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EXCELLENT RINGS, ACCEPTABLE RINGS ETC.

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The local ring of a singular point of an algebraic or an analytic space may still have some good properties, e.g. CM (= Cohen Macaulay), Gorenstein, CI (= complete intersection). In this talk we will consider these properties in the framework of pure algebra. All rings are assumed to be noetherian.

§0. E. Noether's ascending chain condition is without doubt one of the most successful axioms in the whole history of mathematics. The theory of noetherian rings is rich in beautiful theorems. Still, not all noetherian rings are so good as polynomial rings. For instance, there are normal local rings with non-normal completion. Therefore it is sometimes desirable to work in a more restricted class of rings. Grothendieck was the first to recognize the importance of the following two problems:

1) openness of loci,
2) nature of formal fibres.

The first is geometric, the second is purely algebraic, but actually they are closely connected.

§1. Openness of loci. Let P be a property concerning local rings. For any ring A, put $P(A) = \{P \in \text{Spec } A \mid A_P \text{ is } P\}$ and call it the P-locus of A. Grothendieck asked the problem: when is $P(A)$ open is Spec(A) ?

The following topological lemma of Nagata (cf.[10] 22.B) is, though easy to prove, very useful in this problem.

TOPOLOGICAL LEMMA. Let A be a noetherian ring and U be a subset of Spec(A). In order that U is open, it is necessary and sufficient that (a) for each $P \in U$, $U \cap V(P)$ is thick [1] in V(P), and

(b) U is stable under generalization (i.e. if $P, P' \in \text{Spec}(A)$, $P \subset P'$ and $P' \in U$, then $P \in U$).

Henceforth we assume that our property P satisfies the following conditions:

I. P is stable under generalization (i.e. if a local ring A is P and P is a prime ideal of A, then A_P is P).

II. Regular local rings have the property P.

The following statement, which is called Nagata criterion for P, resembles the topological lemma but is ring-theoretical and has to be checked for each property P.

(NC) If $P(A/P)$ is thick in Spec(A/P) for each $P \in P(A)$, then $P(A)$ is thick in Spec(A).

I, II, (NC) hold for P = regular, CM, Gorenstein, CI.[2)]

Let A be an excellent ring (see below). Then Reg(A/P) is open and non-empty for each $P \in$ Spec(A). If P satisfies I, II and (NC), then $P(A/P) \supset$ Reg(A/P) and so $P(A)$ is open in Spec(A) by (NC).

We say that a noetherian ring A is

P if $P(A)$ = Spec(A),[3)]

P-0 if $P(A)$ is thick in Spec(A),

P-1 if $P(A)$ is open (may be empty) in Spec(A),

P-2 if every finitely generated A-algebra B is P-1.

Valabrega[21] considers the following "quotient condition".

(QC) If A is P, then A/P is P-0 for every $P \in$ Spec(A).

(QC) holds for P= CM, Gor, CI, but not for P= regular.[4)]

§2. Formal fibres. Let $f: (A, m) \to (B, n)$ be flat local homomorphism of local rings. Put $k = A/m$, $F = B/mB = B \otimes_A k$. We call F the fibre ring of f. One can prove:

$$\dim B = \dim A + \dim F \quad ([7]\ 6.1.2,\ [10]\ 13.B)$$

$$\operatorname{depth} B = \operatorname{depth} A + \operatorname{depth} F \quad ([7]\ 6.3.1,\ [10]\ 21.C).$$

It follows that

B is CM $\Leftrightarrow$ A and F are CM.

Similar proposition holds for Gorenstein also ([18]):

B is Gor $\Leftrightarrow$ A and F are Gor.

For P= regular we have only the following:[5)]

III. (a) B is P $\Rightarrow$ A is P.

III. (b) A and F are P $\Rightarrow$ B is P.

Definition. A ring homomorphism $f: A \to B$ is called a P-morphism if it is flat and its fibres $\{B \otimes_A \kappa(P) \mid P \in \operatorname{Spec}(A)\}$ are geometrically P.[6)]

Note that, for P = CM or P = Gor, "geometrically P" is just the same as P.[7)]

Definition. A ring A is said to be well-fibred for P [8)] if, for every prime ideal P of A, the natural homomorphism $A_P \to (A_P)^\wedge$ is a P-morphism.

