

3-15-2021

## Engaging Learners: Differential Equations in Today's World

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### Recommended Citation

(2021) "Engaging Learners: Differential Equations in Today's World," *CODEE Journal*: Vol. 14, Article 6.  
Available at: <https://scholarship.claremont.edu/codee/vol14/iss1/6>

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# Engaging Learners: Differential Equations in Today's World

A Special Issue of the CODEE Online Journal, in memory  
of William E. Boyce

Volume 14, Issue 1, 2021



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## DEDICATION

**William Boyce** (1931-2019) taught from 1957-1997 at Rensselaer Polytechnic Institute, where he was ever active in curriculum innovation and reform, receiving Distinguished Faculty and Emeritus Professor awards. Bill was a charter member of CODEE in 1992. He quickly understood the revolution that computer graphics made possible in the teaching of differential equations, and was one of the first to insert those ideas into his books. He was forever a treasured member of CODEE, contributing much, in his very quiet way, to all our meetings and workshops. He also wrote chapters of the CODEE software package ODE Architect, which in 1998 won a Forbes Magazine award as “One of the Nine Best Digital Projects on the Planet” (in any field, not just mathematics)!



Bill Boyce authored (with Richard DiPrima, and more recently with Douglas Meade) a best-selling textbook for differential equations (11 editions!), as well as numerous papers in the subject. He also coauthored (with CODEE founders Robert Borrelli and Courtney Coleman) *A Differential Equations Laboratory Workbook* (Wiley 1992), which received the EDUCOM Best Mathematics Curricular Innovation Award in 1993.



Figure 1: What are the dotted curves on the cover on the left? These are *isoclines* for the solid-line solutions – at the time of the first edition, 1965, the best graphical technique was to sketch, usually by hand, a direction field from these loci of equal slopes. Now a numerical solver draws solutions in fractions of a second, as a simple click on the graph determines an initial condition.

Bill’s leadership is sorely missed. It is most fitting that we dedicate to his memory this Special Issue on *Engaging Learners: Differential Equations for Today’s World*.



## FOREWORD

So, what a year this has been! No one will ever forget 2020 and what it brought to our personal lives, our profession, and higher education in general. In the middle of the pandemic and all else going on in the world, CODEE asked for submissions to our second special issue *Engaging Learners: Differential Equations in Today's World*. The results of this call is an issue that focuses on the application of Ordinary Differential Equations (ODEs) to real world authentic situations. However, it also promotes situations that are motivating to students in our courses. Why did we do this? ODEs are basic in much of the applied mathematics at a university, yes, but also to many other disciplines in our current technological world. Students can be excited to learn mathematics that is truly relevant to what is going on in their lives and the world. This helps them achieve better in the class, but also provides reasons for them to take more math. Research especially tells us that for women, mathematics to solve life problems provides significant motivations for them to learn and use the mathematics!

In this special issue, our goal is to provide you, the reader, with some interesting and powerful ideas for applications of differential equations that you can use to transform, either in little ways, or in significant ways, your differential equations class. Don't we want our course to be a place students come to help solve the world's problems and issues?

This special issue has two components. There are five papers that are from faculty that present examples of using real life situation in differential equation, or systems of differential equations. Additionally, we are fortunate enough to include 7 student papers from one of our author's calculus class. These papers are honestly very interesting as the students have thought hard about what is going on in our world and how we can model that with mathematics, particularly differential equations. I encourage you to read both the faculty and student papers as they show two sides of the same coin about bringing our world into the mathematics classroom.

The five faculty papers have some very nice applications that are excellent ways to bring new problems to engage our students in Differential Equations. The year 2020 brought many new challenges because of the pandemic. One of the articles speaks to that exact crisis. *Facing the Pandemic Together: Forming a Collaborative Research Group* by Michael Barg talks about how he used the issues from COVID-19 to form a collaborative team to study systems of differential equations. Considering a different perspective, instead of using differential equations to investigate the ongoing pandemic, two other authors, Francesca Bernardi and Manuchehr Aminian, provide an engaging way to bring history into differential equations. *Epidemiology and the SIR Model: Historical Context to Modern Applications* presents the students in their classes, and us as we read, opportunities to look at how mathematics has served as a tool to learn about and predict the spread of epidemics (and pandemics) over the last 120 years.

*Modeling the Ecological Dynamics of a Three-Species Fish Population in the Chesapeake Bay* by Iordanka Panayotova and Maila Brucal-Hallare, is a perfect example of how going "local" can be engaging to students. Along similar lines, Glenn Ledder's article *Qualitative Analysis of a Resource Management Model and Its Application to the Past and Future of Endangered Whale Populations* describes a wonderful modeling example about Whales!

Finally, the article that actually opens the special issue, *Engaging Students Early by Internationalizing the Undergraduate Calculus Course*, describes how one faculty member, Chemenye Ofodile, introduced international contexts to his Calculus 2 course; his students use the basic  $dy/dt = ky$  differential equation as a springboard to bring real contexts into their experience in the course. Our student papers are those assigned in the class; each shows how the students investigated what is happening in the world and brought it to bear on solving the basic differential equation.

After reading all of these papers, what comes to my mind is the question of the value of mathematics. Many of us believe that mathematics is elegant and valuable on its own and CODEE affirms that. But in 2021, mathematics is also the most important tool we have to solve the many significant problems of the world. This can go a long way to bringing students into our mathematics world. This can also go a long way towards providing instructors with new tools to engage their students and become active learners of mathematics. I encourage you to have fun with these papers, think about these papers, and find ways to engage your students in similar ways (maybe even have them write papers like those in this special issue). Also, if you use something similar, write it up and send it to CODEE! We are looking for more manuscripts about the amazing work that differential equations educators are doing!

Karen Keene, CODEE Editor-in-Chief  
Embry-Riddle Aeronautical University

## PREFACE

At the Joint Mathematics Meetings 2020 in Denver, CODEE celebrated the global reach of the CODEE Journal. Our first Special Issue, *Linking Differential Equations to Social Justice and Environmental Concerns*, had been published less than a year earlier and had already been downloaded in over 130 (of 190 listed on the internet) countries. Every paper has been popular, with an average of over 1000 downloads each, two years after publication.

So, we sat down and planned this, our second Special Issue, to concentrate on engaging students by turning to current real-world applications. Little did we know then, January 2020, that ‘today’s world’, all of it, would be massively disrupted by the COVID-19 pandemic, on top of many other crises. Nevertheless we have accomplished the task. Although it is shorter-than-expected because everyone has been totally overloaded, it makes good contributions to the theme *Engaging Learners: Differential Equations in Today’s World*.

We dedicate this issue, as we did our JMM 2021 Sessions, to Bill Boyce, a devoted and so-appreciated charter member of CODEE. As you can see on the dedication page, Bill’s lengthy textbook career (over half a century) moved right along with our earliest abilities to use computer graphics to see the behaviors of solutions to differential equations, even those (the majority) that cannot be solved analytically in terms of elementary functions. Those of us (including generations of students) who had the pleasure of working with Bill Boyce are so lucky that we did! He researched applications of differential equations, and would have really enjoyed this Special Issue.

We are pleased to begin this volume with an article that especially contributes to the anti-racism promise of our mathematical organizations, including MAA, AWM, and CODEE.

- The first paper is authored by Professor Chinenye Ofodile of Albany State University in Albany, Georgia, a traditionally black university that has a fascinating history of engaging students by “internationalizing” courses, in many different departments across the University. Mathematics joined this effort a few years ago. Albany State’s very successful internationalization program has been flying under our radar for far too long – their publications have been in the journals of the National Association of Mathematicians (NAM), the organization for black mathematicians, and too many of us have been too otherwise occupied to see them. Fortunately, at JMM 2020 in Denver, Ofodile gave a talk outside the NAM sessions, and it caught our attention: *Engaging Students by Internationalizing the Calculus Course*. His talk and his subsequent paper in this CODEE 2020 Special Issue involve the simplest of differential equations,  $y' = ky$ . We have included some of the diverse student papers that have resulted; the contexts are the Refugee Crisis in Greece, Unemployment in Nigeria, Cardiovascular Diseases in the USA, HIV-AIDS in Georgia, Abortion in Georgia, Coronaviruses (plural, historical, and worldwide) and Home Runs! You will see that the topic of internationalization gives a deeper lesson on *Engagement*.

The subsequent papers in this Special Issue focus on modeling with systems of ordinary

differential equations, addressing issues that are part of *Today's World*. We hope these will encourage more to come.

- One of the simplest systems is the Lotka-Volterra model for predator-prey. Iordanka Panayotova of Christopher Newport University and Maila Brucal-Hallare of Norfolk State University study three species of fish in Chesapeake Bay (practically on their doorsteps). They use this model, and several variations, with their students to address two of today's ecological problems: the overfishing of Atlantic Menhaden, and the invasion of non-native blue catfish. They show that linear harvesting is sufficient to limit the growth of the invasive catfish population, but not to save the striped bass (the third species) from becoming extinct. They introduce several mathematical topics while illustrating the fundamental ecological principle of competitive exclusion. Although details are delayed to a future paper, it is worth noting that their students also learn to write reports to address the issue with local government authorities, a feature also found in Chinenye Ofodile's Albany State University project in our first paper.
- Glenn Ledder, University of Nebraska at Lincoln, increases the level of modeling, in several different ways, using more sophisticated assumptions about consumer/resource interactions. He explores why data of whale populations often do not follow the Lotka-Volterra equations, but rather seem unable, after their population has been decimated, to recover in any large numbers. This careful study sheds light and suggests new strategies to address harvesting of resources.

The final two papers move to modeling epidemics, certainly a hot topic in *Today's World*.

- Francesca Bernardi, Worcester Polytechnic Institute, and Manucher Aminian, California Polytechnic University at Pomona, teach their students the SRI model through its historical development, using original sources for the India Plague epidemic in the early 20th century. Then they move to the modern Ebola epidemic where they can have students fit the model to available datasets and confront assumptions and subtleties. Their injection of historical and social context makes students aware of tools they can use to serve society and advocate for social justice.
- Michael Barg of Niagara University, with his students, shows extensions of the SIR model in ways they found necessary to try to follow the COVID-19 epidemic, which was just being realized mid-semester in March 2020. In an unusual spur-of-the-moment change of course projects, most of his small class of students chose to dig in to researching the current evolving "event". They saw the need to create more variables, examine assumptions, make choices, and struggle with evaluating parameters. This unprecedented experience was enormously positive for this group, in multiple ways beyond the mathematics.

We hope that these papers will inspire more modeling of issues in *Today's World*, as a wonderful way to *Engage Learners*.

We gratefully acknowledge Karen Keene, Editor-in-Chief, and Ami Radunskaya, former Editor-in-Chief, whose ideas inspired this topic; all the CODEE Editorial Board members who reviewed proposals and articles and made so many very helpful suggestions; the

authors of the articles, who persevered through COVID-19 and provided so much food for thought; and Webmaster Darryl Yong, who managed the final stage of publication. It has been another fascinating endeavor of the CODEE Community; we look forward to more in the future.

Samer Habre, Editor for this Special Issue  
Beverly West, Assistant Editor  
January, 2021

# *Engaging Students Early by Internationalizing the Undergraduate Calculus Course*

Chinenye Ofodile  
*Albany State University*

**Keywords:** Student engagement, internationalization, calculus, culture, research  
Manuscript received on May 17, 2020; published on March 15, 2021.

**Abstract:** Today's world is global. However, despite increasing numbers and diversity of participants in Study Abroad programs, only 10% of U. S. college students get that experience. There is an ever-growing need for students to become aware of and experience other cultures, to understand why others think and act differently. Internationalization is the conscious effort, begun nearly 40 years ago, to integrate an international, intercultural, and global dimension into the purpose, functions, and delivery of post-secondary education.

Albany State University began a Global Program Initiative in the 1990s. In 2016, we extended into mathematics the curriculum innovations of this program. The result has engaged students in a serious way, both in mathematical modeling and in cultural research. We have introduced students to new skills of research and presentation. For the past few years we have offered one section of Calculus II in traditional mode and one in internationalized mode, and we have compared results. In this article, we give details of the process and highlight the success of the program. We end with more recent examples from Spring 2020.

## 1 Introduction

Engaging students is my passion. Albany State University in Albany, Georgia, has been focusing for some years on new teaching strategies aiming, among other things, to engage students in an international context and to keep them from switching out of STEM majors (science, technology, engineering, and mathematics). At Albany State, many students are the first in their families to attend college and to have had the educational opportunities and the resources that are taken for granted amongst students elsewhere. I myself came from Nigeria at age 8; I was very fortunate to have mentors who got me to where I am today. I want to pass on this preparatory knowledge so that more students will be able to pursue higher education on an equal footing with their peers. In 2016, I was invited by the Office for Global Programs to join one of their workshops for Internationalizing the Curriculum: Faculty Development Training ([https://](https://www.codee.org/)

[//www.asurams.edu/academic-affairs/oie/oie-publications.php](http://www.asurams.edu/academic-affairs/oie/oie-publications.php)). I had already successfully added research and presentation components to various early mathematics courses so that students would not be waiting until senior theses to learn these skills. Adding an internationalization focus made for a much richer experience. In 2020, I brought some of my students to the Joint Mathematics Meetings in Denver to give them more experience and further raise their horizons.



Figure 1: Chinenye Ofodile (far right) with students brought to Denver for JMM 2020. From left to right: Robert Lavender, Erria Gates, Amiralca Johnson, Darlena Mills, Tavis Jackson. (Photo by B. West.)

At those meetings I gave a talk on my successful effort to “Internationalize the Undergraduate Calculus II Course,” which led to an invitation to contribute to the CODEE Journal. Here I will explain how this program came into being and report on the results of the first two years, Spring 2018 and Spring 2019. Finally, I will conclude by sharing the work of the most recent students enrolled in Spring 2020.

## 2 Global Program Structure

The term “internationalization,” originally used in political science and governmental circles, emerged for education some 40 years ago as the conscious effort to integrate an international, intercultural, and global dimension into the purpose, functions, and delivery of post-secondary education [5, p. 4]. Jane Knight (University of Toronto) has been at the forefront of defining this movement from the beginning and has worked with universities, governments, and UN agencies in over 70 countries.

Albany State University came on board early, in the 1990s, with the establishment of a Global Program Initiative, and has expanded it ever since. We had very few students interested in the study abroad program, not even enough of them to apply for available funds. So we decided instead to bring the global experience into the classroom.

Internationalizing the curriculum became a major focus. Early departments to join included History, English, Finance, Music, and Education (teacher preparation). Later, Mathematics was added to the list. Interested faculty were required to attend intensive professional development workshops for new teaching strategies.

That 2016 workshop was key for me. The outstanding speakers, Dr. James Hill, Dr. Michael Smithee, and Dr. Nneka Nora Osakwe, all shared insights from their many years of experience, and we learned about internationalization efforts in other courses. Dr. Osakwe asserted “Albany State is at the forefront of publishing what faculty members are doing in curriculum internationalization. There are many publications with information on internationalizing the curriculum on campuses, but not many about what faculty actually do in the classroom. You see lots of publications that talk about what should be done or about principles.”

The workshop included the following topics:

- Infusing Internal Perspectives in Courses
- The Internationalization of Curriculum at Albany State: Reflections on a Disparate Evolution
- Internationalization: Learning Outcomes and Assessment

We also discussed revisions to syllabi, assignments, activities, course assessments, and the inclusion of presentation and manuscript components.

### 3 Introducing Global Components into Calculus II

Subsequently, we compared and analyzed two sections of Calculus II offered at Albany State, a traditional section and an internationalized one whereby real world problems were added to the first differential equation they encountered,  $y' = ky$ .

The students in the internationalized section, working either as individuals or in pairs, were asked at the beginning of the course to pick a real world topic where a population seemed to be increasing (or decreasing) at an ever greater (or slower) rate. They were assigned to

- research the culture and background for that topic.

When they reached  $y' = ky$  at the end of the course, they were then asked to

- solve the DE and find parameters to fit the data found in their research and predict future behavior;
- suggest interventions that might diminish the rate of change.



Students really dug into their topics, did the math, and thought hard specifically about the last requirement. In addition to writing up their work as a proper research report, they had to present it in class and later at regional meetings. In 2018 and 2019 we were able in some cases to have the teams report their ideas to appropriate local agencies (yet another skill!) and have agencies respond that they would try some of the interventions.

## 4 The Internationalized Sections of 2018, 2019

Below are important details about the internationalized sections in 2018 and 2019:

- Class size: 17 in 2018; 12 in 2019
- Instructional practices: Group Work, Quizzes, Exams
- Assessment practices: Exams (4), Quizzes (4), Problem Solving (daily applications)
- Writing Assignments, Homework, Research Component
- Internationalizing content areas:
  - Students were assigned to a group by the professor.
  - Each group was assigned a topic by the professor. (Later, students were given the option to choose their own topic.)
  - Each group researched the culture and way of life of the country or region for its chosen topic, prior to application and content).
  - Real world applications/examples were distributed to each group, containing various content areas.
  - Each group was able to master the content area by relating the culture to its application.
  - Each group then used their data to fit parameters to the solution of  $y' = ky$ .
  - Each group then could predict and analyze future behavior and suggest interventions.
  - All students were assessed by their doing a presentation in front of their peers; they were assessed both as groups and individually. See Figures 2, 3, 4. (Students later also presented their work at regional conferences.)
- International research topics for 2018 and 2019:
  - Refugee Crisis in Greece;
  - The Unemployment Epidemic in Nigeria;
  - Health Problems in Lower Socioeconomic Areas of the US;
  - Increasing Rates of Human Immunodeficiency Virus in Dougherty County;

- Economic Growth & Development in The City of Albany and Dougherty County;
- The Effect of the One Child Policy in China.

Comparing the internationalized class with the traditional class, we found promising benchmarks:

- There was an increase of class performance incorporating internationalization.
- The passing rate was higher in the course in which internationalization was incorporated.

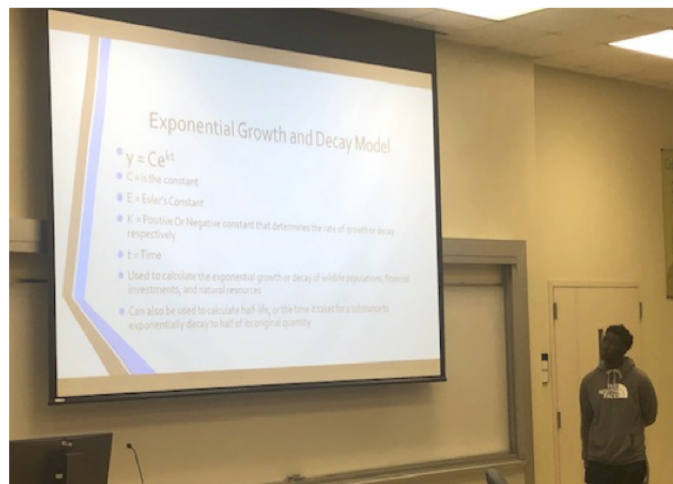


Figure 2: Explaining the mathematics (Class photo).

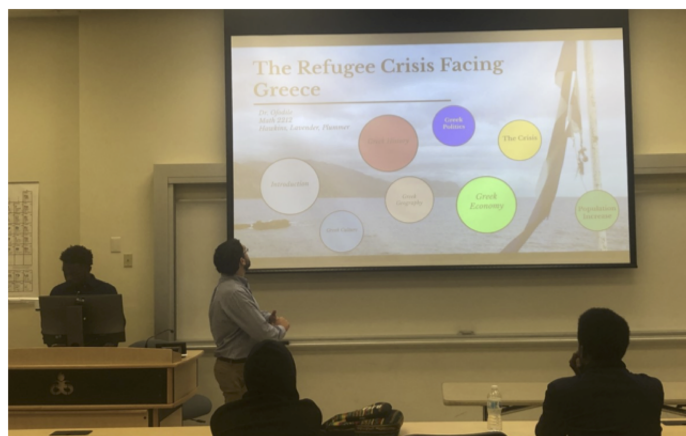


Figure 3: A slide from the Greek Refugee Crisis presentation (Class photo).

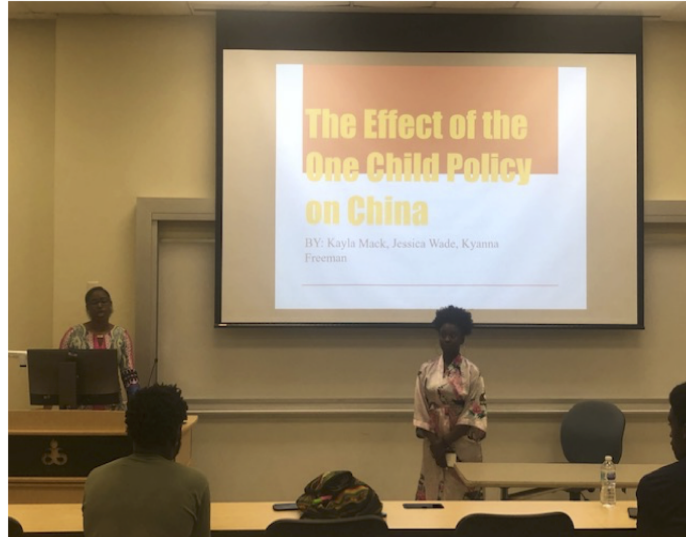


Figure 4: A slide from the One Child Policy in China presentation. Note how this student went all-out and wore Chinese clothing to emphasize the Chinese culture. (Class photo).

## 5 Changes in Spring 2020

After the Denver Joint Mathematics Meetings in January 2020, and after the Spring semester began, COVID-19 disrupted almost all of our plans. The switch to online instruction was especially difficult for students without resources at home; we lost the chance to help them get their papers into LaTeX, to present them to varied audiences, and to work together over the summer on polishing the drafts for publication. We did succeed, however, before the semester ended, in getting preliminary drafts of the papers in progress. These drafts were subsequently finished as much as possible given the many hurdles of reorganizing the students' lives.

The international research topics for 2020 included some previous topics and some new ones:

- i. The Refugee Crisis in Greece;
- ii. The Unemployment Epidemic In Nigeria;
- iii. Cardiovascular Disease in the United States;
- iv. The Impact of HIV in Georgia, USA;
- v. The Effects of Abortions in Georgia, USA;
- vi. Prevention and Control of Coronavirus Outbreaks in the Last Twenty Years;
- vii. The Increased Rate of Homeruns by Year in Major League Baseball;
- viii. Employing Youths Aging out of Foster Care.

Exactly as the previous students had done, the 2020 students enthusiastically dug into their topics, applied the math, and thought hard about that last requirement to suggest ways to modify the exponential trends. Their diverse choice of topics is fascinating. Section 7 presents a summary of several 2020 projects with links to the students' papers, somewhat edited. Due to the COVID-19 disruption, we did not have a chance to truly refine them.

## 6 Preparing Student Papers

Please keep in mind that these are written by beginning college students, with little of the prior research and mathematical writing experience of many of their peers. What should come across is the total engagement, and diligence at research, that is precipitated by asking a key question about exploring the simplest differential equation.

When I introduce  $y' = ky$  in class, I give the usual presentation of the general and particular solutions. I use the half-life example to discuss exponential growth versus decay, depending on whether  $k$  is positive or negative. Then we spend considerable time, over several class periods, analyzing student bank accounts. Students each receive a monthly check and are instructed to deposit it in a bank. Some students withdraw money to buy textbooks, etc., and others just leave the money to grow. They then bring in three months of bank statements and compare with their individual growth rates.<sup>1</sup> They had to do research and analyze their accounts, and then project the growth from the 3 months (one quarter) to 12 months (1 year). The individualization of this task really engaged them, while teaching them skills they would need for their international research projects. Because the solving of this DE was worked out so thoroughly in class, you will note that many of the papers tend to work directly from its solution,  $y(t) = Ce^{-kt}$ .

You may think some of the students' research conclusions and suggestions are ingenuous; the point we are making, however, is how thoroughly the students become engaged. They have learned many new skills, including research, writing, and presenting to others, even in their very first contact with differential equations. If there had been more time in the semester (and the summer) for discussion and feedback, students would have modified their models. Nevertheless, they have greatly broadened their horizons and been introduced to possible studies and careers they never had imagined. They are better prepared to pursue future modeling and research.

## 7 Student Papers, Spring 2020

All of these student papers can be found at <https://scholarship.claremont.edu/codee/vol14/iss1/1/>.

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<sup>1</sup>When I asked the students "How did you determine which bank to use?" I gave three choices: "parents' bank, the most convenient, or the bank with the highest interest?" The most frequent answer was "parents". But this question made them think about other options.

## 7.1 Refugee Crisis in Greece

Refugees have been streaming into Greece in ever-increasing numbers, creating huge pressure on the Greek economy, healthcare, and tourism. The steadily rising queue of applications for asylum has created an ever-increasing backlog. Robert Lavender and James Hawkins delved into Greek history and culture, tabulated data, and using the exponential solution to  $y' = ky$ , projected into the future. They suggest that the strong character of the Greek people would prevail. A key would be to process the asylum claims more quickly – then new immigrants could be employed to stimulate the economy.

## 7.2 The Unemployment Epidemic in Nigeria

Unemployment numbers in many African nations are among the highest in the world. Nigeria has one of the top rates, and is often seen as having no opportunities. Tavis Jackson maintains a global perspective as he digs into the background of the problem from many angles. With extensive research (23 references!), he studies the history, traditions, and economy of Nigeria. He lists several avenues to lower the unemployment rate – concentration on infrastructure (water, electricity, roads) will improve living standards; access to education (schools and universities) must include emphasis on training with new technologies. These corrections will lead to more and better jobs.

## 7.3 Cardiovascular Disease in the United States

Cardiovascular diseases (CVD), which include heart diseases, strokes, and other problems involving narrowing or closed blood vessels, are the leading cause of death in the United States. They affect men and women of all ethnic and racial populations. Risk factors include age, inheritance, and many lifestyle choices. AunJrae Barnes details all of this, and concludes that targeting these risk factors can make a great difference. It was much harder than expected to gather data, because every website seemed to give data for only a year or two at a time, and used a different measure (with different combinations of diseases under the CVD label). Eventually a site was found that gave CVD deaths (all in the same measure) from 1950-2017; this data was used to predict future levels.

## 7.4 The Impact of HIV in Georgia, USA

Of the southern states, Georgia has had the highest rate of HIV/AIDS cases, and good data is available. Consuela Blue researched the disease and its effects on people, then using data from 2013-2017 predicted forward. She points out that personal decisions are extremely important in fighting this disease, and proposes ways to help people make better decisions.

## 7.5 The Effects of Abortions in Georgia, USA

For more than a decade, Georgia, USA has had one of the highest abortion rates in the United States. Jazlyn Meeks discusses all the aspects of this controversial topic – physical,

psychological, social — and the difficulties encountered by many women. Giving women support and education had been successfully lowering the annual numbers of abortions from 2010 to 2015. But then the data for 2016-2018 showed disturbing increases. By fitting parameters to the data from 2012 and 2013, the graph shows a gradual decline, with a projected continuation to 2022. More attention to the mechanisms that had been successful until 2015 should further lower the curve.

## 7.6 Prevention and Control of Coronavirus Outbreaks in the Last Twenty Years

Coronaviruses that affect humans include those responsible for the SARS and MERS epidemics, for which we can find much data, and the unprecedented pandemic of COVID-19, for which data is still evolving. Increased contact with wild animals and the ever-expanding travel issues in this age of globalization seem to be prime causes. Mohammed Najeeb wrote his paper in Spring 2020, when COVID-19 was just beginning in the United States; his data ends during April. After examining the data and information on how SARS and MERS were contained, he concentrates on outbreaks in Georgia, U.S.A. and how “shelter in place” and social distancing seem to be key to bringing the numbers down.

## 7.7 The Increased Rate of Homeruns by Year in Major League Baseball

Baseball was the subject that engaged Pierce Thomas, who looked at the increasing number of home runs every year, most recently after the era of steroid scandals. He investigates what might be the causes of recent spikes, and suggests modifications to the pitchers’ mound to restore more excitement to the game.

# 8 Conclusions

Internationalizing our Calculus II course has been a very positive move, but it is only the beginning. When I was approached with an interest in internationalizing a course, I first envisioned internationalizing a particular course, Calculus II. However, while teaching that course, my vision for internationalizing expanded into other courses and included collaboration with different disciplines on research that incorporates both content based and pedagogical skills.

As a result, workshops and seminars across the university have been created with an intent to internationalize more courses. In addition, relationships across disciplines have also been created. We plan internationalization projects in Calculus III, Differential Equations, and other higher level courses, so that students will be thoroughly prepared when they begin their senior theses. Our big picture is to get all professors involved.

We are building partnerships with colleagues at Georgia State University, The Georgia Institute of Technology, Kennesaw State University, Florida State University, and other universities. We are also entering partnerships with other universities for the National Science Foundation REU (Research Experiences for Undergraduates) programs to expand

the students' experience. (Two of our mathematics students had applied to the 2020 REU at University of California, Berkeley, but COVID-19 cancelled that.)

The students in our internationalized Calculus II courses have greatly broadened their horizons, at the beginning of their college experience; they have been introduced to possible studies and careers they never had imagined. That trip to Denver in January 2020 was the first time some of them had left the city of Albany, GA. They are now eager to move forward. At the JMM meetings we met with various exhibitors, such as NSA, and forged new relationships, such as with University of Pittsburgh (computational biology), and with IBM in North Carolina (corporate sector).

Before COVID-19 interrupted everyone's lives, we were planning spring break trips abroad to increase students' exposure to the cultures they were researching. Online teaching has been far less successful than our former methodologies, but when the pandemic is behind us, we expect to be able to again move forward with our original ideas.

Other obstacles were faced prior, unrelated to the pandemic. The unfortunate problem with the last paper (Employment for Youths Aging out of Foster Care) brings a word of caution in general. Collecting appropriate data was unexpectedly difficult for many of the students. What they needed sounded so simple – just a list of annual figures over a span of at least five years. All too often that was not readily available – many sites tended to give data for only one or two years, and worse yet, these isolated bits of data were often to different measures, as already mentioned for cardiovascular diseases, so the figures given were not comparable from year to year.

We have found two ideas that can help with the data problem:

- Writing to “Ask a Librarian” at the Library of Congress. Be warned, however, that this may not be a very quick fix. It can result in multiple exchanges, and more hours than students (or professors) have available.
- Asking the college or university library to assign a class librarian to the course. This can greatly increase the efficiency of the searches, because this person would understand the course goals, and the students would have a single contact for search assistance. We have a colleague in humanities who has taught online courses for many years with research components; when Harper College (near Chicago) started the class librarian concept a few semesters ago, the quality of student papers there immediately showed vast improvement.

Our internationalized students are definitely becoming better prepared to pursue future modeling and research. Already some of them from the first years of the program have continued with higher level math courses, and some have fulfilled my vision by heading to graduate school. One student has completed a master's degree in Data Studies at Columbia University and returned to Georgia for a good job. Others are pursuing Engineering at University of Illinois and Computer Science at University of Alabama. Most recently, three students from Spring 2020 have already moved from Albany State to Georgia Tech to pursue Engineering.

It has been a good beginning.

## References

- [1] Søren Antonius. Modelling and applications—competences and democratic potential. In Ingvill Stedøy, editor, *Mathematics Education: The Nordic Way*, pages 22–31. 2004. Pre ICME-10 production.
- [2] Marcia Ascher. *Mathematics Elsewhere: An Exploration of Ideas Across Cultures*. Princeton University Press, Princeton, NJ, 2003.
- [3] Jinfa Cai and Victor Cifarelli. Thinking mathematically by chinese learners: A cross-national comparative perspective. In *How Chinese Learn Mathematics: Perspectives from Insiders*, pages 77–106. 2004.
- [4] Jane Knight. Internationalization remodeled: Definition, approaches, and rationales. *Journal of Studies in International Education*, 8:5–31, 2004. Available at <https://doi.org/10.1177/1028315303260832>.
- [5] Nneka Nora Osakwe. International courses: A faculty development process. *International Research and Review: Journal of Phi Beta Delta*, 6(2):1–30, 2017. Available at <http://www.phibetadelta.org/internationalResearch.php>.



# Refugee Crisis in Greece

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**Keywords:** Greece, refugees, asylum, Greek economy  
Manuscript received on April 25, 2020; published in March, 2021.

**Abstract:** Greece has a long history of dealing with issues that threaten their way of life. Currently there is a single issue that has Greece again backed in the corner. The refugee crisis currently facing Greece and Europe is one that doesn't appear to be slowing down any time soon. Through diligent research we find that the crisis is affecting many aspects of the Greek way of life. Economy, healthcare and tourism are a few examples.

## 1 The Driven Purpose of Our Research

There are many statistical tables available to represent the data on refugees coming into Greece. We have used this data from several sources and compiled a series of tables based on calculated constants used to predict the exponential growth of the refugee population in Greece through 2024. We have also explored the anticipated increase of asylum applicants being granted legal asylum. We express these values in an exponential growth model using a calculated constant derived from our research. We believe that granting asylum to these innocent humans is not only the best way to move them through Europe but also a great way to stimulate Greece's economy. As the number of refugees increases at the calculated rate, Greece must decide how to cope or resolve these issues. There is a choice between granting asylum and employing the new citizens, or continuing to stockpile humans in unsanitary camps. Greece will find a way, it always has.

## 2 The Current Situation

Since 2010 Greece has faced adversity from all directions. Some of the more prominent issues Greece is facing are issues with their economy, and issues concerning the refugees that continue to pour into their shorelines. However, due to the history of the Greek culture and their test of time, it is clear that Greece will persevere; it is just unclear when and if it will be soon enough. The refugees entering Greece are from war-torn countries and fleeing from persecution. When they arrive, it might seem that the living conditions aren't much different. The camps housing the refugees are unsanitary due to their over-

crowding, and healthcare isn't readily available, especially for the children. As one could imagine these conditions fuel a stressful environment. Outside of the camps there is the issue of the Economy. Greece, having been 'bailed-out' several times in the past few years, is unable to sustain a healthy economy due to poor spending habits in the past. Greece relies heavily on tourism and their trade markets to help stimulate their economy. With the slumping economy, the refugee crisis seems to be magnified. Not having a means to provide food and services adds to the unsatisfactory living conditions facing the refugees. Through the fog there is light however. The light being the ancient, resilient Greek culture and people who have stood the test of time. A people of strong faith and values that go back thousands of years. It is their faith and perseverance that will find a way for them to once again be a leader among cultures and show the world how to face adversity.

### 3 Ancient History

In order to understand Greek culture and the current economic and sociological state of Greece, it is important to grasp the origins of Greece and its very important role in human history. The cultures in ancient Greece have molded our views on politics, art, literature, and philosophy for thousands of years.

The first traces of human life in Greece date back to the Paleolithic Age (The Stone Age) approx. 120,000–10,000 B.C. [11].

From here we look forward about 3000 years to the Neolithic Age (approximately 7000–3000 B.C.) when early Greek architecture begins to take form [11].

Following the Neolithic Age and the Dark Ages of Greece, the Greek Renaissance years began (9<sup>th</sup>–8<sup>th</sup> Century B.C.). During this time, the formation of Greek city-states took place, the Greek alphabet was invented, and the Homeric epics were composed [11]. The Greek alphabet is a vital part of mathematics and physics used in hundreds of formulas and used as variables in equations. The Homeric epics, basic to many courses of literature worldwide, are also a critical element of Greece's past that are popular to this day. As we approach modern day Greece, we find more and more influential aspects associated with philosophy, arts, literature, and politics.

The Hellenistic Period (3<sup>rd</sup> century – 1<sup>st</sup> century B.C.) gave birth to world-famous philosophers such as Aristotle, Plato, Socrates, and Pythagoras [9]. Their philosophies and teachings are still studied to this day across the world in the subjects of math, physics, and philosophy.

Following the Hellenistic Period, Greece was occupied by the Roman Empire (1<sup>st</sup> century B.C. – 3<sup>rd</sup> Century A.D.). During the Roman occupancy of Greece, Roman architecture, which can be seen throughout much of Greece today, gained momentum; so did Christianity. Prior to Christianity, the major religion in Greece was Dodekathemon, which is the belief that there are twelve gods [9]. The twelve gods appear in the Homeric epics side-by-side with many Greek mythological stories. Christianity was spread throughout Greece by one of the apostles, Paul, during the 1<sup>st</sup> century A.D. [11].

## 4 Modern History

Modern Greece looks very different from ancient Greece, aside from the ancient architecture that can still be seen today.. An article on the history and background of Greece notes that “Greece today occupies about 131,957 square miles, approximately the size of Alabama. The Greek Islands make up one-fifth of this territory. Although there are about 2,000 islands, only 170 are inhabited; the largest is Crete” [5]. Greece’s population in 2019 was 11,129,227, ranked 85<sup>th</sup> in the world [6]. With the rise in the numbers of refugees entering Greece and the closing of borders of neighboring countries, the population of Greece could be expected to grow more rapidly. It is said (2018) that refugees account for about 10% of the population at this time [12].

Due to the fact that Greece is surrounded by water on three sides, it is susceptible to receiving refugees fleeing countries from war and persecution. The flood of refugees impacts the already weakened economy of Greece in a negative way. Greece’s current form of currency is the Euro, which was not adopted until 2001, even though Greece joined the European Economic Community in 1981 [7]. In this same year, Greece became a Presidential Parliamentary Democracy [11]. Up until 2001, Greece’s economic state did not meet the standard set forth in a treaty signed by the additional 12 members of the Community [7].

Following the Wall Street collapse in 2008, Greece has experienced economic turmoil. In 2010, Greece was nearing bankruptcy and sought a bailout from the so-called troika—the International Monetary Fund, the European Central Bank and the European Commission. The troika issued the first two bailouts of Greece totaling 240 billion euro [2].

While Greece is still trying to rebound from bailouts and other government measures, the refugee crisis poses a serious threat to their economy and culture. Public spending to shelter the refugees, as well as a drop of income from tourism, threatens the Greek economy [14].

## 5 Greek Culture

Understanding the Greek culture aids in also understanding how the country has reacted to masses of people seeking asylum. When looking into some of the culture of the Greeks, similarities to western world cultures can be seen, as well as many differences. People of Greece pride themselves on their traditions, food, music, art, literature, and wine as well as their ethnic and national sense of belonging [8].

The spoken language of Greece is an Indo-European language. There have been many phases of dialect throughout the 34-century history of the language. “Approximately 15,000,000 people worldwide speak the language” [13].

The legal system in Greece is very much like that of the Western culture. Their belief in democracy and even trial by jury is similar to that of the United States [8].

Religion in Greece is mostly Christian Orthodox which is followed by 98% of population. The Christian Orthodox Church is the third largest in the worldwide Christian Community, next to Roman Catholicism and Protestantism. Small percentages of the Greek population are of Muslim, Catholic, and Jewish faiths [18]. Since the rise of refugees, there have been other religious and non-religious affiliates documented entering Greece, including atheist, Baha'i, Druze, and Yazidi [19].

## 6 Impact of Refugees

Considering the religious and family-oriented culture of Greece, it is not hard to understand why they are accepting of the task laid before them in aiding the refugees. Much of the Christian Orthodox faith is centered around love for one's neighbor, the widow, the orphan, and the "least of these." As Greece continues to sort out the economic issues that strain their country and aid in helping asylum seekers, much aid and careful planning is needed for the future. The exponential increase of refugees into Greece causes an array of other issues within the infrastructure of this ancient culture. Consideration must be given to issues such as overcrowded camps, pollution, and continued stress on an already wounded economy, to name a few.

The number of refugees that are entering Europe are not evenly dispersed throughout the remainder of the continent, causing some countries to become overly populated. According to Eleanor Paynter, Greece is unfit to process its many transients, due to having 850,000 transients who landed at its shore in 2015 [16]. Refugee camps have become overcrowded establishments occupied by many asylum seekers. Unlike Greece, other countries have not been as welcoming to the refugees. However, Germany welcomed refugees under the condition of their applying for asylum, an example of protection by another sovereign authority.

Asylum, as defined by the European Union, is a legal status granted to refugees fleeing their native countries due to persecution, war, and torture. The process of applying for asylum in Greece is inefficient, making it difficult for a refugee to obtain asylum status there.

This refugee crisis was a result of journeys to Europe in hopes of better living conditions and economy, but it killed the already weak Greek economy and environment. This left the country worried about tourism, its main source of revenue. In 2015, due to war in the Middle East, an influx of people traveled through Greece in order to flee from the war: nearly 1 million people within the span of a year. The following year, 2016, saw a major decline in the number of refugees coming into Greece. That year had the highest number of deaths while crossing the Mediterranean Sea, tallying up over 5,000 deaths, and many other people reported as missing. Post-2016, the refugee population is predicted to grow exponentially, but outside factors such as death rates, poor living conditions, and limited resources could easily lead to another reduction of refugees in Greece.

## 7 Available Data

We collected from the internet three types of annual data, as shown in Table 1, and graphed in Figure 1.

Year	Total Refugees	Total Asylum Applications	Refugees Granted Asylum
2010	57,428	55,724	1,444
2011	45,810	43,922	1,573
2012	38,527	36,183	2,100
2013	73,027	49,830	3,485
2014	42,879	31,929	10,304
2015	51,825	26,141	24,838
2016	86,376	39,965	46,427
2017	83,176	44,188	38,948
2018	137,512	76,066	61,460
2019	186,144	105,690	80,468

Table 1. Annual figures for total refugees in Greece, those applying for asylum, and those granted asylum [17].

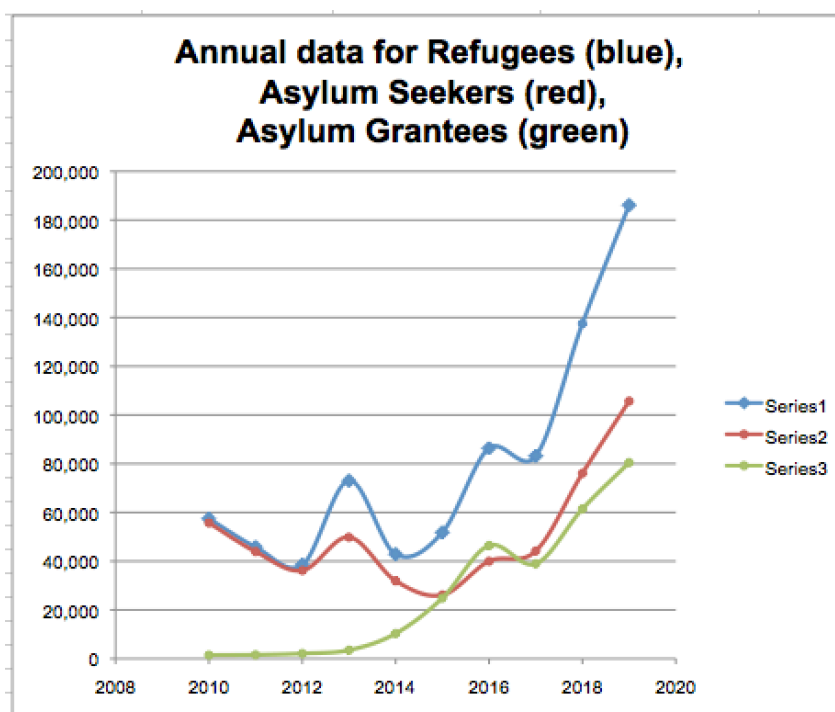


Figure 2. Color-coded graphs of the data in Table 1.

