3.10}$$

If one expands asymptotically both sides of (3.10) into inverse power series of z^2 and takes account of (2.16), (3.7) and (3.9), it is immediate to see that $C_n^{(\ell)}$ can be expressed by a certain finite linear combination of gradients of the conserved quantities $\{I_\ell\}$. In fact we have

$$C_n^{(j)} = \left(\frac{\partial}{\partial b_n} + \frac{\partial}{\partial b_{n+1}} \right)(2I_o - I_j) \qquad \text{for } j = 1, 2,$$

$$C_n^{(3)} = \left(\frac{\partial}{\partial b_n} + \frac{\partial}{\partial b_{n+1}} \right)(4I_o + I_1 - 2I_2 - I_3) \tag{3.11}$$

and so on. In general $C_n^{(\ell)}$ takes the form $(\partial/\partial b_n + \partial/\partial b_{n+1})D_n^{(\ell)}$ with $D_n^{(\ell)}$ being some linear combination of I_j's, $j = 0, 1, \ldots , \ell$.

In this way we conclude that Eq. (2.7) which can be solved by the inverse scattering method of Eq. (1.1) is transformed into

$$db_n/dt = \partial H/\partial b_{n+1} - \partial H/\partial b_{n-1}, \tag{3.12}$$

where

$$H = \sum_{\ell=1}^{M+1} A_-^{(M+1-\ell)} D^{(\ell)} \tag{3.13}$$

We may interprete H as the Hamiltonian, since Eq. (3.12) can be written in the canonical form

$$db_n/dt = \{b_n, H\}, \tag{3.14}$$

if we define the Poisson bracket of two functions F and G of b_n by

$$\{F, G\} = \sum_{n=-\infty}^{\infty} \frac{\partial F}{\partial b_n} \left(\frac{\partial G}{\partial b_{n+1}} - \frac{\partial G}{\partial b_{n-1}} \right). \tag{3.15}$$

Equations (1.3) and (1.5) written in the canonical form have the Hamiltonians

$$H = D^{(1)} = 2I_o - I_1$$

and

$$H = -2D^{(1)} + \frac{1}{2}D^{(2)} = -3I_o + 2I_1 - \frac{1}{2}I_2,$$

respectively.

The Poisson bracket of $\alpha(z)$ and $\beta(z')$ is calculated by the use of (3.2) and becomes

$$\{\alpha(z), \beta(z')\} = -\frac{\lambda\lambda'(\lambda^2 + \lambda'^2 - 4)}{2(z - z^{-1})(z' - z'^{-1})(\lambda^2 - \lambda'^2)} \alpha(z)\beta(z')$$
$$- \frac{\pi}{2} \frac{z + z^{-1}}{z - z^{-1}} \alpha(z)\beta(z')\{\delta(\xi - \xi') - \delta(\xi + \xi' - \pi)\}, \tag{3.16}$$

where we have set $z = \exp(i\xi)$ and $z' = \exp(i\xi')$. Similarly we have

$$\{\alpha(z), \alpha(z')\} = \{\beta(z), \beta(z')\} = 0. \tag{3.17}$$

We define a set of new variables by

$$P(\xi) = i(\tan\xi) \ln[\alpha(e^{i\xi})\alpha(e^{-i\xi})],$$
$$Q(\xi) = \frac{1}{2\pi} \ln[\beta(e^{i\xi})/\beta(e^{-i\xi})]. \tag{3.18}$$

Since $\alpha(z)$ and $\beta(z)$ are even functions of z, $P(\xi)$ and $Q(\xi)$ are periodic functions with period π, so that we may restrict the range of the variable ξ to the interval $0 \leqq \xi < \pi$. These variables obey the Poisson bracket relations

$$\{P(\xi),\ Q(\xi')\} = -\{\delta(\xi-\xi') - \delta(\xi+\xi'-\pi)\},$$
$$\{P(\xi),\ P(\xi')\} = \{Q(\xi),\ Q(\xi')\} = 0. \tag{3.19}$$

When $\alpha(z)$ has simple zeros z_k, $k = \pm 1, \pm 2, \ldots, \pm N$, the analytic continuation of (3.2) yields

$$\frac{\partial}{\partial b_n}\beta_k = \frac{\exp(b_n/2)}{z_k - z_k^{-1}}\phi_{n,k}\bar{\psi}_{n-1,k}. \tag{3.20}$$

