数理解析研究所講究録
332巻 1978年 131-145

Remarks on the Theory of Perturbed Solitons*

Yoshi H. ICHIKAWA
Institute of Plasma Physics
Nagoya University, Nagoya

Abstract

The Bogoliubov-Mitropolsky perturbation method has been applied to the study of a perturbation on soliton solutions to the nonlinear Schrödinger equation with nonlinear Landau damping. Results of the present analysis are discussed in comparison with the numerical observation of Yajima et al and with the theoretical results obtained by Karpman and Maslov.

§1. Introduction

Extensive researches on various types of exactly solvable nonlinear evolution equations have established the notion of solitons on firm mathematical ground provided by the inverse scattering method.[1∼2] The solitons may be regarded as nonlinear normal modes in which the exact solution of given physical system can be expanded. Thus, it is natural to proceed to investigate the interaction effects between solitons and other degrees of freedom of motion in the system.

---

*) The complete version of this report will be published in Physica Scripta under the coauthorship with Professors Weiland and Wilhelmsson.

Since the standard nonlinear equation such as the Korteweg-de Vries equation is the lowest order approximation of the weakly dispersive system, we have undertaken to examine higher order corrections to the KdV soliton in the physical systems such as the ion acoustic wave and the shallow water wave.[3∿6] Complete analysis of the higher order secularities has been carried through by Taniuti and Kodama[7], who have shown that renormalization of the secular terms is accomplished by addition of derivatives of the higher order conserved quantities to the Korteweg-de Vries equation. On the other hand, Kaup[8], Karpman and Maslov,[9∿11] and Keener and McLaughlin[12] have examined simple examples of perturbed solitons by applying the inverse scattering method to perturbed nonlinear evolution equations.

Here, we will present an example of explicit analysis of perturbed envelop soliton for the nonlinear Schrodinger equation with nonlinear Landau damping[13]

$$i \frac{\partial \Psi}{\partial t} + p \frac{\partial^2 \Psi}{\partial x^2} + q|\Psi|^2\Psi = \varepsilon R(\Psi)$$

where

$$R(\Psi) = -\frac{P}{\pi} \int_{-\infty}^{\infty} \frac{|\Psi(x', t)|^2}{x - x'} dx' \; \Psi(x, t)$$

Examining numerical solutions of this equation, Yajima et al[14] have observed that a unperturbed envelop soliton initially at rest starts to run under the action of the nonlinear Landau damping, sufferring asymmetric deformation of its shape. Referring to their observation, we have studied a temporal evolution of the one soliton on the basis of the Bogoliubov-Mitropolysky perturbation theory.[15]

§2. Nonlinear Schrödinger Equation with real coefficients and Perturbation R(Ψ)

We write the perturbation nonlinear Schrödinger equation as

$$i \frac{\partial \Psi}{\partial t} + p \frac{\partial^2 \Psi}{\partial x^2} + q|\Psi|^2\Psi = \varepsilon R(\Psi) \qquad (1)$$

where $\varepsilon << p, q, 1$ and we assume $p \cdot q > 0$. We are going to consider a perturbation around the unperturbed one soliton solution

$$\Psi_0 = 2\nu \mathrm{sech}[(\frac{q}{2p})^{\frac{1}{2}} 2\nu(x-2\mu t)] e^{i[\frac{\mu}{p}(x-2\mu t)+(\frac{\mu^2}{p} + 2q\nu^2)t]} \qquad (2)$$