We can see that all three curves since 2014–2015 are rising, while the numbers granted asylum are falling off a bit. It is our strong belief that a solution to Greece’s struggling economy and the refugee crisis as a whole is to speed up the asylum application and granting process.

## 8 Mathematical Modeling

To predict future growth for each of the three increasingly rising populations tabulated and graphed in the previous section, we use an exponential growth model,

$$y = C e^{kt},$$

where ,

$k$  = exponential *growth* constant,

$C$  = constant,

$t$  = time elapsed since the beginning.

For each population we use the last two years of data to evaluate  $C$  and  $k$  . A sample calculation, for the refugee population, is as follows:

- Starting in 2018, with  $t = 0$ , we have

$$y = 137,512 = C e^0 = C .$$

- Then in 2019, with  $t = 1$ , we get  $y = 186,144 = 137,512 e^k$ , so

$$k = \ln(186,144/137,512) = \ln(1.353656408) = 0.3028.$$

- Thus, to fill the spreadsheet column for predicting number of refugees, we use

$$y = 137,512 e^{0.3028t}.$$

To show that the exponential model is reasonable, for each of the three populations we graph both the given data (to 2019, from Table 1) and the projected curve from the exponential model. See Figures 2, 3, and 4.

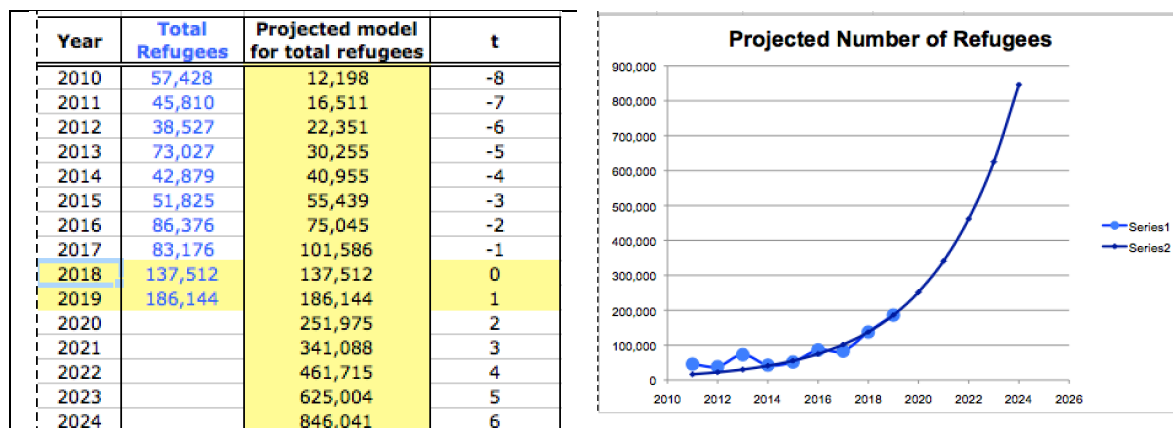


Figure 2. Table and graphs for projecting number of refugees.

The same process produces predictions for asylum figures in Figure 3, and for refugees granted asylum in Figure 4. For asylum seekers,

$$C = 76,066,$$

$$k = \ln(105690/76066) = \ln(1.38945\dots) = 0.3289,$$

$$y = 76066 e^{0.3289t}.$$

Year	Total Asylum Applications	Projected Asylum Seekers	t
2010	55,724	5,476	-8
2011	43,922	7,608	-7
2012	36,183	10,571	-6
2013	49,830	14,688	-5
2014	31,929	20,409	-4
2015	26,141	28,357	-3
2016	39,965	39,401	-2
2017	44,188	54,745	-1
2018	76,066	76,066	0
2019	105,690	105,690	1
2020		146,851	2
2021		204,042	3
2022		283,507	4
2023		393,919	5
2024		547,332	6

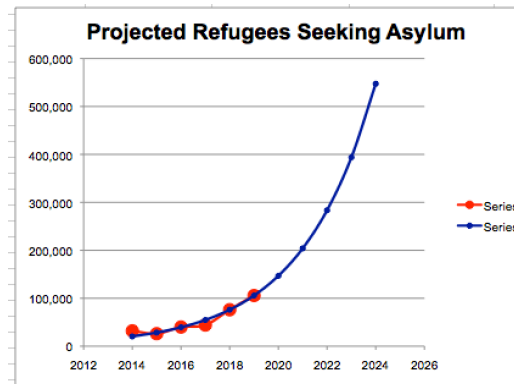


Figure 3. Table and graphs for projecting number of asylum seekers.

For refugees granted asylum,  $C = 61460,$

$$k = \ln(80468/61460) = \ln(1.30927\dots) = 0.269473,$$

$$y = 61460 e^{0.269473t}.$$

Year	Refugees Granted Asylum	Projected Refugees Granted Asylum	t
2010	1,444	7,118	-8
2011	1,573	9,319	-7
2012	2,100	12,201	-6
2013	3,485	15,975	-5
2014	10,304	20,916	-4
2015	24,838	27,384	-3
2016	46,427	35,853	-2
2017	38,948	46,942	-1
2018	61,460	61,460	0
2019	80,468	80,468	1
2020		105,355	2
2021		137,938	3
2022		180,599	4
2023		236,454	5
2024		309,583	6

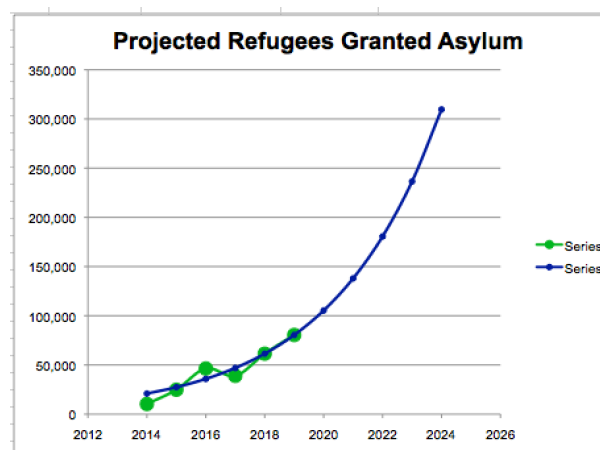


Figure 4. Table and graphs for projecting number of refugees granted asylum.



## 9 Conclusions

If the application process were to become more efficient and asylum easier to obtain, the number of refugees inhabiting the camps and draining the economy would decline. Instead of wearing down the economy, the refugees would be able to obtain employment status and contribute to the economy.

Asylum seekers seem to hold the future of Greece in the palm of their hands. With continued assistance from the EU and careful decision making moving forward, the current status of Greece can slowly begin to reverse. Educating asylum applicants, teaching them the culture, and preparing them for life beyond the walls of refugee camps is the first step. As our research shows, asylum seekers will not stop seeking refuge anytime soon. According to the numbers we projected, it is possible that Greece could collapse unless measures are taken. Therefore, it is crucial to the process that applications be submitted and assessed in a timely manner. Currently the standards of granting asylum are being evaluated to hopefully once again make it easier to obtain. It is our strong belief that granting asylum is the key to building the workforce in Greece and to Greece once again becoming the strong independent country we once knew.

## References

- [1] Adhikari, Saugat. (2019, Apr. 02). Top 10 Ancient Greek Philosophers. <https://www.ancienthistorylists.com/greek-history/top-10-ancient-greek-philosophers/>.
- [2] Alderman, Liz, et al. (2016, Jun. 17). Explaining Greece's Debt Crisis. *The New York Times*. <https://www.nytimes.com/interactive/2016/business/international/greece-debt-crisis-euro.html>.
- [3] Anagnostopoulos, Dimitris. (2019, Mar. 29). Country Report: Greece. *European Society for Child and Adolescent Psychiatry* <https://www.asylumineurope.org/reports/country/greece>.
- [4] Geography. (2019). <https://www.visitgreece.gr/en/geography>.
- [5] Greece: History and Background. (2019). <https://education.stateuniversity.com/pages/541/Greece-HISTORY-BACKGROUND.html>.
- [6] Greece Population. (2019, Apr. 01). <https://worldpopulationreview.com/countries/greece/>.
- [7] Greece's Debt: 1974-2018. (2019). *Council on Foreign Relations*. <https://www.cfr.org/timeline/greeces-debt-crisis-timeline/>.



- [8] Greek Culture Facts. (2019). [https://www.softschools.com/facts/world\\_history/greek\\_culture\\_facts/836/](https://www.softschools.com/facts/world_history/greek_culture_facts/836/).
- [9] Greek Pantheon. (2013, Jan. 22). [https://www.touristorama.com/en/greek\\_pantheon-02106](https://www.touristorama.com/en/greek_pantheon-02106).
- [10] Hellenic Republic Ministry of Citizen Protection General Secretariat for Migration Policy, Reception and Asylum, Asylum Service. (n.d.). Who Can Apply for Asylum. [https://asylo.gov.gr/en/?page\\_id=66](https://asylo.gov.gr/en/?page_id=66).
- [11] History. <https://www.visitgreece.gr/en/history>.
- [12] Kotsiou, Ourania S., et al. (2018, Aug. 20). Impact of the Refugee Crisis on the Greek Healthcare System: A Long Road to Ithaca. *International Journal of Environmental Research and Public Health* 15(8): 1790.
- [13] Lejune, Michael, et al. Greek Language. (2018, Apr. 13). <https://www.britannica.com/topic/Greek-language/>.
- [14] Marans, Daniel. (2016, Feb. 03). Greece's Economy Is Getting Crushed between Austerity and the Refugee Crisis. *Huffington Post*. [https://huffpost.com/entry/greece-refugee-crisis-economy\\_n\\_56b12f1de4b04f9b57d7b7d4](https://huffpost.com/entry/greece-refugee-crisis-economy_n_56b12f1de4b04f9b57d7b7d4).
- [15] McDonald-Gibson, Charlotte. (2016). *Cast Away: Stories of Survival From Europe's Refugee Crisis*. London, Portobello Books.
- [16] Paynter, Eleanor. (2019, Jan. 30). Europe's Refugee Crisis Explains Why Border Walls Don't Stop Migration. <https://theconversation.com/europes-refugee-crisis-explains-why-border-walls-dont-stop-migration-110414>.
- [17] Refugee and Asylum Seeker Populations by Country of Origin and Destination, 2000-2019. (2019). <https://www.migrationpolicy.org/programs/data-hub/charts/refugee-and-asylum-seeker-populations-country-origin-and-destination>.
- [18] Religion in Greece. (n.d.). <https://www.greeka.com/greece-culture/religion/>.
- [19] Religious Identity and the Refugee Crisis. (2016, Jun. 20). *Religion Factor*. <https://www.rug.nl/research/centre-for-religious-studies/religion-conflict-globalization/blog/religious-identity-and-the-refugee-crisis-20-06-2016?lang=en>.

# The Unemployment Epidemic in Nigeria

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**Keywords:** Unemployment, Nigeria, Rates, Solutions, Culture

Manuscript received on June 17, 2020; published in March, 2021.

**Abstract:** Over the years, unemployment has become a worldwide epidemic that has caused many individuals to experience poverty and a lack of education. It has become such a problem in Nigeria that the numbers for this country increase by millions each year. Year by year, the statistics for unemployment grow exponentially. Due to these high rates, it has become necessary to examine the unemployment dynamics to better understand the contributing factors and how they play a role in the statistical data. The purpose of this study is to use differential equations to create a mathematical model that will predict how quickly unemployment rates will increase exponentially over the next ten quarters, beginning with the third quarter of 2019. Afterwards, I will propose effective solutions that will cause the population of unemployed Nigerians to decrease.

## 1 Introduction

Unemployment has become one of the most prominent issues across the entire globe, especially in some countries outside of the United States. Studies have shown that many countries in Africa have millions who are unemployed. Of those, Nigeria's unemployment rate has been reported to be one of highest. Nigeria's data give the impression of it being an area with no opportunities. Every country has issues and unemployment is just one of Nigeria's. The country has traditions and customs, and the people that inhabit it have lifestyles just like residents of any other country. So, what are some of the typical traditions/customs like?

## 2 Culture and Traditions

In many African countries, there are a plethora of tribes that inhabit the area. When it comes to Nigeria, there are over 250 tribes that all have their own traditions. For example, the Omugwo tradition is very prevalent among the Igbo tribe but is also used by the Yorubas, Igalas, and the Annrangs. The Yorubas call it *Itoju-omo*, Igalas call it *Iwagwala-*

*oma*, and the Annarangs refer to it as *Umaan*. Omugwo involves the mother-in-law taking care of the newborn and preventing the mother from even lifting a finger for three months [17].

Another Nigerian tradition relates to growth and wealth. It is called the *Nwaboy*, and it is prominent among the Igbo people. This tradition is used to educate young Nigerian men in business and prepare them for the future. The young man leaves his family's home with an older, rich relative or family friend and is entrusted with activities ranging from errands to supervision and transacting business. After the training is over, the participant is given some funds to start a business to sustain himself and his family [17].

However, many traditions are not tied to specific groups but are more general. For example, many Nigerians find it very disrespectful to eat, greet, or hand over things with the left hand unless the person is left-handed or has no right arm [17]. Furthermore, with some of these traditions, certain dress styles are worn to celebrate them. According to "Nigerian Traditional Clothes", traditional attire is worn daily in many areas of Nigeria, but in other areas traditional attire is worn on special occasions. For different tribes in Nigeria, like the Igbo, the traditional clothing for women is decorated with some designs for day-to-day activities and others for formal occasions. Formal attire consists of wraps that are made of a more expensive material imported from another country. For men, the traditional attire is cotton wrappers, pants, shirts, and sandals for daily wear, and the formal attire consists of wraps made from better material than the ones used for what they wear daily [15]. Fashion in Nigeria is part of everyday life, and it symbolizes how free the people are to wear whatever they want.

Food is another important part of Nigerian culture. Many Nigerians in the northern region have diets based on beans, sorghum, and brown rice. In contrast, other tribes like the Hausa, love to eat meat in the form of *tserere* (kebabs, which are chunks of roasted, skewered meat). Many Nigerians live by old mealtime customs. A plethora of them rise as early as 5 A.M. and eat a small breakfast that can consist of rice and mangoes followed by lunch (which is the most important meal of the day) at 11 A.M.; typical lunches are *efo stew* or *moin-moin* [9].

When it comes to preparing these meals, the wife is responsible while the husband is only responsible for the funds. Many unemployed Nigerian women are restricted from working by their husbands because their husbands do not want them to stress and instead want them to live comfortably [11].

### 3 Finance and Government

As for the currency that is used, Nigerians use the Nigerian Naira, and its symbol is ₦ [13]. The average USD dollar exchanges to 361.03 Naira in Nigeria (US). The term *Naira* was coined by Obafemi Alowo when he was serving as the federal commissioner between 1967 and 1971. His daughter, Tokunbo Awolowo said, "he just took the name of Nigeria and collapsed it to Naira when he was the Federal Commissioner for Finance" [23].

Before the Nigerian Naira, there was another currency called the Nigerian Pound, introduced in 1907 under the British mandate, with the minor unit being Shillings.

However, the Central Bank of Nigeria decided to change to decimal currency due to the recommendations of the Decimal Currency Committee in the 1960s [23].

The use of Pounds instead of Naira was because of the ownership of the country at the time. Nigeria was not always an independent country; it was not until October 1, 1960, that Nigeria officially got its independence [10]. In 1973, the Naira was introduced, and it replaced the Nigerian Pound at a rate of two Nairas equal to one Pound [23].

Four years after the introduction of Naira, a new banknote denomination of twenty Naira was issued on Friday, February 11, 1977. It became the first currency to bear the portrait of a Nigerian citizen, General Murtala Ramat Mohammed. He was the late head of state and torchbearer of the Nigerian Revolution in July 1975. This new denomination of twenty Naira was issued on the first anniversary of his assassination as a tribute to the most “illustrious son of Nigeria”. Two years later, three more denominations were introduced. On July 2, 1979, the ₦1, ₦5, and ₦10 were introduced and were given distinctive colors and portraits of three “eminent and courageous Nigerians” to help in identification. Unfortunately, five years later, the colors on not just those three but all the banknotes were changed due to the currency trafficking prevalent at the time. There would not be any notable changes to the system until 1991 when the Niara were coined. Then the ₦100, ₦200, ₦500, and ₦1000 banknotes were introduced in December 1999, November 2000, April 2001, and October 2005 respectively [7].

The government of Nigeria is very similar to that of the United States. It is a federal republic with a presidential system. Just like the U.S., it has three branches of government: legislative, judicial, and executive. The current constitution was adopted in 1999; it was the fourth constitution following independence from the United Kingdom that restored democratic rule to Nigeria [12]. Its emergence ended 16 years of consecutive military rule [14]. Despite all accomplishments, the government and even the country itself still faces many issues that perpetuate.

## 4 Unemployment Situation

Nigeria has multiple problems that heavily influence the exponential growth of unemployment rates, such as education, infrastructural challenges, and safety problems.

In Nigeria, illiteracy levels are very high. According to Okwuagbala Uzochukwu Mike, a television report on September 9, 2014 stated that “10.5 million Nigerian children are out of school” [18]. The education system suffers from internal and external factors. For example, many lecturers are bribed for good grades, and even the conditions of the schools do not currently match in quality to not only Western nations, but other African countries like Ghana and South Africa. During the late 1990s, the federal government expenditure on education was below 10% of the overall budget, and later, in 2013, it was reported that the budget was ₦426.53 billion which amounts to only 8.8% of the total budget (₦4.92 trillion). The universities and institutions just are not in good shape and are not receiving the support needed to promote education [18].

Furthermore, the infrastructural challenges the Nigerians face daily discourages education. The power sector is corrupt and mismanaged, which causes many academic and business institutions to undergo power outages and limited access to electricity [18].

Along with the poor conditions of the roads, there is just a lack of support for the well-being of Nigerians. Many Nigerians often do not feel safe in their own homes. Some say that they feel as if “they can no longer walk around their own neighborhoods unharmed” [18]. Kidnapping has become very prevalent, so much so that it has turned into a business where individuals can profit from it. The kidnapping epidemic got so big that on December 10, 2017, Chukwudi Onuamadike, the most notorious high-profile kidnapper, was arrested; this was the biggest kidnapping news in Nigeria during 2017 [18].

In summary, Nigerians are daily forced to deal with poor education, terrible infrastructure, limited transportation options, and no support. It is these types of issues that cause/promote unemployment in Nigeria.

Unemployment in Nigeria is one of the most prominent issues that has seen significant increases in the last couple of years. Unemployment is defined as an economic condition in which a person is looking for a job but is unable to find one, and it is expressed a percentage of the total labor force [3]. There are multiple types of unemployment in Nigeria: structural, frictional, cyclical, and classical. Structural unemployment occurs when the technology replaces the workers with machinery and causes layoffs in the economy. Frictional unemployment occurs when a person does not have the credentials for a job. Cyclical unemployment (also defined as Keynesian) occurs when the demand of the economy is not enough to give jobs to everyone who wants to work. It is caused by the total supply of goods exceeding the total demand, which can discourage production and reduce the number of workers in the process [2]. Classical unemployment can occur when the wages a person deserves is more than the amount the employer is willing to pay [4].

In Nigeria, the rate for these unemployment types is attributed to numerous factors like the huge number of school graduates with no opportunities for matching employment to their major course of study, or the freeze on employment in many public and private sector institutions [2]. However, there have been some intervention programs made to help combat these. In Northern Nigeria, there was a three-day workshop titled *Click-On Kaduna* that aimed to equip unemployed Nigerians with skills necessary to apply for online jobs through practical training on how to set up an online profile, establish a personal brand, and land the first job [8]. Furthermore, the Nigerian government’s initial reaction was to draft the youth into agriculture programs, but they decided to make a more rational choice [1]. A 2012 program called *SURE-P* focused on management and investment of federal government savings and offered the opportunity for one-year internships in banks and agencies [1]. Due to the prevalence of unemployment in Nigeria, it became a necessity to examine the dynamics to further understand why the rates are becoming so high, to predict where they will be in the next couple of years, and to find a solution to the issue.

## 5 Mathematical Modeling

The Exponential Growth and Decay model,  $y' = ky$ , was used for this experiment because of its utility. It has a very practical ability to capture the exponential growth or decay of populations. However, it can be also be used for other quantities such as samples, resources, bank investments, and many more [5]. The solution for the Exponential Growth model is

$$y = Ce^{kt},$$

where  $C$  is the constant,  $e$  is the Euler's constant,  $k$  is the positive constant that determines the rate of growth, and  $t$  is the time. For the Exponential Decay model,  $k$  is a negative constant that determines the rate of decay [5]. For this experiment, the Exponential Growth model was used to capture the pattern of growth for unemployment rates in Nigeria. First, we gathered data from the Nigerian National Bureau of Statistics (NBS), as reported by Vanguard [16], and displayed in Table 1, for Q1 2017 to Q3 2018. Then we were able to construct, from two data points (highlighted in yellow), a model that we could extend to multiple concrete points.

<b>t</b>	<b>quarter</b>	<b>population data</b>
-9	Q1 2017	<b>13,230,000</b>
-6	Q2 2017	<b>13,590,000</b>
-3	Q3 2017	<b>15,990,000</b>
0	Q4 2017	<b>17,670,000</b>
3	Q1 2018	<b>19,250,000</b>
6	Q2 2018	<b>20,340,000</b>
9	Q3 2018	<b>20,930,000</b>

Table 1. Nigerian unemployment populations, from Nigerian National Board of Statistics (NBS) [16].

We will measure  $t$  in months. The initial value of  $t = 0$  months was decreased by 3 months when moving down a quarter and increased by 3 when moving up a quarter.

For our initial conditions, the population of unemployed Nigerians during the fourth quarter of 2017 (Q4 2017) was 17,670,000. Therefore, setting  $t = 0$  months will give  $C = 17,670,000$ . Next, after applying the population of 20,930,000 during the third quarter of 2018 (Q3 2018), where for 3 quarters  $t = 9$  months, the rate constant was found. The result is  $k = \ln(1.15)/9 = .02$ , a positive constant that proves that exponential growth is taking place.

Thus, we have a modeling equation of

$$y = 17,670,000e^{0.02t}.$$

The first quarter of 2017, with  $t = -9$  months, was used as a starting point for the modeling experiment. The result was that the model predicted some 14,759,225 unemployed Nigerians in Q1 2017. The following results were consistent with the data that was shared in other articles from Q4 2017 to Q3 2018. See Figure 1.



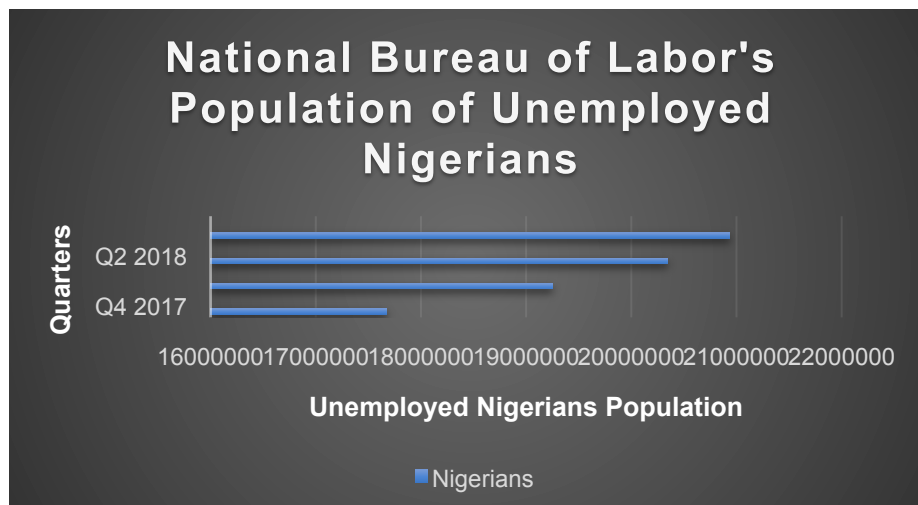


Figure 1. Our graph of the NBS data from Table 1, 4<sup>th</sup> Quarter 2017 to 3<sup>rd</sup> Quarter 2018 .

After using the data prior to this table and comparing it to the NBS data, it was proven that this modeling experiment can produce concrete values. From this conclusion, it was safe to find the data from Q1 2017 to Q2 2019. Because our model set  $t = 0$  months in Q4 2017, Q1 2017 will have  $t = -9$  months. After performing the calculations, the modeled population at Q1 2017 came out be equal to 14,759,225. From there, it was found that the total population increases by over 900,000 each year and that by the first quarter of 2019, there will be 23,852,005 unemployed Nigerians. The results are listed below in Figure 2.



Figure 2. Graphing our modeling results from 1<sup>st</sup> Quarter 2017 to 2<sup>nd</sup> Quarter 2019.

From here, it was safe to move forward and find what the rates would be like in 10 quarters from Q3 2019. We increased  $t$  by 3 months for each quarter. From our results, we were able to conclude that in every quarter year in Nigeria, the unemployment rates would increase by an average of 900,000 Nigerians, and in Q3 2021, there would be 43,461,187 unemployed Nigerians. See Figure 3.



Figure 3. Predicting further into the future with our model.

In summary, we give Table 2 listing the modeling results, accompanied by a graph superimposing the original data (blue) on the modeled data (red).

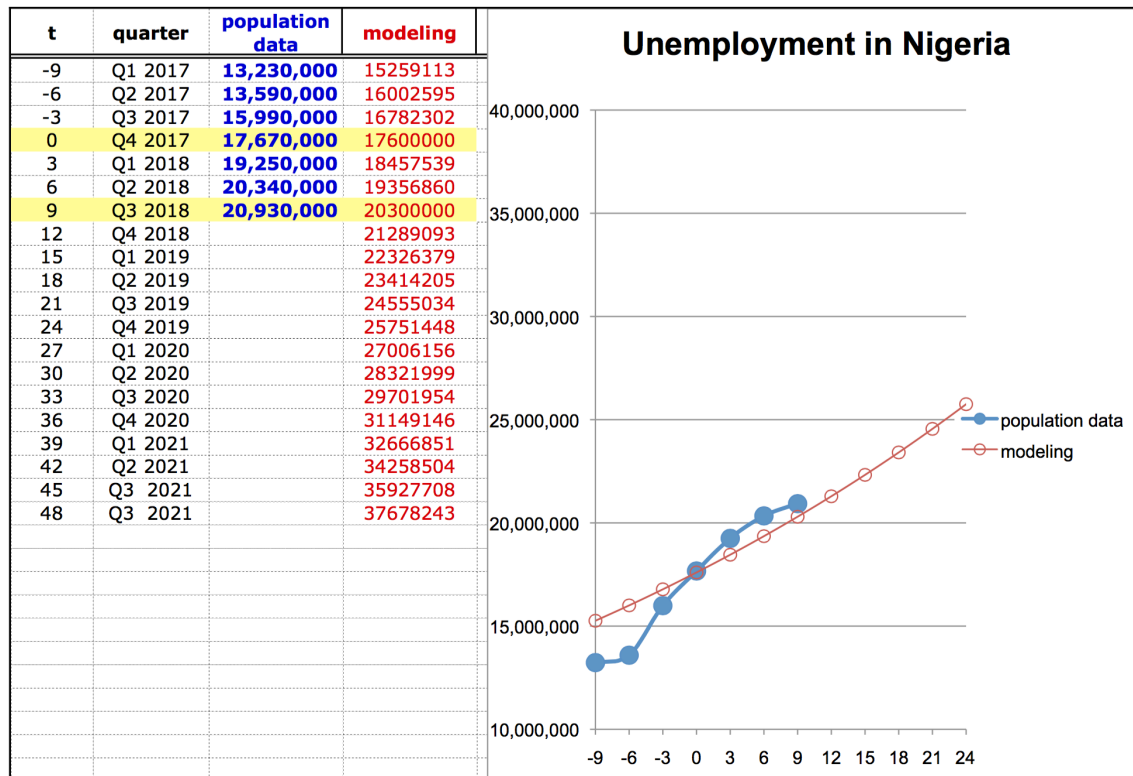


Table 2: Modeling and comparing with given data.



## 6 Ways to Lower Unemployment

However, despite all the negative factors discussed above that play a role in the growth, these rates can be reduced significantly. The absolute best place to start would have to be the infrastructure of many institutions in the cities of Nigeria. These are places where people are educated and gain the skills necessary to get a job. That is why it is essential for the government to rid the country of its lack of water supply and electricity.

It has been reported that it is difficult to get funds from the government due to limited resources and competing needs [20]. But the government does have the money (₦4.92 trillion in 2013) to do so. If enough were disbursed to fix the roads and schools, more jobs could be created. From this, the jobs would increase hired workers' purchasing power and allow for more patrons to support businesses [20]. After fixing the conditions of the roads and schools, the children would be able to get to their schools more easily, and the schools would have conditions that would encourage them to move forward. However, many Nigerians, especially among the youth, do not have the carpentry, plumbing, and bricklaying skills that are needed to rebuild these infrastructures [19]. To fix this, the government sector would have to fund educational institutions for additional curriculum and programs [20].

Along with the low numbers of Nigerians in the lower education systems, there are many graduates from universities without job opportunities. As suggested by “10 Ways To Reduce Unemployment”, if the curriculum of the universities change to give undergraduates the skills that they lack for today’s labor market, that could lead to more possibilities for them in the workforce [20]. With subjects like computer animation, robotics, web courses, data capture, and other technology-related courses, the Nigerians would have skills that are in demand right now [20]. Technology is a discipline that is always advancing and in need of workers. After changing the curriculum, more programs could be added as well to help train them after school hours. With more programs, children will be encouraged to engage more with academics and gain the soft skills through workshops that will help them acquire the credentials necessary to participate in the workforce and bring more money into the economy. If these solutions are brought to fruition, then unemployment will decrease, and Nigeria will see a change for the better.

## 7 Conclusion

In short, Nigeria is a country where the inhabitants are either rich or poor. There is no in-between within the social classes. The poor heavily outnumber the rich and residents often lack the resources they need to move up in the social classes. If nothing is done to improve the economy in Nigeria, the rate of unemployed citizens will exponentially increase to even more staggering numbers. It is imperative that the country find effective solutions to providing some of the unemployed with jobs and/or credentials to get the skills they need to get employed.

## References

- [1] Akande, Tunji. (2016, Jul. 19). Youth Unemployment in Nigeria: A Situation Analysis. <http://www.brookings.edu/blog/africa-in-focus/2014/09/23/youth-unemployment-in-nigeria-a-situation-analysis/>.
- [2] Akeju, Kemi Funlayo. (2015). Unemployment and Economic Growth in Nigeria. *International Journal of African and Asian Studies*. <https://www.iiste.org/Journals/index.php/JAAS/article/view/23940>.
- [3] Bolaji, Francis. (2018, Apr. 11). Unemployment in Nigeria: Interesting Facts You Should Know. Nigeria News. <https://www.legit.ng/1162604-types-unemployment-nigeria.html>.
- [4] Classical Unemployment. (n.d.). *Money*. <https://www.money-zine.com/definitions/career-dictionary/classical-unemployment/>.
- [5] College Algebra. (n.d.). *Lumen Learning*. <https://courses.lumenlearning.com/ivytech-collegealgebra/chapter/model-exponential-growth-and-decay>.
- [6] Falae, Vivian. (2018, Nov. 06). What Do Different Ethnic Groups in Nigeria Wear? *Nigeria News*. <https://www.legit.ng/1200779-ethnic-groups-nigeria-mode-dressing.html>.
- [7] History of Nigerian Currency. (n.d.). *Central Bank of Nigeria*. <https://www.cbn.gov.ng/Currency/historycur.asp>.
- [8] In Northern Nigeria, Online Skills Help Youth, Women Tap New Opportunities. (2018, Sep. 20). *World Bank*. <https://www.worldbank.org/en/news/feature/2018/09/20/in-northern-nigeria-online-skills-help-youth-women-tap-new-opportunities>.
- [9] Nigeria. (n.d.). <https://www.foodbycountry.com/Kazakhstan-to-South-Africa/Nigeria.html>.
- [10] Nigeria. (2018, Jun. 08). <https://www.sahistory.org.za/place/nigeria>.
- [11] Nigerian Food Culture. (n.d.). <https://allnigerianfoods.com/nigerian-food-culture>.
- [12] Nigerian Government. (n.d.). *Nigeria Business Directory*. [https://www.nigeriagalleria.com/Nigeria/Nigeria\\_Government.html](https://www.nigeriagalleria.com/Nigeria/Nigeria_Government.html).
- [13] Nigerian Naira. (n.d.). <https://www.oanda.com/currency/iso-currency-codes/NGN>.
- [14] Nigerian Politics. (n.d.). <https://www.globalsecurity.org/military/world/nigeria/politics.htm>.

- [15] Nigerian Traditional Clothes. (n.d.). <https://interesting-africa-facts.com/African-Garb/Nigerian-Traditional-Clothing.html>.
- [16] Nnorom, Nkiruka, and Elizabeth Adegbesan. (2018, Dec. 20). Unemployed Nigerians Increase by 3.3m to 20.9m - NBS. *Vanguard News*. <https://www.vanguardngr.com/2018/12/unemployed-nigerians-increase-by-3-3m-to-20-9m-nbs/>.
- [17] Okoroafor, Cynthia. (2017, Jun. 18). 10 Traditions Only Nigerians Can Understand. *Culture Trip*. <https://theculturetrip.com/africa/nigeria/articles/10-traditions-only-nigerians-can-understand/>.
- [18] Okwuagbala, Uzochukwu Mike P. (n.d.). Possible Solutions to Challenges in Nigeria Today. *Soapboxie*. <https://soapboxie.com/world-politics/challenges-in-Nigeria-and-solutions>.
- [19] 10 Ways to Eradicate Unemployment in Nigeria. (2018, Jun. 02). <https://infoguidenigeria.com/ways-eradicate-unemployment-nigeria/>.
- [20] 10 Ways to Reduce Unemployment in Nigeria. (2019, Jan. 16). <https://infoguidenigeria.com/reduce-unemployment/>
- [21] 20 Popular Nigerian Food. (n.d.). *Afro Tourism*. <https://afrotourism.com/travelogue/20-popular-nigerian-food>.
- [22] US Dollar Nigerian Naira Exchange Rate [\$1 = ₦361] . (n.d.). *Mataf*. <https://www.mataf.net/en/currency/converter-USD-NGN>.
- [23] Wahab, Bayo. (2018, Aug. 27). This Is How This Term Became Nigeria's Official Currency. *Pulse Nigeria*. <https://www.pulse.ng/lifestyle/money/naira-this-is-how-this-term-became-nigerias-official-currency/bxd4k2c>.

# Cardiovascular Disease in the United States

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**Keywords:** Cardiovascular disease, CVD, heart disease, symptoms  
Manuscript received on February 27, 2020; published in March, 2021.

**Abstract:** Cardiovascular disease (CVD) refers to the symptoms caused by narrowing or closed blood vessels; these symptoms include heart attacks, strokes, and chest pain. Cardiovascular disease is generally associated with the term heart disease. CVDs are the leading cause of death in the United States for both men and women of most ethnic and racial groups. In 2016, CVDs were responsible for 840,767 deaths in the country [13]. There are many risk factors ranging from age, ethnicity, lifestyle choices, etc. that can cause a person to develop a CVD. Even though cardiovascular disease causes many deaths throughout the America, it is still preventable. It is our goal to decrease the rate at which people in the United States contract and suffer from CVDs by targeting the risk factors.

## 1 Introduction

Death is a part of life and is the result of many factors, and most of them, in today's society, are due to medical issues and diseases. One such disease is cardiovascular disease (CVD), which is medical terminology for heart disease or diseases associated with the cardiovascular system.

The purpose of the cardiovascular system is to circulate blood throughout the body within a closed circuit, and it is comprised of vessels called arteries and veins. Arteries circulate oxygen-enriched blood from the heart to the rest of the body while veins return the blood back to the heart so that the process can be repeated on a continuous loop. The ends of arteries and veins become very small and are connected through a system of tiny vessels, which are called capillaries, that supply blood to directly to organs and tissues. The continuous flow of blood within the body is ensured by one of the most important organs, the heart. "The mechanical activity of the heart is a regular and timely succession ... responsible for the heart pumping ... about 7000 litres over 24 hours" [4]. In cardiovascular disease, these vessels are narrowed or blocked, causing heart attacks and other symptoms.

CVD greatly affects the population of the United States and the rest of the world. In an article submitted to the U.S. Department of Human and Health Services in 2012, it was reported that heart disease is the leading cause of death in the United States [8]. This

trend has continued, for in 2016 cardiovascular disease was again reported as the leading cause of death in the United States, responsible for 840,678 deaths that year [13].

## 2 Symptoms of Cardiovascular Disease

The symptoms of CVD vary with each individual from mild chest pain to the most severe, death. The same disease may be present in two different individuals, but it can also be experienced in two different ways. Some symptoms are present and persist more than some others. Adrian Chenzbraun [4] says, “Chest pain, palpitations, shortness of breath, fainting, and leg swelling are associated with heart disease”. These symptoms are not precursors for the each other, and each comes with its own level of awareness and caution.

“Chest pain is a very frequent complaint encountered in many conditions from a tense chest muscle to heart disease” [4]. It is a “defining symptom of angina and heart attack” [4] and as such brings about concern. This symptom is branded with the idea of heart disease and raises caution in individuals.

Palpitations can be described as a disturbance in the rhythm of the heart. Individuals can confuse the normal changes in heart rhythm with this symptom, causing it to be a common complaint. “As a rule of thumb, the occurrence of rare, well-tolerated, and isolated palpitations in a young person without any evidence of heart disease is a benign symptom that does not warrant treatment or investigation” [4].

Another symptom encountered with heart disease is shortness of breath, “where it can be related to heart failure” [4]. Shortness of breath is not specific to heart conditions and is more associated with lung disease and obesity.

A loss of consciousness or fainting is yet another “frequent reason for a cardiology consultation” [4]. This symptom may indicate a serious disorder in a person, but some instances are just trivial. For example, a loss of consciousness can be caused by witnessing a stressful event, suddenly changing position under certain circumstances, or giving blood, and these actions can be replicated or reproduced causing the same risk for fainting.

Lastly, swelling of the legs is a symptom frequently used as a reason to refer a patient for cardiac evaluation because of suspected heart failure. “Leg swelling is a common feature of advanced heart failure because of water retention and a rise in the venous pressure, resulting in passage of blood plasma out of the vessels. ... Leg swelling, however, can be a symptom of other known non-cardiac conditions” [4].

## 3 Risk Factors

The epidemic of cardiovascular disease is not a supernatural phenomenon because it is caused by the way we eat and live day to day. The personal characteristics for each individual that are responsible for developing a CVD are called risk factors. They range from factors that can be controlled, such as what you eat, how much you exercise, and how you interact with others, to factors that are unalienable, such as ethnic group, familial genes or traits, sex, age, etc.

Individuals can control certain risk factors, yet they may still practice bad habits causing them to develop cardiovascular diseases. For example, the way someone eats directly correlates with risk factors such as hypertension, high blood pressure, and high cholesterol. These can be maintained through efficient dieting. Dieting helps a person manage what they ingest daily, and this process allows them to keep their health, which in turn limits the possibility of them developing cardiovascular disease. “High blood pressure is a major modifiable risk factor for heart attacks ... and premature cardiovascular death” [13]. Obesity is another risk factor that can lead to cardiovascular disease. Obesity can be stalled or halted through regular exercise or physical activity. Obesity can cause shortness of breath, and is a condition that worsens other factors such as hypertension, cholesterol, and diabetes. Hypertension is an easily influenced factor because the causality of high blood pressure and heart disease is closely connected, and heart disease has many risk factors. “Uncontrolled high blood pressure can result in hardening and thickening of your arteries, narrowing the vessels through which blood flows” [8]. The choices people make every day from purposely putting themselves in stressful situations to addicting habits like smoking affect hypertension and their chances of developing heart disease. These factors can be regulated in everyone’s daily life if they are willing to take steps and are physically able to do it.

There are risk factors that you can’t control because you are born with them and cannot change them. Factors like age and sex or gender play a part in calculating the chance of cardiovascular disease diagnosis. “Heart disease is the leading cause of death in the United States, accounting for 261 deaths per 100,000 men and 273 deaths per 100,000 women” [12]. It can be inferred that women have a slightly higher death rate from cardiovascular disease than men in the United States. Age plays a significant role in determining the probability that a person can develop a CVD. The older a person becomes, the more susceptible to cardiovascular disease they become. Older people are also more likely to be diagnosed with other illnesses like hypertension or diabetes, which in turn increases their probability for contracting heart disease. The ethnic group a person is born into is also a risk factor that cannot be changed or monitored because their genes determine that they may have elevated blood pressure, cholesterol, or slower than average metabolic rates. Whether a person is Black, Mexican, or Caucasian, their ethnic group is equally affected by CVDs in their respective communities. “Among Hispanics, heart disease is the leading cause of death, representing 29% of all mortality and accounting for 192.4 deaths per 100,000 men and 129.1 deaths per 100,000 women in 2007” [12]. From the data, it can be deduced that Hispanic men suffer from CVDs more than Hispanic women. These traits characterize an individual from birth and are present with them throughout life and they pass them on to the next generation.

## 4 Data and Mathematical Modeling

Cardiovascular disease is the leading cause of death in the United States, and a major issue around the world. From an internet search we found some annual data for CVD deaths in the United States [5], as shown in Figure 1.

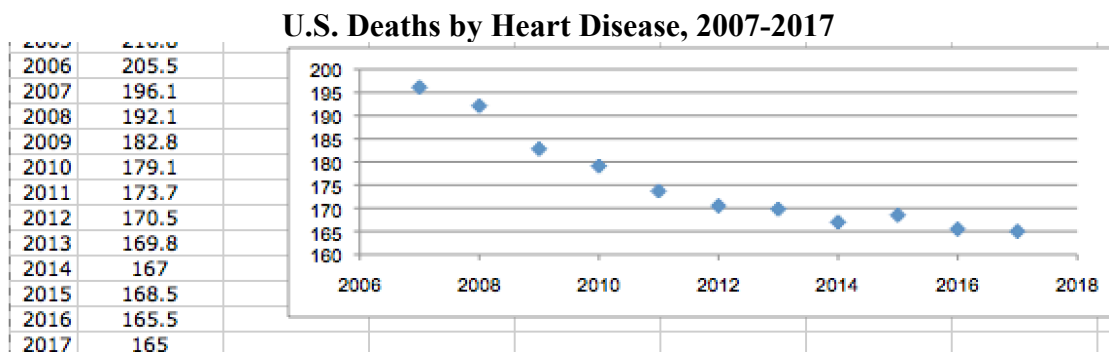


Figure 1. Spreadsheet graph for heart diseases death data 2005-2017 [5].