Grothendieck has proved that complete local rings are well-fibred for P = regular ([6] 22.3.3, [10] 30.D). From this difficult theorem it follows that, for P which satisfies I, II, III, one can replace 'prime ideal' by 'maximal ideal' in the above definition.

Suppose P satisfies I, II, III. If A is well-fibred for P, then all finitely generated A-algebras are well fibred for P([7] 7.4.4, cf. also [10] 33.G).

If A is a local ring which is well fibred for P, and if π: Spec($\hat{A}$) $\to$ Spec(A) is the natural map, then π^{-1}(Reg(A)) = Reg($\hat{A}$). On the other hand π is submersive, i.e. a subset U of Spec(A) is open iff π^{-1}(U) is open in Spec($\hat{A}$) ([10] 6.H). Thus A is P-1 iff $\hat{A}$ is so. Suppose that P satisfies (NC). Since $\hat{A}$ is excellent, $\hat{A}$ is P-1 and so A is P-1. By similar arguments one can show that A is even P-2.

§3. Classes of rings.

Definition. A ring A is quasi-excellent if it is P-2 and well fibred for P with P = regular. If furthermore A is universally catenary [9)], then A is said to be excellent.

Definition (R. Sharp). A ring A is acceptable if it is universally catenary and P-2 and well-fibred for P with P = Gor.

In these definitions, the condition P-2 is superfluous when A is a local ring.

Definition. A ring A is Nagata (or pseudo-geometric, or universally Japanese) if, for every prime ideal P of A and for every finite extension L if the quotient field of A/P, the integral closure of A/P in L is finite over A/P.

Let C be the class of excellent (or acceptable, or Nagata) rings. Then:

$$A \in C \Rightarrow \begin{cases} A[X] \in C & \\ A/I \in C & \text{for any ideal } I \text{ of } A, \\ S^{-1}A \in C & \text{for any multiplicative set } S \subset A. \end{cases}$$

It can be shown that excellent $\Rightarrow$ acceptable
$\Rightarrow$ Nagata.

Complete local rings are excellent. All homomorphic images of Gorenstein rings are acceptable ([16] for finite dimensional case, [4] in the general case). On the other hand there exists a regular local ring R of dimension 2 containing a field K of characteristic zero and a prime element z such that z is a square in $\hat{R}$. Therefore A = R/zR is analytically ramified, and consequently R is not Nagata ([13] App.E7, cf.[10] Th.70). Thus acceptable $\not\Rightarrow$ Nagata. I do not know whether the opposite implication is true.

Sharp[17] shows that a ring which has a dualizing complex is acceptable. Rotthaus [15] gives an example of a regualar local ring of dimension 3 (and also a normal local ring of dimension 2) which contains a field of characteristic zero and which is Nagata but not quasi-excellent.

We list a few important problems which are not yet completely solved.

Problem 1. (Completion). $A \in C \Rightarrow A[[X]] \in C$?

This is equivalent to asking whether the I-adic completion $(A,I)^\wedge$ of A with respect to an arbitrary ideal I is in C, because $(A,I)^\wedge = A[[X_1,\ldots,X_n]]/(X_1-a_1,\ldots,X_n-a_n)$ if $I = (a_1,\ldots,a_n)$, and conversely $A[[X]] = (A[X], XA[X])^\wedge$.

A stronger problem is

Problem 1'. (Lifting). Let B be a ring and I be an ideal of B. Suppose B is I-adically complete and $B/I \in C$. Is $B \in C$?

(Note that a homomorphic image of $A[[X_1,\ldots,X_n]]$, $A \in C$, is a ring of the type of B.)

Problem 1' was proved for Nagata rings by J. Marot [12]. His proof depends on a difficult theorem of Y. Mori about the integral closure of a noetherian domain, which was later simplified and generalized by J. Nishimura [14].