As can be seen this solution depends on two independent parameters, ν and μ. We write the perturbed solution as

$$\Psi = 2\nu(t) \ \mathrm{sech}[(\frac{q}{2p})^{1/2} z(t)] e^{i[\frac{\mu z(t)}{p\nu(t)} + \delta(t)]}$$

$$z = 2\nu(t)(x - 2\int_0^t \mu(t)dt) \qquad (4a)$$

$$\delta = \int_0^t (\frac{\mu^2(t)}{p} + 2q\nu^2(t)) \ dt \qquad (4b)$$

Using the Bogoliubov-Mitropolsky method we expand

$$\rho = \rho_0(z,\tau_0,\tau_1) + \varepsilon\rho_1(z, \tau_0) + \cdots \qquad (5a)$$

$$\sigma = \sigma_0(z,\tau_0,\tau_1) + \varepsilon\sigma_1(z,\tau_0) + \cdots \qquad (5b)$$

where

$$\rho_0 = 2\nu \mathrm{sech} \ \alpha z, \qquad \sigma_0 = \frac{\mu}{2\nu p} z + \delta \ ,$$

and

$$\alpha = (\frac{q}{2p})^{\frac{1}{2}}$$

The time derivative is expanded as

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau_0} + \frac{\partial}{\partial \tau_0}\frac{\partial}{\partial z} + \varepsilon\frac{\partial}{\partial \tau_1} + \varepsilon\frac{\partial}{\partial \tau_1}\frac{\partial}{\partial z} \quad , \tag{6}$$

where $\tau_0 = t$, $\tau_1 = \varepsilon t$ $\cdots$. Introducing now

$$\frac{\partial z}{\partial x} = 2\nu \quad \text{and} \quad \frac{\partial z}{\partial \tau_0} = -4\nu\mu \quad ,$$

we obtain to order zero

$$\frac{\partial \rho_0}{\partial \tau_0} - 4\nu\mu\frac{\partial \rho_0}{\partial z} + 8p\nu^2\frac{\partial \rho_0}{\partial z}\frac{\partial \sigma_0}{\partial z} + 4p\nu^2\rho_0\frac{\partial^2\sigma_0}{\partial z^2} = 0 \tag{7a}$$

$$\rho_0\frac{\partial \rho_0}{\partial \tau_0} - 4\nu\mu\rho_0\frac{\partial \sigma_0}{\partial z} + 4p\nu^2\rho_0\left(\frac{\partial \sigma_0}{\partial z}\right)^2 - 4p\nu^2\frac{\partial^2\rho}{\partial z^2} - q\rho^3 = 0 \tag{7b}$$

The first order equations become considerably more complicated but may abbreviated in the form

$$\frac{\partial \rho_1}{\partial \tau_0} + L_I(\rho_1,\sigma_0) = M_I(\rho_0,\sigma_0) \tag{8a}$$

$$\frac{\partial \sigma_1}{\partial \tau_0} + L_R(\rho_1,\sigma_1) = M_R(\rho_0,\sigma_0) \tag{8b}$$

where

$$M_I = -\left[\frac{\partial \rho_0}{\partial \tau_1} + \frac{\partial z}{\partial \tau_1}\frac{\partial \rho}{\partial z}\right] + \text{Im}\,\{R(\rho_0,\sigma_0)e^{-i\sigma_0}\} \tag{9a}$$

$$M_R = -\left[\frac{\partial \sigma_0}{\partial \tau_1} + \frac{\partial z}{\partial \tau_1}\frac{\partial \sigma_0}{\partial z}\right] - \frac{1}{\rho_0}\,R_e\,\{R(\rho_0\sigma_0)e^{-i\sigma_0}\} \tag{9b}$$

and

$$L_I = 8p\nu^2\left[\frac{\partial \sigma_0}{\partial z}\frac{\partial \rho_1}{\partial z} + \frac{\partial \rho_0}{\partial z}\frac{\partial \sigma_1}{\partial z}\right] + 4p\nu^2\left[\frac{\partial^2\sigma_0}{\partial z^2}\rho_1 + \rho_0\frac{\partial^2\sigma_1}{\partial z^2}\right] + \frac{\partial z}{\partial \tau_0}\frac{\partial \rho_1}{\partial z} \tag{10a}$$

$$L_R = \frac{\partial z}{\partial \tau_0}\frac{\partial \sigma_1}{\partial z} + \frac{\rho_1}{\rho_0}\left(\frac{\partial \sigma_0}{\partial \tau_0} + \frac{\partial z}{\partial \tau_0}\frac{\partial \sigma_0}{\partial z}\right) + 4p\nu^2\frac{\rho_1}{\rho_0}\left(\frac{\partial \sigma_0}{\partial z}\right)^2 + 8p\nu^2\frac{\partial \sigma_0}{\partial z}\frac{\partial \sigma_1}{\partial z} - 4p\nu^2\frac{1}{\rho_0}\frac{\partial^2\rho_1}{\partial z^2} - 3q\rho_0\rho_1 \tag{10b}$$