Because the decreasing curve of Figure 1 is vaguely exponential, a reasonable model can be made using the differential equation  $y' = ky$ , which assumes that the rate  $y'$  of deaths is proportional to the population  $y$ . The solution is an exponential function,

$$y = Ce^{kt}$$

which has two parameters,  $C$  and  $k$ .

We found a reasonable fit to the data curve by setting  $t = 0$  in 2011, when  $y_0 = 173,700$ , and  $t = 1$  in 2012, when  $y_1 = 170,500$ . The following calculations determine  $C$  and  $k$ .

$$173,700 = Ce^{k(0)} = C,$$

and

$$173,700 = 170,500e^{k(1)},$$

$$e^k = 173,700/170,500$$

$$k = \ln(173,700/170,500) = -0.01859.$$

Therefore

$$y = 173,700 e^{-0.01859t}.$$

The results are shown in Table 1 and Figure 2.

year	deaths per 100,000 population	t	predicted value
2007	196.1	-4	187.1
2008	192.1	-3	183.7
2009	182.8	-2	180.3
2010	179.1	-1	177.0
2011	173.7	0	173.7
2012	170.5	1	170.5
2013	169.8	2	167.4
2014	167.0	3	164.3
2015	168.5	4	161.3
2016	165.5	5	158.3
2017	165.0	6	155.4

Table 1. Annual data for CVD [5], and predicted values from model.



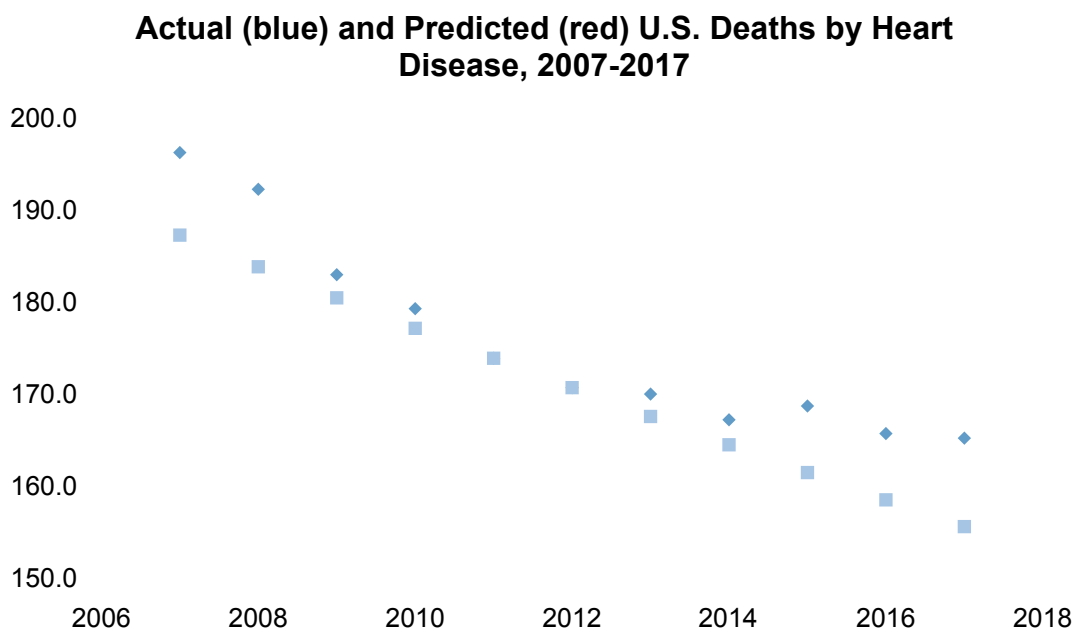


Figure 2. Graphs from spreadsheet for actual and predicted heart diseases U.S. death data 2007-2017 [5].

We note in Figure 2 that the annual death totals do *not* continue the downward exponential trend. While the data still is decreasing overall, it is at a slower rate. We will offer suggestions that could bring the decreasing rate down faster again.

## 5 To Continue Bringing Down the Numbers

A solution to continue and even improve the downward trend of the graph in Figure 2 would be to adjust the accessibility of healthy foods from places like farmers' markets and neighborhood gardens in the country. This is a subtle solution that requires the commitment of all people concerned about their own health and the health of others. Gardens and farmers' market create places where individuals would learn how to use their resources appropriately and make healthier food choices, instead of buying fast food because cooking costs too much. This solution decreases risk of hypertension and high cholesterol, which are some the main reasons for cardiovascular disease. The upkeep of the gardens creates more jobs and a sense of community within neighborhoods, which is much needed in the state of harsh, brutal societal tension that exists today.

Another solution is to mandate that children in schools be in at least one active club and that businesses create exercise regimes for their employees. This solution can be seen as radical, but the extra exercise and effort needed to completely participate in these activities decrease the chance of high cholesterol in individuals. The curve of the graph would decrease by getting the large and growing number of obese people in the country to



become more active; it would be a means to be a better participant in society and to operate successfully in their lives.

There is a continuing epidemic of heart disease cases in the United States and steps need to be taken to save the lives of people now and be proactive for others in the future. Without any effort taken, the number of people dying from this disease will continue to be too high because of the neglect of a simple issue. The health of all individuals is important, and we need stay healthy and make wise decisions to continue living prosperous lives.

## References

- [1] Brown, Arleen F., et al. (2018). Trends in Racial/Ethnic and Nativity Disparities in Cardiovascular Health Among Adults Without Prevalent Cardiovascular Disease in the United States, 1988 to 2014. *Annals of Internal Medicine*, 168(8): 541-549. <https://doi: 10.7326/M17-0996>.
- [2] Cardiovascular Diseases Affect Nearly Half of American Adults, Statistics Show. (2019, Jan.31). <https://www.heart.org/en/news/2019/01/31/cardiovascular-diseases-affect-nearly-half-of-american-adults-statistics-show>.
- [3] Casagrande, Sarah S., Andy Menke, and Catherine C. Cowie. (2016). Cardiovascular Risk Factors of Adults Age 20–49 Years in the United States, 1971–2012: A Series of Cross-Sectional Studies. *PLoS ONE*, 11(8):1-16. <https://doi.org/10.1371/journal.pone.0161770>.
- [4] Chenzbraun, Adrian. (2010). *Heart Disease*. Oxford: OUP.
- [5] Elflein, John. (2019, Nov.06). Deaths by heart diseases in the U.S. 1950-2017. <https://www.statista.com/statistics/184515/deaths-by-heart-diseases-in-the-us-since-1950/>.
- [6] Dray-Spira, Rosemary, Tiffany L. Gary, and Frederick L. Brancati. (2008). Socioeconomic Position and Cardiovascular Disease in Adults with and without Diabetes: United States Trends, 1997-2005. *JGIM: Journal of General Internal Medicine*, 23(10): 1634–1641. <https://www.ncbi.nlm.nih.gov/pmc/articles/PMC2533376/>.
- [7] Fryar CD, Chen T-C, Li X. (2012). Prevalence of uncontrolled risk factors for cardiovascular disease: United States, 1999–2010 . NCHS data brief, no. 103. Hyattsville, MD: National Center for Health Statistics.
- [8] Heart Disease. (2018, Mar. 27). <https://www.mayoclinic.org/diseases-conditions/heart-disease/symptoms-causes/syc-20353118>.
- [9] Heart Disease Facts. (n.d.).*The Heart Foundation*. <https://theheartfoundation.org/heart-disease-facts-2/>.

- [10] McWilliams, J. Michael, et al. (2009). Differences in Control of Cardiovascular Disease and Diabetes by Race, Ethnicity, and Education: U.S. Trends From 1999 to 2006 and Effects of Medicare Coverage. *Annals of Internal Medicine*, 150(8): 505–515. <https://doi: 10.7326/0003-4819-150-8-200904210-00005>.
- [11] Mensah, George A., and David W. Brown. (2007). An Overview Of Cardiovascular Disease Burden In The United States.” *Health Affairs* 26(1): 38–48. <https://doi: 10.1377/hlthaff.26.1.38>.
- [12] Morales, Leo S., et al. “Risk Factors for Cardiovascular Disease among Mexican-American Adults in the United States and Mexico: A Comparative Study. (2014). *Salud Pública de México*, 56(2): 197–205. <https://doi: 10.21149/spm.v56i2.7335>.
- [13] Webb, Sherrie R. (2019, Feb.19). AHA 2019 Heart Disease and Stroke Statistics. *American College of Cardiology*. <https://www.acc.org/latest-in-cardiology/ten-points-to-remember/2019/02/15/14/39/aha-2019-heart-disease-and-stroke-statistics>.

# The Impact of HIV/AIDS in Georgia, USA

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**Keywords:** HIV, AIDS

Manuscript received on February 26, 2020; published in March, 2021.

**Abstract:** Georgia is known as the “Peach State” and is located in the Southern Region of the United States. Of the southern states, Georgia has the highest rate of HIV/AIDS cases, 24.9 per 100,000 people. There the rate for males is more than double the amount for females. My research will show the impact HIV has had on Georgia in the last five years. I will show some comparisons from 2013 to 2017 and predict an estimate number of HIV cases from 2018 to 2024. I will be looking at solutions that were made to determine whether or not those solutions were effective. I will create better and effective solutions based on the numbers for the next five years and show how the rate of HIV in Georgia can decrease.

## 1 Introduction to HIV/AIDS

The study and speculation about Human Immunodeficiency Virus (HIV) has been going on for decades. Through considerable research, scientists have put the pieces together that trace the Human Immunodeficiency Virus back to monkeys in Africa: the red-capped mangabeys and mustached guenons, which were hunted by chimpanzees. “Different viruses were passed from these monkeys to the chimpanzees that ate them” [5]. People in Western Africa would contract HIV (also known as Simian Immunodeficiency Virus, or SIV) from contact with blood from infected animals, or from eating chimpanzee meat.

After diseases of the heart and lungs, HIV/AIDS is the leading cause of death. HIV stands for Human Immunodeficiency Virus, and is also known as the virus that causes AIDS, the Acquired ImmunoDeficiency Syndrome. Unlike other colds, diseases, and viruses, the body cannot get rid of HIV totally. Even if a person catches it at an early stage, no treatment will get rid of HIV. Once a person contracts HIV, they will have it their entire life. HIV can be carried in blood or blood transfusions, semen, fluids from the vagina, and breast milk. Immunodeficiency means the immune system does not work properly for the body. Consequently, HIV is the type of disease that attacks the immune system, typically attacking the CD4-cells or T-cells. CD4-cells help the immune system fight off infections, illnesses, and pathogens. When HIV is not treated, the number of CD4-cells in a person’s body gets lower. Over time, HIV destroys the T-cells that fight infections in the body. As a result, infections or cancers take advantage of the weak immune system, letting the person

know they are in the last stage of HIV infection, AIDS. Currently there is no cure for HIV/AIDS. However, with the proper care, this disease can be controlled.

People diagnosed with HIV may take up to 10 pills a day. There are many factors that determine how many pills a person must take, such as the severity of the case, how far the disease has spread, their T-cell count, and other chronic health conditions. Medicines include Integrase strand transfer inhibitors (INSTIs), Nucleoside reverse transcriptase inhibitors (NRTIs), Non-Nucleoside reverse transcriptase inhibitors (NNRTIs), Cytochrome P4503A Inhibitors (CYP3A), and Protease inhibitors (PIs). “Before the advent of highly active antiretroviral therapy (HAART), patients with HIV infection developed physical manifestations related to the viral infection and AIDS. Patients infected with HIV treated with HAART frequently develop body physical changes that have an important psychosocial burden” [1].

Contracting HIV affects the person mentally, physically and emotionally. Symptoms of HIV may appear within 5 months or even 5 years. Patients will develop changes such as “lean body mass with preservation, herpes simplex, oral candidiasis, hairy leukoplakia, molluscum contagiosum, and rare tumors such as Kaposi’s sarcoma” [1]. Lipodystrophy syndrome has been around for a long time. This disease will start to become recognizable when someone is treated with HAART. “This set of changes includes the loss of fat in peripheral areas (face, buttocks, arms and legs) and the gain of fat in central portions of the body (abdomen and neck)” [7]. Some patients may also have mental symptoms during the time of their sickness. “Differential diagnosis of major depression in HIV-infected patients is complicated because several symptoms including fatigue, sleep disturbance, and weight loss, are frequent symptoms of HIV disease” [6]. HIV has early and late stages. In the early stages signs of HIV may or may not show. A person may feel achy, sick, and feverish. These symptoms are flu-like, which may be the body’s first reaction to the disease. The late stage is when a person develops AIDS.

Someone can contract or transmit HIV in various ways. The most common way that HIV will spread is by “having anal or vaginal sex with someone who has HIV without using a condom” or “sharing needles or syringes, rinse water, or other equipment used to prepare drugs for injection with someone who has HIV” [3]. Rare ways of contracting HIV can be from mother to child (during pregnancy or breastfeeding) or being stuck with a contaminated needle or object. A person cannot get HIV by hugging, shaking hands, mosquitoes, ticks, or by breathing in air. HIV is only contracted or transmitted by certain body fluids that carry HIV including “blood, semen, pre-seminal fluid, rectal and vaginal fluids, and/or breast milk” [3]. The most dangerous way of contracting or transmitting HIV is anal sex. During anal sex, either partner is at risk of contracting HIV. However, the person receiving anal sex is at a higher risk. “The bottom’s risk is very high because the lining of the rectum is thin and may allow HIV to enter the body during anal sex” [3]. A less risky way of getting HIV is through vaginal sex. As with anal sex, either partner is capable of contracting HIV. “When women have vaginal sex with someone who is HIV-positive, the virus can enter her body through the mucous membranes that line the vagina and cervix” [3].

To prevent a person from becoming HIV-positive, they must make a conscious effort to make the right decision. There are many prevention programs and tools for people to take great advantage of. Therefore, a person cannot say they were not aware of how to prevent such action. Not only are there programs, but a person can prevent themselves from

contracting the disease. People can practice strategies such as: “abstinence, limiting the number of sexual partners, never sharing needles, and using condoms the right way every time you have sex” [3]. Two major prevention drugs that people are using to their advantage are pre-exposure prophylaxis (PrEP) and post-exposure prophylaxis (PEP). The difference between (PrEP) and (PEP) is that (PrEP) “is the administration of antiretroviral drugs to an uninfected person before potential HIV exposure to reduce the risk of infection and continued during risk” [4]. This antiretroviral drug can be administered by a gel being applied to the vagina or rectum or in the form of a pill. This drug is very useful because it prevents a person from contracting HIV. “So, the moment the virus enters the body, HIV replication is inhibited, and HIV is not able to establish permanent infection” [4]. A person should consider taking PrEP if they are having sexual intercourse with someone with HIV, if they are not using contraceptives, and if they are sharing needles, or syringes. For complete protection, PrEP is taken for 7 consecutive days for receptive anal sex. However, for receptive vaginal sex and injection drug use, PrEP is taken for 21 consecutive days for full protection. “Studies have shown that PrEP reduces the risk of getting HIV from sex by about 99% when taken daily. Among people who inject drugs, PrEP reduces the risk of getting HIV by at least 74% when taken daily” [3]. For post-sexual exposure, PEP is given to HIV- infected blood or potentially infectious bodily fluid patients. So, this prevention tool is for patients who are HIV-positive or people who may think they have contracted HIV from another person. PEP also “is a comprehensive management which includes first aid, counseling, risk assessment, relevant laboratory investigations based on informed consent of the source and exposed person” [4]. A patient can take PEP if they have been exposed to someone who is HIV-positive up to 72 hours after exposure; then the patient will have to take the drug for about 30 consecutive days.

## 2 Data and Mathematical Model

The most prominent, modern example of how the government, policies, and public health are intertwined is HIV/AIDS. The HIV/AIDS rate in Georgia continues to increase year by year. The increasing rates in Georgia solely belong to a certain group – young, black males who have sex with other males. Between 2013 and 2017, males have held the lead for the number of cases and new cases every year. The rate continues to increase yearly, as shown in Table 1 and Figure 1.

year	living with HIV/AIDS
2012	50,436
2013	51,510
2014	53,230
2015	54,754
2016	56,789
2017	58,808
2018	60,346

Table 1

**Annual totals of people in GA living with HIV/AIDS 2012-2018**

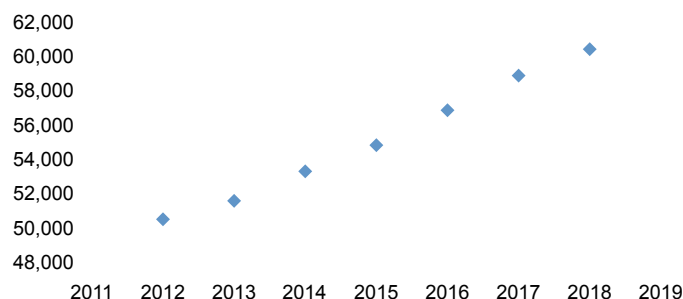


Figure 1. Annual data for people in Georgia, USA living with HIV/AIDS. [2]

We observe that the yearly increase is growing (ever so slightly), so we will make a model from the differential equation  $y' = ky$ , which has the exponential solution

$$y = Ce^{kt}.$$

In 2013, there were 51,510 cases (overall) of HIV/AIDS in Georgia. This number will be the initial point to determine the value of  $C$  in the exponential function. Then in 2014, a year later, the number of HIV/AIDS cases increased to 53,230 in Georgia. Therefore, the initial condition is identified: at  $t = 0$ ,  $y = 51,510$  and at the second point,  $t = 1$ ,  $y = 53,230$ . We can now formulate the exponential function:

$$\begin{aligned} y &= Ce^{kt}, \quad t = 0, \quad y = 51,510 \\ \Rightarrow 51,510 &= Ce^{k(0)} = Ce^0 = C, \end{aligned}$$

and

$$\begin{aligned} y &= 51,510 e^{kt}, \quad t = 1, \quad y = 53,230 \\ \Rightarrow 53,230 &= 51,510 e^{k(1)} = 51,510 e^k \\ \Rightarrow \frac{53,230}{51,510} &= e^k \\ \Rightarrow \ln \left( \frac{53,230}{51,510} \right) &= \ln e^k \\ \Rightarrow \ln \left( \frac{53,230}{51,510} \right) &= \ln e^k = k \\ \Rightarrow k &= \ln \left( \frac{53,230}{51,510} \right) = 0.0328461836. \end{aligned}$$

Thus

$$y = 51,510e^{0.0328461836t}$$

Now we can graph predicted values, listed in Table 2 and shown in Figure 2 on the next page, and compare with the data we have.

year	living with HIV/AIDS	t	predicted living with HIV/AIDS
2012	50,436	-1	49,846
2013	51,510	0	51,510
2014	53,230	1	53,230
2015	54,754	2	55,007
2016	56,789	3	56,844
2017	58,808	4	58,742
2018	60,346	5	60,704
2019		6	62,731
2020		7	64,826
2021		8	66,990
2022		9	69,227
2023		10	71,539
2024		11	73,927

Table 2: Yellow lines show that 2013 and 2014 were used to compute  $C$  and  $k$  for exponential model.

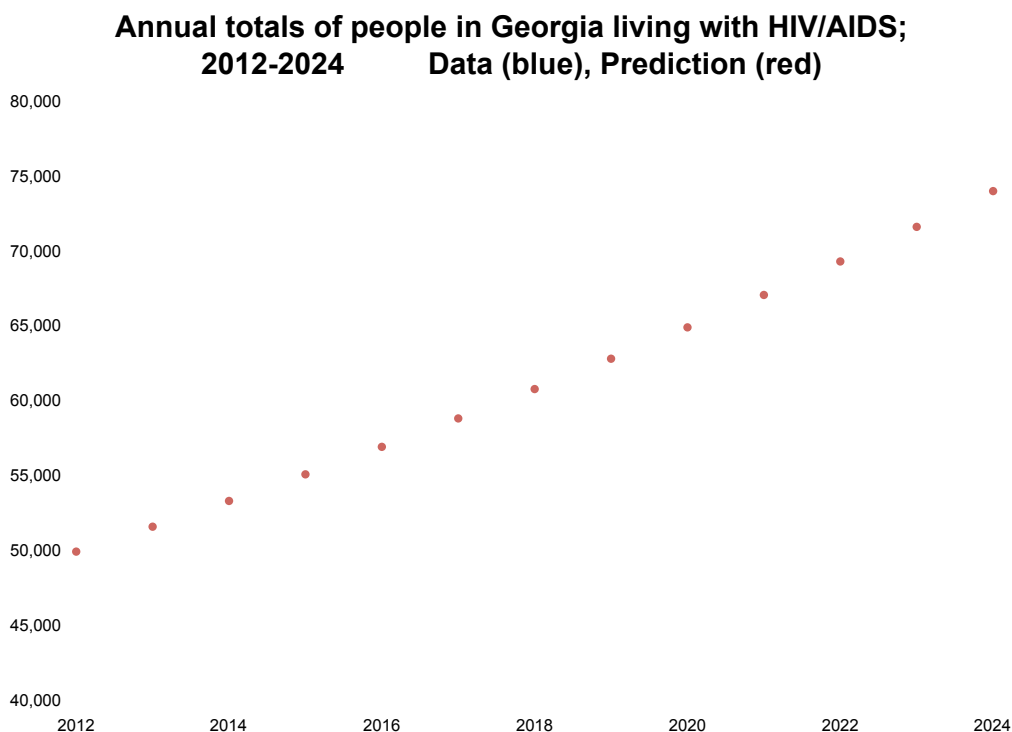


Figure 2: Anticipated annual totals (red) for people in Georgia, USA living with HIV/AIDS. Data values from Figure 1 added in blue.

As shown in Figure 2, the exponential model (red) closely approximates the data (blue diamonds) from 2012 to 2018, and indicates how over the next few years the HIV/AIDS rate may continue to increase. According to the graph, there are approximately 2,000 new cases of HIV/AIDS every year. Therefore, the solutions that have been put in place previously will not decrease the number of cases in the next five years. New solutions must be implemented to decrease those.

### 3 Suggested Interventions

A solution that could decrease the number of HIV/AIDS would be to have clinic workers go to different organizations, give speeches, and host workshops for the community. This will ensure that people know exactly what HIV/AIDS is, how a person can contract it, and the many ways one will remain HIV positive. This also gives a person the opportunity to ask questions about things that they didn't fully understand. The clinic workers could bring in people who have HIV and allow them to share their stories on how they contracted HIV.

An alternative solution that would help keep the number of HIV cases low is to implement a course curriculum for all high schools worldwide for 11<sup>th</sup> and 12<sup>th</sup> graders. The class should be year-round and a required course to graduate. In today's time, students at this age (or younger) are starting to become sexually active and getting tattoos and piercings. This solution could help those students who aren't being given this information at home. It will allow students at an early age to make sure they are taking care of their bodies for the right reason.

### 4 Conclusion

The wide spread of HIV/AIDS has been around for years and has impacted the lives of many people and their families, some of whom may have even died from this terrible disease. Families, communities, school systems, and churches must take the time out to educate themselves and others around them on how this disease can ultimately change their lives for the worst. This epidemic must be prevented so people can live healthy and prosperous lives.



## References

- [1] Cabrero, Esther, et al. (2010). Prevalence and Impact of Body Physical Changes in HIV Patients Treated with Highly Active Antiretroviral Therapy: Results from a Study on Patient and Physician Perceptions. *AIDS Patient Care and STDS*, 24(1): 5–13. <https://pubmed.ncbi.nlm.nih.gov/20095903/>.
- [2] Georgia HIV Surveillance Summary. (n.d.). <https://dph.georgia.gov/epidemiology/georgias-hivaids-epidemiology-section/georgia-hiv-surveillance-data>.
- [3] HIV Prevention. (2019, Dec. 02). <https://www.cdc.gov/hiv/basics/prevention.html>.
- [4] Marfatia, Yogesh S., et al. (2017). Pre- and Post-sexual Exposure Prophylaxis of HIV: An Update. *Indian Journal of Sexually Transmitted Diseases and AIDS*. 38(1): 1-9. <https://pubmed.ncbi.nlm.nih.gov/28442797/>.
- [5] McPartland, Randall. (2015). *HIV and AIDS*. New York: Cavendish Square Publishing LLC.
- [6] Perkins, D. O., et al. (1995). Somatic Symptoms and HIV Infection: Relationship to Depressive Symptoms and Indicators of HIV Disease. *The American Journal of Psychiatry*, 152(12): 1776–1781. [https://www.researchgate.net/publication/14674736\\_Somatic\\_symptoms\\_and\\_HIV\\_infection\\_Relationship\\_to\\_depressive\\_symptoms\\_and\\_indicators\\_of\\_HIV\\_disease](https://www.researchgate.net/publication/14674736_Somatic_symptoms_and_HIV_infection_Relationship_to_depressive_symptoms_and_indicators_of_HIV_disease).
- [7] Santos, Claudia Paula, et al. (2005). Self-Perception of Body Changes in Persons Living with HIV/AIDS: Prevalence and Associated Factors. *AIDS*, 19 (Suppl 4). [https://journals.lww.com/aidsonline/Fulltext/2005/10004/Self\\_perception\\_of\\_body\\_changes\\_in\\_persons\\_living.3.aspx](https://journals.lww.com/aidsonline/Fulltext/2005/10004/Self_perception_of_body_changes_in_persons_living.3.aspx).
- [8] Willis, Haisten. (2019, Apr. 18). Attacking AIDS in Atlanta. *U.S. News - The Civic Report*, C10–C14. <https://www.usnews.com/news/healthiest-communities/articles/2019-04-18/trump-plan-to-fight-hiv-aids-meets-skepticism-in-atlanta>.

# Effects of Abortion in Georgia, USA

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**Keywords:** Abortion, pregnancy, contraception, choice

Manuscript received on February 23, 2020; published in March, 2021.

**Abstract:** The legal definition of the term ‘abortion’ is the ‘termination of pregnancy by various methods, including medical surgery, before the fetus is able to sustain independent life’. Georgia has had one of the highest abortion rates in the United States, with a rate of 16.9%, as of 2017 [6]. In this research, I will be comparing the abortion rates in Georgia from 2010 to 2018. The results of this topic will include the understanding of factors behind abortions and solutions to this ongoing issue. In the end, this research can be used nationally and locally by abortion clinics.

## 1 Abortion and its Consequences

Abortion has been practiced in every culture known to mankind since the beginning of the civilization. Kaplan, Schauer, and Chare observe that Ancient Assyria, now known as Northern Iraq, was the first country to create a law to condemn abortion as a crime. Early Hebrew law considered abortion as a crime, unless the procedure was necessary to save the mother’s life. The Greeks allowed abortion, but Hippocrates, the well-known physician, disapproved of the procedure; he believed “It [abortion] violated a doctor’s responsibility to heal” [3]. Abortion is a well-known controversial topic about which everyone’s opinion is divided; there is no one side to this never-ending issue.

“Abortion is defined as the ending of a woman’s pregnancy due to the death of the embryo (fetus) in the mother’s womb” [7]. More than a hundred years ago, abortion was considered a criminal act in the U.S. “In 1859, the AMA passed a resolution condemning abortion as a criminal act” [3]. Then in 1973 the United States Supreme Court decided the *Roe v. Wade* case, declaring that the Constitution of the United States protects a pregnant woman’s right to make the decision of having an abortion without immoderate restriction from the Government. Before this case, there was a movement within the American Medical Association (AMA) against abortion. *Roe v. Wade* was the case that changed the situation in the United States. Yet even though there was a Supreme Court ruling on abortions, some states still have their own restrictions regarding abortion. One-fourth of the world has laws that restrict women from making it their choice to obtain an abortion.

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<sup>1</sup> This research was supported by a STEM IV Initiative grant from BOR under the supervision of Dr. Roosta as PI.

Abortion comes with the consequences of psychological, physical, and social side-effects. For decades, abortion has been known for causing mental instability, and sometimes even suicide. Some women who have received abortions can experience a form of Post-Traumatic Stress Disorder (PTSD) called Post-Abortion Stress Syndrome (PASS). This term has yet to be accepted by either the American Psychiatric Association or the American Psychological Association, because it is believed to have been made up by people who are pro-life (against abortions) to further their platform politically. “Women who suffer from PASS experience the same symptoms as someone who is suffering from PTSD such as guilt, anxiety, depression, flashbacks, suicidal thoughts, etc.” [4].

One may experience guilt after getting an abortion because one may feel like they made a mistake or that they were not more thoughtful about the situation prior. One can also experience guilt from society, from their families, their religion, their culture, etc. Guilt from having an abortion comes from the fear of what others may think about the decision made. Anxiety comes into play after getting an abortion because one never knows what to expect. Some people feel anxious about getting an abortion because they are worried about having fertility issues or not being able to conceive a child afterwards. Anxiety about abortions can also come from experiences in the past, such as prior abortions, which can trigger one’s anxiety in their present situation. Depression is a symptom one can exhibit for multiple reasons after getting an abortion. The woman could be sad about losing her baby, the thought of going through with the abortion, or letting go of the thought process of the decision. Flashbacks are another symptom of post-abortion stress since although abortion is considered surgery, the woman is conscious the entire time. A flashback in this instance would be remembering how one felt the moment after receiving an abortion, causing a trigger.

Women seek abortions based on personal circumstances such as socioeconomic status, age, health, parity, and marital status. Biggs, Gould, and Foster report that the reasons that women seek abortions fall into 11 broad themes, but the main themes are “financial reasons (40%), timing (36%), partner related reasons (31%), and the need to focus on other children (29%). Most women (64%) reported multiple reasons for seeking an abortion crossing over several themes” (954 women from 30 different abortion facilities in the United States were included in this sample) [1]. Other reasons why women seek abortions, also included in this research, are the new baby interfering with future opportunities, not being emotionally or mentally prepared, health-related reasons (health of the fetus, drug, tobacco, or alcohol use, birth control use, etc.), lack of maturity or independence, and a plethora of additional reasons. Some women are unemployed, don’t have health insurance, and don’t qualify for Government Assistance Aid, which is the reason why getting an abortion is the solution to their problem.

There are numerous ways to end a pregnancy such as a medical abortion, with the use of drugs or a surgical procedure. Abortions can happen naturally in miscarriages or by inducing the pregnancy intentionally; both may be considered early abortions. “In early medical abortion, up to nine weeks, drugs are used. In the United States, the two drugs used are methotrexate and misoprostol” [7]. Methotrexate is “given by injection and stops the embryo’s cells from dividing” [7]. Misoprostol, a medication used to prevent the occurrence of stomach ulcers and decrease the risk of ulcer complications such as bleeding, is then “inserted into the vagina, often by the woman at home, five to seven

days after the methotrexate injection. ... The woman experiences cramping and bleeding and the embryo is usually expelled within a week” [7].

The most common method used for abortions for women who are in later stages of pregnancy, typically about five to twelve weeks of pregnancy, are the vacuum method, medically known as suction aspiration. “During a vacuum aspiration abortion, a thin tube is eased into the uterus through the cervix (the passage that links the vagina to the womb). By using a pump, the contents of the uterus pass out of the womb and into the tube” [7].

The surgical method used for women who are fifteen to nineteen weeks pregnant is called dilation and evacuation. Since the baby is too large to be removed from the womb by suction at this stage of the pregnancy, the doctor has to give the woman either general anesthesia or a shot through the abdomen, if later in the second trimester, to stop the fetus’ heart. Another surgical method used for women who are twenty to twenty-four weeks pregnant is called a surgical two-stage abortion, medically known as a medical induction, where two operations are used to remove the fetus from the womb. “During a medical induction, the fetal heart is usually stopped, and then the doctor uses drugs to induce early labor” [7].

As stated before, abortion is such a controversial topic worldwide that the debate has caused two groups to form to project their thoughts and feelings about abortion issues, Pro-life and Pro-choice. Pro-life is defined as “those in favor of outlawing abortions, and of empowering the government to determine whether specific types of abortion should be allowed” [5]. People who are pro-life believe that human life begins at conception and “all abortions involve the killing of innocent children” [7]. Pro-choice is defined as “those in favor of allowing women to make all relevant decisions regarding conception, contraception, pregnancy and abortion” [5]. Some women believe that they should be able to make their own decisions, especially when it comes to abortion, because it’s their body. “Pro-choice supporters believe that abortion is not wrong in itself and should be far more widely available throughout the world” [7]. People who are pro-choice believe that no one should be able to tell a woman what she can and cannot do with her own body; getting an abortion is a woman’s freedom of choice and action.

## 2 Abortion Data in Georgia

Georgia has had one of the highest abortion rates in the United States.

It is not easy to find yearly totals to compare, but Table 1 gives one list, which shows the numbers decreasing from 2010 to 2014, then rising again. We will model the portion of the curve from 2010 to 2015 where the decline appears to be exponential (as shown in Figure 1).

**Georgia Abortions by year**

year	abortions
2010	31,315
2011	29,558
2012	28,036
2013	27,456
2014	26,485
2015	26,612
2016	29,551
2017	27,453
2018	28,544

Table 1: Abortions Performed in Georgia 2010-2018 [2].

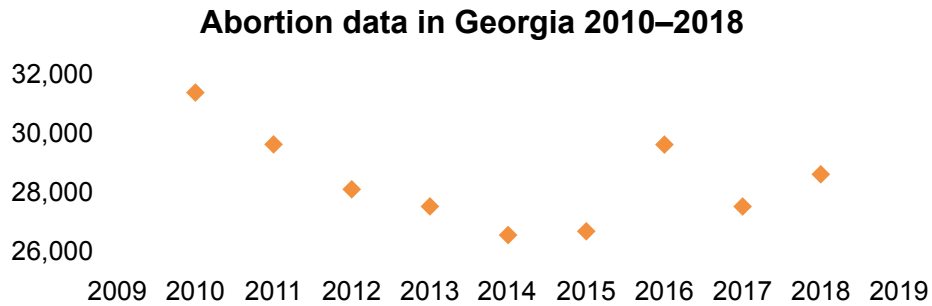


Figure 1 Graph of abortion data 2010–2018 as reported in [2].

### 3 Calculations to Model Decline 2010 to 2015

For the years 2010-2015, we see a curve suggestive of exponential decline, from the differential equation  $y' = ky$ , for which the solution is  $y = Ce^{kt}$ . Thus we choose two points of data from which we can calculate  $C$  and  $k$ .

If we set  $t = 0$  in 2012, we get  $y = 28,036$  and  $C = 28,036$ .

Then in 2013,  $t = 1$ , and the data gives us  $y = 28,036e^{kt}$ , or

$$y = 27,456 = 28,036e^k$$

$$\ln(27,456) = k + \ln(28,036)$$

$$k = \ln(27,456/28,036) = \ln(0.979312312) = -0.020904675.$$

Thus the general solution is

$$y = 28,036 e^{-0.020904675t}.$$

A graph of this solution is shown in Figure 2.

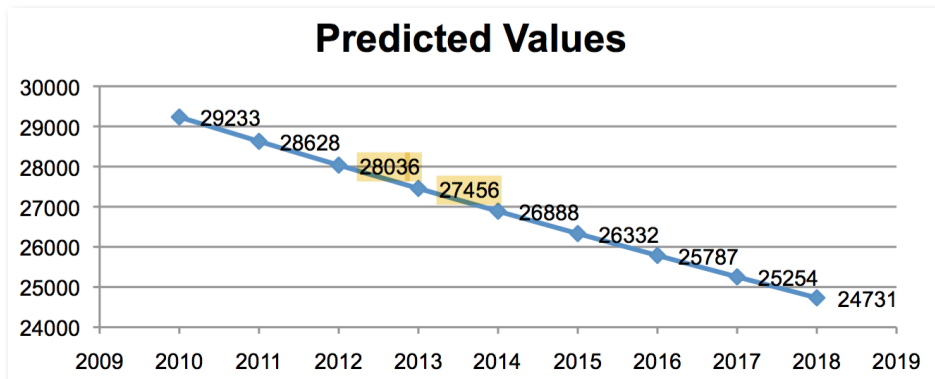


Figure 2. Predicted values calculated from  $y = 28,036e^{-0.020904675t}$ , which was computed from the data for 2012 and 2013.

The lowest number of predicted cases, based on this graph, would be 24,731 in 2018, if the exponential decrease continued.

## 4 Comparing Predictions with Data

According to Figure 2, each year from 2010 to 2018, the number of abortion cases should have decreased. But this trend is not realized in the data for the years 2016-2018, as shown in Table 1. The graph of Figure 3 merges Figures 1 and 2 to highlight this discrepancy.

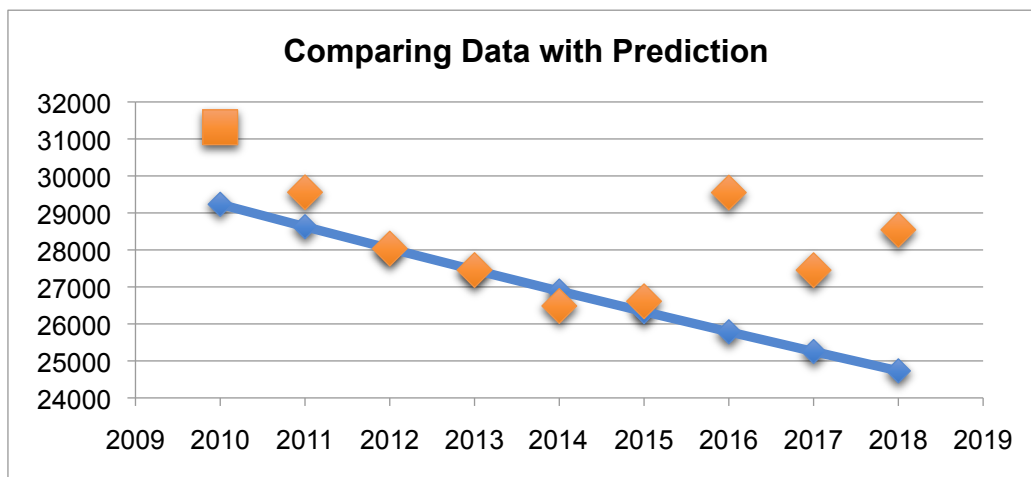


Figure 3: The blue line is the graph of our solution,  $y = 28,036e^{-0.020904675t}$ , as shown in Figure 2.; the orange diamonds are the data points from Table 1 and Figure 1.

As we see in Figure 3, the number of actual abortions has not continuously decreased since 2014. Since the number of abortions *increased* after 2014, it shows that the previous solutions used to reduce these numbers are no longer working, which is why I created two solutions that could continue to lower the number of abortions.

## 5 Possible Solutions to Stop Numbers Rising

A solution that could decrease the number of abortions in Georgia would be getting local churches more involved with women. Churches could come up with a program that helps women with making their decision on whether to get an abortion. Instead of performing the abortions or referring the women to abortion clinics, the churches would be there for guidance for the women while making this tough decision.

Another solution that could decrease the number of abortions in Georgia would be making access to abortions less necessary to women. What this means is that in order to reduce the large number of unexpected pregnancies, we, as a whole, need to show dedication to obtaining the accessibility to thorough sexuality education, which will include medically correct information concerning abstinence and contraception. This also means greater access to emergency contraception (Plan B, birth control, etc.).

In conclusion, abortion is defined as the ending of a woman's pregnancy due to the death of the embryo (fetus) in the mother's womb. It is such a world-known controversial topic that the debate has caused two groups to form to project their thoughts and feelings

about abortion issues, Pro-life and Pro-choice. Women who seek abortions base their decision on their own personal circumstances. Regardless of Pro-life and Pro-choice, women still have the right to decide what they want to do with their bodies.

## References

- [1] Biggs, M.A., H. Gould, and D.G. Foster. (2013). Understanding Why Women Seek Abortions in the US. *BMC Women's Health* 13(29) .  
<https://bmcwomenshealth.biomedcentral.com/articles/10.1186/1472-6874-13-29>.
- [2] Johnston, W. Robert. (2019, Dec. 01). Historical Abortion Statistics, Georgia (USA).  
<http://www.johnstonsarchive.net/policy/abortion/usa/ab-usa-GA.html>.
- [3] Kaplan, Clair, Leslie V. Schauser, and Paul J. Chare, Jr. (2019). Abortion. *Magill's Medical Guide* (Online Edition).
- [4] Pourreza, Abolghasem, and Aziz Batebi. (2011). Psychological Consequences of Abortion among the Post Abortion Care Seeking Women in Tehran. *Iranian Journal of Psychiatry* 6(1): 31-36. <https://pubmed.ncbi.nlm.nih.gov/22952518/>.
- [5] Rich, Alex K., and Geraldine Wagner. (2019). Abortion debate. Salem Press Encyclopedia.
- [6] State Facts about Abortion: Georgia. (2020, Mar.). <https://www.guttmacher.org/fact-sheet/state-facts-about-abortion-georgia>.
- [7] Walters, Mike.(2016) *Abortion*. Broomall, PA: Mason Crest Publishers. ProQuest Ebook Central.



# Prevention and Control of Coronavirus Outbreaks in the Last Twenty Years

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**Keywords:** coronavirus, SARS, MERS, COVID-19, quarantine, lockdown, social distancing.

Manuscript received on February 23, 2020; published in March, 2021.

**Abstract:** Coronaviruses are a group of viruses that infect mammals and birds, resulting in mainly respiratory problems. Human coronaviruses are known to cause the common cold, but there are some that can be lethal, to the point of causing a pandemic. In fact, out of the seven known human coronaviruses, three have caused global pandemics in just the last 20 years. Increased human contact with wild animals and an ever-increasing population of international travelers have made it possible for new strains to emerge and spread from country to country. Even the least contagious viruses can prove devastating in the current global economy. This paper will discuss the methods being used to control current coronavirus outbreaks and prevent future outbreaks in the state of Georgia.

**Note:** Remember when reading that this paper was written from February into April, 2020, at the very early stages of the COVID-19 pandemic.

## 1 Introduction

Coronaviruses are not new viruses. These viruses have been infecting humans for decades; however, it's only been recently that strains of coronaviruses have been infecting and killing humans at an unprecedented rate. There are currently three reported types of fatal coronaviruses. These are Severe Acute Respiratory Syndrome, Middle East Respiratory Syndrome, and the newly discovered Severe Acute Respiratory Syndrome Coronavirus 2 (now called COVID-19) that is the cause of the 2019-2020 coronavirus disease outbreak.