The gradient of z_k is calculated by the perturbation method[6] and found to be

$$\frac{\partial z_k}{\partial b_n} = \frac{\exp(b_n/2)}{(z_k - z_k^{-1})\dot{\alpha}(z_k)}\phi_{n,k}\psi_{n-1,k}. \tag{3.21}$$

Therefore the Poisson brackets of z_k and $\beta_{k'}$ are given as follows :

$$\{z_k,\ \beta_{k'}\} = \frac{z_k(z_k + z_k^{-1})}{2(z_k - z_k^{-1})}\beta_{k'}(\delta_{k,k'} + \delta_{k,-k'}),$$
$$\{z_k,\ z_{k'}\} = \{\beta_k,\ \beta_{k'}\} = 0. \tag{3.22}$$

If the variables are changed into the new ones defined by

$$p_k = \ln(z_k + z_k^{-1}),$$
$$q_k = -2\ln\beta_k, \tag{3.23}$$

the Poisson brackets (3.22) become

$$\{p_k, q_{k'}\} = -\delta_{k,k'},$$
$$\{p_k, p_{k'}\} = \{q_k, q_{k'}\} = 0. \tag{3.24}$$

The Poisson brackets between continuous and discrete variables all vanish.

§ 4. Representation of Hamiltonian in action-angle variables

The function $\hat{\alpha}(z)$ defined by

$$\hat{\alpha}(z) = \alpha(z^{-1}) \prod_{k=1}^{N} (z^2 - z_k^2)/(1 - z^2 z_k^2) \tag{4.1}$$

is analytic and has neither zeros nor poles inside the unit circle. The function $\ln \hat{\alpha}(z)$ is analytic for $|z| < 1$, $|\hat{\alpha}(z)| = |\alpha(z^{-1})|$ for $|z| = 1$, and by virtue of Schwarz's integral formula it can be represented as

$$\ln \hat{\alpha}(z) = i \arg \alpha(0) + \frac{1}{2\pi} \int_0^{2\pi} \ln|\alpha(e^{-i\xi})| \frac{e^{i\xi} + z}{e^{i\xi} - z} d\xi. \tag{4.2}$$

Then the conserved quantities $\{I_\ell\}$ given by (2.16) are evaluated as the integrals

$$I_\ell = \frac{1}{2\pi i} \oint z^{-2\ell-1} \ln|\alpha(z^{-1})| dz$$

$$= \frac{1}{2\pi i} \oint z^{-2\ell-1} \ln|\hat{\alpha}(z)| dz - \frac{1}{2\pi i} \sum_{k=1}^{N} \oint z^{-2\ell-1} \ln \frac{z^2 - z_k^2}{1 - z^2 z_k^2} dz, \tag{4.3}$$

where the contour integral is taken over a closed path around the origin and inside the unit circle. It follows from (4.2) and (4.3) that

$$I_o = \frac{1}{2\pi}\int_0^{2\pi} \ln|\alpha(e^{-i\xi})|\, d\xi - \sum_{k=1}^{N} \ln z_k^2 \tag{4.4}$$

and

$$I_\ell = \frac{1}{2\pi}\int_0^{2\pi} e^{-2i\ell\xi} \ln|\alpha(e^{-i\xi})|^2\, d\xi - \sum_{k=1}^{N} \frac{z_k^{2\ell} - z_k^{-2\ell}}{\ell} \tag{4.5}$$

for $\ell = 1, 2, \ldots$.

Since I_ℓ's and hence the Hamiltonian H also depend only on the action variables $P(\xi)$ and p_k, the canonical equation (3.14) is completely integrable. Thus each of the nonlinear differential-difference equations (2.7) describes a completely integrable Hamiltonian system. For instance the Hamiltonian of the Volterra system is

$$H = -\frac{1}{\pi}\int_0^{\pi} (e^{-2i\xi} - 1)P(\xi)d\xi + \sum_{k=1}^{N} (z_k^2 - z_k^{-2} - 4\ln z_k), \tag{4.6}$$

which yields

$$\begin{aligned} dP(\xi)/dt &= \{P(\xi), H\} = 0, \\ dQ(\xi)/dt &= \{Q(\xi), H\} = -\frac{1}{\pi}(z^2 - z^{-2}), \\ dp_k/dt &= \{p_k, H\} = 0, \\ dq_k/dt &= \{q_k, H\} = 2(z_k^2 - z_k^{-2}). \end{aligned} \tag{4.7}$$