Equations (8) may be expressed as

$$\frac{\partial}{\partial\tau_0}\begin{pmatrix}\rho_1\\ \sigma_1\end{pmatrix} + L\begin{pmatrix}\rho_1\\ \sigma_1\end{pmatrix} = M \quad . \tag{11}$$

By introducing explicitly the zeroth order solution

$$\frac{\partial\sigma_0}{\partial z} = \frac{\mu}{2\nu p} \; ; \quad \frac{\partial\sigma_0}{\partial\tau_0} = \frac{\partial\delta}{\partial\tau_0} = \frac{\mu^2}{p} + 2q\nu^2; \quad \frac{\partial z}{\partial\tau_0} = -\,4\nu\mu$$

we obtain

$$L = \begin{pmatrix} 0 & 8p\nu^3\left[2\,\dfrac{\partial\,\mathrm{sech}\alpha z}{\partial z}\,\dfrac{\partial}{\partial z} + \mathrm{sech}\alpha z\,\dfrac{\partial^2}{\partial z^2}\right] \\ -\,\dfrac{2\nu}{\mathrm{sech}\alpha z}\left[p\,\dfrac{\partial^2}{\partial z^2} - \dfrac{q}{2} + 3g\,\mathrm{sech}^2\alpha z\right] & 0 \end{pmatrix} \tag{12}$$

where

$$\alpha = \left(\frac{q}{2p}\right)^{\frac{1}{2}} \quad \text{and} \quad M = \begin{pmatrix} M_I \\ M_R \end{pmatrix}$$

Since M contains parts that depend only on the slow time scale τ_1 , we will obtain secular contributions to $\begin{pmatrix}\rho_1\\ \sigma_1\end{pmatrix}$ on the time scale τ_0 unless M does not contain a part that is orthogonal to $L\begin{pmatrix}\rho_1\\ \sigma_1\end{pmatrix}$ where $\begin{pmatrix}\rho_1\\ \sigma_1\end{pmatrix}$ is unknown. This may be written as

$$\int_{-\infty}^{\infty}\Psi(z)M\,dz = 0 \qquad \text{if} \qquad \int_{-\infty}^{\infty}\Psi(z)L\begin{pmatrix}\rho_1\\ \sigma_1\end{pmatrix}dz = 0$$

The last condition may be rewritten as

$$\int_{-\infty}^{\infty}\Psi(z)\;L\begin{pmatrix}\rho_1\\ \sigma_1\end{pmatrix}dz = \int_{-\infty}^{\infty}\begin{pmatrix}\rho_1\\ \sigma_1\end{pmatrix}L^+(\Psi)dz = 0 \tag{13}$$

where L^+ is the operator adjoint to L.

In order to satisfy (13) for arbitrary $\begin{pmatrix}\rho_1\\ \sigma_1\end{pmatrix}$ we must impose the condition

$$L^{+}(\Psi) = 0 \tag{14}$$

The operator L^{+} is obtained from L by transposition and partial integration yielding

$$L^{+} = \begin{pmatrix} 0 & -2\nu p \frac{\partial^2}{\partial z^2}\operatorname{sech}\alpha z + \frac{q\nu}{\operatorname{sech}\alpha z} - 6g\nu \operatorname{sech}\alpha z \\ 8p\nu^3 \frac{\partial}{\partial z}\left[-2\frac{\partial \operatorname{sech}\alpha z}{\partial z} + \frac{\partial}{\partial z}\operatorname{sech}\alpha z\right] & 0 \end{pmatrix} \tag{15}$$