The first fatal case of coronavirus was severe acute respiratory syndrome, also called SARS, which originated in the Guangdong Province in China in 2003. On March 12, 2003, the World Health Organization (WHO) issued a global warning for a disease similar to pneumonia. Two days later, on March 14, the CDC activated their Emergency Operation Center and then proceeded to name the virus SARS. SARS quickly spread to over 37 different countries in a matter of weeks, mainly through travelers coming into



said countries from China, or coming into contact with people who had been to that region of China [2]. According to the CDC, SARS Response Timeline, on July 5<sup>th</sup>, the WHO announced that the outbreak of SARS was globally contained [6]. There had been, “8,273 confirmed cases of infection, of which 775 (9%) were fatal” [2]. Since then, there have been no reported cases of SARS, including at the time of writing this paper, February 23, 2020.

The second lethal coronavirus was Middle East Respiratory Syndrome, or MERS. Though MERS was first reported in Saudi Arabia in September of 2012, retrospective investigations have found that it may have originated in Jordan in April 2012. According to WHO, since September 2012, there have been 2,494 confirmed cases of MERS throughout the world reported from 27 different countries. A large majority of these cases are mainly from countries in the Arabian Peninsula, such as Qatar, Oman, and Jordan. Saudi Arabia had the highest number of occurrences, at 2,102 confirmed cases of MERS. Of those infected, 858 died due to MERS-related effects, with 780 of those deaths occurring in Saudi Arabia. This gave MERS a higher fatality rate than SARS, coming in at around 34.4% [9].

The third and most recent case of a lethal coronavirus is severe acute respiratory syndrome coronavirus 2, also called SARS-CoV-2 or COVID-19. Just like the original SARS, this virus originated in China. However, this time, COVID-19 appears to have come out of Wuhan, China, in Hubei Province. This coronavirus appears to have the largest case count out of all 3 fatal coronaviruses. A WHO situation report published on February 22, 2020 states that so far, there are 77,794 confirmed cases of COVID-19 worldwide from over 29 different countries (See Figure 1). Of these cases, 76,392 are from China. See Figure 2 for the serious outbreak locations. Also, 2359 deaths have been confirmed, of which 2348 are also from China [14]. This death count is higher than any previous coronaviruses. Both the case count and death count are subject to change as the number of cases appears to be continually rising.

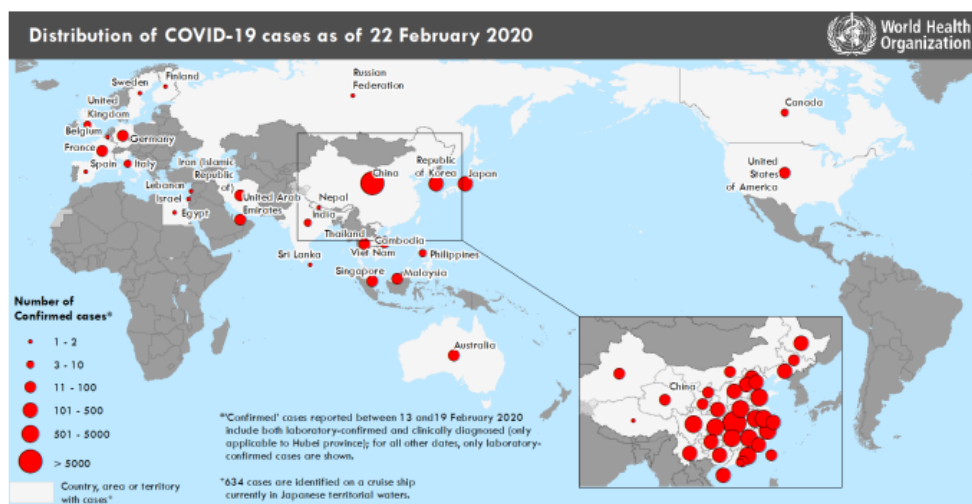


Figure 1: Countries, territories, or areas with reported confirmed cases of COVID-19, Feb 22, 2020



Figure 2: Map of China, with COVID-19 outbreaks circled.

## 2 Control of Coronavirus Outbreaks

Controlling the outbreaks of coronaviruses in a timely manner has proved essential in stopping their spread. One of the main tools to control a spreading virus is quarantine. During SARS, there was no vaccine or cure to treat infected individuals, so quarantining them in spaces away from any human contact seemed to be the best way to slow the spread of the disease. Infected individuals would usually get quarantined in hospitals and receive the appropriate aid; meanwhile, individuals suspected of carrying the virus were isolated in their homes and not allowed much human contact until they had been cleared by health officials. Quarantine was also implemented during the MERS outbreak. Though there were promising results for a vaccine for MERS, there is currently no officially accepted cure or vaccine for MERS. Therefore, slowing the spread by quarantine was very much required. During an outbreak of MERS in South Korea in May 2015, the South Korean government quarantined around 16,000 individuals. Even though there were only 186 cases of MERS and the government lost roughly 8 billion U.S. dollars, the epidemic came to a close only 2 months after it had started [12]. There were 38 fatalities, and there might very well have been more if the proper measures had not been taken.

Another tool that is useful in controlling viruses is vaccines. As stated before, there currently are no cures for any of the three above mentioned coronaviruses. However, there is a large amount of research dedicated to finding vaccines. Currently, there is some progress in developing MERS vaccines. A U.S. Army-led trial developed and tested a

vaccine in a human for the first time in 2016. No negative reactions were reported and the subject appeared to have developed a strong immune system response to MERS. Most potential SARS treatments were shelved after no new cases of SARS were reported. However, many researchers are considering going back to these tabled treatments in an effort to try and control COVID-19 [13].

There are many other ways to help control viruses. Sanitation is a major step in lowering the rate of the spread of viruses. It is mainly effective in diseases that can spread orally, and coronaviruses generally do fall in that category. Another potential tool is antiviral chemotherapy. There are different types of antiviral chemotherapeutic agents, and they may prove useful in the current outbreak; however, further development is required in this field. Another possible means is using viruses against each other. When one virus enters a cell, it creates something called an interferon. Interferons appear to reduce viral infection when introduced to non-infected cells [7].

### 3 Prevention of Coronavirus Outbreaks

Prevention of coronaviruses is also very important in order to reduce further new outbreaks. The main way to prevent them is to prevent transmission of the viruses from person to person. Quarantine is one of the most obvious ways to do this, as was previously mentioned before. However, another important step in preventing transmission is limiting or completely stopping travel to and from infected countries. For example, during the 2003 SARS outbreak, the CDC issued a “Health Alert Notice” for any travelers coming out of Hong Kong, China, in Guangdong Province. They also issued Health Alerts from other countries where SARS cases were reported. Eventually, after there were no more new cases, the CDC removed their Health Alerts. This was also the case during the MERS outbreak. Travel to and from the Middle East was incredibly regulated. During both outbreaks, the CDC repeatedly sent quarantine staff to any planes that were coming in from the infected area [1].

Another way to prevent coronaviruses is to shut down wet markets that sell wildlife. Researchers have found that all three fatal coronaviruses may be zoonotic. This means that the viruses originated in animals, which somehow transmitted them to humans. SARS was suspected to be from bats, which transferred the virus to civets. Civets are small, weasel-like creatures, and they were being sold as food in markets in Guangdong Province. Researchers suspected the virus was transferred to humans who were in contact with civet meat. Therefore, the CDC banned all imports of civets from China, and the ban is still in effect to this day [1]. MERS is also suspected to have originated from bats. However, the animals that transmitted the virus to humans are thought to be camels, a very important resource in the Middle East used for transportation throughout the desert, as a food source, and occasionally even in trading. Researchers suspect that somehow, bats infected the camels, and because of the close contact between camels and humans, the virus spread to humans. The current coronavirus, SARS-CoV-2, is also suspected to have originated in bats, which passed the disease to pangolins which were being sold in an animal market in China. Therefore, shutting down the markets where the carriers of the virus seem to be so prevalent would theoretically help in slowing infection rates.

## 4 Control of Coronavirus Outbreak in Wuhan, China

The virus that causes COVID-19 is spread very easily from person-to-person. The virus spreads from droplets that are produced when an infected person coughs, sneezes, or talks. These droplets can land in the mouths or noses of people who are nearby or can be inhaled into the lungs. A person can also get COVID-19 by touching a surface or object that has the virus on it and then touching their own mouth, nose, or eyes. Recent studies have also shown that the virus may be spread by people who are not showing symptoms. All of these factors make SARS-CoV-2 a particularly dangerous pathogen.

Because the virus spreads so easily between people, the only way to prevent the disease is by isolation, social distancing, and quarantine. China implemented all three strategies in order to control the spread of SARS-CoV-2 and to prevent more cases. Isolation requires separating ill people from the general public. Those with mild symptoms were required to stay at home, and those with severe symptoms were isolated in a hospital or health center. Social distancing is a strategy used to reduce the interactions between. Because the disease is spread when people are in close proximity to one another, social distancing is an especially important component of control. Examples for social distancing include closure of schools, office buildings, and shopping centers and cancellation of large gatherings. Quarantine is the most extreme and effective method of controlling the spread of a disease. Quarantine requires sealing off a city or a disease-infected area; the residents are not allowed to leave, and outsiders are not allowed to enter [15].

The Chinese government enforced quarantine on the city of Wuhan on January 23. Transport into and out of the city was closed, with no exceptions even for personal and medical emergencies. Schools, offices, and all shops were shut down except those selling food or medicine. Public and private transport was out of the question as everyone was forced to isolate at home. In some areas, police limited outings to one family member every two days to buy necessities, and in other areas residents were completely prohibited from leaving their homes, requiring them to order in food and other supplies. Within 2 months of this complete lockdown, the number of new cases per day in the Hubei province had stabilized (Figure 3) [8]. Wuhan had managed to “flatten the curve.” This is the idea of slowing a virus' spread so that fewer people need to seek treatment at any given time. Similar quarantines were enforced in cities throughout China. The Chinese government's implementation of isolation, social distancing, and quarantine was effective in reducing the peak number of COVID-19 infections and controlling the spread of the virus within the country (Figure 4) [11].

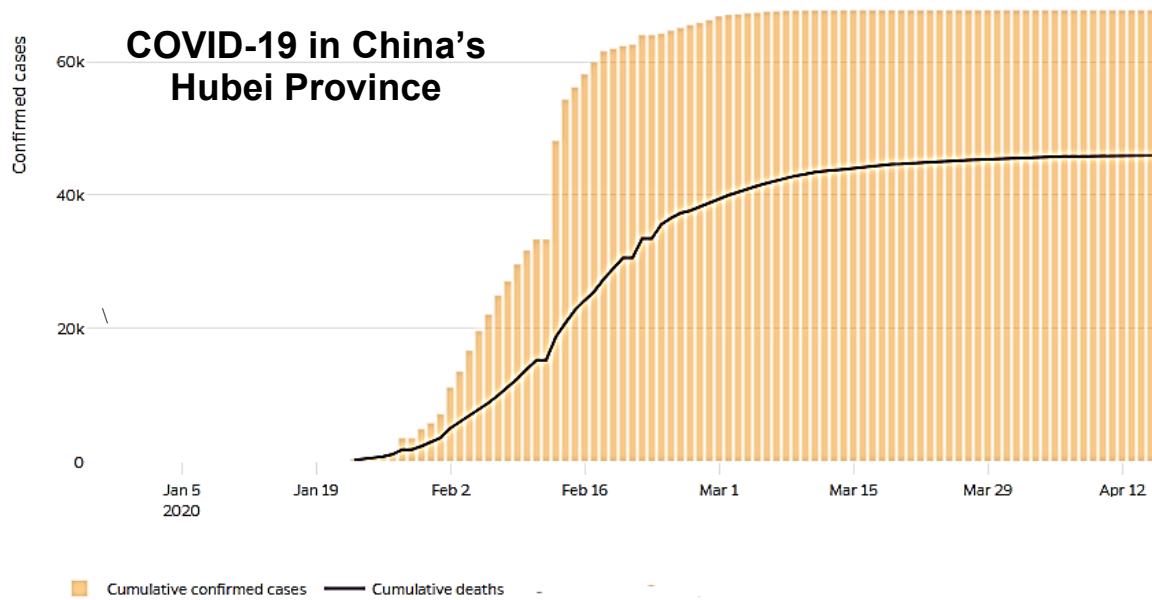


Figure 3: Confirmed COVID-19 Cases and Deaths in Hubei Province [8].

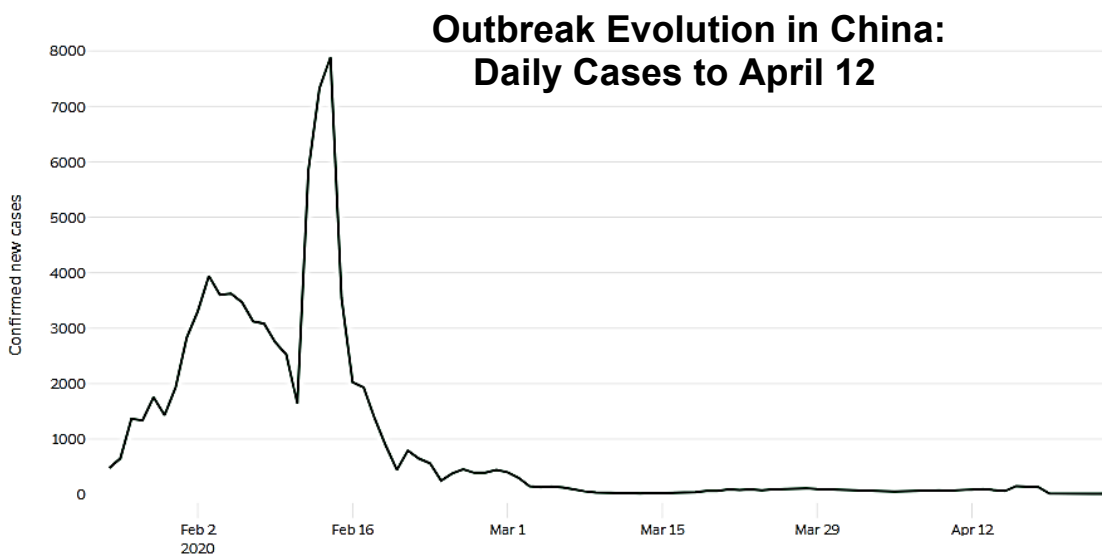


Figure 4: Outbreak evolution in China, showing new cases each day [11].

## 5 Controlling the Coronavirus Outbreak in Georgia, USA

The first reported cases of COVID-19 in Georgia occurred on March 2, 2020 [6]. On the next page, Table 1 and Figure 5 show data and graph for the course of the epidemic through April 28, 2020, the end date for this research project.

### COVID-19, GA

date	cumulative cases
1-Mar	2
2-Mar	3
3-Mar	5
4-Mar	7
5-Mar	8
6-Mar	8
7-Mar	11
8-Mar	17
9-Mar	26
10-Mar	39
11-Mar	55
12-Mar	88
13-Mar	115
14-Mar	139
15-Mar	191
16-Mar	240
17-Mar	283
18-Mar	378
19-Mar	493
20-Mar	615
21-Mar	700
22-Mar	855
23-Mar	1,076
24-Mar	1,258
25-Mar	1,539
26-Mar	1,841
27-Mar	2,365
28-Mar	2,590
29-Mar	2,833
30-Mar	3,222
31-Mar	4,235
1-Apr	4,805
2-Apr	5,442
3-Apr	5,930
4-Apr	6,352
5-Apr	6,648
6-Apr	7,385
7-Apr	8,946
8-Apr	10,076
9-Apr	10,688
10-Apr	11,622
11-Apr	12,041
12-Apr	12,199
13-Apr	13,285
14-Apr	14,129
15-Apr	14,816
16-Apr	15,809
17-Apr	16,719
18-Apr	17,070
19-Apr	17,575
20-Apr	18,418

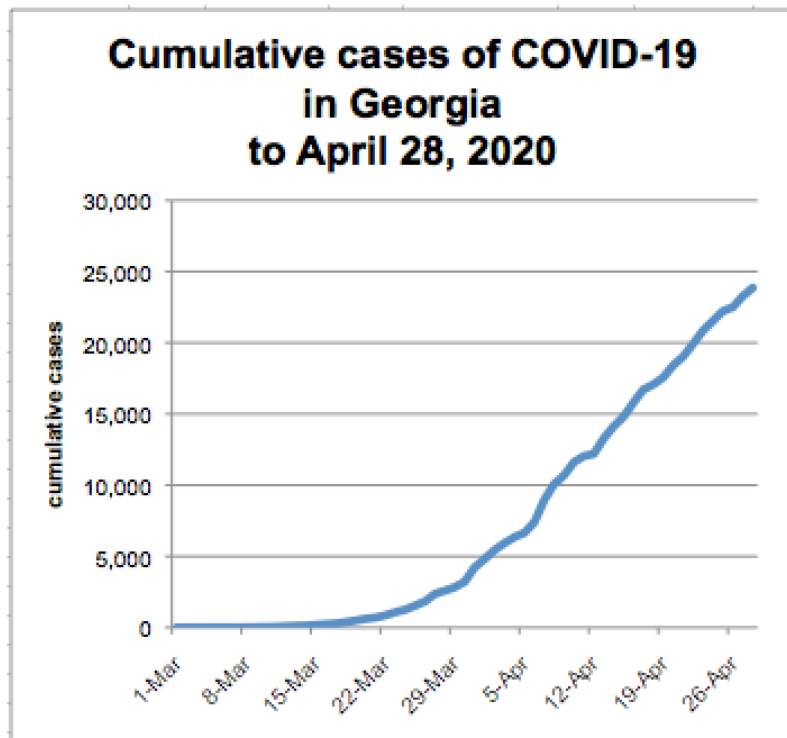


Figure 5: Rise of total cases in Georgia for first 2 months [15].

We noted that the cumulative case graph was rising at an ever steeper rate till the end of March, and then did not continue to get steeper. Thus the March portion of the graph looked like  $y' = ky$ , with solutions

$$y = C e^{kt}.$$

We found a reasonable fit for the data curve (in March) using

$$t = 0, y = 191 \text{ on March 15, and}$$

$$t = 17, y = 4805 \text{ on April 1.}$$

Thus the final mathematical model that was derived is

$$y = 191 e^{0.1897t}.$$

Table 2 on the next page gives the results, and Figure 6 shows the comparison of this model with the given data of Table 1 through April 20 when the scale of the model ceases to obscure the curve of data.

Table 1: Cumulative COVID-19 cases in GA USA, over the first two months [15].



**Modeling with**

$$y = 191 e^{0.1897t}$$

date	t= #days	predicted value using Mar 15, Apr 1
1-Mar	-14	13
2-Mar	-13	16
3-Mar	-12	20
4-Mar	-11	24
5-Mar	-10	29
6-Mar	-9	35
7-Mar	-8	42
8-Mar	-7	51
9-Mar	-6	61
10-Mar	-5	74
11-Mar	-4	89
12-Mar	-3	108
13-Mar	-2	131
14-Mar	-1	158
15-Mar	0	191
16-Mar	1	231
17-Mar	2	279
18-Mar	3	337
19-Mar	4	408
20-Mar	5	493
21-Mar	6	596
22-Mar	7	721
23-Mar	8	871
24-Mar	9	1,053
25-Mar	10	1,273
26-Mar	11	1,539
27-Mar	12	1,861
28-Mar	13	2,250
29-Mar	14	2,720
30-Mar	15	3,288
31-Mar	16	3,975
1-Apr	17	4,805
2-Apr	18	5,809
3-Apr	19	7,022
4-Apr	20	8,489
5-Apr	21	10,263
6-Apr	22	12,407
7-Apr	23	14,998
8-Apr	24	18,132
9-Apr	25	21,919
10-Apr	26	26,498
11-Apr	27	32,034
12-Apr	28	38,726
13-Apr	29	46,816
14-Apr	30	56,596
15-Apr	31	68,419
16-Apr	32	82,712
17-Apr	33	99,991
18-Apr	34	120,880
19-Apr	35	146,132
20-Apr	36	176,659

Table 2: Predicting from model, using March 15 and April 1 to evaluate  $C, k$ .

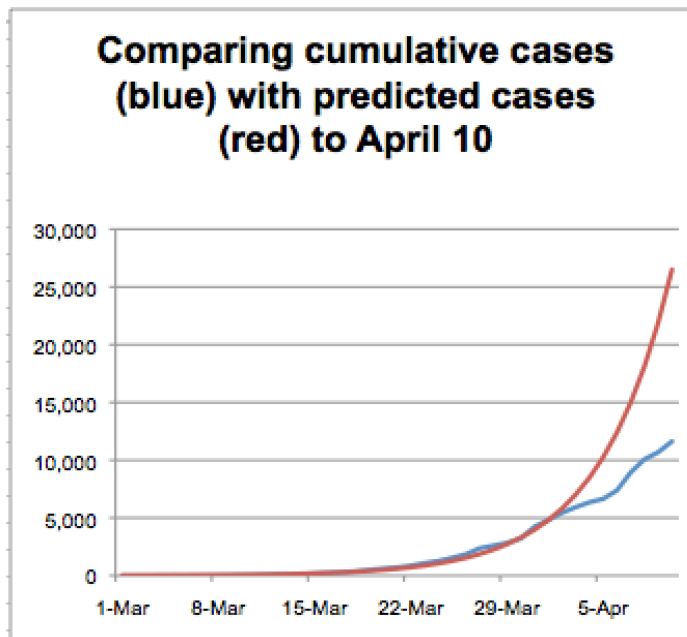


Figure 6: Comparing predicted values (Table 2) with data (Table 1) to April 10, 2020.

Based on Figure 6 and Table 2, we can see that the projected number grows exponentially in April. However, back in Figure 5, we can see that the actual number of cases in Georgia stopped increasing so rapidly at the end of March.

Georgia, on March 20<sup>th</sup>, banned most social gatherings including churches and most non-essential businesses. Then a “Shelter-in-Place” order began on April 3<sup>rd</sup>. So we see that although the actual number of cases began to exponentially increase in March, we also see that eventually, thanks to the quarantine and the stay at home order, the curve had begun to flatten in early April.

If the “Shelter-in-Place” order and social distancing are enforced further, Georgia should avoid the exponential rise in cases shown in the red line of Figure 6.

## References

- [1] CDC SARS Response Timeline. (2013, Apr. 26) *Centers for Disease Control and Prevention*. <https://www.cdc.gov/about/history/sars/timeline.htm>.
- [2] Coleman, Christopher M., and Matthew B. Frieman. (2014). Coronaviruses: Important Emerging Human Pathogens. *Journal of Virology* 88(10): 5209–5212. <https://doi:10.1128/jvi.03488-13>.
- [3] Coronavirus. (2020, Jan. 26). *Centers for Disease Control and Prevention*. <https://www.cdc.gov/coronavirus/index.html>.
- [4] Coronavirus. (n.d.). *World Health Organization*. <https://www.who.int/health-topics/coronavirus>.
- [5] Coronavirus (COVID-19). (n.d.). <https://www.cdph.ca.gov/Programs/CID/DCDC/Pages/Immunization/nCOV2019.aspx>.
- [6] COVID-19 Status Report. (2020, Apr. 28). *Georgia Department of Public Health*. <https://dph.georgia.gov/covid-19-daily-status-report>.
- [7] Goldenthal Karen L., Karen Midthun, and Kathryn C. (1996). Control of Viral Infections and Diseases. In S. Baron, editor. *Medical Microbiology*. 4th edition. Galveston: University of Texas Medical Branch at Galveston. Chapter 51. PMID: 21413344.
- [8] Hubei Timeline. *Johns Hopkins Coronavirus Resource Center*. (n.d.). <https://coronavirus.jhu.edu/data/hubei-timeline>.
- [9] Middle East Respiratory Syndrome Coronavirus (MERS-CoV). (2020, Jan. 23). *World Health Organization*. <https://www.who.int/emergencies/mers-cov/en/>.
- [10] Nazario, Brunilda. (2020, Jan. 22). Coronavirus Symptoms, Causes, Treatments, Types. <https://www.webmd.com/lung/coronavirus#1>.
- [11] New Cases of COVID-19 in World Countries. (n.d.). *Johns Hopkins Coronavirus Resource Center*. <https://coronavirus.jhu.edu/data/new-cases>.
- [12] Oh, Myoung-Don, et al. (2018). Middle East respiratory syndrome: What We Learned from the 2015 Outbreak in the Republic of Korea. *The Korean Journal of Internal Medicine*. 33(2): 233-246. <https://doi:10.3904/kjim.2018.031>.
- [13] Shen, Xuejuan, et al. Vaccine against Middle East Respiratory Syndrome Coronavirus. (n.d.). *The Lancet Infectious Diseases* 19(10): 1053–1054. [https://doi:10.1016/s1473-3099\(19\)30476-1](https://doi:10.1016/s1473-3099(19)30476-1).



- [14] Situation Report 33-Novel Coronavirus (2019-NCoV) Situation Reports. (2020, Feb. 22). *World Health Organization*. <https://www.who.int/emergencies/diseases/novel-coronavirus-2019/situation-reports/>
- [15] Wilder-Smith, A. and D O Freedman. (2020). Isolation, Quarantine, Social Distancing and Community Containment: Pivotal Role for Old-style Public Health Measures in the Novel Coronavirus (2019-nCoV) Outbreak. *Journal of Travel Medicine* 27(2). <https://doi.org/10.1093/jtm/taaa020>.

# The Increased Rate of Home Runs by Year in Major League Baseball

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**Keywords:** baseball, home runs

Manuscript received on February 23, 2020; published in March, 2021.

**Abstract:** Looking at the last ten years and including the current year, there has been an increase in the number of home runs in major league baseball. This is especially the case with two of the last three years having records being set for single seasons home runs by the league. This is surprising because these records of recent years have been more than those of the steroid era. The second-place record broke the old record by nearly 500 home runs, and the current record of last season broke that record by over 600 home runs.

While there has not been a complete positive trend of an increase in home runs in the past ten years, there have been numbers put up that can compare with no other season. The last three years have had higher home runs per game by nearly 2.3, which is again interesting given that the players during the steroid era were not able to achieve such a stat line.

The purpose of this investigation is to try to determine what caused the spike in home runs and to give potential solutions to make the number of home runs trend back in the direction more representative of the ideal average.

## 1 Background

Baseball is considered by many to be America's national pastime. This game has been around since 1869, and over the years, the game has changed. The ideals and basis of the game have remained the same; but with new ideology, understanding, players, and play style, the game of baseball in today's world is far from what it was even ten years ago.

The premise of baseball can be boiled down to scoring more runs than the other team. The most efficient way to accomplish this goal is to have the hitter hit the ball over the fence. He is then awarded all four bases; thus, giving his team one run, (plus up to three more if the other bases had previous hitters on them). While home runs are exciting and very effective in giving teams runs, they were not the most common method of scoring. According to the Baseball Almanac, 5,042 home runs were hit in the year 2009. This is

compared to the 6,776 that were hit in 2019, or about 58 more home runs per team in the entire league [6]. This is the beginning and end of the trend, but in the years between 2009 and 2019, the rate of home runs increased slowly but surely until 2017, when there was a steep increase that has continued the last three seasons.

This rise in offensive power has been very good for the popularity of the league, especially in a time when the NBA and NFL are getting more attention across the nation. The question that is trying to be answered by not only the players, coaches, and fans, but also the head of baseball, is why has there been such an increase in home runs over the years, and what is causing the exponential growth over the past three seasons?

## 2 What is causing the increase?

Baseball Commissioner Rob Manfred decided to hire investigators to look into the home run situation and determine a possible cause for the change. A team of scientists (Table 1) conducted experiments to determine to what the rise in home runs could be attributed [1].

<ul style="list-style-type: none"> <li>• <b>ALAN NATHAN</b> (Chairman) - Professor of Physics Emeritus, University of Illinois</li> <li>• <b>JIM ALBERT</b> - Professor of Statistics, Bowling Green State University</li> <li>• <b>JAY BARTROFF</b> - Professor of Mathematics, University of Southern California (USC)</li> <li>• <b>ROGER BLANDFORD</b> - Professor of Physics, Stanford University</li> <li>• <b>DAN BROOKS</b> - Owner of BrooksBaseball.net</li> <li>• <b>JOSH DERENSKI</b> - Ph.D Student, Marshall School of Business, USC</li> <li>• <b>LARRY GOLDSTEIN</b> - Professor of Mathematics, University of Southern California</li> <li>• <b>PEKO HOSOI</b> - Professor of Mechanical Engineering, MIT</li> <li>• <b>GARY LORDEN</b> - Professor Emeritus of Mathematics, California Institute of Technology</li> <li>• <b>LLOYD SMITH</b> - Professor of Mechanical &amp; Materials Engineering, Washington State University</li> </ul>
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Table 1. Members of the Committee studying home run rates in Major League Baseball.

One of their findings related to something that many people had speculated could be a cause, the baseballs themselves. There had been rumors and complaints from MLB pitchers that the balls used during the season had been “juiced” in order to make it easier for hitters to hit homeruns; thus, making the game more interesting to watch and giving the MLB more money. It was not known if this was being done intentionally or not, or even at all. In the report led by Jim Albert, there was not enough evidence to say definitively that the ball was the cause for the rise in home run levels. However, there was also not enough evidence to disprove the idea that the balls were the cause for change [1]. The Rawlings plant in Costa Rica, the MLB provider of baseballs, was under some scrutiny for changing the design of their balls between 2018 and 2019. This was taken into account in the investigation done by the MLB, but there was no definitive evidence to lean one way or another.

Even with the uncertainty, many players and fans find it difficult to explain the shift in correlation between home runs in the minor leagues and major leagues. According to Ben Lindeburgh and Rob Arthur, MLB sports writers, there has been a steady positive correlation between home runs in the minor leagues and home runs in the major leagues dating all the way back to 1990 [7]. In 2015, the minors and majors began using different baseballs and there has been a negative correlation in home run levels ever since.

Apart from the baseballs, there has been a different mindset and approach when it comes to hitting that may be a cause for more home runs. In today's game, pitchers are borderline unhittable. ESPN tells how in the 2019 season, the top 21 pitchers in opponent batting average, were all under .250. This means less than one out of every four hitters was getting a hit, making it very difficult for teams to string together base knocks and rally to score runs. They were better off gambling trying to hit a home run, which is a guaranteed one run minimum.

With this new mindset about hitting, baseball gave birth to a new wave of hitting style called *launch angle*, which can best be described as the trajectory at which the ball leaves the bat. Many players were under the impression that MLB fielders can field a ground ball and make a routine play. They also knew that a ball hit parallel to the ground had a 0% chance of going over the fence. Tom Verducci, MLB analyst, describes the idea of launch angle: "use a slightly upward path when swinging, hit the bottom third of the ball, hit the ball in the air no matter what" [11]. This style of play can be seen in players like Pete Alonso, who led the MLB in home runs in 2019 with 53, and Jorge Soler who was third in home runs in 2019 with 48.

This mindset of 'hit home runs, and swing for the fences' has also had the effect of more strikeouts. Since there are more strikeouts, there is a lower average among hitters overall. This phenomenon can be described with a simple statistic. In 2010, Josh Hamilton of the Texas Rangers hit .359. The closest anyone has come since is .348. Also, in the last two seasons the average leader in the national league has hit below .330, or .029 points lower than Hamilton. Undoubtedly the new style of play has had a lot to do with the new numbers that have spiked in the past ten years and last three in particular.

Another idea on the cause of change has been the weather. Olivia Miltner, ACCU weather staff writer, claims that part of the reason for the improved home run numbers can be associated with warmer weather [8]. With global warming being proven more and more every day, there is no doubt the air is warmer, thus making it less dense. Less dense air means the ball has less air resistance on it, making it travel further. A prime example of the effects of air density causing an increase in homeruns can be seen with the Colorado Rockies, the team that plays at Coors Field in Denver, well above sea level (by a bigger margin than almost any other field). The high elevation makes the air less dense; it has been proven that it is easier to hit home runs in this stadium. This can be backed by the fact that two of the three longest home runs that have been tracked have been at Coors Field, per USA Today [10]. Trevor Story hit a ball 505 feet, and Giancarlo Stanton hit a ball 504 feet, which coincidentally both occurred in the last three years, with the second longest home run between them taking place in 2016. Now that the planet is heating up, every stadium is becoming more and more like Coors Field. A study conducted by Tyler Ashoff, an MIT undergrad engineer, found that in the MLB, on days when the air was less humid or dense, that more home runs were hit [2]. He also found that in recent years, there have been more so-called 'low density air' days. Sports writer Wendy Trum also explains

how the increase in temperature has caused higher home run levels [9]. She focused her numbers on the temperature (in degrees Fahrenheit) and less on the air density, but the ideas go hand in hand. The summer months have had more 4+ home run games than any other month. The top ten games with the most home runs have been in summer months, all but one having a temperature above 70° Fahrenheit. This study was conducted in the 2012 season.

The issue with the increase in home runs is that the game is becoming inconsistent and ironically, slower paced. The strikeouts are not fun to watch, and there is less emphasis on what made the game so fun and strategy-driven in the first place. Baseball Commissioner Rob Manfred is looking to bring the game of baseball back to where it was before the home runs started going off the charts. There is a strong effort to normalize the game by finding the cause and fixing it. There are different solutions that could fix the problem, but the issue is picking the one with the least negative repercussions.

The ability or inability of the commissioner to do this impacts how the players and fans will enjoy the game for the future. Some like the old school nature of the game and want it to go back to how it was. Others like the newer play style and want the home run trend to continue even further. Manfred believes there is a happy medium that will satisfy both players and fans. He has a very difficult job at hand and that is why there have been so many different people looking into the cause. Once the cause is determined then the solution is that much easier.

### 3 Calculations to predict the future

In order to accurately represent the situation at hand, we first note the trend in annual home run totals, over the entire history of Major League Baseball, as shown by Ben Cooper in Figure 1 [4]. The data from 1901 to 2019, given in Table 2 ranges from a low of 245 in 1907 to a high of 6776 in 2019, with an overall steadily increasing upward trend [6].

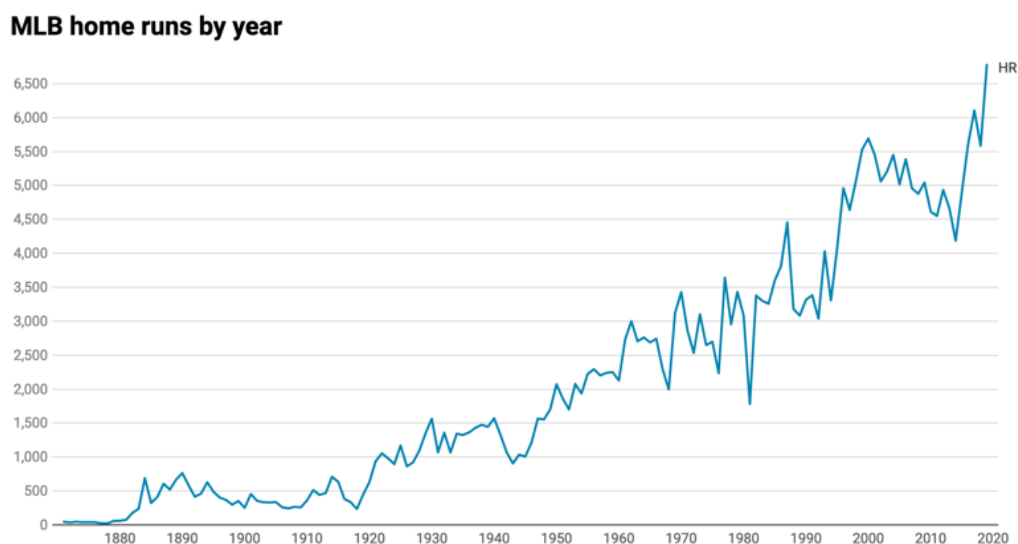


Figure 1. MLB home runs by year, 1859–2019. (Chart by Ben Cooper [4]).

Because the trend is ever more steeply increasing, an exponential model,  $y' = ky$ , with solution  $y = Ce^{kt}$ , is appropriate for predicting total home runs  $y$  in the years to come. One of the biggest jumps was from the year of 2018 when 5585 homeruns were hit to the year 2019 when 6776 homeruns were hit (1191 more).

If we set  $t = 0$  at year 2018, we can calculate  $C = 5585$ . Then for  $t = 1$  at year 2019, we use  $y = 6776$  and find  $k = 0.19330254$ . The calculations follow.

year	total MLB home runs
2000	5,693
2001	5,458
2002	5,059
2003	5,207
2004	5,451
2005	5,017
2006	5,386
2007	4,957
2008	4,878
2009	5,042
2010	4,613
2011	4,552
2012	4,934
2013	4,661
2014	4,186
2015	4,909
2016	5,610
2017	6,105
2018	5,585
2019	6,776

Table 2 Annual home run figures 2000–2019, from [6].

Using the data from 2018 and 2019, we start with

$$y = Ce^{kt}, t=0, y(0) = 5585.$$

This gives  $5585 = Ce^{k(0)}$ ,  $C = 5585$ , and  $y = 5585e^{kt}$ .

When  $t = 1$ ,  $y = 6776$ , so we now have

$$6776 = 5585e^k$$

$$6776/5585 = e^k$$

$$\ln(6776/5585) = \ln e^k = k$$

$$k = \ln(6776/5585) = 0.19330254$$

$$\text{Solution: } y = 5585e^{(0.19330254)t}$$

In Figure 2 we graph in red this solution function from 2010 to 2024 ( $t = -8$  to  $t = 6$ ) and compare it with blue data points for the MLB data in Table 2.

The graph of Figure 2 predicts that the number of homeruns each MLB season will increase. While it would be interesting to see 18,000 homeruns in a 2024 MLB season, this is not really possible. There is a limit to what people can achieve, as they are human after all, and the model could be calculated from a different pair of data points. However, with that said, there is a serious upward-facing trend that the graph does accurately model.

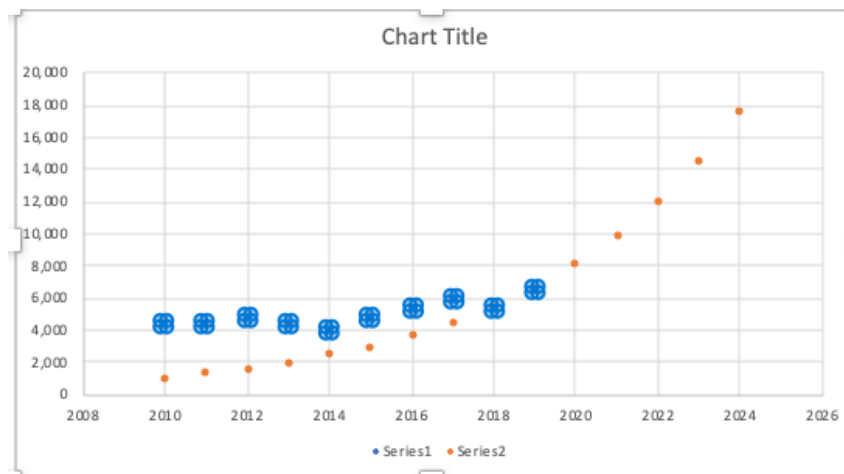


Figure 2. Potential number of home runs (red) in the MLB by year, compared with MLB data to 2019 (in blue).

## 4 Possible Ways to Harness the Home Runs

One very easy solution to this problem is raise the pitching mound. Ironically, the mounds used were raised in 1968 to promote more offense. The higher mounds gave the pitchers an opportunity to stride farther down the mound, thus decreasing the time the hitters had to react to the pitch as it was being released closer to the plate. There was also more of a plane created on the pitch which made it more difficult for the hitter to square a ball up. Lastly, breaking balls moved more as they have more time to drop and change direction making hitting that much more difficult. Raising the mounds again would lead to lower hitting stats all around, not just in home runs.

Another solution that may be more difficult is moving the mound closer by about two feet. An average MLB fastball is about 90 miles per hour, giving the hitter roughly 300 milliseconds to react and swing. Shortening the distance the ball has to travel between the mound and the plate will give the hitter less time to react (ten milliseconds less). The decreased reaction time makes hitting that much harder because there is already so little time to react in the first place.

These two solutions are the best possible because it is impossible to control things like the weather. They offer the ability to make changes over things that are manageable. The issue is not necessarily with the number of homeruns alone, but the negative aspects that come along with it.

The increased rate of home runs needs to be monitored because the game is becoming completely centered around homeruns. Currently there are people in the league who are not very good compared to other players, but they are strong and can hit homeruns, so they play. Fans wanted more offense, but all the homeruns have had the opposite effect and made the game more boring. Key elements of the game like stealing bases, situational hitting, and defense are all being neglected for the sake of hitting homeruns. The game is becoming either a home run or strikeout for a majority of 'at bats', which most fans find boring. Either of these possible solutions will make the game go back to how it used to be played and give

all aspects of the game an equal level of importance again; therefore, putting the thrill of the game back into baseball.

## References

- [1] Albert, Jim, et.al. (2018, May 24). Report of the Committee Studying Home Run Rates in Major League Baseball. [https://www.mlb.com/documents/7/9/4/278128794/Full\\_Report\\_of\\_the\\_Committee\\_Studying\\_Home\\_Run\\_Rates\\_in\\_Major\\_League\\_Baseball\\_052418.pdf](https://www.mlb.com/documents/7/9/4/278128794/Full_Report_of_the_Committee_Studying_Home_Run_Rates_in_Major_League_Baseball_052418.pdf).
- [2] Ashoff, Tyler. (2018). Understanding the Relationship between Weather Conditions and Homerun Rates in the MLB. <https://dspace.mit.edu/handle/1721.1/120271>.
- [3] Baseball Reference. (n.d.). <https://www.baseball-reference.com/>.
- [4] Cooper, Ben. (2019, Oct. 1). A Look into Major League Baseball's Massive Home Run Surge in 2019. *Capital News Service*. <https://cnsmaryland.org/2019/10/01/a-look-into-major-league-baseballs-massive-home-run-surge-in-2019/>.
- [5] Lahman, Sean. 2019 Pitcher Stats. *MLB Database*. <https://www.seanlahman.com/baseball-archive/statistics/>.
- [6] League by League Totals for Home Runs. (n.d.). <https://www.baseball-almanac.com/hitting/hihr6.shtml>.
- [7] Lindbergh, Ben and Rob Arthur. (2016, Jul. 20). Are Juiced Baseballs the New Steroids? <https://fivethirtyeight.com/features/are-juiced-balls-the-new-steroids/>.
- [8] Miltner, Olivia. (2019, Jul. 01). Is Weather Playing a Role in MLB's Record-setting Home Run Spike this Season? *Accuweather*. <https://weatherfanatics.blogspot.com/2017/08/is-weather-playing-role-in-mlbs-record.html>.
- [9] Trum, Wendy. (2012, Jun. 21). As Temperatures soar, so Do the Homeruns. *Fangraphs*. <https://blogs.fangraphs.com/as-temperatures-soar-so-do-the-home-runs/>.
- [10] Watch MLB's 10 Longest Home Runs of the Statcast Era. (2019, Aug. 02). USA Today Sports. <https://www.usatoday.com/story/sports/mlb/2019/08/02/mlb-longest-home-runs-statcast/1900372001/>.
- [11] Verducci, Tom. (2018, Mar. 21). Countdown to Liftoff. *Sports Illustrated*. <https://www.si.com/mlb/2018/03/21/evolution-swing-home-run-opening-day>.