Part II

§ 5. Formulation of the problem

We write simply v_n to denote the column matrix $(v_{1,n}\ \ v_{2,n})^T$. As in the previous paper,[6] in order to normalize the scattering data we consider

$$u_n = v_n \prod_{k=-\infty}^{n-1} [(1 - S_k T_k)/(1 - Q_k R_k)]^{\frac{1}{2}} \tag{5.1}$$

instead of v_n and transform Eqs. (1.7) into

$$\begin{aligned} u_{1,n+1} &= [(z + R_n S_n)u_{1,n} + (Q_n + z^{-1} S_n)u_{2,n}]/\Lambda_n, \\ u_{2,n+1} &= [(z^{-1} + Q_n T_n)u_{2,n} + (R_n + zT_n)u_{1,n}]/\Lambda_n, \end{aligned} \tag{5.2}$$

where $\Lambda_n = [(1-Q_nR_n)(1-S_nT_n)]^{\frac{1}{2}}$.

The Jost functions ϕ_n, $\bar{\phi}_n$, ψ_n, $\bar{\psi}_n$ are defined as solutions of Eqs. (5.2) satisfying the boundary conditions

$$\begin{aligned} &\phi_n(z) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} z^n, \qquad \bar{\phi}_n(z) \sim \begin{pmatrix} 0 \\ -1 \end{pmatrix} z^{-n} \qquad \text{as } n \to -\infty, \\ &\psi_n(z) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} z^{-n}, \qquad \bar{\psi}_n(z) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} z^{n} \qquad \text{as } n \to \infty. \end{aligned} \tag{5.3}$$

For rapidly decaying potentials the functions $z^{-n}\phi_n$ and $z^n\psi_n$ are analytic in $|z| > 1$, and $z^n\bar{\phi}_n$ and $z^{-n}\bar{\psi}_n$ are analytic in $|z| < 1$. Two sets of functions $\{\phi, \bar{\phi}\}$ and $\{\psi, \bar{\psi}\}$ are linearly dependent and we have

$$\phi_n(z) = \alpha(z)\overline{\psi}_n(z) + \beta(z)\psi_n(z),$$
$$\overline{\phi}_n(z) = -\overline{\alpha}(z)\psi_n(z) + \overline{\beta}(z)\overline{\psi}_n(z), \tag{5.4}$$

which define the scattering data $\alpha(z)$, $\beta(z)$, $\overline{\alpha}(z)$ and $\overline{\beta}(z)$; they are given explicitly by

$$\alpha(z) = W_n(\phi, \psi), \qquad \beta(z) = -W_n(\phi, \overline{\psi}),$$
$$\overline{\alpha}(z) = W_n(\overline{\phi}, \overline{\psi}), \qquad \overline{\beta}(z) = W_n(\overline{\phi}, \psi), \tag{5.5}$$

where

$$W_n(u, u') = u_{1,n}u'_{2,n} - u_{2,n}u'_{1,n} \tag{5.6}$$

is the Wronskian of two solutions u_n and u'_n of Eqs. (5.2) corresponding to eigenvalues z and z', respectively. The scattering data satisfy a relation normalized to unity owing to (5.1) :

$$\alpha(z)\overline{\alpha}(z) + \beta(z)\overline{\beta}(z) = 1.$$

The functions $\alpha(z)$ and $\overline{\alpha}(z)$ are time-invariant. The function $\alpha(z)$ ($\overline{\alpha}(z)$) is analytic for $|z| > 1$ ($|z| < 1$) and assumed to have simple zeros z_k ($\overline{z}_k$), $k = 1, 2, \ldots, N$ ($\overline{N}$) outside (inside) the unit circle At $z = z_k$ we have from (5.5)

$$\phi_{n,k} = \beta_k\psi_{n,k}$$

and similarly at $z = \overline{z}_k$

$$\overline{\phi}_{n,k} = \overline{\beta}_k\overline{\psi}_{n,k},$$

where $\beta_k = \beta(z_k)$ etc.

We calculate the gradients of the Jost function ϕ_n (or ψ_n) with respect to the potentials by solving Eqs. (5.2) iteratively towards $n \to -\infty$ (or $n \to \infty$) to get

$$\frac{\partial\phi_n}{\partial Q_n} = \frac{\partial\phi_n}{\partial R_n} = \frac{\partial\phi_n}{\partial S_n} = \frac{\partial\phi_n}{\partial T_n} = 0,$$

$$\frac{\partial\psi_n}{\partial Q_n} = \frac{R_n\psi_n}{2(1-Q_nR_n)} + \begin{pmatrix}1\\0\end{pmatrix}\frac{T_n\psi_{1,n+1} - \psi_{2,n+1}}{\Lambda_n},$$