We now want to find a solution $\Psi = \begin{pmatrix} r \\ \theta \end{pmatrix}$ satisfying (14). We then obtain

$$-2\nu p \frac{\partial^2}{\partial z^2}\left(\frac{\theta}{\operatorname{sech}\alpha z}\right) + \frac{\theta}{\operatorname{sech}\alpha z}(q\nu - 6g\nu \operatorname{sech}\alpha z) = 0 \tag{16a}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}(r \operatorname{sech}\alpha z) - 2r\frac{\partial \operatorname{sech}\alpha z}{\partial z}\right) = 0 \tag{16b}$$

with the solution

$$r = K \operatorname{sech}\alpha z \tag{17a}$$

$$\theta = -\ \gamma \operatorname{sech}\alpha z \ \frac{\partial}{\partial z}\operatorname{sech}\alpha z \tag{17b}$$

where K and γ are arbitrary constants.

The conditions for nonsecularity now are

$$\int_{-\infty}^{\infty} r\, M_I\ (\rho_0,\sigma_0)dz = 0 \tag{18a}$$

$$\int_{-\infty}^{\infty} \Theta\, M_R\ (\rho_0,\sigma_0)dz = 0 \tag{18b}$$

By introducing

$$\frac{\partial z}{\partial \tau_1} = \frac{z}{\nu}\frac{\partial \nu}{\partial \tau_1} - 4\nu\int_0^{\tau_0}\frac{\partial \mu}{\partial \tau_1}\, d\tau_0 \quad ,$$

we may write (18a) as

$$\int_{-\infty}^{\infty}\left[2\frac{\partial \nu}{\partial \tau_1}\text{sech}^2\alpha z + \left(\frac{z}{\nu}\frac{\partial \nu}{\partial \tau_1} - 4\nu\int_0^{\tau_0}\frac{\partial \mu}{\partial \tau_1}d\tau_0\right)2\nu\,\text{sech}\alpha z\,\frac{\partial\,\text{sech}\alpha z}{z}\right]dz$$

$$= \text{Im}\int_{-\infty}^{\infty} R(\rho_0,\sigma_0)e^{-i\sigma_0}\ \text{sech}\alpha z dz$$

Integrating the left hand side we now obtain

$$\frac{\partial \nu}{\partial \tau_1} = \frac{\alpha}{2}\ \text{Im}\int_{-\infty}^{\infty} R(\rho_0,\sigma_0)e^{-i\sigma_0}\ \text{sech}\alpha z\, dz \qquad (19a)$$

In order to evaluate (18b) we rewrite M_R as

$$M_R = -\left(\frac{z}{2\nu p}\frac{\partial \mu}{\partial \tau_1} + \frac{\partial \delta}{\partial \tau_1} - \frac{2\mu}{p}\int_0^{\tau_0}\frac{\partial \mu}{\partial \tau_1}d\tau_0\right) - \frac{\text{Re}\{R(\rho_0,\sigma_0)e^{-i\sigma_0}\}}{2\ \text{sech}\alpha z}$$

From (18b) we then obtain

$$\frac{\partial \mu}{\partial \tau_1} = -\alpha\ p\ \ \text{Re}\int_{-\infty}^{\infty}\frac{\sinh\alpha z}{\cosh^2\alpha z}\ R(\rho_0,\sigma_0)e^{-i\sigma_0}\ dz \qquad (19b)$$

We have thus obtained the variation in time on the time scale τ_1 of the independent parameters ν and μ . In order to write our solution as a function of a single time we make the replacements $\tau_0 = t$ and $\tau_1 = \varepsilon t$. Our results give the same variation in time of ν and μ as those of Karpman and Maslov[11]. They, however, introduce two more independent variables, $\delta(t)$ and $\xi(t) = \int_0^t 2\mu(t)dt$. Their results for these quantities differ from the present ones by the appearance of terms that are one order higher in ε.
A similar discrepancy is present between the resutls of Karpman and Maslov for the Korteweg de Vries equation[9] and the results

by Ott and Sudan[16)] and seems to be due to the larger freedom in the choice of number of independent parameters in the Karpman Maslov approach.