# *Modeling the Ecological Dynamics of a Three-Species Fish Population in the Chesapeake Bay*

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**Keywords:** Mathematical modeling, fish population dynamics, prey, predator, invader

Manuscript received on May 17, 2020; published on March 15, 2021.

**Abstract:** We present an inquiry-based project that is designed for a mathematical modeling class of undergraduate junior or senior students. It discusses a three-species mathematical model that simulates the biological interactions among three important fish species in the Chesapeake Bay: the prey Atlantic menhaden and its two competing predators, the striped bass and the non-native blue catfish. The model also considers the following ecological issues related to these three species: the overfishing of menhaden, the invasiveness of the blue catfish, and the harvesting of blue catfish as a method to control the population. A series of modeling scenarios are considered based on some simplifying assumptions to demonstrate the application of theoretical concepts to actual fisheries in the Chesapeake Bay. Analysis involves elementary skills such as finding the roots of polynomial equations, computing eigenvalues and eigenvectors, and some advanced topics such as Routh-Hurwitz criteria and the Hartman-Grobman Theorem. Numerical simulations via MATLAB are utilized to produce graphical simulations and analyze long-time behaviors. Our model predicts that if no serious measures are taken to prevent the spread of the invasive blue catfish, the native predator species will be seriously affected and may even become extinct. The model also shows that linear harvesting is sufficient to limit the growth of the invasive catfish population; however, it is not sufficient to save the striped bass from becoming extinct. The results of this study illustrate the fundamental ecological principle of competitive exclusion, according to which two competing species that attempt to occupy the same niche in an ecosystem cannot co-exist indefinitely and one of the two populations will either go extinct or will adapt to fill a different niche.

# 1 Introduction

Chesapeake Bay, the largest estuary in the United States, is an extremely complex ecosystem. The fisheries of the Chesapeake Bay play a very important role in the ecosystem but have declined significantly as a result of habitat loss, deterioration in water quality, and the effects of the changing climate, only amplified by commercial over-fishing.<sup>1</sup>

One particular example of a seriously affected native fish is the Atlantic Menhaden (*Brevoortia tyrannus*). Menhaden is important for both ecological and commercial reasons to the Atlantic region. Often called “the most important fish in the sea,” they are filter feeders that play an essential role in cleaning the water. They are also a critical link in the Chesapeake Bay food chain as they transfer nutrients from lower to higher trophic levels. The number of young menhaden in the Chesapeake Bay dropped dramatically in the early 1990s and still remains low.<sup>2</sup> On May 4, 2020, the Bay Journal announced:

*The Virginia Marine Resources Commission unanimously adopted a new regulation that requires this year’s Bay menhaden harvest be cut by nearly half from what it was in 2019. Effective June 17, 2020, there will be a moratorium on fishing for menhaden in Virginia state waters. [8]*

Another native fish that declined significantly is the striped bass (*Morone saxatilis*), a sought-after commercial and recreational catch and a key predator in the Bay food web. Menhaden is the main prey to the striped bass. Due to the decrease of Menhaden, some striped bass now struggle from poor nutrition, which is linked to a decrease in growth rates and an increase in sensitivity to disease.<sup>1</sup> Due to a critical decline of the bass, the Atlantic States Marine Fisheries Commission (ASMFC) has determined that conservation measures are needed for the 2020 fishing season. The Striped Bass Technical Committee and the ASMFC Board announced on February 4, 2020:

*In addition to a cut in the commercial quota, targeting of striped bass by the recreational sector will be prohibited starting April 1 — including a prohibition of trolling — and the spring trophy season will be delayed until May 1. The department will also move forward with conservation options for the summer and fall seasons in the 2020 Implementation Plan. Measures in the summer and fall are designed to reduce mortality caused by high temperatures and low oxygen in the water.<sup>3</sup>*

Another severe ecological impact on the native species in the Bay has been recently identified as a result of predation by invaders such as the blue catfish (*Ictalurus furcatus*). Brought to the region in the beginning of the 1970s for recreational purposes, the blue and flathead catfish now make up to 75 percent of total fish biomass in the Chesapeake Bay.<sup>4</sup>

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<sup>1</sup><https://www.cbf.org/issues/fisheries/>, retrieved March 13, 2021

<sup>2</sup><https://www.cbf.org/about-the-bay/more-than-just-the-bay/chesapeake-wildlife/menhaden/index.html>, retrieved March 13, 2021

<sup>3</sup><https://www.proptalk.com/striped-bass-conservation-regulations-set-spring-2020>, retrieved March 13, 2021

<sup>4</sup><https://www.fisheries.noaa.gov/feature-story/bay-invaders-blue-catfish-fishery>, published March 20, 2018, retrieved March 13, 2021

As novel apex predators that feed on important fishery resources, including native and anadromous fish, blue and flathead catfish have the potential to exert severe ecological harm to the region. Some preliminary studies have already documented harmful effects to native species; however, the full extent of negative impacts from catfish invasions remains poorly understood. In 2011, the Atlantic States Marine Fisheries Commission approved a resolution expressing concern about the impacts of blue and flathead catfish on Atlantic Coast migratory fish species. The resolution suggested that “all practicable efforts should be made to reduce the population levels and ranges of non-native invasive species” [2]. In 2018, the Fisheries News published by NOAA stated:

*The NOAA Chesapeake Bay Office (NCBO) and other Chesapeake Bay organizations identified invasive catfish as a challenge facing the Chesapeake ecosystem several years ago. The commercial and recreational fisheries are helping to control the population. NOAA Fisheries and our partners are supporting the development of these fisheries and working to identify further solutions to control the growth of this invasive population.*<sup>4</sup>

The Chesapeake Bay management jurisdiction tries to engage the public to minimize the ecological and economic effects of the catfish invasion by promoting harvesting. However, it is not clear if such a strategy will have any practical effects on the growing catfish population. Fisheries in the Bay are managed as single species entities, yet the multi-species nature of the ecosystem has to be taken into account when making management plans or taking management decisions about the Chesapeake Bay.

We create a three-species mathematical model to investigate the relationship between the three fish species in the Chesapeake Bay ecosystem: menhaden, striped bass, and catfish. We ignore interactions with other species, that is, we assume that there are no other predators, no other prey, and no other competitors. We also do not consider environmental or societal factors such as temperature, seasons, illegal fishing, etc. We use a system of ordinary differential equations and utilize linear stability theory, graphical analysis, and numerical simulations.

This inquiry-based project is developed for a Mathematical Modeling class with junior and senior undergraduate students. The model naturally builds upon the classic Lotka-Volterra equations for two interacting species, such as predator-prey interactions or two species competing for the same resources. Students are expected to have already taken linear algebra and differential equations. This project provides an opportunity to apply theoretical concepts to real-world problems related to fisheries in the Chesapeake Bay.

Using  $x(t)$  to denote the prey population and  $y(t)$  to denote the predator population over time  $t$ , the Lotka-Volterra predator-prey model introduces students to modeling the growth of two interacting populations:

$$\begin{cases} x' &= Ax - Bxy \\ y' &= -Dy + Cxy \end{cases}$$

where the parameters  $A, B, C, D$  are all positive. Each term in this system is a rate; in particular, the term  $Ax$  represents the linear growth rate of prey  $x$ , the term  $-Bxy$  represents the predation rate of the predator  $y$  on the prey  $x$  during their interactions, the term  $-Dy$

represents the rate at which the predator  $y$  will decrease without the prey as food, and the term  $Cxy$  represents the rate at which the the predator  $y$  grows in the presence of the prey as food. When students understand the functional relationships between  $x(t)$  and  $y(t)$ , they are able to build upon simple models such as these.

Although the Lotka-Volterra model is elementary and only uses basic knowledge of ecology, it may be used skillfully to encourage discussion and inquiry about theoretical ecology and real ecological issues. For example, the following guiding questions may inspire curiosity and help students reflect on the power of differential equations in modeling population dynamics:

1. What are the ecological assumptions in this model?
2. What happens to the prey population if the predator is absent, and vice-versa?
3. Does the predator rely only on the given prey for food and its existence? Does the model consider the various appetites of the predator and its competitors? What does the prey eat? Why do we assume that the rate of change of the prey population is proportional to its size?
4. How does the model consider the growth and death rates of the prey, the predator, and the competitor?
5. How does the model capture the strength of competition between competing predators? Will a stronger competition between predators benefit the prey?
6. Can we utilize the given model to any season, habitat condition, or environmental situation? How do we capture human, technology, and societal effects on the predator and prey populations?
7. How does one construct a model that considers a longer food chain and/or a more intricate food web?

In particular, the model that we present here builds upon the Lotka-Volterra model to analyze the dynamics and the ecological issues among the Atlantic menhaden (see Figure 1a), the striped bass (Figure 1b), and the blue catfish (Figure 1c). A schematic representation of the food web connection and the ecological issues considered in this project are given in Figure 2 and are explained in detail in the next section.

This paper is organized as follows. Section 2 presents the assumptions, variables, parameters, and differential equations that make up our mathematical model. It also contains preliminary analysis on the system's invariant sets. Section 3 investigates the three-species system with equal interference between two predators competing over one prey. This section records the computations for the equilibria and their stability analyses. It also presents a brief review of fundamental results in dynamical systems. Section 4 builds upon the system analyzed in Section 3 by considering harvesting scenarios. Section 5 continues to investigate the original system by varying the competition strength between the predators. Section 6 presents numerical simulations and analyses to support theoretical computations and to further uncover interesting behavior. Section 7 presents a list of

suggestions on how students may complete a report on their mathematical modelling project and a possible grading rubric. Section 8 restates the phenomenon being considered, summarizes the important mathematical results, and provides evidence-based answers to the ecological issues that motivated this study. Also, in many places throughout the paper, we include *Student Exploration Exercises* to supplement discussion and analysis in a mathematical modelling classroom.

## 2 Mathematical modeling of the fish dynamics

Let us analyze the biological interactions of a system that has two predators competing over one prey and two ecological concerns, namely, over-fishing of the menhaden and the invasion of the catfish on the Chesapeake Bay. Let  $x = x(t)$  be the population of the Atlantic menhaden,  $y = y(t)$  be the population of striped bass, and  $z = z(t)$  be the population of blue catfish. We assume the following:

1. the prey  $x$  grows exponentially at a rate proportional to its size,
2. the two predators  $y$  and  $z$  are in competition, both preying on  $x$ ,
3. the predator  $y$  declines at a rate proportional to its size in the absence of  $x$ ,
4. the predator  $z$  is an invasive species and grows exponentially at a rate proportional to its size, and
5. the prey  $x$  and the predator  $z$  are harvested.

A. The most general system of differential equations describing the three-species biological interactions under the above assumptions is given by:

$$\begin{cases} x' &= ax - bxy - cxz - h_1x \\ y' &= -dy + bxy - \alpha yz \\ z' &= ez + cxz - \beta yz - h_2z. \end{cases} \quad (\text{A})$$



(a) Atlantic Menhaden, *Brevoortia tyrannus*, (a.k.a. Alewife, Bunker, Pogy, Bugmouth, Fat-Back)



(b) Striped Bass, *Morone saxatilis*, (a.k.a. Rockfish, Rock, Striper)



(c) Blue Catfish, *Ictalurus Furcatus*, (a.k.a. Hump-back blue)

Figure 1: The three fish species in Chesapeake Bay in this project, their scientific and common names. Pictures from Chesapeake Bay Program, [www.chesapeakebay.net](http://www.chesapeakebay.net).

**Ecosystem Connection and Issues  
Between the Three Fish Species**

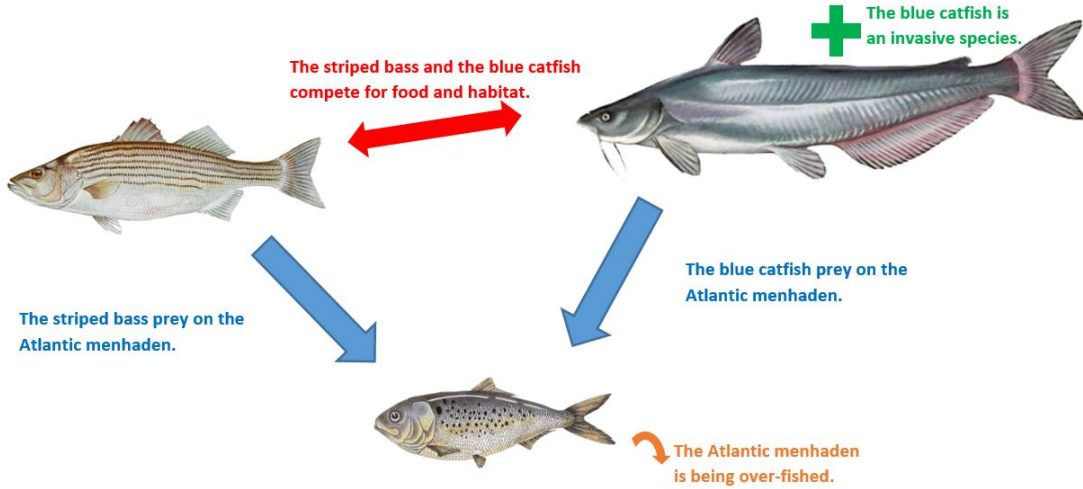


Figure 2: The food web connection and ecological issues among the three fish species. The one-sided blue arrows represent predation towards prey; the red double-arrow represents competition; the green plus sign represents invasiveness; and the orange outward arrow represents over-fishing.

$x(t)$	population of prey
$y(t)$	population of one predator
$z(t)$	population of another (invasive) predator
$a$	growth rate parameter of $x$
$b$	preying effect of $y$ on $x$
$c$	preying effect of $z$ on $x$
$d$	death rate parameter of the predator $y$ in the absence of prey $x$
$e$	growth rate parameter of $z$
$h_1$	harvesting effect on $x$
$h_2$	harvesting effect on $z$
$\alpha$	competition effect of $z$ on $y$
$\beta$	competition effect of $y$ on $z$

Table 1: The functions and parameters in model (A). All parameters are positive.

We call this model (A) and explain the notation in Table 1.

B. To simplify computations and analysis of model (A), we will proceed by analyzing simpler models and then building the results up. First, we will consider the case when there is *no harvesting* and there is equal competition between the predators, that is,  $h_1 = 0 = h_2$



and  $\alpha = \beta$ ; call this model (B):

$$\begin{cases} x' = ax - bxy - cxz \\ y' = -dy + bxy - \alpha yz \\ z' = ez + cxz - \alpha yz. \end{cases} \quad (\text{B})$$

Investigations of this model can be found in Sections 2.1, 2.2, and 3.

C. Then, in Section 4, we consider the effects of harvesting. Section 4.1 covers the case when only the menhaden is harvested and there is equal competition between the predators, that is,  $h_1 \neq 0$  and  $\alpha = \beta$ ; call this model (C):

$$\begin{cases} x' = ax - bxy - cxz - h_1x \\ y' = -dy + bxy - \alpha yz \\ z' = ez + cxz - \alpha yz. \end{cases} \quad (\text{C})$$

D. Section 4.2 covers the case when only the catfish is harvested and there is equal competition between the predators, that is,  $h_2 \neq 0$  and  $\alpha = \beta$ ; call this model (D):

$$\begin{cases} x' = ax - bxy - cxz \\ y' = -dy + bxy - \alpha yz \\ z' = ez + cxz - \alpha yz - h_2z. \end{cases} \quad (\text{D})$$

E. Next, Section 5 considers the case when there is unequal competition between the predators. Sections 5.1 and 5.2 cover the case when there is no harvesting, that is,  $h_1 = 0 = h_2$  and  $\alpha \neq \beta$ ; call this model (E):

$$\begin{cases} x' = ax - bxy - cxz \\ y' = -dy + bxy - \alpha yz \\ z' = ez + cxz - \beta yz. \end{cases} \quad (\text{E})$$

F. Finally, Section 5.3 considers the case when the prey is not harvested but one of the predators is harvested and there is unequal competition between the predators, that is,  $h_1 = 0 \neq h_2$  and  $\alpha \neq \beta$ ; call this model (F):

$$\begin{cases} x' = ax - bxy - cxz \\ y' = -dy + bxy - \alpha yz \\ z' = ez + cxz - \beta yz - h_2z. \end{cases} \quad (\text{F})$$

*Student Exploration 1: Understanding the models and parameters.*

1. The predator effect rates of  $y$  and  $z$  on the prey  $x$  are given by  $\pm bxy$  and  $\pm cxz$ , respectively. How should the system be modified to consider the reality that the effect of the presence of either predators on  $x$  may not be the same as the effect of the absence of prey on the predators?
2. If one predator is a stronger competitor than the other predator, what is the corresponding relationship between  $\alpha$  and  $\beta$ ?

3. In any of the models,  $z$  is an invasive species that will not die in the absence of  $x$ . What does it mean when  $ez$  replaced by  $-ez$ , where  $e > 0$ ?
4. If one assumes logistic growth for either the prey or one of the predators, how is the system modified?
5. How does one consider various harvesting scenarios, such as constant, periodic, and delayed harvesting?

## 2.1 Invariant planes

Since populations are non-negative, we will restrict the domain for  $x$ ,  $y$  and  $z$  to the non-negative octant  $\{(x, y, z) | x \geq 0, y \geq 0, z \geq 0\}$ . By definition, a surface is invariant with respect to a system of differential equations if every solution that starts on the surface does not escape the surface. In our case we will show that coordinate planes are invariant with respect to the system (B). The biological interpretation of invariant planes is that if a species is extinct, then it will not reappear. To prove that a surface is invariant we will use the following theorem from [1].

**Theorem 2.1.** *Let  $S$  be a smooth closed surface without boundary in  $\mathbb{R}^3$  and*

$$\begin{cases} x' = f_1(x, y, z) \\ y' = f_2(x, y, z) \\ z' = f_3(x, y, z) \end{cases} \quad (2.1)$$

where  $f$ ,  $g$ , and  $h$  are continuously differentiable functions. Suppose that  $\mathbf{n}$  is a normal vector to the surface  $S$  at  $(x, y, z)$ , and for all  $(x, y, z) \in S$  we have that

$$\mathbf{n} \cdot (x', y', z') = 0.$$

Then  $S$  is invariant with respect to the system (2.1).

Consider the system (B) and the problem of finding its invariant planes. Let  $S$  be the plane  $z = 0$ , then the vector  $\mathbf{n} = (0, 0, 1)$  is always normal to  $S$ , and at any point  $(x, y, 0) \in S$  the following is true:

$$(x', y', z') \cdot \mathbf{n} = (ax - bxy, -dy + bxy, 0) \cdot (0, 0, 1) = 0$$

and consequently, by the above theorem, the plane  $z = 0$  is invariant with respect to the system (B).

*Student Exploration 2: Computing invariant planes.*



1. Show that the planes  $x = 0$  and  $y = 0$  are invariant with respect to the system (B).
2. Analyze the the invariant planes of systems (A) to (F).
3. Show that, if there is no competition between  $y$  and  $z$ , the plane defined by  $S = \{(x, y, z) : ax - dy + ez = 0\}$  is an invariant plane.

## 2.2 Reduced planar systems

A standard method of analyzing our three-species model is by considering the resulting two-species models; in particular, what happens to the system (B) when  $z = 0$ ,  $y = 0$ , or  $x = 0$ ?

### No catfish

Next, we will look into each of the three reduced planar systems. Assume that there is only one predator by setting  $z = 0$ . Then the system (B) reduces to the classic Lotka-Volterra prey-predator system.

$$\begin{cases} x' &= x(a - by) \\ y' &= y(-d + bx) \\ z' &= 0. \end{cases} \quad (2.2)$$

This system is well-studied. It has two equilibria, namely,  $(0, 0, 0)$  and  $(d/b, a/b, 0)$ , and it can be shown that system (2.2) has closed trajectories centered at  $(d/b, a/b, 0)$ . For details on this analysis, the reader is referred to [5].

### No striped bass

In the case  $y = 0$ , or when a trajectory starts on the plane  $y = 0$ , the system (B) reduces to the following equations:

$$\begin{cases} x' &= x(a - cz) \\ y' &= 0 \\ z' &= z(e + cx). \end{cases}$$

From equation  $z' = z(e + cx) > 0$ , as the derivative is always positive, it follows that  $z(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , which is consistent with the fact that  $z$  is the invader. Expressing  $\frac{dz}{dx} = z'/x'$  gives the following separable differential equation

$$\frac{dz}{dx} = \frac{z(e + cx)}{x(a - cz)}$$

which can be solved in the  $x - z$  plane implicitly

$$\int \left( \frac{a}{z} - c \right) dz = \int \left( \frac{e}{x} + c \right) dx$$

$$a \ln z - cz + K = e \ln x + cx, \quad (2.3)$$

where  $K$  is a constant. Figure 3 illustrates the trajectories' behavior in the  $x$ - $z$ -plane for some particular values of the parameters. The graph shows that as time  $t \rightarrow \infty$ , the invader  $z \rightarrow \infty$  while the prey gets depleted  $x \rightarrow 0$ .

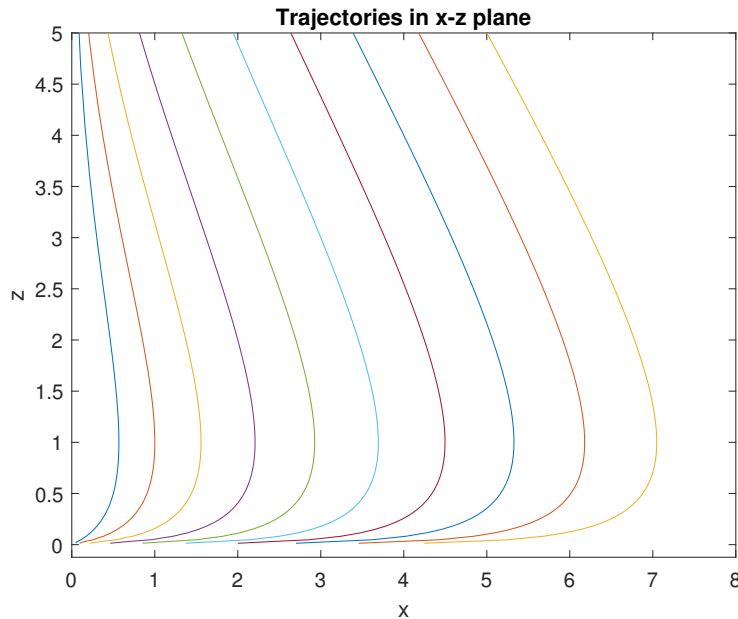


Figure 3: A family of trajectories in the  $x$ - $z$ -plane of the implicit solution (2.3) with coefficients  $a = c = e = 1$ .

*Student Exploration 3. Analyzing resulting two-species systems.*  
Consider system (B).

1. Biologically, what does it mean when  $x = 0$ ? when  $y = 0$ ?  $z = 0$ ?
2. Express the reduced system when  $x = 0$  and solve the resulting separable equation for  $\frac{dy}{dz}$ . Create a graph that illustrates the trajectories in  $y - z$ -plane

### 3 Three-species model without harvesting

Here, we will consider the system given by (B), which we write again here:

$$\begin{cases} x' = ax - bxy - cxz \\ y' = -dy + bxy - \alpha yz \\ z' = ez + cxz - \alpha yz \end{cases} \quad (B)$$

### 3.1 Equilibria for (B)

Let us compute the set of equilibrium points  $(x, y, z)$  of the system (B). These points, also called fixed points, stationary solutions, or zeroes, are solutions that do not change with time, that is,  $x' = y' = z' = 0$ . We have:

$$\begin{cases} x = 0 & \text{or} & a - by - cz = 0 \\ y = 0 & \text{or} & bx - d - \alpha z = 0 \\ z = 0 & \text{or} & e + cx - \alpha y = 0 \end{cases}$$

Observe that computing equilibrium points requires solving equations, a general theme in mathematics. Besides the equilibrium  $E_0 = (0, 0, 0)$ , we obtain four more equilibria:

$$E_1 = \left(0, \frac{e}{\alpha}, \frac{-d}{\alpha}\right), E_2 = \left(\frac{-e}{c}, 0, \frac{a}{c}\right), E_3 = \left(\frac{d}{b}, \frac{a}{b}, 0\right),$$

$$E_4 = \left(\frac{cd - be + \alpha a}{2bc}, \frac{cd + be + \alpha a}{2\alpha b}, \frac{\alpha a - (cd + be)}{2\alpha c}\right).$$

the last of which arises from solving the system of linear equations

$$\begin{cases} a - by - cz = 0 \\ bx - d - \alpha z = 0 \\ e + cx - \alpha y = 0 \end{cases}$$

Let us call  $E_0, E_1, E_2, E_3$  our boundary equilibrium points because at least one of the three components is equal to 0. We observe that equilibria  $E_1$  and  $E_2$  are not biologically relevant because each contains a negative component. Hence, we will only look at  $E_0, E_3$ , and  $E_4$  more closely.

Now, we want the components of  $E_4$  to be all positive. Observe that the  $y$ -component is always positive. For the  $x$ -component to be positive, we require

$$cd - be + \alpha a > 0 \quad \text{or} \quad \frac{be - cd}{a} < \alpha. \quad (3.1)$$

For the  $z$ -component to be positive, we require

$$\alpha a - (cd + be) > 0 \quad \text{or} \quad \frac{cd + be}{a} < \alpha. \quad (3.2)$$

Since it is true that  $(be - cd)/a < (be + cd)/a$  for all positive parameters  $b, c, d, e$ , it follows that by choosing  $\alpha > \alpha_{\text{crit}}$ , where

$$\alpha_{\text{crit}} = \frac{cd + be}{a},$$

all components of  $E_4$  are positive. Let us summarize our results into a theorem:

**Theorem 3.1.** *Suppose  $\alpha > \alpha_{\text{crit}} = (cd + be)/a$ . Then the system (B) has three biologically relevant equilibria, namely,  $E_0 = (0, 0, 0)$ ,  $E_3 = (d/b, a/b, 0)$ , and*

$$E_4 = \left(\frac{cd - be + \alpha a}{2bc}, \frac{cd + be + \alpha a}{2\alpha b}, \frac{\alpha a - (cd + be)}{2\alpha c}\right).$$

Biologically, this means that there exists a possibility when all three species co-exist if both predators have equal competition effects on each other and when this competition parameter  $\alpha$  is greater than  $\alpha_{\text{crit}}$ . Also observe also that when  $\alpha = \alpha_{\text{crit}}$ , this equilibrium  $E_4$  reduces to equilibrium  $E_3$ . In this case, we say that  $\alpha_{\text{crit}}$  is a bifurcation value. Some students may be ready to proceed with a deeper study of bifurcation analysis, a good reference for this topic is [3].

In the next section, we will analyze the stability of these three equilibria using linear stability analysis.

### 3.2 Elementary dynamical systems theory

We will first state some basic definitions and results from dynamical systems which will be then used to analyze the stability of the three equilibria of (B).

**Definition 3.2.** An equilibrium point  $\bar{w} = (\bar{x}, \bar{y}, \bar{z})$  is called **stable** if a solution  $w(t) = (x(t), y(t), z(t))$  based nearby remains close to  $\bar{w}$  for all time. Otherwise, the point  $\bar{w}$  is called an unstable equilibrium point. An equilibrium point is (locally) **asymptotically stable** if it is stable and the state of the system converges to the equilibrium point as time increases.

These definitions of the stability of equilibria may be used as a good discussion point about the use of real analysis in dynamical systems. It may be a good exercise to ask students to write these definitions in terms of limit statements. The definitions capture the essence of the concept of stability but they do not indicate computational steps to determine the stability of equilibria.

In order to classify the stability of our equilibria, the definition requires that we characterize the behavior of solutions near our equilibria. This is accomplished by linearizing our system (B). Students have been introduced to the concept of linearization from their Differential Calculus courses; they know that the tangent line provides to a nonlinear function at a point in its domain a good approximation of the graph of the function. Analogously, the behavior of nonlinear systems such as (B) near the equilibrium point is similar to the behavior of linear systems near that equilibrium point. This is the so-called Hartman-Grobman Theorem and we refer to [1], [3], [7] for proof and the precise statement.

Towards this end, we consider the linearized system

$$\dot{w}(t) = J(\bar{w})w, \quad (3.3)$$

where  $w = (x(t), y(t), z(t))$  and  $J(w)$  is the 3-by-3 Jacobian matrix given by

$$J(x, y, z) = \begin{pmatrix} a - by - cz & -bx & -cx \\ by & bx - d - \alpha z & -\alpha y \\ cz & -\alpha z & e + cx - \alpha y \end{pmatrix}. \quad (3.4)$$

Entries of the Jacobian matrix  $J = (a_{ij})$  are obtained by taking the partial derivatives,

$$a_{ij} = \frac{\partial f_i}{\partial x_j}, \quad \text{where} \quad (x_1, x_2, x_3) = (x, y, z).$$

For example,

$$a_{11} = \frac{\partial f_1}{\partial x} = a - by - cz, \quad a_{12} = \frac{\partial f_1}{\partial y} = -bx, \quad a_{13} = \frac{\partial f_1}{\partial z} = -cx.$$

When  $J(\bar{w})$  has no eigenvalues with zero real part, then  $\bar{w}$  is called a **hyperbolic equilibrium point** and the asymptotic behavior of solutions near  $\bar{w}$  is determined by the linearization (3.3). If any one of the eigenvalues of  $J(\bar{w})$  has a zero real part, then interesting behaviors may occur. Linear stability analysis is accomplished via a fundamental theorem [1], [3], [7]:

**Theorem 3.3.** *An equilibrium point  $\bar{w}$  is asymptotically stable if all the eigenvalues of  $J(\bar{w})$  have negative real parts. If at least one of the eigenvalues of  $J(\bar{w})$  has a positive real part, then the equilibrium point is unstable.*

Suppose that the three eigenvalues of  $J(\bar{w})$  are  $\lambda_1 < 0$ ,  $\lambda_2 = 0$ , and  $\lambda_3 > 0$  with corresponding eigenvector  $v_1, v_2, v_3$ . Solutions which start close to the equilibrium point  $\bar{w}$  will decay in the direction of  $v_1$  and will grow in the direction of  $v_3$ . Since  $\lambda_2 = 0$ , we cannot conclude from linearization whether the solutions decay or grow in the direction of  $v_2$ . The spaces spanned by the eigenvectors  $v_1, v_3$  are called the local stable and local unstable manifolds of  $\bar{w}$ , respectively; the space spanned by the eigenvector  $v_2$  is called the local center manifold of  $\bar{w}$ . At this point, a manifold may be briefly described as a generalization of a linear subspace of  $\mathbb{R}^n$ . Points, lines, planes, arcs, and spheres are examples of manifolds. Students who become interested in this topic may want to look into the area of mathematics called Differential Geometry.

To make a direct conclusion about the stability of an equilibrium, one has to analyze the nature of the eigenvalues, that is, the roots of the characteristic equation  $\det(J(\bar{w}) - \lambda I) = 0$ . In some cases, finding the roots of the characteristic equation may be obtained by hand. Indeed, for the system that we are considering in this project, the characteristic equation is a third-degree polynomial with real coefficients. In case solving the roots is not possible, then there are some analytic, algebraic, and computational options. In this paper, we will demonstrate these options.

The following theorem summarizes the existence of stable, center, and unstable manifolds for a three-dimensional system and is oftentimes referred to as the Center Manifold Theorem. For the precise statement and proof, we refer to [1], [3], [7].

**Theorem 3.4.** *Suppose  $f$  is a smooth vector field in  $\mathbb{R}^3$ . Assume that the system  $\dot{w} = f(w)$  has an equilibrium point  $\bar{w}$  and its linearization at  $\bar{w}$  is given by  $\dot{w} = J(\bar{w})w$ . Suppose  $\sigma_s, \sigma_c, \sigma_u$ , contain the negative, zero, and positive eigenvalues of  $J(\bar{w})$ , respectively. Let  $E^s, E^c, E^u$  be the eigenspaces associated with  $\sigma_s, \sigma_c, \sigma_u$ , respectively. Then there exist smooth stable  $W^s$ , center  $W^c$ , and unstable  $W^u$  manifolds tangent to  $E^s, E^c, E^u$ , respectively at  $\bar{w}$ . The stable and unstable manifolds are unique, but  $W^c$  need not be.*

### 3.3 Stability for (B)

We have seen that there are three biologically relevant equilibrium points. Let us call  $E_0$  the trivial equilibrium point,  $E_3$  the catfish-free boundary equilibrium point, and  $E_4$

the interior co-existence equilibrium point. In this section, we will show that they are all unstable.

### Stability Analysis for $E_0$

The equilibrium point  $E_0$  is unstable. To see this, observe that the matrix  $J(E_0) = J(0, 0, 0)$  is given by

$$J(0, 0, 0) = \begin{pmatrix} a & 0 & 0 \\ 0 & -d & 0 \\ 0 & 0 & e \end{pmatrix}$$

with eigenvalues  $a > 0$ ,  $-d < 0$ ,  $e > 0$ , with corresponding eigenvectors  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$ . According to Theorem 3.3,  $E_0$  is an unstable equilibrium. By Theorem 3.4, we know that the negative eigenvalue has a corresponding one-dimensional stable manifold, or curve, tangent to the eigenvector  $(0, 1, 0)$  at the equilibrium  $(0, 0, 0)$ . This stable curve is actually the  $y$ -axis which means that  $y(t) \rightarrow 0$  as  $t \rightarrow \infty$ ; this is biologically relevant as in the absence of  $x$ , the prey, the predator  $y$  will become extinct. Corresponding to the two positive eigenvalues, there is a two-dimensional unstable manifold or a plane, tangent at the equilibrium  $(0, 0, 0)$  and spanned by the vectors  $(1, 0, 0)$  and  $(0, 0, 1)$ . This two-dimensional unstable manifold is the  $xz$ -plane which from a biological perspective means that both the prey and the invader will grow without a boundary in the absence of  $y$ , i.e.  $x(t) \rightarrow \infty$  and  $z(t) \rightarrow \infty$ .

Since  $J(E_0)$  has two positive eigenvalues and one negative eigenvalue,  $E_0$  is a hyperbolic equilibrium point. Moreover, in this case, we say that  $E_0$  is a hyperbolic saddle point.

### Stability Analysis for $E_3$

The equilibrium  $E_3 = \left(\frac{d}{b}, \frac{a}{b}, 0\right)$  is an unstable non-hyperbolic equilibrium point. To see this, the Jacobian at  $E_3$  is given by the following matrix

$$J\left(\frac{d}{b}, \frac{a}{b}, 0\right) = \begin{pmatrix} 0 & -d & -cd/b \\ a & 0 & -\alpha a/b \\ 0 & 0 & (cd + be - \alpha a)/b \end{pmatrix},$$

with characteristic equation  $g(x) = (x - \lambda_s)(x^2 + ad)$  where  $\lambda_s = \frac{cd + be - \alpha a}{b}$ . The three eigenvalues are  $\lambda_s$  and the two purely imaginary eigenvalues  $\pm i\sqrt{ad}$ .

Since we assume that  $\alpha > \alpha_{\text{crit}}$ , observe that  $\lambda_s$  is negative and hence by Theorem 3.4, there is a one-dimensional stable manifold at  $E_3$  tangent to the eigenvector spanned by

$$(\beta_1\beta_3 - \beta_2d, a\beta_1 + \beta_2\beta_3, \beta_3^2 + ad),$$

where  $\beta_1 = -cd/b$ ,  $\beta_2 = -\alpha a/b$ , and  $\beta_3 = \lambda_s$ .

Corresponding to the purely imaginary eigenvalues  $\pm i\sqrt{ad}$ , there exists a two-dimensional center manifold at  $E_3$  tangent to the space spanned by the eigenvectors

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \pm \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \sqrt{\frac{d}{a}} i.$$

The eigenspace spanned by these vectors is the  $xy$ -plane. As not all eigenvalues have negative real parts, this equilibrium is unstable.

Stability Analysis for  $E_4$

Next, we will examine the stability of the equilibrium  $E_4 = (x^*, y^*, z^*)$ , where

$$x^* = \frac{cd - be + \alpha a}{2bc}, y^* = \frac{cd + be + \alpha a}{2\alpha b}, z^* = \frac{\alpha a - (cd + be)}{2\alpha c}.$$

Recall that we assume that  $\alpha > \alpha_{\text{crit}}$  so that  $x^*, y^*, z^*$  are always positive. The Jacobian matrix (3.4) at this point is given by

$$J(x^*, y^*, z^*) = \begin{pmatrix} 0 & -bx^* & -cx^* \\ by^* & 0 & -\alpha y^* \\ cz^* & -\alpha z^* & 0 \end{pmatrix}.$$

where  $(x^*, y^*, z^*)$  is the solution to the system

$$\begin{cases} a - by - cz = 0 \\ bx - d - \alpha z = 0 \\ cx + e - \alpha y = 0. \end{cases}$$

To find the eigenvalues  $\lambda$ , we need to analyze the roots of the characteristic equation

$$-\lambda^3 - (b^2x^*y^* + c^2x^*z^* - \alpha^2y^*z^*)\lambda + 2bc\alpha x^*y^*z^* = 0,$$

that is,

$$\lambda^3 + B\lambda + C = 0, \tag{3.5}$$

where  $B = b^2x^*y^* + c^2x^*z^* - \alpha^2y^*z^*$  and  $C = -2bc\alpha x^*y^*z^*$ . Solving for the closed-form of the roots of (3.5) is not a straightforward computation.

Since we are not able to compute the exact roots of the characteristic equation, let us present an analytic method of analyzing the roots using the Intermediate Value Theorem from Calculus I. Consider  $g(x) = x^3 + Bx + C$ . Since  $y = g(x)$  has a positive leading coefficient for  $x^3$ , it follows that the graph of  $y = g(x)$  is ultimately increasing, positive, and unbounded. This implies that there exists  $\gamma > 0$  such that  $g(\gamma) > 0$ . However, we also observe that  $g(0) = C < 0$ . Thus, it follows from the Intermediate Value Theorem that there exists  $0 < \beta < \gamma$  such that  $g(\beta) = 0$ . The existence of this positive real root for  $y = g(x)$  tells us that one of the eigenvalues is a positive real number. Hence, the equilibrium  $E_4$  is unstable!

Summarizing our results on the behavior of the equilibria of the system (B), we have

**Theorem 3.5.** Suppose  $\alpha > \alpha_{crit} = (cd + be)/a$ . Then, system (B) has two boundary equilibrium points and one interior equilibrium point. They are all unstable:

$$E_0 = (0, 0, 0), E_3 = \left(\frac{d}{b}, \frac{a}{b}, 0\right), E_4 = \left(\frac{cd - be + \alpha a}{2bc}, \frac{cd + be + \alpha a}{2\alpha b}, \frac{\alpha a - (cd + be)}{2\alpha c}\right).$$

*Student Exploration 4.* Using algebra and calculus to analyze characteristic equations. Consider the cubic polynomial  $g(x) = x^3 + Bx + C$  where  $C < 0$ . Recall from the Fundamental Theorem of Algebra that  $g$  will have at most three roots; furthermore,  $g$  can either have three real roots or one real root and a pair of complex conjugate roots.

1. Draw possible pictures of  $y = g(x)$  where  $g$  has three real roots.
2. Draw possible pictures of  $y = g(x)$  where  $g$  has one real root and a pair of complex roots.

### 3.4 Phase portrait near co-existence equilibrium for (B)

We can also use computer power to analyze the roots of equation (3.5). Using MATLAB, we can calculate the eigenvalues and the eigenvectors, we can create a time-evolution of the functions, and we can also create a phase portrait. Consider the following particular values of the parameters:

$a$	growth rate parameter of $x$	0.5
$b$	preying effect of $y$ on $x$	0.01
$c$	preying effect of $z$ on $x$	0.01
$d$	death rate parameter of the predator $y$ in the absence of prey $x$	0.5
$e$	growth rate parameter of $z$	0.5

Table 2: The parameter values used in the simulation are summarized in this table.

Furthermore, we assume in this section that  $\alpha = \beta = 0.1$ . With these parameter values, the interior coexistence equilibrium point is  $E_4 = (275, 30, 20)$  with eigenvalues  $-1.6046, -0.8398, \text{ and } 2.4444$ . The bifurcation value is  $\alpha_{crit} = (cd + be)/a = 0.02$ .

Figure 4 shows the phase portrait consisting of trajectories near the interior coexistence equilibrium point  $E_4$ , which is represented in the plot as a red dot. Starting from initial conditions close to the equilibrium  $E_4$ , the phase portrait shows that the trajectories approach what appears to be orbit trajectories on the  $xy$ -plane around the boundary equilibrium point  $E_3$ , represented by a yellow dot. This illustrates that the coexistence equilibrium is highly unstable. Equilibrium is possible for some critical positive populations of the three species, and while each species maintains that population level, they may co-exist. If at any instant the population levels are not at the critical sizes, then the effects of competition will drive some of the species to extinction.



## 4 Three-species model with harvesting

### 4.1 Overfishing of menhaden

Now, we modify the model (B) to include harvesting of the prey (Atlantic menhaden)  $x(t)$ . Here, we assume that harvesting of the menhaden occurs at a linear rate,  $h_1x$ , for some positive constant  $h_1$ , that is, the harvesting of  $x$  happens in direct proportion with its growth:

$$\begin{cases} x' &= ax - bxy - cxz - h_1x \\ y' &= bxy - dy - \alpha yz \\ z' &= ez + cxz - \alpha yz. \end{cases} \quad (\text{C})$$

When harvesting of the prey is accomplished at a higher rate than the prey's growth, we say that overfishing occurs. Defining  $A = a - h_1$ , we see that overfishing happens when  $A < 0$ . In case  $A > 0$ , we observe that we recover system (B) with  $a$  replaced by  $A$ .