Problem 1 has been solved for acceptable rings in [21]. For excellent rings it is still open. But if A is a finitely generated k-algebra, where k is a 1-dimensional excellent domain of characteristic zero or an arbitrary field, then $(A,I)^\wedge$ is excellent for any I ([20]).

Problem 2. Find useful sufficient conditions for a ring to be excellent.

In characteristic p, Kunz [8] proved the following useful theoem: If A is reduced and is finite over $A^p = \{a^p \mid a \in A\}$, then A is excellent. For instance, if k is a field of characteristic p with $[k:k^p] < \infty$, then $k[X_1,\ldots,X_n][[Y_1,\ldots,Y_m]]$ is excellent by this theorem.

In characteristic zero, Nagoya group (Mizutani, Nomura and myself) found that Jacobian matrices can be used effectively in proving excellence. Let R denote a regular ring, let $P \in \mathrm{Spec}(R)$ and $\mathrm{ht}P = r$. Then we say that weak Jacobian condition (WJ) holds at P if there exist $D_1,\ldots,D_r \in \mathrm{Der}(R)$ and $f_1,\ldots,f_r \in P$ such that $\det(D_i f_j) \notin P$. We say (WJ) holds in R if it holds at every prime ideal P.
We say (UWJ) holds in R if (WJ) holds in $R[X_1,\ldots,X_n]$ for all n.

THEOREM. A regular ring R which satisfies (UWJ) is excellent.

If R contains $\mathbb{Q}$ then (WJ) implies (UWJ). In characteristic p it doesn't.

Let R be a regular ring containing $\mathbb{Q}$. We say (SJ) holds at $P \in \mathrm{Spec}(R)$ if there exist $D_1,\ldots,D_r \in \mathrm{Der}(R)R_P$, and $x_1,\ldots,x_r \in P$, where r = ht P, such that $[D_i, D_j] = 0$ and $D_i x_j = \delta_{ij}$.
We say that (SJ) holds in R if it holds at every maximal ideal. (Then it holds at every prime ideal.)

THEOREM. ([11]). Let C_0 be the class of regular rings containing $\underline{Q}$ and satisfying (SJ). Then:

$R \in C_0 \Rightarrow R[X], R[[X]] \in C_0$, any localization of $R \in C_0$, and a regular ring which is a homomorphic image of R is in C_0.

The convergent power series rings $R = k\{X_1,\ldots,X_n\}$, where $k = \underline{R}$ or $\underline{C}$, are examples of rings satisfying (SJ). For any ideal I of R, the I-adic completion $(R,I)^\wedge$ is in C_0, hence excellent.

In characteristic p, one should at least add to (SJ) the condition $D_i^p = 0$. But that doesn't seem to be enough.

§4. Proof of (NC) for Gorenstein.

We will outline the interesting proof given in [4].

LEMMA. Let (A,m,k) be a local ring with a unique minimal prime ideal P. Suppose that $\operatorname{Ext}_A^i(A/P, A) = 0$ $(i > 0)$, $= A/P$ $(i = 0)$. Then A is Gor iff A/P is Gor.

Proof. If $0 \to A \to I^\cdot$ is an injective resolution of the A-module A, then $0 \to A/P \to \operatorname{Hom}_A(A/P, I^\cdot)$ is exact by the assumption, and it gives an injective resolution of the A/P-module A/P. It follows that $\operatorname{Ext}_A^i(k, A) = \operatorname{Ext}_{A/P}^i(k, A/P)$ for all i. Moreover, dim A = dim A/P. Our assertion follows from these.

(NC) for Gor. Let A be a noetherian ring such that Gor(A/P) is thick in Spec(A/P) for each $P \in$ Gor(A). Then Gor(A) is open in Spec(A).

Proof. Let $P \in$ Gor(A). By virtue of the topological lemma we have only to prove that Gor(A) $\cap$ V(P) is thick in V(P). Let ht P = n. Since A_P is CM we can choose $x_1,\ldots,x_n \in P$ in such a way that they define an A_P-regular sequence. From now on we replace A by rings of the form A_f, $f \notin P$ (hence Spec(A) by a neighborhood of P) several times.