§3. Nonlinear Landau damping

The additional term in the nonlinear Schrödinger equation due to nonlinear Landau damping was derived by Ichikawa and Taniuti[13)]. It may be written

$$R(\rho_0,\sigma_0) = -\frac{P}{\pi}\int_{-\infty}^{\infty} \frac{\rho^2(z',t)}{z-z'}\,dz' \quad \rho(z,t)e^{i\sigma(z,t)}$$

Introducing this nonlinear, nonlocal term into (19a) and (19b) we obtain

$$\frac{\partial\nu}{\partial t} = 0 \tag{20a}$$

$$\frac{\partial\mu}{\partial t} = 8\varepsilon\alpha p\nu^3 \frac{P}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sinh\alpha z}{\cosh^2\alpha z'\cosh^3\alpha z}\frac{dzdz'}{z-z'} \tag{20b}$$

A numerical calculation of the double integral yields

$$P\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sinh z}{\cosh^2 z'\cosh^3 z}\frac{dzdz'}{z-z'} \approx 1.4615 \quad .$$

An analytical calculation is included in Appendix 1. We may then reduce (20b) to

$$\frac{\partial\mu}{\partial t} \approx 3.7217\ \varepsilon p\nu^3 \tag{21}$$

The results (20a) and (20b) may also be obtained by introducing the solution with time dependent ν and μ into the expressions for time derivative of number of quanta, momentum and energy given by Ichikawa and Taniuti[13)]. The conservation of number of quanta is a well known feature of nonlinear Landau damping, which just causes a wave quanta of a higher frequency to turn into one with a lower frequency.

From (21b) we observe that a soliton initially at rest will start to move due to the nonlinear Landau damping. This is in qualitative agreement with the numerical results of Yajima et al[14]. Effects of nonlinear Landau damping on solitons has recently also been observed experimentally by Watanabe[17]. Integrating (21) twice with respect time, we get for the trace of the maximum point of envelope soliton $\xi(t)$,

$$\xi(t) = \xi(0) + 1.46(\frac{\varepsilon}{\pi}) \ p \ (2\nu)^3 \ t^2. \qquad (22)$$

In Fig.1, we compare the theoretical result of (22) with the observed results of Yajima et al for the two values of $\varepsilon = 0.2\pi$ and $\varepsilon = 0.5\pi$. Although the present analysis is restricted only for the variation of the 0-th order soliton core, we notice agreements between the perturbation calculations and the numerical experimental observation are remarkable.

§4. Concluding Remarks

The analysis developed in the preceeding sections illustrates behaviour of perturbed envelope soliton. The motion of envelope soliton observed by Yajima et al are in good agrrement with our analysis, yet the asymetric distortion of the envelope soliton calls for further investigation of the first order components ρ_1 and σ_1. Analysis of contribution of similar term for the K-dV soliton has been carried out by Karpman and Maslov, recently.

We also remark that McLaughlin and Scott[19] have suggested to investigate fluxon interactions on the basis of a perturbed sine-Gordon equation.

In conclusion, we emphasize that the perturbation theory of solitons or envelope solitons provides us promissing tools to investigate the physical system composed with strongly interacting many degrees of freedom.

References

1) A.C. Scott, F.Y.F. Chu, and D.W. McLaughlin; Proc. IEEE 61, 1423 (1973).

2) Proceedings on a National Science Foundation Conference on Solitons, edited by H. Flaschka and D.W. McLaughlin; Rocky Mountain Mathematics Journals (1977).