#### Equilibria for (C)

As before, setting the derivatives equal to zero in (C) solves the equilibrium points. We obtain

$$E_0 = (0, 0, 0), E_1 = \left(0, \frac{e}{\alpha}, \frac{-d}{\alpha}\right), E_2 = \left(\frac{-e}{c}, 0, \frac{a - h_1}{c}\right), E_3 = \left(\frac{d}{b}, \frac{a - h_1}{b}, 0\right),$$

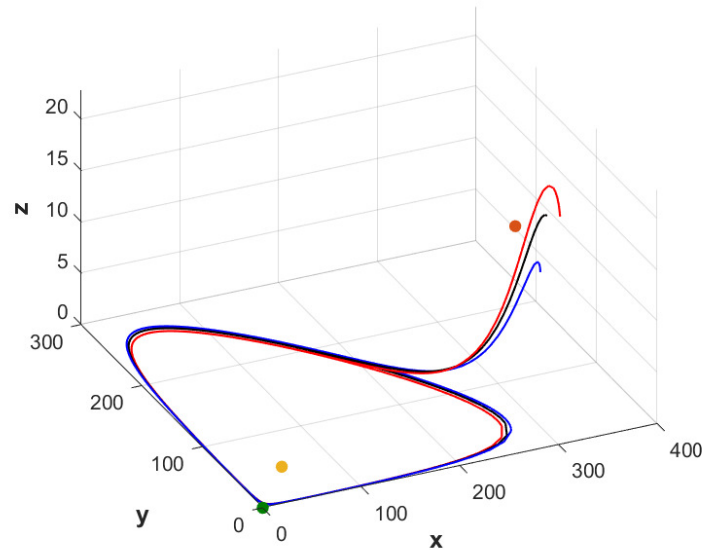


Figure 4: Trajectories behavior for model (B) near equilibrium point  $E_4 = (275, 30, 25)$  with initial conditions  $(300, 30, 15)$  in blue,  $(310, 35, 20)$  in black, and  $(320, 30, 20)$  in red. The red dot is the equilibrium  $E_4$ , the yellow dot is  $E_3 = (50, 50, 0)$ , and the green dot is  $E_0 = (0, 0, 0)$ . See Table 2 for parameter values.

and, as in the previous model, a fourth equilibrium

$$E_4 = \left( \frac{cd - be + \alpha(a - h_1)}{2bc}, \frac{cd + be + \alpha(a - h_1)}{2ab}, \frac{\alpha(a - h_1) - (cd + be)}{2\alpha c} \right).$$

Let us observe that  $E_1$ ,  $E_2$ , and  $E_3$  and  $E_4$  are not biologically relevant as some of their components are negative due to the assumption  $a - h_1 < 0$ . In particular, when overfishing of menhaden happens, the interior co-existence equilibrium point does not exist.

**Theorem 4.1.** *Suppose  $h_1 > a$ . Then the system (C) has only one biologically relevant equilibrium point,  $E_0 = (0, 0, 0)$ .*

### Linearization and stability analysis for (C)

The linearization of (C) is represented by the matrix

$$J(x, y, z) = \begin{pmatrix} a - by - cz - h_1 & -bx & -cx \\ by & bx - d - \alpha z & -\alpha y \\ cz & -\alpha z & cx + e - \alpha y \end{pmatrix}.$$

Let us now analyze the behavior of the system at the only biologically relevant equilibrium point,  $E_0$ . Observe that

$$J(0, 0, 0) = \begin{pmatrix} a - h_1 & 0 & 0 \\ 0 & -d & 0 \\ 0 & 0 & e \end{pmatrix}$$

with eigenvalues  $a - h_1 < 0$ ,  $-d < 0$ ,  $e > 0$  and the corresponding eigenvectors are  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$ . Since we assume that  $h_1 > a$ , we have two negative eigenvalues and one positive eigenvalue. Thus,  $E_0$  is unstable. Moreover, using Theorem 3.4, there is an invariant two-dimensional stable manifold that is tangent to the space spanned by eigenvectors  $(1, 0, 0)$  and  $(0, 1, 0)$ , which is the  $xy$ -plane. On the  $xy$ -plane, solutions approach the equilibrium  $E_0 = (0, 0, 0)$ , meaning that  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$  and  $y(t) \rightarrow 0$  as  $t \rightarrow \infty$ . This is biologically relevant as if the prey is overfished, then in time both the prey and the predator will become extinct. In addition, there is also an one-dimensional unstable manifold spanned by the eigenvector  $(0, 0, 1)$ , or the  $z$ -axis, on which  $z(t) \rightarrow \infty$  or the catfish will grow without limit.

*Student Exploration 5. What happens when the harvesting rate is equal to the growth rate? Analyze the system (C) when  $h_1 = a$ .*

1. Show that there are only two biologically relevant equilibria,  $(0, 0, 0)$  and  $(d/b, 0, 0)$ .
2. Show that both equilibria are unstable.
3. Sketch a phase portrait.

## 4.2 Harvesting of catfish

In this section, we ask, will harvesting of the catfish solve its invasive effect on the Chesapeake Bay? In other words, does it help to *eat the invaders* [2]?

Let us modify the model (B) to include harvesting of one of the predators, the catfish. Assume that harvesting of the catfish occurs at a linear rate  $h_2z$ , for some positive constant  $h_2$ , that is, the harvesting of  $z$  happens in direct proportion to its growth. We get

$$\begin{cases} x' &= ax - bxy - cxz \\ y' &= -dy + bxy - \alpha yz \\ z' &= ez + cxz - \alpha yz - h_2z, \end{cases} \quad (\text{D})$$

which is equivalent to

$$\begin{cases} x' &= ax - bxy - cxz \\ y' &= -dy + bxy - \alpha yz \\ z' &= Ez + cxz - \alpha yz \end{cases} \quad (4.1)$$

where  $E = e - h_2$ . Note that unlike the other parameters,  $E$  may be positive, negative, or zero.

### Equilibrium points for (D)

We set the derivatives equal to zero and find four boundary equilibrium points

$$E_0 = (0, 0, 0), E_1 = \left(0, \frac{E}{\alpha}, \frac{-d}{\alpha}\right), E_2 = \left(\frac{-E}{c}, 0, \frac{a}{c}\right), E_3 = \left(\frac{d}{b}, \frac{a}{b}, 0\right),$$

and one interior co-existence equilibrium point

$$E_4 = \left(\frac{cd - bE + \alpha a}{2bc}, \frac{cd + bE + \alpha a}{2\alpha b}, \frac{\alpha a - (cd + bE)}{2\alpha c}\right).$$

For the equilibrium  $E_4$  to exist, we need to require that its components are positive:

$$\begin{cases} cd - bE + \alpha a &> 0 \\ cd + bE + \alpha a &> 0 \\ \alpha a - (cd + bE) &> 0. \end{cases}$$

Now, let us study parameter intervals using these inequalities. Manipulation of the inequalities depends on the parameter that one wishes to investigate. As previously, let us find the bounds for the competition parameter  $\alpha$ . Toward this end, we have

$$\begin{cases} \alpha a &> bE - cd \\ -\alpha a &< bE + cd \\ \alpha a &> bE + cd. \end{cases}$$

Or, upon combining these inequalities, we have

$$\left| \frac{bE + cd}{a} \right| < \alpha. \quad (4.2)$$

The equilibrium point  $E_4$  exists when the competition parameter  $\alpha$  satisfies this inequality.

Next, let us analyze model **D** with respect to the sign  $E = e - h_2$ , where  $e$  is the growth rate and  $h_2$  is the harvesting parameter of the catfish.

**Case 1.**  $E < 0$ . This is the case when  $e < h_2$ . We shall call this the overfishing of the catfish population, when the rate of harvesting exceeds the growth rate of the species  $z$ . From the analysis above, the equilibrium  $E_4$  is biologically relevant as long as the competition parameter is sufficiently large. If the parameter  $\alpha$  is not big enough, then the equilibrium  $E_4$  will not have positive components and hence  $E_4$  will not be biologically relevant. This means that even when there is overfishing of the catfish, all three species may still co-exist. If the competition strength between the striped bass and the catfish is below the critical value  $|bE + cd|/a$ , then the three species will not co-exist. Hence, when  $E < 0$ , there are three boundary equilibrium points,  $E_0, E_2, E_3$  and one interior co-existence equilibrium point  $E_4$ .

**Case 2.**  $E > 0$ . We shall call this case (a plain) harvesting. Then the equilibrium  $E_4$  exists as it will have positive components as long as the competition is at a sufficiently large value. This means that even if there is harvesting of catfish, all three species may co-exist. When  $E > 0$ , there are three equilibrium points  $E_0, E_3$ , and  $E_4$ .

**Case 3.**  $E = 0$ . In this case the harvesting rate is exactly equal to the growth rate. Then the equilibrium  $E_4$  exists, that is, it will have positive components provided  $-cd + \alpha a > 0$ . As a bonus, we can see here that with respect to the competition parameter  $\alpha$ , a bifurcation value is  $\alpha = cd/a$ . When  $E = 0$ , the equilibrium points are  $E_0, E_2, E_3$ , and  $E_4$ .

These results on the equilibria are summarized in the following theorem.

**Theorem 4.2.** Suppose  $\left| \frac{bE + cd}{a} \right| < \alpha$ , where  $E = e - h_2$ . Then (4.1) has 5 equilibrium points:

$$E_0 = (0, 0, 0), \quad E_1 = \left( 0, \frac{E}{\alpha}, \frac{-d}{\alpha} \right), \quad E_2 = \left( \frac{-E}{c}, 0, \frac{a}{c} \right), \quad E_3 = \left( \frac{d}{b}, \frac{a}{b}, 0 \right),$$

$$E_4 = \left( \frac{cd - bE + \alpha a}{2bc}, \frac{cd + bE + \alpha a}{2\alpha b}, \frac{\alpha a - (cd + bE)}{2\alpha c} \right).$$

If  $E \leq 0$  then  $E_0, E_2, E_3, E_4$  are biologically relevant. If  $E > 0$  then  $E_0, E_3, E_4$  are biologically relevant. In particular, the interior co-existence equilibrium point  $E_4$  always exists for any value of  $E$ , provided the competition parameter  $\alpha$  is large enough.

This theorem highlights the interesting result that when harvesting on the invasive species happens, competition between the two predators must be high enough in order for the three species to co-exist. Competition, in this case, is essential and beneficial to the ecological dynamics of this system.

### Linearization and stability analysis for (D)

The linearization is given by the 3-by-3 matrix  $\dot{w}(t) = J(\bar{w})w$  where  $w = (x(t), y(t), z(t))$  and  $J(w)$  is the 3-by-3 Jacobian matrix given by

$$J(x, y, z) = \begin{pmatrix} a - by - cz & -bx & -cx \\ by & bx - d - \alpha z & -\alpha y \\ cz & -\alpha z & E + cx - \alpha y \end{pmatrix}.$$

Equilibrium Point	Characteristic Equation	Eigenvalues	Stability Result						
$E_0(0, 0, 0)$	$(\lambda - a)(\lambda + d)(\lambda - E) = 0$	$a, -d, E$	This equilibrium point exists and is unstable for all values of $E$ .						
$E_2\left(\frac{-E}{c}, 0, \frac{a}{c}\right)$	$(\lambda + \lambda_{t+})(\lambda^2 - aE) = 0,$ $\lambda_{t+} = \frac{bE + cd + \alpha a}{c}$	$-\lambda_{t+}, \pm\sqrt{aE}$	<table border="1"> <tr> <td><math>E &gt; 0</math></td> <td>not an equilibrium</td> </tr> <tr> <td><math>E = 0</math></td> <td>unstable</td> </tr> <tr> <td><math>E &lt; 0</math></td> <td>unstable</td> </tr> </table>	$E > 0$	not an equilibrium	$E = 0$	unstable	$E < 0$	unstable
$E > 0$	not an equilibrium								
$E = 0$	unstable								
$E < 0$	unstable								
$E_3\left(\frac{d}{b}, \frac{a}{b}, 0\right)$	$(\lambda - \lambda_{t-})(\lambda^2 + ad) = 0,$ $\lambda_{t-} = \frac{bE + cd - \alpha a}{c}$	$\lambda_{t-}, \pm i\sqrt{ad}$	This equilibrium point exists and is unstable for all values of $E$ .						
$E_4(x^*, y^*, z^*)$ $x^* = \frac{cd - bE + \alpha a}{2bc},$ $y^* = \frac{cd + bE + \alpha a}{2\alpha b},$ $z^* = \frac{\alpha a - (cd + bE)}{2\alpha c}$	$\lambda^3 + B\lambda + C = 0,$ $B = b^2x^*y^* + c^2x^*z^* - \alpha^2y^*z^*,$ $C = -2bc\alpha x^*y^*z^*$	The closed-form of roots is not straightforward to compute.	This equilibrium point exists and is unstable for all values of $E$ .						

Table 3: The table presents a summary of the equilibria of model (D) and their stability when harvesting of the catfish happens, where  $E = e - h_2$ .

As described and demonstrated in the previous section, stability analysis requires analyzing the roots of the resulting characteristic equation for  $J(\bar{w})$  where  $\bar{w} \in \{E_0, E_2, E_3, E_4\}$ . Here, we assume that the competition parameter  $\alpha$  satisfies  $\alpha > |bE + cd|/a$ . A summary of the equilibria and their stability result is presented in Table 3, with  $E = e - h_2$ , as stated in Theorem 4.2.

As a result, we can say that the policy that recommends eating the catfish to control its invasive growth [2] may not be able to accomplish the following: drive the catfish to extinction, eradicate competition with the striped bass, or affect the menhaden's population growth adversely; however, mathematical analysis shows that overfishing the catfish will still allow all three species to coexist.

## 5 Mathematical model when $\alpha \neq \beta$

In this section, we investigate the system (A) when the interference between the two predators are not equal,  $\alpha \neq \beta$ :

$$\begin{cases} x' &= ax - bxy - cxz \\ y' &= -dy + bxy - \alpha yz \\ z' &= ez + cxz - \beta yz. \end{cases} \quad (\text{E})$$

The interference measures the competition effect of one predator over another: when the catfish is more aggressive than the striped bass then  $\alpha > \beta > 0$  and when the striped bass is more aggressive than the catfish then  $0 < \alpha < \beta$ . Observe that if  $\beta$  were negative, then the term  $-\beta yz$  in the third equation would mean that the species  $z$  acts as a predator on the species  $y$  and in this case, the system would model the food chain scenario:  $z$  preys on  $y$  which preys on  $x$ , which is a completely different system than what we are studying. Indeed, a slight modification in the terms of a system of differential equations, such as a change in the sign of the parameters, may pertain to a different biological phenomenon!

## 5.1 Equilibria for (E)

Setting the derivatives equal to zero yields four boundary equilibrium points and one interior equilibrium point for (E):

$$Q_0 = (0, 0, 0), Q_1 = \left(0, \frac{e}{\alpha}, \frac{-d}{\alpha}\right), Q_2 = \left(\frac{-e}{c}, 0, \frac{a}{c}\right), Q_3 = \left(\frac{d}{b}, \frac{a}{b}, 0\right), \quad (5.1)$$

and

$$Q_4 = \left(\frac{(cd + \alpha a)\beta - abe}{(\alpha + \beta)bc}, \frac{cd + be + \alpha a}{(\alpha + \beta)b}, \frac{a\beta - (cd + be)}{(\alpha + \beta)c}\right). \quad (5.2)$$

Note that  $Q_1$  and  $Q_2$  have negative components and are not biologically possible.  $Q_0$  is the trivial equilibrium and  $Q_3 = E_3$  is the equilibrium in the case when  $\alpha = \beta$ . For the equilibrium point  $Q_4$  to be biologically relevant, all of its components must be positive. Observe that its  $y$ -component is always positive while its  $z$ -component is positive provided

$$a\beta > cd + be \quad \text{or} \quad \beta > \frac{cd + be}{a}.$$

Observe that the right hand side of the above inequality is the critical value  $\alpha_{\text{crit}} = \frac{cd + be}{a}$  obtained in Section 3.1. If  $\beta > \alpha_{\text{crit}}$  the  $z$ -component of  $Q_4$  will be positive. Let us prove that the condition  $\beta > \alpha_{\text{crit}}$  is sufficient for the  $x$ -component of  $Q_4$  to be also positive. Using the fact that  $\beta > \alpha_{\text{crit}}$ , substitute  $\beta$  in the numerator of the  $x$ -component as follows

$$(cd + \alpha a)\beta > \frac{(cd + \alpha a)(cd + be)}{a}.$$

The right-hand side of this inequality can be simplified to

$$\frac{(cd)^2 + cdbe + aacd + aabe}{a} = \frac{(cd)^2 + cdbe + aacd}{a} + abe > abe.$$

Thus,  $(cd + \alpha a)\beta > abe$ , so the first component of  $Q_4$  is positive. If  $\beta$  is larger than some critical value  $\alpha_{\text{crit}}$ , then there exists an interior equilibrium point. Biologically, this means that co-existence of the three species is possible.

## 5.2 Linearization and stability analysis for (E)

The linearization of the system (E) is similar to the Jacobian (3.4) except for the third row as given by the following matrix:

$$J(x, y, z) = \begin{pmatrix} a - by - cz & -bx & -cx \\ by & bx - d - \alpha z & -\alpha y \\ cz & -\beta z & cx + e - \beta y \end{pmatrix}. \quad (5.3)$$

At the trivial equilibrium  $Q_0$ , the Jacobian matrix has a diagonal form

$$J(0, 0, 0) = \begin{pmatrix} a & 0 & 0 \\ 0 & -d & 0 \\ 0 & 0 & e \end{pmatrix}$$

and has the same eigenvalues and eigenvectors discussed in section 3.2 for the equilibrium  $E_0$ . Consequently,  $Q_0$  is also unstable. Furthermore, observe that the Jacobian  $J(Q_3)$  is similar to the Jacobian at the equilibrium  $E_3$  except for one term in the third row:

$$J\left(\frac{d}{b}, \frac{a}{b}, 0\right) = \begin{pmatrix} 0 & -d & -cd/b \\ a & 0 & -\alpha a/b \\ 0 & 0 & (cd + be - \beta a)/b \end{pmatrix},$$

with one real eigenvalue  $(cd + be - \beta a)/b$  and the purely imaginary eigenvalues  $\pm i\sqrt{ad}$ . For  $\beta > \alpha_{\text{crit}}$ , the real eigenvalue is negative and by the Center Manifold Theorem, there is a one-dimensional stable manifold tangent at  $(d/b, a/b, 0)$  to the corresponding eigenvector. Since not all eigenvalues have negative real parts, the equilibrium  $Q_3$  is also unstable.

Finally, let us analyze the stability of  $Q_4$  via its Jacobian  $J(Q_4)$ . Like before, the computation of the roots of the resulting characteristic equation is not straightforward. Here, let us demonstrate another technique in analyzing the roots of polynomials via an algebraic result called the Routh-Hurwitz Criteria. This theorem gives a necessary and sufficient condition for a polynomial to have only zeros with negative real parts [9]. Here, we include this theorem's version for the particular case of a third-degree polynomial:

**Theorem 5.1. (Routh-Hurwitz Criteria)** *For a third-degree polynomial of the form  $\lambda^3 + A\lambda^2 + B\lambda + C = 0$ , the three roots will have negative real parts if and only if the following three conditions are satisfied*

$$A, C > 0 \quad \text{and} \quad AB - C > 0.$$

From this theorem follows that if at least one of the coefficients  $A$  or  $C$  is not strictly positive, then it is not possible for all three roots to have negative real parts and hence the equilibrium point will be unstable.

*Student Exploration 6. Using algebraic results to investigate characteristic equations. Analyze the stability of the interior co-existence equilibrium point  $Q_4$ .*



1. Verify that the characteristic equation corresponding to the equilibrium point  $Q_4$  is

$$\lambda^3 + (b^2x^*y^* + c^2x^*z^* - \alpha\beta y^*z^*)\lambda - bc(\alpha + \beta)x^*y^*z^* = 0.$$

2. Use the Routh-Hurwitz Criteria to verify that  $Q_4$  is unstable.

**Theorem 5.2.** Suppose  $\beta > \frac{cd + be}{a}$ . Then the system (E) has the following biologically relevant equilibria:

$$Q_0 = (0, 0, 0), Q_3 = \left(\frac{d}{b}, \frac{a}{b}, 0\right), \quad \text{and} \quad Q_4 = \left(\frac{(cd + \alpha a)\beta - \alpha be}{(\alpha + \beta)bc}, \frac{cd + be + \alpha a}{(\alpha + \beta)b}, \frac{a\beta - (cd + be)}{(\alpha + \beta)c}\right).$$

They are all unstable.

### 5.3 Harvesting of catfish

In this part, let us analyze the model when the competition interference between the striped bass and the catfish is not equal with the additional condition that the catfish is also being harvested at a linear rate  $h_2z$ . In fact, we assume here that  $\alpha > \beta > \frac{cd + bE}{a}$ , that is, between the two competing predators, the catfish causes more harm towards the striped bass' population. Here,  $E = e - h_2$ . We have,

$$\begin{cases} x' = ax - bxy - cxz \\ y' = -dy + bxy - \alpha yz \\ z' = Ez + cxz - \beta yz. \end{cases} \quad (\text{F})$$

Like before, a bifurcation occurs when  $e = h_2$ ; the results are summarized in Table 4.

## 6 Numerical simulations

While the previous sections utilized theoretical analysis to investigate the local stability of equilibrium points, here we will use MATLAB to numerically simulate the systems. Numerical simulations and their graphs offer numerical and visual perspectives on the long term behavior of the three fish species.

### 6.1 Discovery via numerical time-evolutions

Figure 5 shows the long-term behavior of the fish populations for some particular values of the parameters  $a, b, c, d, e$  given in Table 2 with initial conditions (100, 10, 10). Panel (a) illustrates the populations when  $\alpha = \beta$  and the two predators are equally competitive. The catfish (represented by the blue curve) grows without bound. The other two species die out: the striped bass (represented by the green curve) dies out earlier than the menhaden (represented by the red curve). This behaviour among the three species is still reflected

Equilibrium Point	Characteristic Equation	Eigenvalues	Stability Result						
$E_0(0, 0, 0)$	$(\lambda - a)(\lambda + d)(\lambda - E) = 0$	$a, -d, E$	This equilibrium point exists and is unstable for all values of $E$ .						
$E_2\left(\frac{-E}{c}, 0, \frac{a}{c}\right)$	$(\lambda + \lambda_{t+})(\lambda^2 - aE) = 0,$ $\lambda_{t+} = \frac{bE + cd + \alpha a}{c}$	$-\lambda_{t+}, \pm\sqrt{aE}$	<table border="1"> <tr> <td><math>E &gt; 0</math></td> <td><b>not an equilibrium</b></td> </tr> <tr> <td><math>E = 0</math></td> <td><b>unstable</b></td> </tr> <tr> <td><math>E &lt; 0</math></td> <td><b>unstable</b></td> </tr> </table>	$E > 0$	<b>not an equilibrium</b>	$E = 0$	<b>unstable</b>	$E < 0$	<b>unstable</b>
$E > 0$	<b>not an equilibrium</b>								
$E = 0$	<b>unstable</b>								
$E < 0$	<b>unstable</b>								
$E_3\left(\frac{d}{b}, \frac{a}{b}, 0\right)$	$(\lambda - \lambda_{t-})(\lambda^2 + ad) = 0,$ $\lambda_{t-} = \frac{bE + cd - \beta a}{c}$	$\lambda_{t-}, \pm i\sqrt{ad}$	This equilibrium point exists and is unstable for all values of $E$ .						
$E_4(x^*, y^*, z^*)$ where $x^* = \frac{(cd + \alpha a)\beta - abE}{(\alpha + \beta)bc},$ $y^* = \frac{cd + bE + \alpha a}{(\alpha + \beta)b},$ $z^* = \frac{a\beta - (cd + bE)}{(\alpha + \beta)c}$	$\lambda^3 + B\lambda + C = 0,$ where $B = b^2x^*y^* + c^2x^*z^* - \alpha\beta y^*z^*,$ $C = -bc(\alpha + \beta)x^*y^*z^*$	The closed-form of roots is not straightforward to compute.	This equilibrium point exists and is unstable for all values of $E$ .						

Table 4: The table presents a summary of the equilibria of model (F) and their stability in the case there is unequal competition interference between predators,  $\alpha \neq \beta$ , and when harvesting of the catfish happens  $E = e - h_2$ .

in the case when  $\alpha > \beta$ , that is, the catfish is a stronger competitor than the striped bass as shown in panel (b). If we assume that the catfish is less competitive or less aggressive than the striped bass  $\alpha < \beta$  then as shown in panel (c), the catfish becomes extinct while the menhaden and the striped bass co-exist. Furthermore, in this case, we observe periodic shifted behavior between menhaden and bass that is typically found in two-species predator-prey models.

However, in reality the blue catfish is much more competitive in the following sense. According to [6], about 75% of all fish biomass in the Virginia Coastal waters consists of the blue catfish and the catfish actually also preys on many other fish species, not just the menhaden. Hence, it is biologically more realistic to assume that  $\alpha > \beta$ . Our model (see Figure 5 (b)) predicts that ultimately, the blue catfish grows in an unbounded manner compared with the native species, the striped bass and the Atlantic menhaden. If no serious measures are taken to prevent the widespread of the invasive blue catfish, the native species will be seriously affected and may even become extinct.

Now, panel (d) in Figure 5 presents a simulation of the time-evolution of the three species' populations when there is harvesting of the catfish with the more realistic case of  $\alpha > \beta$ . If the harvesting term is assumed to be a linear rate  $h_2z$ , and the harvesting parameter  $h_2$  is high enough, then it is possible for the catfish and the menhaden to

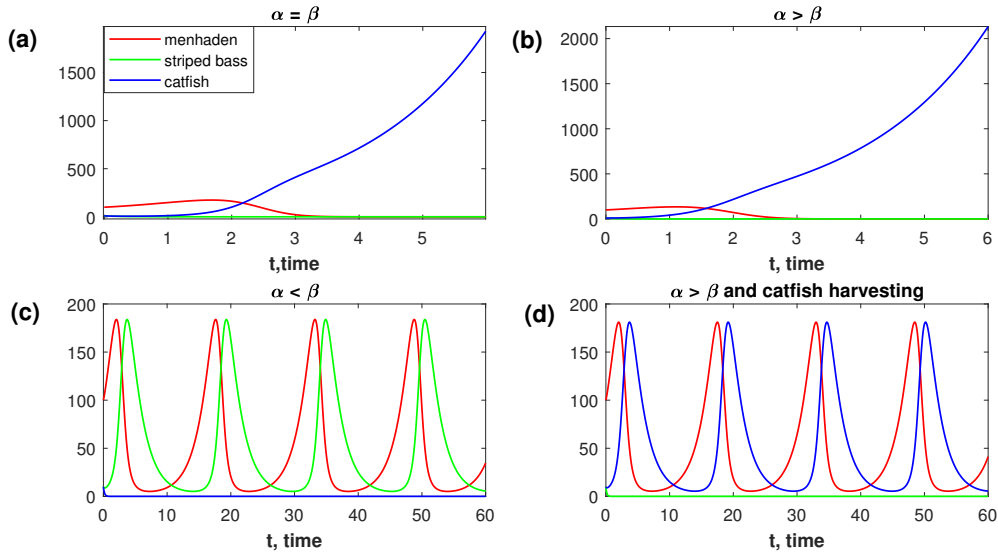


Figure 5: Menhaden, striped bass, and catfish distributions over time for the following values of the parameters:  $a = d = e = 0.5$ ,  $b = c = 0.01$  with initial conditions  $(100, 10, 10)$ . Panels (a)  $\alpha = \beta = 0.5$ , (b)  $\alpha = 0.8$  and  $\beta = 0.2$ , (c)  $\alpha = 0.2$  and  $\beta = 0.8$ , (d)  $\alpha = 0.8$ ,  $\beta = 0.2$  and harvesting parameter of the catfish is  $h_2 = 1$ .

co-exist, and they will exhibit the typical predator-prey phase-shifted periodic behavior. However, the striped bass dies out.

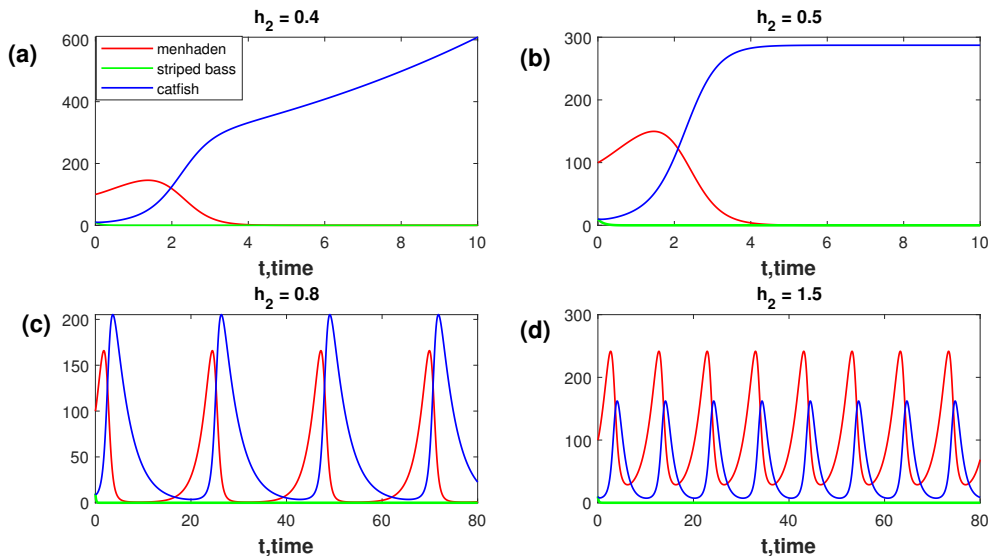


Figure 6: Time-evolutions on the population of menhaden, striped bass, and catfish when there is harvesting of catfish. The following values of the parameters were used:  $a = d = e = 0.5$ ,  $b = c = 0.01$ ,  $\alpha = 0.8$ ,  $\beta = 0.2$  with initial conditions  $(100, 10, 10)$ . Panels (a)  $h_2 = 0.4$ , (b)  $h_2 = 0.5$ , (c)  $h_2 = 0.8$ , (d)  $h_2 = 1.5$ .

## 6.2 Further numerical investigations: harvesting of aggressive catfish

Continuing with the more biologically realistic assumption that  $\alpha > \beta$ , Figure 6 shows the effect of harvesting on the catfish. Here, we would like to investigate if harvesting is able to control the invasive effect of the catfish on the Chesapeake Bay. In these simulations, we will keep  $e = 0.5$  constant, where  $e$  is the growth parameter for the catfish.

Panel (a) shows that a harvesting rate  $h_2 = 0.4$  is not sufficient to control the population growth of the catfish. Panel (b) shows that with a harvesting rate of  $h_2 = 0.5 = e$ , the (blue) population curve for the catfish exhibits bounded growth. Thus, we say that in this case, harvesting is able to control the growth of the catfish. Unfortunately, the menhaden eventually dies out. The second predator (the striped bass) dies even before the menhaden, due to the decreasing population of its prey and the aggressive behavior of its competitor.

Now, what happens when the harvesting rate of the catfish exceeds its growth rate, that is, when overfishing of the catfish happens? Panels (c) and (d) show two cases, with  $h_2 = 0.8$  and  $h_2 = 1.5$  respectively.

Panel (c) shows that with aggressive harvesting  $h_2 = 0.8 > 0.5 = e$ , a periodic phase-shifted behavior becomes evident for the catfish and its menhaden prey. The striped bass, which is a second predator and also a competitor to the catfish, dies out.

Panel (d) shows that with even more aggressive harvesting  $h_2 = 1.5 > 0.5 = e$ , a periodic phase-shifted behavior also occurs between the predator and prey. The striped bass also dies out in this case. Hence, we can conclude that it is possible to control the exponential growth of the invasive species with aggressive harvesting.

Finally, observe that between panels (c) and (d) in Figure (6), we notice two interesting observations as  $h_2$  increases, that is, when harvesting of catfish becomes more aggressive: (1) the local maximum values of the menhaden exceed the local maximum values of the catfish; and (2) the periods for both menhaden and catfish as they exhibit their phase-shifted predator-prey behavior decrease. These observations may give rise to further inquiries on how aggressive harvesting affects the resulting predator-prey relationship between menhaden and catfish.

## 6.3 Further numerical investigations: harvesting of the catfish with equally competitive striped bass

Another interesting long-term behavior of the three fish species in this project is visualized in Figure 7. Here, we consider the case when the competition effects are equal on both predators. Also, we assume that there is harvesting of the catfish. The values for  $h_2$  are  $h_2 = 0.4, 0.5, 0.8$ , respectively for panels (a), (b), (c) in Figure 6 and Figure 7. Comparing the graphs, we observe similar time-evolution behaviors among these three species! This means that when the competition effect of the catfish is at least as strong as the striped bass  $\alpha \geq \beta$ , the bass loses and dies out even when harvesting of the catfish happens up to a certain point.

Panel (d) in Figure 7 illustrates a very different scenario. When the competition strength between the striped bass and the catfish are equal, and when the harvesting rate  $h_2 = 1.2 > 0.5 = e$  is big enough, the model shows that catfish can be driven to extinction while the menhaden and striped bass persist with a regular predator-prey relationship.

In conclusion, this model predicts the extinction of one of the two predators depending on their competitiveness. These results are consistent with the fundamental ecological principle of competitive exclusion, also called the Gause principle, which states that when two competing species that attempt to occupy the same niche in an ecosystem can not co-exist indefinitely, one population will either go extinct or will adapt to fill a different niche [4].

## 7 Writing and assessment of the project

Preparing a mathematical report is an important skill for STEM students. The report must be accurate, well-structured, and thorough, must look professional, and must not be presented like a typical homework assignment that students complete on a daily basis. The readers of the report and the audience of the presentation should be able to understand the report without being burdened by mathematical computations. It can even be said that creating the report is almost as important as the analysis that goes into the process of creating the model and solving the problem.

In this section, we include suggestions on how the mathematical report may be prepared by students and how their work may be assessed by a grading rubric.

### Mathematical report

The mathematical report may be divided into the following sections:

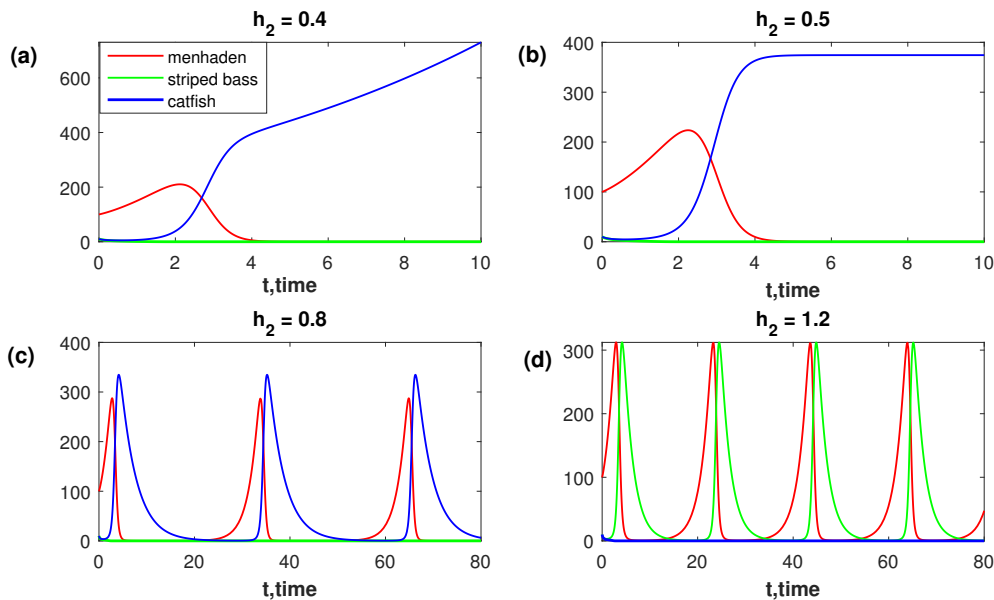


Figure 7: Menhaden, striped bass, and catfish population distributions over time when a linear harvesting term  $-h_2z$  of the blue catfish is included in the model. The following values of the parameters were used:  $a = d = e = 0.5$ ,  $b = c = 0.01$ ,  $\alpha = \beta = 0.5$  with initial conditions (100, 10, 10). Panels (a)  $h_2 = 0.4$ , (b)  $h_2 = 0.5$ , (c)  $h_2 = 0.8$ , (d)  $h_2 = 1.2$ .

1. **Statement of the problem and existing data.** This section contains the natural phenomenon being studied, the significance of the project, and all existing qualitative and quantitative data.
2. **Mathematical model.** This section identifies the variables, the parameters, and the differential equations used to model the problem. It also contains the assumptions and the rationale for them. It should explain each component of the differential equations.
3. **Mathematical analysis.** This section contains some mathematical computations that lead to the specific answers to the questions or scenarios proposed in the given problem. Computational steps may be skipped if necessary, but every key step must be explained.
4. **Numerical simulations.** This section contains the numerical discoveries and analysis through visualizations. The graphs should be explained clearly.
5. **Conclusions.** This section summarizes the final results. It should contain the answers to the motivating problem, and it must be presented in the language that is relevant to the phenomenon being studied.

When presented to clients or to an audience who are not necessarily mathematically-trained in differential equations, the project must be presented in a language that can be understood. The report may have informal language to help develop intuition and understanding.

## Grading rubric

The students' work on this project may be graded with respect to the following tasks:

1. Describe the observed phenomenon in Chesapeake Bay as reported in media, websites by environmental agencies, and/or government offices.
2. Enumerate and distinguish assumptions and simplifications.
3. Define the variables and parameters. Create the system of ordinary differential equations.
4. Solve for the equilibrium points. Analyze the linear stability of each equilibrium point.
5. Create time-evolutions and phase portraits using a numerical and visualization software such as MATLAB.
6. Interpret theoretical results, graphical simulations, and numerical computations.

## 8 Conclusion

Inquiry-based projects enable students to actively discover information while the instructor guides students through posing research questions, designing methods, and interpreting data with the main goal of helping students develop critical thinking and deeper understanding of the material. In this paper, we present an inquiry-based project that is designed for a mathematical modeling class of undergraduate junior or senior students. This project discusses a three-species mathematical model that simulates the biological interactions among three important fish species in the Chesapeake Bay: the prey Atlantic menhaden and its two competing predators: the striped bass and the blue catfish. The model also considers the following ecological issues related to these three species: the overfishing of menhaden, the invasiveness of the blue catfish, and the harvesting of blue catfish as a method to control the population. This model builds naturally upon the classical Lotka-Volterra prey-predator models and allows students to apply their theoretical knowledge from previous classes in calculus, linear algebra, and differential equations for investigating real-world ecological problems.

We use a series of modified modeling scenarios based on some simplifying assumptions to demonstrate the application of theoretical concepts to actual fisheries in the Chesapeake Bay. We start with a three-species system (B) including prey menhaden  $x(t)$  and two competing predators striped bass  $y(t)$  and blue catfish  $z(t)$  and assume no harvesting effect and equal competition interference between  $y$  and  $z$ . Then we build this system up by including harvesting terms on the menhaden (C) and harvesting terms on the catfish (D). Then we consider a case when competition interference is not equal (E) and finally, a system that has unequal competition with overfishing of the catfish (F).

To study the local stability of the equilibrium points to non-linear systems of differential equations, the linearization theory and the Hartman-Grobman theorem were used through algebraic, analytic, and numerical methods. In particular, when solving the characteristic equation was not straight-forward, the Routh-Hurwitz criteria was applied. It was found that even though the co-existence equilibrium (when all three species exist together), is possible if  $\alpha > \alpha_{\text{crit}}$  for some critical value  $\alpha_{\text{crit}} = (cd + be)/a$ , this equilibrium is highly unstable. If one species exceeds its critical size or fails to achieve it, then the model predicts that at least one species will die out.

Numerical simulations via MATLAB were utilized to study the long-term behavior of the three fish population dynamics. All three case scenarios representing the competitiveness between the two predators  $\alpha < \beta$ ,  $\alpha = \beta$ , and  $\alpha > \beta$  were analyzed. The model shows that in the long run if  $\alpha = \beta$  (or the two predators are equally competitive), the catfish, as an invasive species, would persist while the menhaden and the striped bass would become extinct. If the catfish is assumed to be less competitive, or the case  $\alpha < \beta$ , then catfish become extinct while the menhaden and the striped bass co-exist with the typical phase-shifted periodic behavior. The model shows that this is the only scenario in which striped bass could survive. However, in reality, the catfish is much more competitive ( $\alpha > \beta$ ) as it preys on a variety of fish and makes up almost 75 percent of all fish biomass in the Chesapeake Bay's fisheries. Hence, the model predicts the unlimited growth of the catfish and the extinction of both native species: the menhaden and the striped bass.

The model shows that linear harvesting is sufficient to limit the growth of the invasive

catfish population, provided that the competition between the striped bass and the catfish is strong enough. Moreover, if the harvesting coefficient is sufficiently large, it is possible for the catfish and the menhaden to co-exist with phase-shifted periodic behavior but the striped bass becomes extinct.

In either case, the model predicts extinction of one of the two predators depending on their competitiveness. These results illustrate the fundamental ecological principle of competitive exclusion according to which two competing species that attempt to occupy the same niche in an ecosystem cannot co-exist indefinitely and one of the two populations will either go extinct or will adapt to fill a different niche.

## References

- [1] V.I. Arnold. *Ordinary Differential Equations*. Springer-Verlag, 1992.
- [2] ASMFC Fisheries Focus. ASMFC approves resolution on non-native invasive catfish. Available at <http://www.asmfc.org/uploads/file/septOct2011.pdf> (Page 11, 2011/09/02), 2011.
- [3] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer, 1983.
- [4] G. Hardin. The competitive exclusion principle. *Science, New Series*, 131(3409):1292–1297, 1960.
- [5] M. Olinick. *Mathematical Modeling in the Social and Life Sciences*. Wiley, 2014.
- [6] D. Orth, Y. Jiao, J. Schmitt, and C. Schmitt. Dynamics and role of non-native blue catfish *ictalurus furcatus* in Virginia’s tidal rivers final report, December 2017.
- [7] L. Perko. *Differential Equations and Dynamical Systems*. Springer-Verlag, New York, 2001.
- [8] T. B. Wheeler. Virginia menhaden season starts with reduced harvest cap for Bay. Available at <https://tinyurl.com/wyp959m>, retrieved March 13, 2021., May 2020.
- [9] J. L. Willems. *Stability Theory of Dynamical Systems*. Wiley, New York, 1970.



# *Qualitative Analysis of a Resource Management Model and Its Application to the Past and Future of Endangered Whale Populations*

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**Keywords:** Resource management, consumer-resource model, predator-prey model, phase line, whale populations, bifurcation

Manuscript received on September 9, 2020; published on March 15, 2021.