Step 1. We may assume $x_1,\ldots,x_n$ is an A-regular sequence.
Step 2. We may replace A by $A/(x_1,\ldots,x_n)$, making ht P = 0. Also we may assume that P is the unique minimal prime ideal.
Step 3. Since A_P is Gorenstein we have

$$(\mathrm{Ext}^1_A(A/P, A))_P = \mathrm{Ext}^1_{A_P}(\kappa(P), A_P) = 0,$$

$$(\mathrm{Hom}_A(A/P, A))_P = \mathrm{Hom}_{A_P}(\kappa(P), A_P) = A_P/PA_P.$$

Therefore we may assume that $\mathrm{Ext}^1_A(A/P, A) = 0$, $\mathrm{Hom}_A(A/P, A) = A/P$. There exists a filtration of the A-module P

$$P = M_1 \supset M_2 \supset \cdots \supset M_{r+1} = 0, \quad M_i/M_{i+1} \simeq A/P_i \text{ with } P_i \in \mathrm{Spec}(A).$$

Killing all P_i other than P, we may assume P has such a filtration with $M_i/M_{i+1} \simeq A/P$ for all i. Then $\mathrm{Ext}^1(A/P, A) = 0$ implies $\mathrm{Ext}^1(P,A) = 0$, which implies $\mathrm{Ext}^2(A/P, A) = 0$, and so on. Thus we have $\mathrm{Ext}^i_A(A/P, A) = 0$ for all $i > 0$.

Step 4. We may assume A/P is Gorenstein (by hypothesis).
Step 5. Apply the preceding lemma to conclude that A is Gorenstein.

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Notes.
1) We say that a subset U of an irreducible set V is thick in V if U contains a non-empty open set of V.
2) A local ring A is regular $\Leftrightarrow$ gl.dim $A < \infty$ $\Leftrightarrow$ dim A = rank m/m^2.
A is CM $\Leftrightarrow$ dim A = depth A $\Leftrightarrow$ $\mathrm{Ext}^i_A(k, A) = 0$ for $i <$ dim A.
A is Gorenstein $\Leftrightarrow$ inj.dim $A < \infty$ $\Leftrightarrow$ $\mathrm{Ext}^i_A(k, A) = 0$ for $i <$ dim A and $= k$ for $i =$ dim A. $\Leftrightarrow$ $\mathrm{Ext}^i_A(k, A) = 0$ for some $i >$ dim A.
A is CI $\Leftrightarrow$ the completion of A is of the form R/I, where R is a regular local ring and I is an ideal generated by an R-regular sequence.
Condition II for CI was proved by Avramov [2] who used the characterization of CI by means of André's homology: A is CI iff $H_3(A,k,k) = 0$.
(NC) for regular (due to Nagata, cf.[6]6.12.2, [10] 32.A).
(NC) for CM (cf. [6] 6.11.8), (NC) for Gor and CI (cf.[4]).

3) When P = CI, it is better to say that A is locally CI.
4) CM ([6] 6.11.9 (ii), [9]). Gor ([4]), CI ([4]).
5) Cf. [10] 21.D. For P = CI, the condition III was proved in [2] by means of André homology.
6) Here $\kappa(P)$ denotes the residue field A_P/PA_P of A_P. A fibre $B \otimes \kappa(P)$ is said to be geometrically P if, for every finitely generated extension field L of $\kappa = \kappa(P)$, the ring $(B \otimes_A \kappa(P)) \otimes_\kappa L = B \otimes_A L$ is P.
Usually it is enough to check finite algebraic extension L. (Cf. [6] 6.7.7)
7) Cf.[18] for P = Gor and CI. The case of CM is easy from III.
8) In the terminology of [6] and [21], A is called a P-ring. This name may lead to confusion because, for instance, a CM ring and a P-ring for P = CM are different concepts.
9) We say a ring A is catenary if, for each pair of prime ideals $P \supset P'$, the length of a maximal chain of prime ideals between P and P' depends only on P and P' and not on the choice of the maximal chain. We say A is universally catenary if all finitely generated A-algebras are catenary. Any ring which is a homomorphic image of a regular ring (or more generally, of a CM ring) is known to be universally catenary.

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