3) Y.H. Ichikawa, T. Mitsuhashi and K. Konno; J. Phys. Soc. Japan 41, 1382 (1976)

4) T. Aoyama and Y.H. Ichikawa; J. Phys. Soc. Japan 42, 313 (1977)

5) K. Konno, T. Mitsuhashi and Y.H. Ichikawa; J. Phys. Soc. Japan, 43, 669 (1977)

6) Y.H. Ichikawa, T. Mitsuhashi and K. Konno; J. Phys. Soc. Japan, 43, 675 (1977)

7) Y. Kodama and T. Taniuti; preprint

8) D.J. Kaup; SIAM J. appl. Math, 31, 121 (1976)

9) V.I. Karpman and E.M. Maslov; Phys. Lett. 61A, 307 (1977)

10) V.I. Karpman and E.M. Maslov; Phys. Lett. 61A, 355 (1977)

11) V.I. Karpman and E.M. Maslov; preprint

12) J.P. Keener and D.W. McLaughlin; Phys. Rev. A 16, 777 (1977)

13) Y.H. Ichikawa and T. Taniuti; J. Phys. Soc. Japan 34, 513 (1973)

14) N. Yajima, M. Oikawa, M. Satsuma and C. Namba; Rep. Res. Inst. appl. Mech, XXII, 89 (1975)

15) Bogoliubov and Mitropolsky, Asymptotic Methods in the Theory of Nonlinear Oscillations (Hindustan Publishing Company 1961).

16) E. Ott and R.N. Sudan; Phys. Fluids 12, 2388 (1969)

17) S. Watanabe; J. Plasma Physics 17, 487 (1977).

18) V.I. Karpman and E.M. Maslov; preprint

19) D.M. McLaughlin and A.C. Scott; App. Physics Lett. 30, 545 (1977).

Appendix

Evaluation of the double integral in eq. (25b)

We write

$$I = P \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sinh z}{\cosh^2 z' \cosh^3 z} \frac{dz dz'}{z-z'} \tag{1}$$

Since

$$\frac{\sinh z}{\cosh^3 z} = -\frac{1}{2}\frac{\partial}{\partial z}\operatorname{sech}^2 z,$$ we may write

$$I = -\frac{1}{2} P \int_{-\infty}^{\infty} dz \frac{\partial}{\partial z} \operatorname{sech}^2 z \int_{\infty} \frac{dz'}{z-z'} \operatorname{sech}^2 z' \tag{2}$$

It is convenient to express the last integral in Fourier representation

since $$\int_{-\infty}^{0} e^{ik(z-z')} dk = \pi\delta(z-z') - iP\frac{1}{z-z'}$$

we obtain

$$iP\frac{1}{z-z'} = \frac{1}{2}\left[-\int_{-\infty}^{0} e^{ik(z-z')}dk + \int_{-\infty}^{0} e^{ik(z'-z)}dk\right] =$$

$$= \frac{1}{2}\left[-\int_{-\infty}^{0} e^{ik(z-z')}dk + \int_{0}^{\infty} e^{ik(z-z')}dk\right] = \frac{1}{2}\int_{-\infty}^{\infty}\frac{k}{|k|} e^{ik(z-z')}dk$$

The last integral in (2) is then

$$I_1 = \frac{1}{2}\int_{-\infty}^{\infty} dz' \operatorname{sech}^2 z' \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{k}{|k|} e^{ik(z-z')}dk = \frac{1}{2}\int_{-\infty}^{\infty}\frac{k}{|k|} F(k) e^{ikz} dk$$

where

$$F(k) = \int_{-\infty}^{\infty} \operatorname{sech}^2 z\, e^{ikz} dz = \pi k \operatorname{csech}\left(\frac{\pi}{2}k\right)$$

The total integral is then

$$I = \frac{i}{4}\int_{-\infty}^{\infty} dz \frac{\partial}{\partial z}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(k') e^{-ik'z} dk'\right]\int_{-\infty}^{\infty}\frac{k}{|k|} F(k) e^{ikz} dk =$$

$$= \frac{1}{8\pi}\int_{-\infty}^{\infty} dk' \int_{-\infty}^{\infty} dk k' F(k') F(k) \frac{k}{|k|}\int_{-\infty}^{\infty} e^{i(k-k')z} dz$$

$$= \frac{1}{8\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' F(k)\, F(k') \frac{k}{|k|} k'\; 2\pi\delta\, (k-k') =$$