**Abstract:** Observed whale dynamics show drastic historical population declines, some of which have not been reversed in spite of restrictions on harvesting. This phenomenon is not explained by traditional predator prey models, but we can do better by using models that incorporate more sophisticated assumptions about consumer-resource interaction. To that end, we derive the Holling type 3 consumption rate model and use it in a one-variable differential equation obtained by treating the predator population in a predator-prey model as a parameter rather than a dynamic variable. The resulting model produces dynamics in which low and high consumption levels lead to single high- and low-level stable resource equilibria, respectively, while intermediate consumption levels result in both high and low stable equilibria. The phase-line analysis is made more transparent by applying a particular structure to the function that gives the derivative in terms of the state. By positing a consumption level that starts low, gradually increases through technological change and human population growth, and decreases as a result of public policy, we are able to tell a story that explains the unexpectedly rapid decline of some resources, such as whales, followed by limited recovery in response to conservation. The analysis also offers guidelines for how to establish sustainable harvesting for restored populations. We include a bifurcation analysis and suggestions for how to teach the material with three different levels of focus on the modeling aspect of the study.

## 1 Introduction

Case studies play an important role in student learning by embedding the mathematics into non-mathematical contexts. The more interesting the context is to students, the more engaged they will be in the case study. The example presented here grew out of my interest in constructing a case study for a first-order differential equation that would

use phase-line analysis to obtain qualitative behavior for a model that many students would find interesting. phase-line analysis is (or at least can be) presented early in a differential equations course, which allows the instructor to assign the case study before students reach the point of being overwhelmed by a full schedule. A secondary point of the case study is that it shows the relatively limited value of analytical solution methods for anything more complicated than the linear equations of radioactive decay and mechanical vibration. While an implicit solution formula can be found for any autonomous first-order equation, such a formula is not as useful as phase-line analysis for obtaining qualitative results nor as easy to use for solution graphs as numerical methods.

The key insight that led to the case study presented here is that any predator-prey model can be recast as a resource management model by assuming the ‘predator’ density is a control parameter rather than a dependent variable, as is the case where the number of harvesters in a system is controlled by policy rather than by population dynamics. It was only after doing the analysis that I saw that the results could be incorporated into a narrative setting involving the history of whale populations. There is insufficient data to determine accurate parameter values; instead, the goal of the investigation is to formulate a theory that can potentially explain the qualitative features of the observed population history.

Just what these qualitative features are is not completely clear. Several dates stand out as important in the history of whale populations [9]. Commercial whaling hit a peak in the early 1900s roughly between 1904 and 1916 and subsequently declined because of diminished stocks. The first attempt to regulate whaling was the International Agreement for the Regulation of Whaling of 1937. This agreement made very little impact, but was followed by a set of international regulations approved by the newly formed International Whaling Commission in 1946. In spite of the new regulations, whale populations continued to decline, leading to a public “Save the Whales” campaign instituted by the World Wildlife Fund in 1961. This exerted public pressure on countries to limit whaling, but the effect was still not enough to significantly increase whale populations. In 1986, the International Whaling Commission placed a moratorium on commercial whaling, which was largely successful, although it did not eliminate subsistence whaling or so-called ‘scientific’ whaling. This moratorium has led to significant recovery of some species, but its impact going forward may be limited since Japan withdrew from the commission in 2019.

Data on whale populations prior to 1986 is scarce, but there is a small amount of useful, if incomplete, information from whale catch data. Some species experienced sharp increases in hunting followed by sharp decreases, while others experienced gradual increases followed by sharp decreases [2]. Most whale populations remain severely depleted, but there are isolated examples of significant recovery.

1. Estimates based on mathematical modeling suggest that the historical population of New Zealand right whales was somewhere between 29,000 and 47,000. Hunting in the 19th century depleted the population to a mere 120 by the beginning of the 20th century. The current population estimate of about 2,200 shows that the population is recovering, but still dangerously low [5].
2. Annual catches of humpback whales in the west South Atlantic breeding group peaked at about 12,000 in 1910 before falling sharply to 2,000 in 1913. Modeling

estimates suggest that the population dropped below 1,000 by about 1915 and stayed at that level for several decades. Increased hunting in 1950 reduced the population in a single year to near extinction. A slow recovery began in the 1970s with accelerating growth after the International Whaling Commission’s 1986 moratorium. Data from 2008 and 2012 showed a population approaching 20,000 [10].

Because there is no clear data on whale populations during the population declines, we focus here on using a model to explain qualitative features that are at least consistent with the limited historical record. In particular, it seems likely that there are populations that experienced a rapid decline while being subjected to only a modest increase in hunting and have remained depressed in spite of conservation efforts. This scenario is what we will explain with our model.

We begin by developing a resource management model in Section 2. We scale the model in Section 3 and analyze it in Section 4. The association of the model with the context of our whale population scenario appears in Section 5, followed by a discussion of further work and options for teaching the model in Section 6. All of these topics will be at least somewhat novel for most mathematicians. The model we present in Section 2 is similar to standard models for predator-prey dynamics, but the Holling type 3 component is largely unknown outside of the mathematical ecology community. Preparation for analysis by scaling does not appear in any currently available differential equations book as far as I know, although it does appear in a mathematical biology book [6]. Even the phase-line analysis in Section 4 will be novel to nearly all readers, as it is based on recasting the differential equation using a structure that greatly simplifies the analysis.

## 2 Derivation of A Resource Management Model

Mathematical models for predator-prey systems and consumer-resource systems are identical. While the predator-prey terminology is more common, the consumer-resource terminology is more appropriate in the context of resource management. Hence, we begin with a system consisting of a dynamic resource level  $X(T) \geq 0$  and a dynamic consumer level  $Y(T) \geq 0$ , where  $T$  is time.<sup>1</sup> The dynamic variables are usually thought of as population sizes, but in many cases it is more biologically correct to think of them as biomasses [3], the idea being that predators will eat two half-grown fish as a substitute for one full-grown fish. Thus, we think of  $X$  as the biomass of a resource (whales) and  $Y$  as representing consumers (people) in terms of units based on the total potential consumption (so that two half-size fishing boats are equivalent to one full-size boat). In its most general form, we have the system

$$\frac{dX}{dT} = G(X) - YH(X), \tag{2.1}$$

$$\frac{dY}{dT} = EYH(X) - MY. \tag{2.2}$$

In this formulation,  $G(X)$  represents the growth rate of the resource in the absence of consumers and  $H(X)$ , known to biologists as the ‘functional response’, represents the

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<sup>1</sup>The reason for using  $T$  rather than  $t$  for time will become apparent in Section 3.

harvesting rate per unit of consumers at a given resource level. The parameter  $M > 0$  is the natural decay rate constant of consumer units over time, while the parameter  $E > 0$  is an efficiency of conversion from total harvested resource amount to units of consumers. Various growth and harvesting functions can be chosen, subject to reasonable restrictions. We assume that there is no growth when the resource level is 0 but that there is positive growth for resource levels sufficiently small; thus,  $G(0) = 0$  and  $G'(0) > 0$ . The harvesting rate should be an increasing function of resource level, but 0 if there is no resource; thus,  $H(0) = 0$  and  $H' \geq 0$ .

The simplest and most common example of a consumer-resource model is the Lotka-Volterra model, obtained by choosing linear functions  $G(X) = RX$  and  $H(X) = SX$ . This widely used model is actually a very poor choice for predator-prey or consumer-resource interactions (see Section 6.3), so we will instead develop our own choices based on plausible biological assumptions.

## 2.1 Growth and Harvesting Functions

The natural choice for a growth function is

$$G(X) = RX \left(1 - \frac{X}{K}\right), \quad (2.3)$$

which prescribes logistic growth with maximum rate  $R$  and carrying capacity  $K$ . Our resource will then exhibit logistic growth in the absence of consumers.

The obvious choice for the harvesting function  $H(X)$  is the linear function  $H(X) = SX$ , which is used in the Lotka-Volterra model and a number of other models. It seems reasonable at first thought that the harvesting rate per consumer should be proportional to the resource population. However, this model predicts that there is no limit to how large the harvesting rate can be. Clearly if the resource level is enormous, a consumer can only harvest so much. If you catch a fish every time you cast your line into the water, then you will spend nearly all of your time reeling in the fish, taking it off the hook, storing it, and preparing for your next attempt. Increasing the number of fish in the lake beyond this point is not going to help you catch fish faster. This saturation effect had been observed in predation data, but it remained empirical until given a mechanistic derivation in the 1950s by C.S. Holling [4]. Ecologists today associate Holling's name with a collection of three related functional response models, starting with the linear model, known as *Holling type 1*. Because the choice between the three models is critical, we present here the derivation of the Holling type 2 and type 3 models.

The key insight needed for the mechanistic derivation of both the Holling type 2 and type 3 models is that consumers must divide their time between searching and processing. Harvesting of a resource only occurs during time spent searching, so if the resource discovery rate is  $SX$  and  $T_s$  is the fraction of time spent searching, then the total harvest rate per unit time is

$$H(X) = T_s SX. \quad (2.4)$$

If we assume all harvesting time is spent searching, then  $T_s = 1$ , and we get the linear model. If some time is spent processing the harvest, then the fraction of time spent in

searching is dependent on the harvest rate, which means that we need another equation to supplement (2.4).

Assume that each unit of harvest requires processing time  $B$ . Since consumption occurs at the rate  $H$  units per total time, the associated processing time is  $BH$ . One unit of total time is then partitioned between processing time and searching time; thus,

$$T_s + BH = 1. \quad (2.5)$$

Combining (2.4) with (2.5) yields the Holling type 2 model

$$H(X) = \frac{SX}{1 + BSX}. \quad (2.6)$$

As written, the model uses the biological parameters  $S$  and  $B$ . These are very difficult to measure, so it is more common to do experiments to measure values of  $H$  for various  $X$ . The model can then be parameterized from data, for which purpose it is best to rewrite it in terms of empirical parameters. If we define  $Q = 1/B$  and  $A = 1/BS$ , the model takes the form

$$H(X) = \frac{QX}{A + X}, \quad (2.7)$$

where  $Q$  is the asymptotic limit of consumption per consumer unit and  $A$  is the level of resource for which consumption will be half of the maximum. This is the more common form of the Holling type 2 model.

Most ecological models use either Holling type 1 or type 2. The simplicity of the linear type 1 model makes it preferable for any biological system in which resources are scarce and consumers really do spend most of their time searching rather than processing. Such systems are relatively uncommon, as it is hard for organisms to survive if food is so scarce that they are almost constantly looking for it. Thus, the type 2 model is the one that is usually used in theoretical ecology. However, there is still a problem with the type 2 model that deserves attention. It is appropriate for specialist consumers, who are reliant on one specific resource. Generalist consumers have other options; as an example, bears eat a variety of foods and will decrease the amount of effort they put into fishing during periods when fish are scarce. The Holling type 3 model is designed for cases such as this. Instead of using a search rate  $S$  that is independent of the availability of the resource, suppose we make the search rate proportional to the resource level, with maximum rate  $S$  for a resource level at the carrying capacity  $K$ . This assumption roughly corresponds to the idea that the consumer is dividing its time between different resources and will do more searching for a plentiful resource than a scarce one. Replacing  $S$  with  $SX/K$  in the type 2 mechanistic model (2.6) yields the mechanistic version of the type 3 model:

$$H(X) = \frac{SX^2}{K + BSX^2}. \quad (2.8)$$

As with type 2, it is common to write the model using empirical parameters that can be determined from data rather than biological parameters that are hard to measure. With new parameters defined by  $Q = 1/B$  and  $A^2 = K/BS$ , we obtain the graphical version,

$$H(X) = \frac{QX^2}{A^2 + X^2}. \quad (2.9)$$

The left panel in Figure 1 compares the graphical versions of the Holling type 2 and type 3 models. The type 3 model is initially concave up for small resource levels, which leads to different long-term behavior than would be the case with Holling type 2.

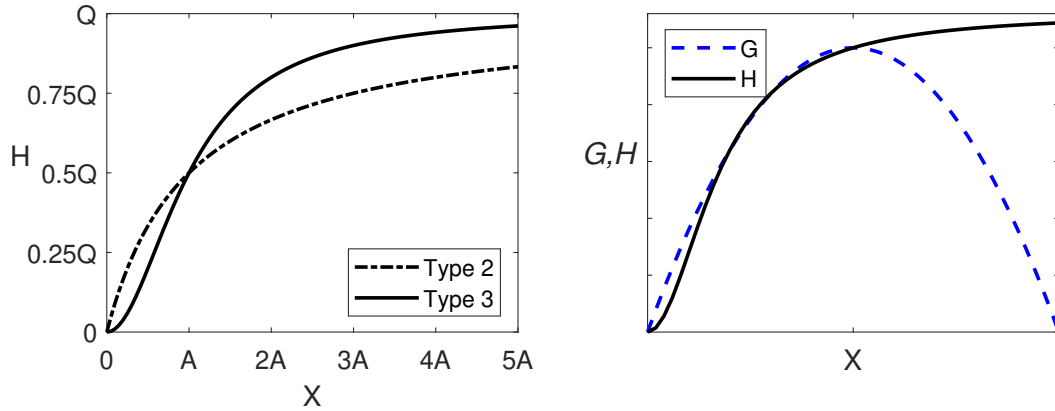


Figure 1: Left: The Holling type 2 and 3 functional responses. Right: A comparison of logistic growth and Holling type 3 harvesting functions. With the given parameter values, the  $G$  and  $H$  curves cross at three points, resulting in different system behaviors for different ranges of  $X$ .

## 2.2 The Resource Management Model

As presented in the usual way (Equations 2.1–2.2), we have a dynamical system of two interacting populations. The consumer level responds to changes in the resource level because harvesting leads to growth in the consumer population. But suppose the consumers are not a biological population, but rather a community of harvesters within a population, such as the community of fishermen within a human population. Then it makes sense to think of the number of harvesters as a parameter determined by public policy rather than a population whose size is determined by population growth. The result is a modified prey equation, with the dynamic variable  $Y$  replaced by a parameter  $C \geq 0$ . In this sense, the model maps the parameter  $C$  to a 1-variable dynamical system for  $X$ , suggesting questions about how the behavior of that system with given functions  $G$  and  $H$  depends on the parameter.

Using the mechanistic version of the Holling type 3 model, we obtain the complete resource management model

$$\frac{dX}{dT} = RX \left( 1 - \frac{X}{K} \right) - C \frac{SX^2}{K + BSX^2}. \quad (2.10)$$

It would make sense to use the graphical version if we had empirical data to help determine  $Q$  and  $A$ . The mechanistic model is preferable here because it facilitates biological interpretation.



Although our model is merely a one-dimensional dynamical system, it is capable of rich behavior. This is suggested by the right panel of Figure 1, which compares the growth and harvesting functions, with parameters carefully chosen to indicate that the location of points where  $G = H$  can be very sensitive to the parameter values in the functions.

### 3 Scaling

Model (2.10) has five parameters (although it could be thought of as having just 4 parameters by using  $CS$  and  $BS$  instead of  $C$ ,  $B$ , and  $S$ ). This is not an issue if our interest is limited to studying the model with one or two sets of known parameter values, but it is far from ideal if our goal is to characterize the full range of model behaviors. We can address this issue by scaling the model; that is, by replacing the dimensional variables  $X$  and  $T$  with dimensionless counterparts that are ratios of the dimensional quantities to carefully chosen representative values. Scaling is ubiquitous in some areas of mathematical modeling, such as fluid mechanics, while being relatively uncommon in population dynamics. This is in part because the choices of scales and dimensionless parameters in population dynamics are subtle (see [6] for a gentle introduction to scaling in population dynamics and [7] for a more thorough guide).

The choice of scales for a model is often tricky, but this is not the case here. We are thinking of scenarios in which the consumption parameters  $C$ ,  $S$ , and  $B$  change gradually over time, so it is best to use scales that come from the growth term rather than the harvesting term. This means we should choose  $K$  for the reference resource level and  $1/R$  for the reference time.

While scaling is usually thought of as a process of constructing new dimensionless variables, I find it helpful to think about it as a process of factoring a dimensional variable into the product of an appropriate dimensional constant and a dimensionless magnitude. Several notation conventions are in use for connecting dimensional variables to their dimensionless counterparts; I find the simplest to be upper case Latin letters for dimensional quantities and lower case Latin letters for dimensionless ones. The minor inconvenience of using  $T$  for dimensional time adds far less confusion than the corresponding inconveniences of other notational systems. Thus, we factor the variables  $X$  and  $T$  in terms of scale factors  $K$  and  $1/R$  and dimensionless variables  $x$  and  $t$  as

$$X = Kx, \quad T = R^{-1}t. \quad (3.1)$$

Making these substitutions and rearranging factors yields

$$x' = x(1-x) - \frac{CS}{R} \frac{x^2}{1 + BSKx^2},$$

with the prime symbol indicating the derivative with respect to dimensionless time.

Notice that the five dimensional parameters have naturally sorted themselves into two dimensionless groupings, allowing us to rewrite the model in terms of two dimensionless parameters. There are multiple ways to do this [7], and the best choice is often clear only after doing the analysis. Our choice here is to factor  $BSK$  out of the denominator to get

$$x' = x(1-x) - \frac{cx^2}{p+x^2}, \quad p = \frac{1}{BSK}, \quad c = \frac{C}{BRK}. \quad (3.2)$$

It is helpful to examine the specific dimensionless parameters to identify a biological interpretation. The quantity  $1/B$  is the rate at which a harvested resource can be processed, while the quantity  $SK$  is the rate of resource discovery when the resource is at its carrying capacity; thus, the parameter  $p$  can be interpreted as the ‘processing to discovery’ ratio. We will focus on small values of this ratio, corresponding to large natural populations. The maximum growth rate of the resource is  $RK/2$ , which is enough production to fully support  $BRK/2$  units of consumers, given that each consumer requires  $B$  units of time to process one unit of resource. A value  $c = 0.5$  therefore means that the maximum productivity is just adequate to support the number of consumers present. Clearly values of  $c$  at or above 0.5 will deplete the resource. Of course the actual results will be more subtle than this, but it is helpful to have some sense of what to expect prior to doing the analysis.

## 4 Analysis

The first step in an analysis plan for modeling is deciding what questions to address. It helps to think of a model as a mapping from the parameter space to some desired outcome. In dynamical system modeling, we typically look for stable equilibrium solutions and their domains of attraction. Our model has two parameters, one that represents the consumption effort ( $c$ ) and one that combines the biological characteristics of the growth and hunting processes ( $p$ ). Over the course of historical time, search speed has increased while handling time has decreased. These trends tend to result in less long-term change in the product  $SB$  that appears in the definition of  $p$ . It is therefore not unreasonable to assume that a particular whale community will have some fixed value of  $p$ , but that these will differ among species and regions. We therefore ask the following question:

- For any given value of  $p$ , how does the pattern of stable and unstable equilibria depend on the parameter  $c$ ?

As a case study for students, more than one variable parameter means there is too much detail; hence, we assume a specific value  $p = 0.01$ . This value is chosen in hindsight because it is in the range where the most interesting behavior occurs. The influence of  $p$  is addressed in Section 6.

There are four types of analysis that we could consider doing with a continuous dynamical system: finding an analytical solution, using the phase-line to determine equilibria and stability, using linearized stability analysis to determine stability, and running simulations. Linearized stability analysis is not as good as phase-line analysis, both because it is more work and because phase-line analysis yields global as well as local stability. Simulations can be run using any numerical differential equation solver, but one can only do examples rather than general cases. This is warranted when trying to match a real data set. These considerations suggest we focus on finding an analytical solution and doing phase-line analysis.



## 4.1 An Implicit Solution Formula

Textbooks often emphasize analytical solution methods. While such methods have their place, it is easy to be misled by the emphasis placed on them into thinking that they have value for a large class of problems. Our model (3.2) is first-order and autonomous, which means that it can be solved by separation of variables. However, the solution has to be expressed implicitly using a definite integral:

$$t = \int_{x(0)}^x \frac{(p + y^2) dy}{y[1 - (c + p)y + y^2 - y^3]}. \quad (4.1)$$

This solution has no practical value, whether for our current investigation or a different one. Using it to determine long-term behavior of solutions would require identifying the range of upper bounds (both less than and greater than  $x(0)$ ) for which the integral converges, which is a daunting problem. Given an initial condition and parameter values, we could plot graphs of the solution by identifying the time corresponding to any particular  $x$ ; however, this requires a numerical method to compute the integral for each point, along with additional code to make sure the particular  $x$  chosen for a given point is one for which the integral converges. It requires far less numerical work to use a numerical differential equation solver on the initial value problem.

## 4.2 The Phase-line

The idea of phase-line analysis is simple: plot a graph that can be used to break the  $x$  axis into regions where  $x$  is increasing and regions where it is decreasing, then use this information to plot an  $x$  axis with arrows showing which equilibria are stable and which are unstable. The naive way to do this is to use a graph of  $x'$  vs  $x$  to determine when  $x$  is increasing. This method appears in nearly all differential equations books written since the initiation of the calculus reform movement around 1990. In the case of our model, the graph of the function

$$x(1 - x) - \frac{cx^2}{p + x^2}$$

depends on the parameters  $p$  and  $c$  in complicated ways. Even with a single fixed value of  $p$ , the analysis is entirely procedural because we can only obtain the graph for specific values of  $c$ .

As an alternative to the naive approach, we can use an approach that preserves our intuition by imposing a structure on the function in the differential equation. The plan is to write the formula for  $x'$  using the structure

$$x' = f(x)[g(x) - h(x)], \quad f(0) = 0, \quad f' > 0, \quad (4.2)$$

where  $fg$  is the growth rate and  $fh$  is the harvesting rate. These requirements do not yield a unique factorization; for example, we could choose

$$x' = x \left[ (1 - x) - \frac{cx}{p + x^2} \right] \quad (4.3)$$

or

$$x' = \frac{x}{p + x^2} [(1 - x)(p + x^2) - cx]. \quad (4.4)$$

Before choosing which factorization to use, it helps to understand how the functions  $f$ ,  $g$ , and  $h$  will be used. For any factorization (4.2), we may draw the following conclusions:

1.  $x = 0$  is an equilibrium point for the differential equation.
2. All equilibria other than  $x = 0$  are points where the graphs of  $g$  and  $h$  intersect.
3. The state variable  $x$  is increasing whenever the graph of  $g$  is above the graph of  $h$  and decreasing whenever the graph of  $g$  is below that of  $h$ .

If possible, the choice of factoring should be made so that the graphs of  $g$  and  $h$  can be sketched by hand. Neither of the options works for both  $g$  and  $h$ , but option (4.4) is more attractive because our principal parameter  $c$  appears only in the simpler function. For any fixed  $p$ , the nonlinear function  $g$  needs to be graphed just once and we can superimpose multiple plots of the linear function  $h$ .

### 4.3 Phase-line Analysis of the Resource Model

The top left plot of Figure 2 shows the graph of the nonlinear function  $g(x)$  with  $p = 0.01$  along with lines of three different slopes. While the graphs of  $g$  and  $h$  are not the growth and harvesting functions, they nevertheless *represent* these functions in a relative sense, that being given by conclusion 3 above.

When  $c = 0.12$ , there is one positive equilibrium, at a value not much less than the environmental capacity 1. This equilibrium is identified by the intersection of the curve and the line and marked as a disk on the  $x$  axis. The point  $x = 0$  is also marked as an equilibrium because this is built into the structure (4.2). For populations between these two equilibrium values, the curve  $g$  is above the line  $h$ . This means that growth outstrips consumption and the population increases, as marked by the arrow pointing to the right. To the right of the positive equilibrium, the graphs are reversed, so the population decreases. The arrows show that the positive equilibrium is globally asymptotically stable, while the extinction equilibrium is unstable. Given  $p = 0.01$ , the specific case  $c = 0.12$  is representative of a range of ‘small’ values of  $c$ . Consumption is relatively low and the stable equilibrium population is relatively high.

When  $c = 0.36$ , as seen in the lower right panel, the situation is similar, except that the consumption level is sufficiently large that the single positive equilibrium is at a very small value of the resource. It is still true that the curve is above the line for values of  $x$  between the two equilibria and below the line when  $x$  is to the right of the positive equilibrium. As before, the positive equilibrium is globally asymptotically stable, and the extinction equilibrium is unstable. The difference is that a moderate value, like  $x = 0.4$ , is now in the ‘large  $x$ ’ region where the curve is below the line. The number of consumers is high, but the actual amount of consumption is not, since the consumption is given as  $fh$  and both  $f$  and  $h$  are small.

The lower left panel shows the more interesting case of an intermediate value of  $c$ . Here there are three positive equilibria, which partition the  $x$  axis into four regions. In

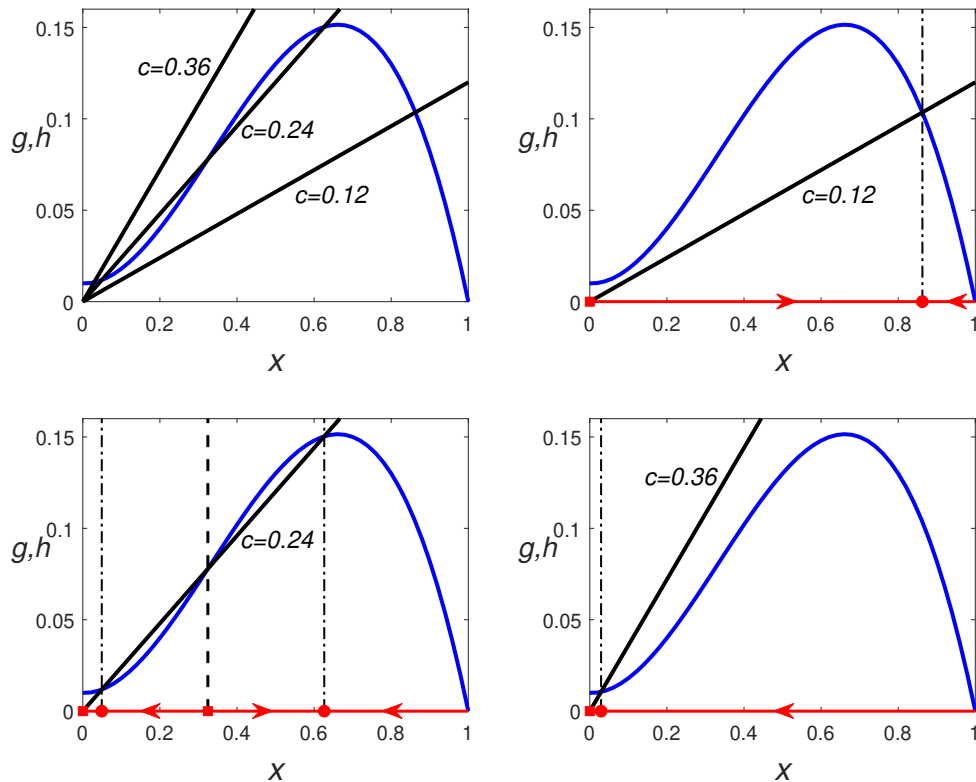


Figure 2: Plots of  $g$  and  $h$  from (4.4) with  $p = 0.01$ . Top Left: Three cases for  $c$ , showing a single large positive equilibrium for  $c = 0.12$ , three positive equilibria for  $c = 0.24$ , and a single small positive equilibrium for  $c = 0.36$ . The other panels show the cases individually, with the phase-line drawn on the  $x$  axis, using disks for stable equilibria, squares for unstable equilibria, and arrows showing the direction of change, which is always to the right when the curve ( $g$ ) is above the line ( $h$ ) and to the left otherwise. The unstable equilibrium at  $x = 0$  is where  $f = 0$ .

the first and third of these, the curve is above the line, so growth exceeds consumption and the population increases. Similarly, the population decreases in the second and fourth regions because the curve is below the line. The result of this analysis is that the largest and smallest of the positive equilibria are locally asymptotically stable, while the middle one and the extinction equilibrium are unstable. The positive unstable equilibrium has special significance, as illustrated by the arrows: it marks the boundary between the domains of attraction of the two stable equilibria. Thus, the ultimate fate of the system depends on whether it starts above or below this critical value. This feature will play an important role in the application of these results to the history of resources that became depleted very quickly and have made only a modest recovery, such as some species of whales.

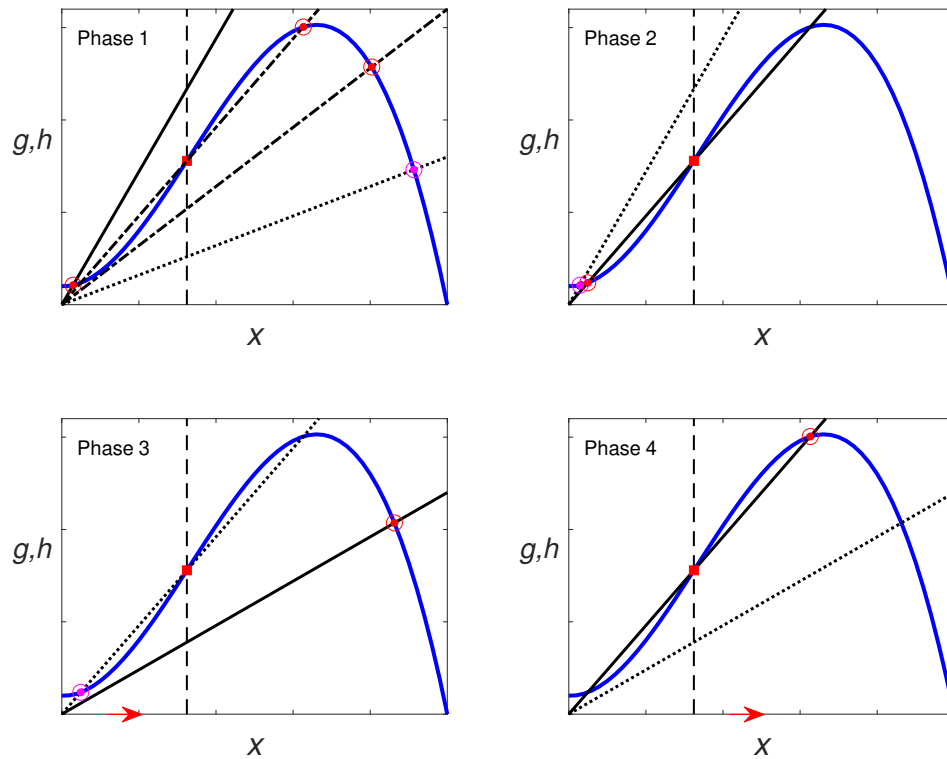


Figure 3: A reconstructed history of whale resource levels using  $p = 0.01$ . Each phase begins with a value of  $c$  corresponding to the dotted line and proceeds to the value of  $c$  corresponding to the solid line, with any intermediate values shown as dash-dot. The bulls-eye markers show the corresponding stable equilibria and the square markers show unstable positive equilibria. The arrows on the  $x$  axis in phases 3 and 4 indicate the direction of change toward a stable equilibrium.

## 5 A Reconstructed History of Whale Populations

Figure 3 shows a possible history of a resource that has suffered a sudden depletion and been difficult to restore. All phases assume  $p = 0.01$ ; hence, the  $g$  curve is unchanged. The parameter  $c$  represents the capacity of humans to harvest whales rather than just the number of consumers. Thus, it increases naturally if unregulated, through a combination of human population growth and technological change. We assume that changes in  $c$  occur slowly enough that the system is able to continually adjust to the new stable equilibrium value, provided that equilibrium depends continuously on  $c$ . (This is not entirely realistic, as technological change can be sudden.) The history assumes that  $c$  increases naturally, represented by a steepening of the straight line, until some point at which it becomes controlled by public policy. Changes in population occur on two different time scales: both the increase in  $c$  and natural population growth occur on a scale of tens of years, while the depletion from unsustainable consumption can be much faster.

## 5.1 Phase 1: Depletion

Prior to the introduction of advanced technology, humans functioned as natural predators of whales. A small value of  $c$ , as indicated by the dotted line in the phase 1 plot, resulted in an equilibrium population only slightly below the environmental carrying capacity. Over a long period of time, the value of  $c$  rose steadily, gradually decreasing the equilibrium  $x$ , but slowly enough as to be unnoticeable over a human lifespan. Eventually, the value of  $c$  rose above the critical value that separates cases 1 and 2. At that point, seen in the second dash-dot line, there would have been two stable equilibria. The existence of a stable low-level equilibrium would have had no effect on the population, as the initial state for each successive consumption level would have been on the positive side of the unstable equilibrium. Only when  $c$  exceeded the critical value that separates cases 2 and 3 would there have been a drastic change. Once that critical value was exceeded, the high-level stable equilibrium disappeared, leaving only the heretofore unobserved low-level equilibrium. At that point, the initial condition was far from the final equilibrium value, so the decrease in population would have occurred on a time scale corresponding to the harvesting, whereas previously the population changes occurred on the much slower time scale corresponding to consumption increase. There is no question that some populations of whales, among other resources, have experienced a population crash at some point without a large contemporary change in consumption. This is different from instances where the consumption rate changed suddenly, such as the depletion of British forests during the rise of industry.

## 5.2 Phase 2: Inadequate Correction

Once a whale population became depleted, hunting them stopped being commercially viable. This would have caused a significant decrease in whale hunting efforts, which would have decreased the consumption capacity. Between that and the initiation of whale conservation efforts,  $c$  might perhaps have been lowered to the solid curve in the plot. By the solid curve, the system is back in case 2; unfortunately, this time the initial condition is at the prior low-level equilibrium, so it is the high-level equilibrium that is unachievable. As with phase 1, the phase 2 pattern is well documented for whales and other resources. This is what seems to have happened between 1961 and 1986, and it was difficult to sustain politically because the payoff was so low. Some stocks of whales are undoubtedly still in this phase, as not all have recovered as well as the west South Atlantic humpback and New Zealand right whales [9]

## 5.3 Phase 3: Strict Conservation and Restoration

As the environmental movement grew, international policy shifted toward further decreases in consumption, helped along by environmental activists such as Greenpeace. In phase 3, the decrease in  $c$  continues until the system is back in case 1. At that point, the low-level equilibrium is lost and the system begins to move toward the high-level equilibrium. Here again, the improvement in some whale populations is well documented and explainable using our simple model. Unfortunately, the time scale for restoration is

much slower than the time scale for the depletion that occurred at the end of phase 1. In the depletion event, consumption outpaced population growth and drove the population decrease. In the restoration phase, the population increase occurs on the slower time scale of natural population growth.

## 5.4 Phase 4: Restoration and Sustainable Management

With a harvesting level corresponding to the solid line in the phase 3 plot and dotted line in the phase 4 plot, the whale population is recovering. If we maintain strict conservation for a while and then return to case 2, the outcome depends on whether the amount of restoration has raised the population above the unstable equilibrium value for the case 2 consumption rate. If so, as represented by the arrow on the  $x$  axis, we will have moved the population into the domain of attraction for the high-level equilibrium and the population will continue to grow in spite of an increase in consumption capacity. If not, we will still be in the domain of attraction for the low-level equilibrium and the population will begin to decline again. Of course the description of our current situation using the phase 3 and phase 4 plots is qualitative at best. We don't know the parameter values and there are flaws in the model that are not serious enough to falsify our narrative but are serious enough to make prediction uncertain. The prudent approach would be to maintain strict conservation until the population recovery rate is large and then increase consumption capacity only gradually. By monitoring the whale population and insisting on further growth, we can make sure that we have entered case 2 on the correct side of the unstable equilibrium. The point, though, is that it is not necessary to maintain strict conservation forever. A moderate consumption level that was not small enough for recovery in phase 3 is small enough for sustainability in phase 4. Based on recorded population data, we are clearly in phase 4 for some whale stocks [10].

# 6 Discussion

Here we consider a mathematical extension of the case study and turn to the practical question of how to teach the case study material in various ways.

## 6.1 Bifurcation Analysis

In our analysis of the model, we saw that the sensitivity of the equilibrium solutions to the parameter  $c$  is small at times and large at others, sometimes so large as to be discontinuous. The dependence of a solution on parameter values is of both mathematical and biological interest. It can be seen in a 1-parameter system by plotting a curve of the equilibrium value against the parameter. In a 2-parameter system, we can plot multiple such curves, using several different values for the second parameter. The left panel of Figure 4 shows such a plot for the model (3.2). The consumption parameter  $c$  is taken as the independent variable on the graph, while the process-to-discovery ratio parameter  $p$  is set at multiple values. The  $p = 0.01$  case we have been studying is the second curve from the left and shows that case 2 requires  $c$  values roughly in the range 0.18 to 0.27. Plots of this sort

are called bifurcation plots; the points where the stability changes on each of the three leftmost curves are called bifurcation points. The bull's-eye marker indicates what we might call a critical point, as it marks the critical value  $p^*$  that is the boundary between smaller  $p$  values, for which bifurcation occurs at some values of  $c$ , and larger  $p$  values that show no bifurcation in  $c$ . The plot shows that the corresponding value  $c^*$  also marks the largest value of  $c$  among bifurcation points.

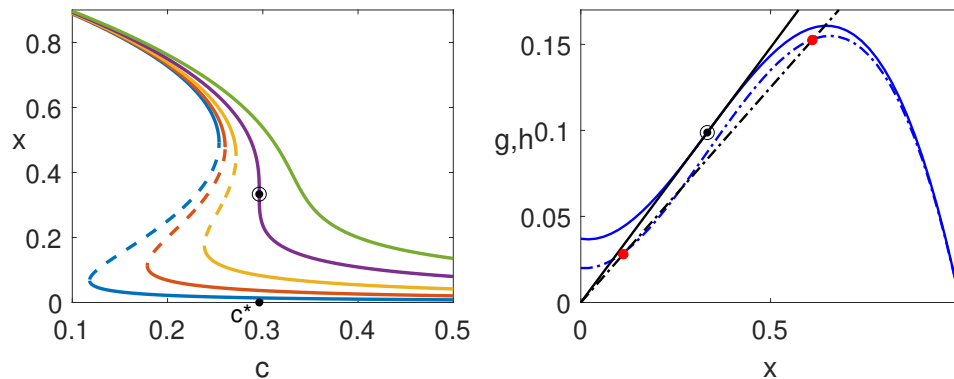


Figure 4: Left: Equilibrium resource levels as a function of consumer parameter, with  $p=0.004, 0.01, 0.02, 1/27$ , and  $0.06$ , from left to right. Equilibria on the solid curves are stable, while those on the dashed curves are unstable. The critical value  $p^* = 1/27$  is that for which the bifurcation curve has a vertical tangent, at the point with a bull's-eye marker. Right: multiple positive equilibria are possible if and only if  $p < p^*$  and  $c$  is in a narrow range below  $c^*$ . The solid curve and solid line are for  $p^*$  and  $c^*$ . The dash-dot curve shows a smaller value of  $p$  and the dash-dot line shows a value  $c < c^*$  corresponding to case 2, with disks marking the stable equilibria.

The critical parameter values can be determined analytically using a combination of calculus and algebra. The key to doing this is to identify the properties of the critical case on a plot of  $g$  and  $h$  from (4.2, 4.4), shown in the right panel of Figure 4. The solid line that marks the critical case is tangent to the graph of  $g$  at the inflection point, at the bull's-eye marker. The dash-dot curve in the plot shows that multiple equilibria occur for  $p < p^*$ , provided  $c$  is smaller than  $c^*$ , but not so much smaller as to be in case 3 of Figure 2.

The critical point is found by solving the set of equations for the properties that the critical case must satisfy, as seen in the right panel of Figure 4:

$$g(x^*) = h(x^*), \quad g'(x^*) = h'(x^*) = c^*, \quad g''(x^*) = 0. \quad (6.1)$$

Setting  $g'' = 0$  yields the result  $x^* = 1/3$ . Then  $g'(x^*) = c^*$  yields  $c^* + p^* = 1/3$ , and finally  $g(x^*) = h(x^*)$  then yields  $p^* = 1/27$ . The discontinuous time histories of Section 5 only occur when natural resource stocks are large enough for the process-to-discovery parameter  $p$  to be sufficiently small.



## 6.2 Finding Equilibria

The plot on the left side of Figure 4 was obtained by solving the equilibrium relation for  $c$  and then computing the  $c$  values for a given set of  $x$  values. If we want to find the equilibrium points for a given set of parameters values, we need to solve the equation

$$(1 - x)(p + x^2) = cx \quad (6.2)$$

numerically. Even with a professional root finder, such as Matlab's `fzero` function, this needs to be done carefully. Any solver works best when it is given a good approximate solution to start with, particularly when there are multiple solutions. The needed approximate solutions can be found graphically without much difficulty.

## 6.3 A Few Words About Modeling

A mathematical model is a collection of one or more variables together with enough mathematical equations to prescribe the values of those variables. Models are based on some actual or hypothetical real-world scenario and created in the hope that they will capture enough of the features of that scenario to be useful. This concept of models has an important consequence. The applications in most mathematics books begin with a listing of 'facts' used to derive the model. Given that models are based on, rather than equivalent to, a real scenario, what are given as 'facts' should be instead identified as *assumptions*, and the misleading term 'application' should be replaced by the more accurate term 'model.'

The distinction between applications and models is more than just nit-picking. One views mathematical results differently depending on whether the problem starts with facts or with assumptions. If it starts with facts, then the certainty of mathematics guarantees the certainty of the results, which must then be true statements about the scenario under study. If it starts with assumptions, then the certainty of mathematics guarantees only that the results follow from the assumptions, which means that their value in understanding the scenario is determined by non-mathematical considerations. In modeling, mathematical results need at least some degree of validation from outside of mathematics. At minimum, the qualitative behavior of the model should match the real scenario. As an example, the Lotka-Volterra model is the simplest instance of the broad class of predator-prey models (2.1–2.2), obtained with the linear functions  $G(X) = RX$  and  $H(X) = SX$ . It predicts that no increase in the predator death rate is sufficient to drive the predator to extinction. This is sometimes stated as a biological outcome, when it is really only a mathematical outcome of a proposed model. Biologically, the result is clearly false in any realistic setting. The correct conclusion to draw is not that predators can't be driven to extinction, but that the Lotka-Volterra model is unsuitable for predator-prey modeling. That even this modest amount of validation is frequently overlooked is obvious from an examination of Google hits on 'Lotka-Volterra model': far more of these are for documents that use the Lotka-Volterra model to (incorrectly) 'explain' biological phenomena than for those that explain why the model should not be used.

Our model (3.2) has been derived from first principles and exhibits unusual qualitative behavior that matches a crude set of observed phenomena. This gives us some reason



to believe that the story we tell about whales is plausible, but it should not be taken as a definitive explanation of real phenomena.

## 6.4 Using the Case Study in a Class

This case study can be used in several different ways, depending on the emphasis the instructor wants to place on modeling as compared to mathematics. We briefly outline several options, with increasing levels of modeling. The reader may also find it useful to have access to a Powerpoint presentation based on the case study [8].

Most differential equations courses today include phase-line analysis. This is always done by using graphs of  $x'$  vs  $x$  to determine whether  $x$  is increasing or not. It takes only a small amount of additional time to show the method based on the factorized form (4.2). The one-parameter equation

$$x' = x(1 - x) - \frac{cx^2}{0.01 + x^2}$$

can be used as an example, with Figure 2 and the accompanying analysis. This can be cited as a resource management model without any of the modeling detail. A brief discussion of bifurcation could be included by using a plot of equilibrium  $x$  vs  $c$ , using just  $p = 0.01$ .