$$\frac{\partial\psi_n}{\partial R_n} = \frac{Q_n\psi_n}{2(1-Q_nR_n)} + \begin{pmatrix}0\\1\end{pmatrix}\frac{S_n\psi_{2,n+1} - \psi_{1,n+1}}{\Lambda_n}, \qquad (5.7)$$

$$\frac{\partial\psi_n}{\partial S_n} = \frac{T_n\psi_n}{2(1-S_nT_n)} + \begin{pmatrix}-z^{-1}\\R_n\end{pmatrix}\frac{\psi_{2,n+1}}{\Lambda_n},$$

$$\frac{\partial\psi_n}{\partial T_n} = \frac{S_n\psi_n}{2(1-S_nT_n)} + \begin{pmatrix}Q_n\\-z\end{pmatrix}\frac{\psi_{1,n+1}}{\Lambda_n}.$$

Similar expressions for the gradients of $\bar{\phi}_n$ and $\bar{\psi}_n$ are also obtained.

§ 6. Main results

Equation (1.9) can be put into a canonical form

$$dQ_n/dt = \{Q_n, H\}, \qquad dR_n/dt = \{R_n, H\},$$
$$dS_n/dt = \{S_n, H\}, \qquad dT_n/dt = \{T_n, H\}, \qquad (6.1)$$

if we define the Hamiltonian by

$$H = \sum_{n=-\infty}^{\infty}(Q_nT_{n-1} + R_nS_n - Q_nT_n - R_nS_{n-1}) \qquad (6.2)$$

and the Poisson bracket of two functions F and G of the potentials by

$$\{F, G\} = \sum_{k=-\infty}^{\infty} [(1 - R_n Q_n)\left(\frac{\partial F}{\partial R_n}\frac{\partial G}{\partial Q_n} - \frac{\partial F}{\partial Q_n}\frac{\partial G}{\partial R_n}\right)$$

$$- (1 - S_n T_n)\left(\frac{\partial F}{\partial T_n}\frac{\partial G}{\partial S_n} - \frac{\partial F}{\partial S_n}\frac{\partial G}{\partial T_n}\right)]. \qquad (6.3)$$

At present both the Hamiltonian and the Poisson brackets are found not so systematically as in Part I but only by intuition. As we remarked in the introduction, however, the Poisson brackets (6.3) are available also for the Hamiltonian formulation of any other nonlinear differential-difference equation of evolution generated by Eqs. (1.6) and (1.7).

On account of (5.5) - (5.7), the Poisson brackets between the logarithms of the scattering data α, β, $\bar{\alpha}$, $\bar{\beta}$ are calculated to be

$$\begin{aligned} \{\ln\alpha(z), \ln\beta(z')\} &= -X - Y, \\ \{\ln\alpha(z), \ln\bar{\beta}(z')\} &= X + Y, \\ \{\ln\bar{\alpha}(z), \ln\beta(z') &= X - Y, \\ \{\ln\bar{\alpha}(z), \ln\bar{\beta}(z')\} &= -X + Y, \end{aligned} \qquad (6.4)$$

where

$$X = \frac{1}{2}\frac{z + z'}{z - z'} \qquad \text{and} \qquad Y = \pi\delta(\xi - \xi')$$

with $\xi = \arg z$ and $\xi' = \arg z'$. We also have

$$\{\alpha(z), \bar{\alpha}(z')\} = \{\beta(z), \bar{\beta}(z')\} = 0. \qquad (6.5)$$

From these results we find a canonical set of variables

$$P(\xi) = \ln[\alpha(e^{i\xi})\overline{\alpha}(e^{i\xi})],$$
$$Q(\xi) = \frac{1}{4\pi}[\beta(e^{i\xi})/\overline{\beta}(e^{i\xi})], \tag{6.6}$$

whose nonvanishing Poisson bracket is

$$\{P(\xi),\ Q(\xi')\} = -\ \delta(\xi - \xi'). \tag{6.7}$$

The corresponding set of discrete variables is given by

$$p_k = \ln z_k, \quad q_k = -\ln \beta_k, \qquad k = 1, 2, \ldots, N,$$
$$\overline{p}_k = \ln \overline{z}_k, \quad \overline{q}_k = -\ln \overline{\beta}_k, \qquad k = 1, 2, \ldots, \overline{N}, \tag{6.8}$$

which have the Poisson brackets

$$\{p_k,\ q_{k'}\} = -\ \delta_{k,k'},$$
$$\{\overline{p}_k,\ \overline{q}_{k'}\} = -\ \delta_{k,k'}. \tag{6.9}$$