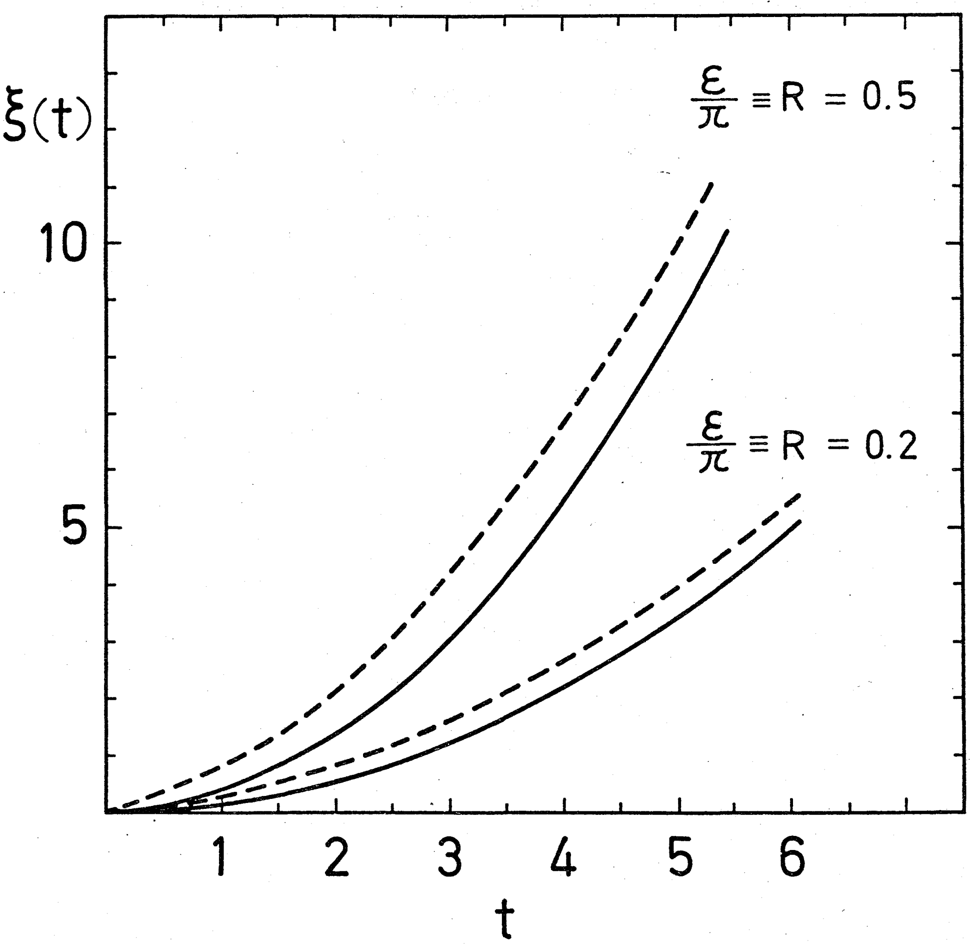
$$= \frac{1}{4} \int_{-\infty}^{\infty} k\, F^2(k)\, dk = \frac{1}{2} \int_0^{\infty} \left(\frac{\pi k}{\sinh(\frac{\pi}{2}\, k)}\right)^2 kdk$$

$$= \frac{8}{\pi^2} \int_0^{\infty} \xi^3 \quad \mathrm{csech}^2\, \xi d\xi \;\; = \frac{2}{\pi^2}\, \Gamma(4)\zeta(3)$$

where $\zeta(3) = \frac{1}{\Gamma(3)} \int_0^{\infty} \frac{t^2}{e^t - 1}\, dt$

with the numerical result

$$I = 1.461525939 \;........$$



The trace of the maximum point of the envelope soliton for two values of the size of the nonlinear Landau damping, $\varepsilon=0.2\pi$ and $0.5\ \pi$. The broken lines are read from Figs.1 a) and 1b) of Yajima et al[14], while the real lines represent the results of (27) for $p=1/2$ and $2\nu=1$.

Fig. 1