Two sets of infinite number of conserved quantities are obtained by the asymptotic expansion of $\ln \alpha(z)$ and $\ln \overline{\alpha}(z)$ outside and inside the unit circle, respectively :

$$\ln \alpha(z) = \sum_{\ell=0}^{\infty} z^{-\ell} I_\ell,$$
$$\ln \overline{\alpha}(z) = \sum_{\ell=0}^{\infty} z^{\ell} \overline{I}_\ell. \tag{6.10}$$

In terms of canonical variables these conserved quantities are expressed as

$$I_0 = \bar{I}_0 = \frac{1}{2}\left[- \sum_{k=1}^{N} p_k + \sum_{k=1}^{\bar{N}} \bar{p}_k + \frac{1}{2\pi}\int_0^{2\pi} P(\xi)\, d\xi \right],$$

$$I_\ell = - \sum_{k=1}^{N} \frac{\exp(\ell p_k)}{\ell} + \sum_{k=1}^{\bar{N}} \frac{\exp(\ell \bar{p}_k)}{\ell} + \frac{1}{2\pi}\int_0^{2\pi} e^{i\ell\xi} P(\xi)\, d\xi, \qquad (6.11)$$

$$\bar{I}_\ell = \sum_{k=1}^{N} \frac{\exp(-\ell p_k)}{\ell} - \sum_{k=1}^{\bar{N}} \frac{\exp(-\ell \bar{p}_k)}{\ell} + \frac{1}{2\pi}\int_0^{2\pi} e^{-i\ell\xi} P(\xi)\, d\xi$$

for $\ell = 1, 2, \ldots$.

The Hamiltonian (6.2) of Eqs. (1.8) becomes

$$H = - I_1 + \bar{I}_1$$

$$= 2 \sum_{k=1}^{N} \cosh p_k - 2 \sum_{k=1}^{\bar{N}} \cosh \bar{p}_k - \frac{i}{\pi}\int_0^{2\pi} \sin \xi\, P(\xi)\, d\xi. \qquad (6.12)$$

Since the Hamiltonian (6.12) contains only the action variables $P(\xi)$, p_k and $\bar{p}_k$, the canonical equations of motion are completely integrable :

$$dP(\xi)/dt = dp_k/dt = d\bar{p}_k/dt = 0,$$

$$dQ(\xi)/dt = - \frac{i}{\pi} \sin \xi,$$

$$dq_k/dt = 2 \sinh p_k,$$

$$d\bar{q}_k/dt = - 2 \sinh \bar{p}_k.$$

The results obtained in this part constitute a counterpart in discrete systems of those for the continuous systems analyzed by Kodama[2)] and by Flaschka and Newell.[14)]

References

1) V. E. Zakharov and L. D. Faddeev, Func. Anal. Appl. 5 (1971), 280.

2) Y. Kodama, Prog. Theor. Phys. 54 (1975), 669.

3) V. E. Zakharov and S. V. Manakov, Theor. Math. Phys. 19 (1974), 551.

4) H. Flaschka and D. W. McLaughlin, Lecture Notes in Mathematics 515, Bäcklund Transformations, ed. R. Miura (Springer-Verlag, Berlin-Heidelberg-New York, 1976), p. 253.

5) D. W. McLaughlin, J. Math. Phys. 16 (1975), 96, 1704.

6) F. Kako and N. Mugibayashi, J. Phys. Soc. Japan 43 (1977), 1783.

7) S. V. Manakov, Soviet Phys. - JETP 40 (1975), 269.

8) S. Fujii, F. Kako and N. Mugiabyashi, J. Phys. Soc. Japan 42 (1977), 335.

9) M. J. Ablowitz and J. F. Ladik, J. Math. Phys. 16 (1975), 598.

10) M. J. Ablowitz and J. F. Ladik, J. Math. Phys. 17 (1976), 1011.

11) F. Calogero and A. Degasperis, Nuovo Cimento B23 (1976), 201.

12) S.-C. Chiu and J. F. Ladik, J. Math. Phys. 18 (1977), 690.

13) R. Hirota, J. Phys. Soc. Japan 35 (1973), 289.

14) H. Flaschka and A. C. Newell, Lecture Notes in Physics 38, Dynamical Systems, Theory and Applications, ed. J. Moser (Springer-Verlag, Berlin-Heidelberg-New York, 1976), p. 253.