If an instructor wants to incorporate some modeling into the course, (s)he can present the two-parameter equation (3.2) with a brief statement that  $x$  is the resource level expressed as a fraction of carrying capacity, that time  $t$  is measured so that 1 unit of time is roughly the amount required for resource growth to be significant (rather than specifically time in days or years), that  $p$  is a parameter that represents the processing speed relative to discovery speed and will be small for large resource stocks because it is easy to discover things that are common, and that  $c$  is a parameter that represents the number of consumers, with  $c = 0.5$  indicating a consumer level that would be sustainable only if maximum resource production could occur even with overutilization of the resource. This is the minimal amount of modeling needed to develop the historical analysis of Section 5. The students could be given the brief historical information in Section 2 or just the listing of articles cited in that section. The bifurcation analysis of Section 6.1 could be added to this option.

A full treatment of the modeling is suitable for courses that are as much focused on modeling as on differential equations, particularly courses based on projects or case studies rather than lists of topics. Such a treatment would include the derivation of the Holling type 3 model and the scaling, as well as all of the analysis. The bifurcation analysis would serve as an illustration of how modelers should try to extract as much understanding as possible from a model, and that this depends on doing a variety of experiments and asking questions about the results.

## References

- [1] F. Brauer and C. Castillo-Chavez. *Mathematical Models in Population Biology and Epidemiology*. Springer-Verlag, 2001.
- [2] R. Gambell. World whale stocks. *Mammal Review*, 6:41–53, 1976. doi: 10.1111/j.1365-2907.1976.tb00200.x.
- [3] W.S.C. Gurney and R.M. Nisbet. *Ecological Dynamics*. Oxford University Press, 1998.
- [4] C.S. Holling. Some characteristics of simple types of predation and parasitism. *Canadian Entomologist*, 91:385–398, 1959.
- [5] J. A. Jackson, E. L. Carroll, T. D. Smith, A. Z. Zerbini, N. J. Patenaude, and C.S. Baker. An integrated approach to historical population assessment of the great whales: case of the new zealand southern right whale. *Royal Society Open Science*, 2016. doi: 10.1098/rsos.150669.
- [6] G. Ledder. *Mathematics for the Life Sciences: Calculus, Modeling, Probability, and Dynamical Systems*. Springer-Verlag, 2013.
- [7] G. Ledder. Scaling for dynamical systems in biology. *Bulletin of Mathematical Biology*, 79:2747–2772, 2017.
- [8] G. Ledder. The past, present, and future of endangered whale populations: Qualitative analysis and modeling. Available at <https://www.math.unl.edu/~gledder1/Talks>, 2020.
- [9] K. Whiting. This is how humans have affected whale populations over the years. Available at <https://www.weforum.org/agenda/2019/10/whales-endangered-species-conservation-whaling/>, 2019.
- [10] A.N. Zerbini, G. Adams, J. Best, P.J. Clapham, J.A. Jackson, and A.E. Punt. Assessing the recovery of an antarctic predator from historical exploitation. *Royal Society Open Science*, 2019. doi: 10.1098/rsos.190368.

# *Epidemiology and the SIR Model: Historical Context to Modern Applications*

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**Keywords:** SIR model, plague, Ebola, epidemic, India, West Africa  
Manuscript received on May 17, 2020; published on March 15, 2021.

**Abstract:** We suggest the use of historical documents and primary sources, as well as data and articles from recent events, to teach students about mathematical epidemiology. We propose a project suitable — in different versions — as part of a class syllabus, as an undergraduate research project, and as an extra credit assignment. Throughout this project, students explore mathematical, historical, and sociological aspects of the SIR model and approach data analysis and interpretation. Based on their work, students form opinions on public health decisions and related consequences. Feedback from students has been encouraging.

We begin our project by having students read excerpts of documents from the early 1900s discussing the Indian plague epidemic. We then guide students through the derivation of the SIR model by analyzing the seminal 1927 Kermack and McKendrick paper, which is based on data from the Indian epidemiological event they have studied. After understanding the historical importance of the SIR model, we consider its modern applications focusing on the Ebola outbreak of 2014-2016 in West Africa. Students fit SIR models to available compiled data sets. The subtleties in the data provide opportunities for students to consider the data and SIR model assumptions critically. Additionally, social attitudes of the outbreak are explored; in particular, local attitudes towards government health recommendations.

## 1 Introduction

It is increasingly evident that the younger generations of students are actively involved in pushing for justice on their college campuses [10, 5, 8]. Students, the general public, and even fellow mathematicians are often skeptical that STEMM (Science, Technology, Engineering, Mathematics, and Medicine) can be a tool for social good. We believe it is part of our mission as faculty members to correct this misconception and guide students in

understanding the role that ethics, equity, and social justice have in mathematics research and education.

Recently, the COVID-19 pandemic has awakened a sudden, growing interest from students in epidemiology and disease modeling from a mathematical point of view as well as from public health and sociological perspectives. Countries' responses to this global crisis have differed widely due to varying access to resources, trust and value given to scientific expertise, and societal norms. Consideration of the local cultural and historical contexts as well as the lessons learned from previous epidemics has been crucial to planning local and country-wide approaches to this recent international threat. For example, differences in culture and traditions centering the collective good over the individual (or vice-versa) had a great impact on policy decisions and approaches to contain the spread of COVID-19 [13, 14].

Students should be trained to consider that epidemiological models were indeed developed to aid containment of disease spreading and to plan public health responses to epidemics. These models have real effects on actual populations, often in real time. While the disease itself will have common characteristics that span countries, regions, ethnicities, religions, and more, certain characteristics affecting epidemics development are geographically focused and subject to local values and access to resources. In this paper, we describe a project for undergraduate students designed to teach them about the Susceptible-Infected-Recovered (SIR) model in an historical and social context, and then let them explore its application to the 2014-2016 Ebola epidemic in West Africa.

In this project, students learn first about the Indian Plague epidemic of the early 1900s through historical primary sources [3]. Then, they are guided through the derivation of the SIR model by analyzing the seminal 1927 Kermack and McKendrick paper [9], which utilizes data from the Indian Plague epidemic itself (Section 2). Students are then asked to consider modern applications of this model by focusing on the Ebola outbreak of 2014-2016 in West Africa. Fitting the SIR model to real, available, compiled datasets [12] leads students to consider the subtleties of working with real data and confronting model assumptions critically. Additionally, students learn about local attitudes towards government health recommendations and how those affected the spread of the Ebola epidemic [4] (Section 3). Finally, we discuss possible implementations and variations of this project and conclude by reflecting on potential improvements and future directions (Section 4).

An Appendix follows the Reference section, giving some of the materials we prepared for students.

## 2 Historical Context: The Indian Plague Epidemic of the Early 1900s and the SIR Model

Our goal is to guide students in understanding the actual applicability of the SIR model, starting from its inception. The 1927 Kermack and McKendrick paper [9] utilizes data from the Indian Plague epidemic of the early 1900s, so students start off the project by learning about this event through historical public health records. Early in the epidemic, the British Empire instituted the so-called Indian Plague Commission to study the spread

of plague in India, understand the causes of the diseases, and help stop the epidemic. We select several key excerpts of the Commission's extensive report (freely available to the public [3]) for students to read and discuss. We provide them with questions (see Appendix A for our *Student Guide*) and supplementary materials from the Centers for Disease Control [6, 7] to reflect on the Commission's epidemiological observations while highlighting the intrinsic imbalance of power between the British colonialists authoring the report, and the Indian population being observed [2]. We help students in building mathematical intuition regarding what key aspects of the epidemic could be modeled and should be considered and what others could be omitted.

Reading a historical primary source will most likely be a new experience for students outside of the context of history or literature courses, especially for those enrolled in STEMM majors. Having them reflect on the Commission's report and related documents achieves four goals:

- Reading primary sources reporting the real, drastic effects of these events on the local population will open the students' eyes to the potential impact of mathematics and epidemic modeling. Given their own recent experiences with the COVID-19 pandemic, this is probably less needed now than it would have been just last year.
- Having students apply critical thinking in a context traditionally associated with a history class, rather than a math class, may provide an extra 'buy-in' to the project for participants. We want them to be challenged in this project in ways they probably haven't been before in a mathematics context.
- Allowing students to read about this epidemiological event without the burden of having to connect it to new mathematical concepts allows them to be braver when it comes to stipulating model assumptions and hypotheses. Students are often subject to anxiety as well as to self-imposed and external pressures when confronting new mathematical challenges [11]; initially separating the two aspects of this project aids them in gaining confidence in themselves.
- The supplementary materials provide guiding questions for students (Appendix A), a brief overview of the disease in question [6, 7], and a way to appreciate the intrinsic biases in the historical writings [2]. It is important for students not to take the readings at face value but rather understand who the authors and the subjects are and how their positioning in the epidemic and colonial contexts affects their public health choices and outcomes. We will return to similar ideas later when working on the 2014-2016 Ebola epidemic in West Africa.

Continuing with the historical part of the project, students navigate through the original 1927 Kermack and McKendrick paper, following prompts and focusing on selected sections of the manuscript (for details and example questions, see Appendix A). As they read through the chosen parts of [9], they are asked to answer questions in writing and follow along some of the mathematical derivations. There are several parts of the manuscript where a knowledge of single-variable calculus is enough to work through the equations. After digesting the Introduction and General Theory of [9], students jump

ahead to one of the Special Cases described in the paper, i.e., *Part B. Constant Rates*. This section includes a figure with data from the Indian plague epidemic, so students are asked to analyze the plot and compare it to what they know from reading the primary source material. This is also where they first see the full SIR system (albeit in 1927 notation). Then, students spend time thinking through more technical aspects of the problem, such as choice of variables and units, the meaning of each term and each equation, and the system end-behaviors. From here, they work towards the modern notation of the SIR model and learn about the basic reproduction number along the way. Finally, they are asked to solve analytically the SIR system for this special case and analyze its behavior as it compares to their earlier qualitative predictions.

### 3 Modern Application: The 2014-2016 Ebola Epidemic in West Africa

We believe that incorporating real data into mathematical models is an essential part of an applied mathematics curriculum in the modern day, and this project provides an excellent opportunity to do so. The core mathematics of what students are asked to do here is applying a few approaches to identifying initial conditions and/or SIR model parameters. However, this part of the project goes well beyond these mathematical tasks. Students get experience in loading and preprocessing data, posing questions and ‘subsetting’ the data, analyzing their results critically, and finally making careful conclusions and considering further analyses.

The mathematics of ‘fitting’ parameters in an ODE model becomes exponentially more difficult as the number of parameters increases, the model dependencies become nonlinear, or both. Depending on the preparation of students, the instructor may consider beginning with a simplification for the early period of the epidemic, where an approximate solution form and simple linear least squares may be used. If the students are more advanced, or a longer term project is intended, the instructor may consider having the students apply a nonlinear least squares solver to work with the full SIR model; this can be an alternative as well as an addition to the linear fit previously discussed. Below we report a possible first approach to fitting the SIR model to data from the 2014-2016 Ebola epidemic in West Africa [12], starting with a simpler SI approximation. Python codes used for fitting and plotting are available in [1].

#### 3.1 The ‘SI’ Approximation

The purpose of this first possible step is to obtain an approximate solution for the ‘Infectious’ population,  $I(t)$ , which is simple enough to allow students the use of ordinary least squares to estimate  $\beta$ , the disease contact rate. To achieve this, a few assumptions need to be made, and students are asked to think them through. Removing the ‘Recovered’ category from the typical SIR essentially results in the model only having a single transfer

between compartments:

$$\frac{dS}{dt} = -\frac{\beta SI}{N} \quad (3.1)$$

$$\frac{dI}{dt} = \frac{\beta SI}{N} \quad (3.2)$$

This is the first simplification for students to grapple with as the missing category can be interpreted as either being included in the ‘Infectious’ group, or being assumed to be so small as to be insignificant (an infectious person has an expected number of days until they recover). In any case, applying the conserved quantity  $S + I = N$ , then solving for the ‘Susceptible’ population and substituting, gives an equation depending only on the Infectious population and two parameters,  $\beta$  and  $N$ . Students enrolled in an ODE course (and even some who have only taken calculus) will have likely encountered this as the well-known logistic equation:

$$\frac{dI}{dt} = \beta \frac{I}{N} (N - I), \quad I(0) = I_0. \quad (3.3)$$

As a side exercise, this can be solved using partial fraction decomposition; one solution form is:

$$I(t) = \frac{Ne^{\beta t}}{e^{\beta t} - 1 + N/I_0}. \quad (3.4)$$

The last step of the derivation here leads students towards the assumptions that, first,  $I_0$  is small relative to  $N$  (that is, the epidemic begins with a small number of infected people in the population), and second, we are interested in fitting the contact rate  $\beta$  during the initial phases of the infection (this meshes well with being able to ignore the ‘Recovered’ compartment for being zero, or close to it). Then, one can make an asymptotic approximation considering  $e^{\beta t} - 1$  to be very small relative to  $N/I_0$ , so that this term can be crossed off. Hence, the expression reduces to what is often understood as the ‘exponential growth’ phase of the SIR model:

$$I(t) \sim I_0 e^{\beta t}, \quad \text{with } t \text{ small.} \quad (3.5)$$

If the instructor or students are time constrained, an alternative approach can be to start at this approximation and argue its reasonableness from a modeling perspective. Recall that the advantage of following this process is to obtain an explicit formula for  $I(t)$ , so that  $\beta$  can be fit to data. When estimating  $\beta$  from available time series data  $(t_k, \log I(t_k))$ , students should apply a log-transformation:  $\log(I) \sim \log(I_0) + \beta t$ , and use a linear least squares fit to find a value of  $\beta$ . Here, there is the option of simplifying this process even further by treating  $I_0$  as a known value; alternatively,  $I_0$  can be viewed as an unknown to be inferred from the data. We expect students to grapple with a few modeling concerns at some point during this process:

1. Defining initial conditions. What does  $t = 0$  refer to here? Similarly, what should  $I_0$  be? One option is to start time at the first observed case, and use initial condition  $I_0 = 1$  (1 person), but these choices can cause fitting difficulties down the road. **The**



**important thing for students to realize here is that there is no perfect answer.** Trying to make an appropriate choice for the time frame of reference and  $I_0$  is a modeling challenge of its own, involving difficult mathematics especially when working with real, noisy data.

2. Short-time approximations. What does “ $t$  small” mean? What does “initial phases of infection” mean? A possible answer from a risky interaction-type model considers this period lasting as long as the chance of two infectious people interacting with each other is very small. Once again, the question of ‘smallness’ is difficult to address, but it can lead to interesting conversations regarding nondimensionalization and epidemiological context.
3. Total population. How do we decide what the total population  $N$  should be? Each choice of  $N$ , whether *ad hoc* or informed by the data (e.g., the population of a neighborhood, city, or country) has critical modeling assumptions built in and comes with implications for data fitting. Depending on the focus and scope of the project, the instructor may guide students to make a suitable choice (e.g., the city’s population) and leave other options as potential directions to explore for a project extension.

## 3.2 Loading and Processing Data

We suggest students work with the Ebola data sets compiled in the Github repository [12] which includes data from the Ministries of Health of several West African countries and data sets from the World Health Organization, among other things. This data includes very fine-grained information which is useful for study by epidemiologists, and to the despair of mathematicians. There are two general routes when working with data in the `case_products` folder of [12] :

1. Work with the Excel file `case_data_consolidated_sl_and_liberia.xlsx`. In Python, we recommend using the `pandas`<sup>1</sup> package to load this file and access its sheets of data. The sheet `Sierra_Leone_transposed`, for example, has daily Ebola case data grouped for various regions throughout the country, and allows students the opportunity to further narrow their focus, or aggregate across the country.
2. The file `country_timeseries.json` is a so-called JSON file, which stores data in a flexible format allowing for potentially heterogeneous, messy data. One may use the `json` package in Python to load it quickly, then take further steps to process it enough to plot and analyze it. Here, data is stored per-day, with daily cases and deaths aggregated by country. This allows students to get started as quickly as possible in a data fitting exercise. The examples presented in the next section utilize this version of the data.

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<sup>1</sup>`pandas` is a fast, powerful, flexible and easy to use open source data analysis and manipulation tool, built on top of the Python programming language.



### 3.3 Some Example Results

Depending on students’ interest and focus, the data analysis can follow many different paths. We discuss a few nuances working with the JSON per-country, per-day case data so that instructor and students have a guidepost in doing their own work. We have made the associated data and code available in our public GitHub repository [1].

The data were loaded and arranged in a pandas DataFrame in Python, with each row being a calendar date. We have transcribed a small sample of this data in Table 1, which was the result of the following processing. Each row has an integer time (days

Date	Day	Cases_Liberia	Cases_SierraLeone	Cases_Guinea
6/1/2014	71	13	79	328
6/3/2014	73	13		344
6/5/2014	75	13	81	
6/10/2014	80	13	89	351
6/16/2014	86	33		398
6/17/2014	87		97	

Table 1: A small sample of Ebola cases by country contained in the JSON data file. Missing values in this Table represent the data source not having a report from a country on a given day. Nigeria and Senegal did not have any cases to report until July 23, which is why we do not include them here.

since a reference), and death and case counts in five different countries spanning about six months. This data is stored entirely as strings, initially, so our loading script casts values to datetime objects (for simplified plotting of dates on the horizontal axis) and to integers for case counts and time for fitting. Imputing empty strings with NaN (“not a number”) allows pyplot, the plotting software, to bypass missing data in a natural way.

Initial examination of the data reveals several facts. First, the behavior is vastly different from country to country and a difference in time of initial outbreak can be observed. More interestingly, countries with significant outbreaks – Guinea, Liberia, and Sierra Leone – show very different growth rates during this period. Given this, we felt further aggregation across some or all of the countries lost too much information.

With cleaned data, we applied the approximations and data fitting methodology described in the previous section. For the purposes of modeling and fitting, we chose to restrict our focus to Sierra Leone and Liberia (Guinea would also be a reasonable choice, but we do not explore it here). We define  $t = 0$  as June 1, 2014. Where data was missing, we ignored the corresponding  $(t, I(t))$  pair by utilizing a mask to restrict focus to time points for each country where data was available. Finally, we obtained fits of the model parameters for the Sierra Leone and Liberia case data as reported in Table 2.

	$\beta$	$\log I_0$	$I_0$ (predicted cases on June 1)	Doubling time (days)
Liberia	0.0494	2.8820	18	14
Sierra Leone	0.0278	4.5892	98	25

Table 2: Example fit of the model parameters for Liberia and Sierra Leone.

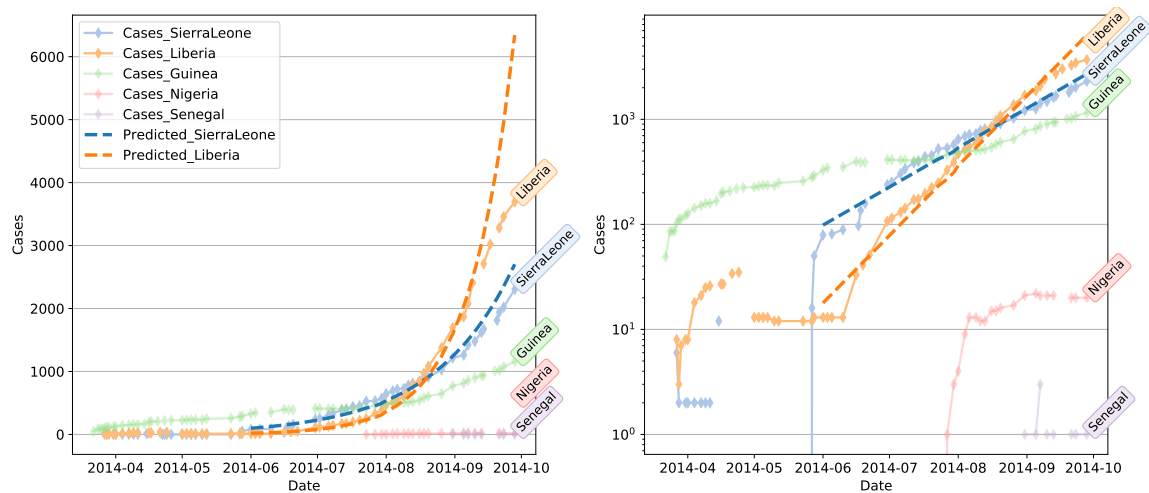


Figure 1: Example of data visualization for Ebola cases count during April-October 2014 in several countries in West Africa. Predicted Ebola cases count for Sierra Leone (shades of blue) and Liberia (shades of orange) starting on June 1, 2014. Left and right figures illustrate the same data with a linear scale (left) and logarithmic scale (right) for the cases. Code to produce this plot is available our own public GitHub repository [1].

We confirm reasonableness of our fits by including the predicted model on top of the actual data in Figure 1. We have observed that this last step is often challenging for students. While obtaining parameters is ultimately a sequence of five commands in the script once the data is cleaned, correctly applying those parameters in a model forces students to tackle a few challenges. To name a few: they should revisit how these parameters appear in the model, they should consider how the choice of  $t = 0$  associates to the model built, and they should understand the relationship between calendar dates and ‘time’ used in the model. If guided past these challenges, students get a deep satisfaction when they see their model curve overlapping well with observed data in a way that merely obtaining a numerical value for a parameter cannot show.

An interesting feature to note in Figure 1 is that the data curves for Sierra Leone and Liberia intersect between August and September 2014. In Sierra Leone there were about a factor of 10 more cases in June 2014, but Liberia had a much shorter doubling time in their cases (which we showed in Table 2), so it passed Sierra Leone in the number of cases in about two months. We do not know the reasons for this; there could have been spatial differences in how the disease was spreading (e.g., higher density areas versus rural areas), or public health policy differences, or a combination of causes. There is an excellent opportunity here to dig deeper, for example within the context of an undergraduate research project.

### 3.4 Putting the Modern Ebola Epidemic in Context

In conjunction with studying this data, as with historical documents on the Indian Plague, we have encouraged students to read related materials to expand their thinking beyond the mathematical exercise. As an example, one such article from the *New York Times*

provides a cultural lens on the Ebola epidemic [4]. Public health efforts coming from primarily Western aid organizations (including the World Health Organization), clashed with local customs in Liberia during the 2014-2016 epidemic, especially in regard to the burning of corpses of those afflicted with Ebola. Local people working in crematories were ostracized by their families and communities for going against Liberian tradition. When reading these documents, students learn about other cultures and are faced with how the complex realities of epidemiological modeling, data collection, and analysis influence public health decisions and policy that potentially affect the lives of millions.

## 4 Possible Implementations and Future Work

This project has been implemented at multiple US institutions in different versions as part of the syllabus and as an extra credit assignment for an introductory ODE course, as well as an independent study project. Most students who worked on this project belonged to STEMM and social sciences majors, but not mathematics majors. A first version of this project was ideated for an introductory modeling course in 2014, and various versions of the project have been used almost every year since. Most recently, in the 2019/2020 academic year, two undergraduate students at Florida State University (majoring in Biological Sciences and Economics and Statistics, respectively) worked on a year-long version of this project as part of the Undergraduate Research Opportunity Program (UROP) on campus.

When asked to think back on their experience working through this project, all students reported enjoying reading primary sources and using them as a basis for mathematical exploration. This was particularly true for students majoring in the life sciences. Students were surprised to realize how many factors need to be considered to design appropriate mathematical modeling to fit the behavior of real-world populations and cultures. Students appreciated the chance to consider the importance of context as well as mathematical modeling in making policy decisions and evaluating whether a chosen approach is working as hoped. Unsurprisingly, participants found reading the Kermack and McKendrick paper [9] most challenging. They often got lost in the details of the manuscript and were not able to follow the steps of the calculation reported in the paper, even when the mathematics involved should have been accessible to them. The challenges of reading a technical mathematics paper (especially one from 1927) became apparent very quickly; this is the part of the project where we as instructors had to step in more consistently during the implementations. Some students liked the data analysis part of the project more than others. In this final part of the project, students were faced with subtleties in the data collection and analysis and with having to confront the reality of messy and incomplete data sets. This helped them realize the stark difference between the pre-arranged class exercises they are used to and the realities of modeling with actual data.

While part of the reason for developing this project is to get students outside of mathematics excited about differential equations and modeling, it would be interesting to see if the more technical aspects of the 1927 Kermack and McKendrick paper could be appreciated and explored further by participants majoring in mathematics. Nonetheless, analyzing the selected parts of the manuscript is enough for students to understand where

the model comes from and do some mathematical experimentation on their own. The solid historical connections between the Indian Plague epidemic of the early 1900s and the seminal SIR paper of 1927 make them an ideal place to start this exploration, but given the wealth of data freely available online, this project can be adapted to include virtually endless other modern applications. Even within our focus of the 2014-2016 Ebola epidemic in West Africa, there are numerous avenues that we leave unexplored. We mentioned one such example at the end of Section 3.3. Additionally, first and second-hand accounts of the difficulties reported on the ground when dealing with health officials handling the epidemics could be further researched. From the data analysis and parameter fitting perspective, estimating multiple parameters in addition to  $\beta$  involves complex, nonlinear data fitting. The identifiability of the recovery rate  $\gamma$  primarily relates to the medium and long-time dynamics of SIR and can only be considered when including the ‘Recovered’ compartment; asymptotic analysis could be applied to compute long-time approximate solutions in a similar fashion to the short-time analysis done for the infection rate  $\beta$ .

Students participating in this project are exposed to clear examples of how mathematics, and STEM more broadly, can be tools in service of public health and social problems. We believe students would be more interested in working towards technical STEM degrees if made aware of the many ways they can use them to serve society. Young students, and in particular those from underrepresented groups in STEM, find strength in helping others and advocating for social justice. We advocate for teaching the younger generations how to use mathematics ethically to serve their broader goals. We believe the approach showcased in this paper incorporating historical and social context can be adopted for a variety of projects focused on ODE modeling. While admittedly not all differential equation models have as rich a history and widespread a use as the SIR, ODEs are so often used to model the real world that they are an ideal avenue for this type of project. We hope our work can be viewed as a guiding example of how to inject some historical and social context in a mathematics classroom.

## References

- [1] Manuchehr Aminian. Supplementary materials for a lesson plan for a math epidemiology project. URL [https://github.com/maminian/codee\\_ebola](https://github.com/maminian/codee_ebola).
- [2] David Arnold. *Colonizing the body: State medicine and epidemic disease in nineteenth-century India*. University of California Press, 1993.
- [3] Indian Sanitary Commissioner. *The Etiology and Epidemiology of Plague – A Summary of the Work of the Plague Commission*. Superintendent of Government Printing, 1908. URL <https://babel.hathitrust.org/cgi/pt?id=uc1.b5626368>.
- [4] H. Cooper. They helped erase Ebola in Liberia. Now Liberia is erasing them. *New York Times*, 2015. URL <http://www.nytimes.com/2015/12/10/world/africa/they-helped-erase-ebola-in-liberia-now-liberia-is-erasing-them.html>.

- [5] Nature Editorial. Teenage activist and an IPCC triumph. *Nature*, 572:283, 2019. URL <https://doi.org/10.1038/d41586-019-02425-7>.
- [6] Centers for Disease Control and Prevention. Plague. URL <https://www.cdc.gov/plague/index.html>.
- [7] Centers for Disease Control and Prevention. Protect yourself from Plague. URL [https://www.cdc.gov/plague/resources/235098\\_Plaguefactsheet\\_508.pdf](https://www.cdc.gov/plague/resources/235098_Plaguefactsheet_508.pdf).
- [8] Wesley C. Hogan. *On the Freedom Side: How Five Decades of Youth Activists Have Remixed American History*. UNC Press Books, 2019.
- [9] William O. Kermack and Anderson G. McKendrick. A contribution to the mathematical theory of epidemics. *Proceedings of the Royal Society of London. Series A*, 115(772):700–721, 1927.
- [10] Dawn Laguens. Planned parenthood and the next generation of feminist activists. *Feminist Studies*, 39(1):187–191, 2013.
- [11] María Isabel Núñez-Peña, Macarena Suárez-Pellicioni, and Roser Bono. Effects of math anxiety on student success in higher education. *International Journal of Educational Research*, 58:36–43, 2013.
- [12] Caitlin Rivers. Data for the 2014 global Ebola outbreak. URL <https://github.com/cmrrivers/ebola>.
- [13] Robert Simon. Is there a trade-off between freedom and safety? A philosophical contribution to a COVID-19 related discussion. *Philosophy*, 10(7):445–453, 2020.
- [14] Jay J. Van Bavel, Katherine Baicker, et al. Using social and behavioural science to support COVID-19 pandemic response. *Nature Human Behaviour*, 4:460–471, 2020.

## A Appendix. The SIR Model in Historical Context: Student Guide

The following is a packet of exercises and readings most recently tailored by Francesca Bernardi to use as the introduction of a year-long Undergraduate Research Opportunity Program (UROP) project on the SIR model at Florida State University in the academic year 2019/2020. It is an expanded version of a shorter document developed by Manuchehr Aminian in 2014. We provide this as an appendix for a more concrete picture of the types of questions that have been asked of students working on this project. Not all readings and options discussed in the main manuscript appear in the appendix.

There is a range of difficulty in questions. Some sections of this packet assume knowledge of basic ordinary differential equations (separable equations and the method of integrating factors, in particular). Students will have an easier time with this document if they have some understanding of how ODEs are manipulated and solved. However, some sections can be used with students without any prior differential equations experience. These typically involve little to no calculation, but rather mostly working with the readings and understanding the model and its modern applications conceptually. These have been used successfully in an introductory mathematical modeling course geared towards students majoring in fields other than mathematics.

For both math and non-math majors, it may be useful to provide a sheet of definitions of technical terms to help them in readings. For an example of this, see section [A.4](#) of the Appendix.

This project revolves around mathematical epidemiology.<sup>2</sup> This document is a starting point for our project; it is a guided exploration of the SIR model for disease spreading. It consists of reading assignments, trying a few mathematical challenges on your own, and keeping track of what you're learning by answering the questions listed to form a written report.

### A.1 The Indian Plague Epidemic of the Early 1900s

In the early 1900s, an outbreak of plague erupted in India which was then part of the British Empire.<sup>3</sup> Take a look at the *Wikipedia* article on the *Third Plague Pandemic*<sup>4</sup> and pay particular attention to the section titled *Political Impact in Colonial India*.

The Indian Plague Commission was instituted by the British government to study the situation, understand the causes of the diseases, and help stop the epidemic. The written report produced by the Commission was very thorough (for full document, see [\[3\]](#)).

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<sup>2</sup>For more information on mathematical epidemiology, see *Computational and Mathematical Epidemiology*, written by Dr. Fred S. Roberts. Science Magazine | Careers, 2004.

<sup>3</sup>For more information on colonial India, see [https://en.wikipedia.org/wiki/Colonial\\_India](https://en.wikipedia.org/wiki/Colonial_India).

<sup>4</sup>See [https://en.wikipedia.org/wiki/Third\\_plague\\_pandemic](https://en.wikipedia.org/wiki/Third_plague_pandemic).



- ★ Please read the excerpt from *The Etiology and Epidemiology of Plague - A Summary of the Work of the Plague Commission* available at [1] (for full version, see [3]). Note that the “General Conclusions” (on page 12 of the linked PDF) are taken from the very end of the report.

When reading, focus your attention on answering the following questions:

1. According to the report, how is the plague transferred to humans?
2. What is definitely not true about the way the plague is transferred to humans?
3. Does any of this surprise you?
4. Do you think there is a parallel between the spreading of the disease among rats and humans?

After this first introduction to the report, choose one Part from it and read it carefully (the table of contents of the full report is on page 7 of the linked PDF). At our next meeting, be ready to present a summary of what you learned to the group. Note that this report was written in 1908 for a technical audience; it is understandable if you don’t grasp everything right away. Take some time to think about it and let it sink in.

This short Plague fact sheet [7] from the CDC (Centers for Disease Control and Prevention) answers a lot of questions regarding the disease itself. If you want more information about it, see the full CDC Plague webpage [6].

## A.2 The SIR Model

Now that you have some historical context of the plague epidemic, we’re going straight to the (mathematical) source.

- ★ Please read the paper that first defined the SIR model as we know it today, by following the instructions below. The manuscript titled *A Contribution to the Mathematical Theory of Epidemics* was written by W.O. Kermack and A.G. McKendrick in 1927 [9].

This is a technical mathematical paper with notation from 1927. You are not supposed to be comfortable with all of its content. It is expected for all students to struggle through the first reading of this manuscript. Skim the entire paper, but pay particular attention to:

- The *Introduction* section.
- The *General Theory* section. This part is long and gets tedious after a while; it’s important to understand through the end of page 703. We will discuss this together.

As you’re reading, try to take notes of what you’re understanding and any question that comes up for you. The ideal way of reading a mathematics paper is to follow the authors’ argument by deriving the equations along with them. This may only be possible for parts of this paper, but give it a try! Please answer the following questions:



5. Summarize, in your own words and in a bullet point format, the evolution of an epidemic from beginning to end, as described in the *Introduction*.
6. Look for any assumptions the authors make in the *Introduction*. Do you think each of them is reasonable? Why or why not?
7. How do these assumptions relate to the assumptions or conclusions drawn in the Report? Does anything jump out at you?

We expect students to come to our meetings with questions and comments about what they read. Don't be discouraged if the reading is tough — it's hard for everyone, including the instructors. Be kind to yourself: this is likely your first time reading a technical mathematics paper, and this is a manuscript from 1927!

### A.2.1 Derivation

While the early part of the paper should go by more quickly, you may need help in deriving some of the equations, so here are some tips.

The process to derive equation (17) is described on pages 703-706; it involves using infinite series, the method of integrating factors to solve first order linear ODEs, and patience. In particular, the first few lines of the calculation require some leaps of faith on your part, but once the paper shows you the expression for

$$x = f_0(t) + \lambda f_1(t) + \lambda^2 f_2(t) + \dots \quad (\text{A.1})$$

at the top of page 706, you should be able to follow along all the way through to successfully derive equation (17). On page 705, the authors mention that they haven't found a way to solve equation (16) explicitly, but they are going to base their solution process on the observation that (16) is a Volterra-like equation (of the second kind).<sup>5</sup> Do not focus on this too much, i.e., believe that this is possible and accept their solution form reported in equation (A.1).

### A.2.2 Special Cases

After agonizing over some of the details of the *General Theory*, we are now jumping ahead to the *Special Cases* section, Part B. *Constant Rates*, from the bottom of page 712. The "constant rates" referenced in the title of the section are the rate of infectivity  $\phi(t) = \kappa$  (pronounced "fee of tee = kappa") and the rate of removal  $\psi(t) = \ell$  (pronounced "psi of tee = elle"), where both  $\kappa$  and  $\ell$  are constants. In particular, note:

- (a) The nonlinear system of three first order ODEs on page 713 (equation (29) of the paper) is essentially the first so-called SIR model. The variables  $x$ ,  $y$ , and  $z$  represent the number of Susceptible, Infected, and Removed/Recovered individuals in the population, respectively. The total population density is:  $x + y + z = N$  (see also *General Theory*).

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<sup>5</sup>This is a link to learn more about the Volterra integral equations: [https://en.wikipedia.org/wiki/Volterra\\_integral\\_equation](https://en.wikipedia.org/wiki/Volterra_integral_equation).

- (b) Pay particular attention to the plot on page 714 and its caption which continues to page 715 (slightly smaller font than the main text). In the plot, the small black dots correspond to the measured data for rat deaths during the Indian Plague epidemic, and the line with the open circles represents the rat deaths per week predicted by this model.

Answer the following questions:

8. Look for any additional assumptions the authors make in the figure on page 714. Do you think these are reasonable? As mentioned, the figure represents rat deaths over time, not human deaths. Does this make sense in the context of the report of the Indian Plague Commission?
9. Overall, is there anything that stood out to you in the parts you read?

Now let's focus on the equations. We have worked on deriving some of them, but their meaning may have gotten lost in the mathematical details. Hence, now we want you to take a step back and really try to understand what the model means. Here are the equations of the SIR (Susceptible, Infected, Removed) model from the 1927 paper by Kermack and McKendrick [9], written using modern notation for the variables, followed by Figure A1, a visualization of the compartment model:

$$\frac{dS}{dt} = -\kappa S(t)I(t), \quad \frac{dI}{dt} = +\kappa S(t)I(t) - \ell I(t), \quad \frac{dR}{dt} = +\ell I(t). \quad (\text{A.2})$$



Figure A1: Visualization of the SIR model described by (A.2). Parameter  $\kappa$  is the rate of infectivity and  $\ell$  is the rate of removal.

Let's look at the equations in more detail. Read below and answer the following questions:

- The dependent variables are  $S$ ,  $I$ , and  $R$ , representing the number of individuals (or rats, as in the paper) in each of the Susceptible, Infected, or Removed group, respectively. The independent variable  $t$  represents time since the beginning of the infection. The chosen unit for time is selected depending on the situation at hand. The parameters  $\kappa$  and  $\ell$  affect how quickly individuals move from one group to the other. Please answer:
  10. What happens if  $\kappa$  is large? Would people get infected more or less quickly than if  $\kappa$  was small? Explain why.
  11. What happens if  $\ell$  is large? Explain your answer.

- The derivatives  $dS/dt$ ,  $dI/dt$ , and  $dR/dt$  represent the net rate of change of the population of each of the three groups due to all the factors taken into account in the model.
- The  $\kappa S(t)I(t)$  term is based on the **Law of Mass Action**.<sup>6</sup> This law states that the rate at which people get infected in a population is dependent on the product of the number of healthy and infected people. Please answer:
  12. If there is a very small number of infected people,  $I$ , and a large number of healthy people,  $S$ , what will the rate of new infections be?
  13. If there is a large number of infected people,  $I$ , and a very small number of susceptible people,  $S$ , what will the rate of new infections be? Does this make sense to you?
- The  $\ell I(t)$  term is more familiar than you think. This assumes that people get removed from the infectious group,  $I$ , continuously at a rate  $\ell$ .
  14. What type of mathematical function would describe this decay accurately?
  15. If  $\ell = 0.5$ , what is the continuous rate at which people are removed from the infected group at each time unit?
  16. What do the positive and negative signs of each term indicate? (See Figure A.2.2 for a hint.)

Now that you have thought through the meaning of all the terms in the equations, please answer the questions below:

17. Summarize with your own words what the second equation is expressing. Use the diagram of the compartment model in Figure A.2.2 to aid your understanding.
18. What does it mean physically if  $dR/dt = 0$ ?
19. What happens if  $\ell = 0$  and  $\kappa \neq 0$ ?
20. What happens if  $\ell \neq 0$  and  $\kappa = 0$ ?

We would like to *modernize* the parameters in the equations as well, not just the variables. In the modern notation of the SIR model:

- $\kappa = \beta/N$ , where  $\beta$  is called the **rate of contact** and takes into account the probability of contracting the disease when there is contact between a susceptible and an infected individual. It is more realistic to consider a force of infection that does not depend on the absolute number of infectious subjects, but rather on their fraction with respect to the total constant population  $N$ .

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<sup>6</sup>For more information on the Law of Mass Action, see [https://en.wikipedia.org/wiki/Law\\_of\\_mass\\_action](https://en.wikipedia.org/wiki/Law_of_mass_action). In particular, see the section about Mathematical Epidemiology.

- $\ell = \gamma$ , where  $\gamma$  is the **rate of recovery or death**. If the duration of the infection is denoted by  $D$ , then  $\gamma = 1/D$ , since an individual experiences one recovery in  $D$  units of time.

Hence, the **modern SIR model** is defined as:

$$\frac{dS}{dt} = -\frac{\beta}{N} S(t)I(t) \quad (\text{A.3a})$$

$$\frac{dI}{dt} = +\frac{\beta}{N} S(t)I(t) - \gamma I(t) \quad (\text{A.3b})$$

$$\frac{dR}{dt} = +\gamma I(t) \quad (\text{A.3c})$$

As mentioned earlier, this is a nonlinear system of three first-order ODEs. In general, it cannot be solved exactly. However, given some assumptions, solutions for special cases can be derived. Let's analyze some key aspects of this system.

21. If we wanted to solve this system exactly, how many initial conditions would we need to fix a value for the constants of integration?
22. Add the equations to one another to verify that:

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0. \quad (\text{A.4})$$

What does this imply about the sum of  $S(t)$ ,  $I(t)$ , and  $R(t)$ ? (Remember that you know what  $S + I + R$  is equal to.)

23. How many equations need to be solved to find an expression for  $S(t)$ ,  $I(t)$ , and  $R(t)$ ?

### A.2.3 The Basic Reproduction Number, $R_0$

The dynamics of the infectious group depends on the **basic reproduction number**,<sup>7</sup> defined as  $R_0 = \beta/\gamma$ . This ratio can be interpreted as the number of cases one case generates on average over the course of its infectious period in an otherwise uninfected population. This is a useful metric because it is understood that for  $R_0 < 1$  the infection will die out (in this case the disease is referred to as a 'dud'), while for  $R_0 > 1$  the infection will spread in a population (and the disease is referred to as an 'epidemic'). That is because for  $R_0 > 1$ , the infection rate is large relative to the recovery rate  $\gamma$ , and the total number of people to be infected is expected to be large. On the other hand, if  $R_0 < 1$  the recovery rate is fast enough that, while a few people may get infected, the spreading is very slow and not considered to be a full-blown epidemic. See Table A1 below from *History and Epidemiology of Global Smallpox Eradication*<sup>8</sup> to get an idea of typical values for  $R_0$  for well-known infectious diseases.

<sup>7</sup>See [https://en.wikipedia.org/wiki/Basic\\_reproduction\\_number](https://en.wikipedia.org/wiki/Basic_reproduction_number).

<sup>8</sup>The *History and Epidemiology of Global Smallpox Eradication* is a module of the training course "Smallpox: Disease, Prevention, and Intervention" from the CDC and the World Health Organization, 2001. The table appears on slide 17.

Disease	Transmission	$R_0$
Measles	Airborne	12-18
Diphtheria	Saliva	6-7
Smallpox	Airborne droplet	5-7
Polio	Fecal-oral route	5-7
Rubella	Airborne droplet	5-7
Mumps	Airborne droplet	4-7
HIV/AIDS	Sexual contact	2-5
Pertussis	Airborne droplet	5-5
SARS	Airborne droplet	2-5
Influenza (1918)	Airborne droplet	2-3
Ebola (2014)	Bodily fluids	1.5-2.5

Table A1: Values of  $R_0$  for well-known infectious diseases. Taken from *History and Epidemiology of Global Smallpox Eradication* (see footnote 7 for more information).

Now it's time to solve the problem! There are a variety of ways to approach the solution to this system. You should have realized in answering the questions above that we only need to solve two differential equations out of the three to obtain an expression for all  $S$ ,  $I$ , and  $R$ . That is because the sum of the three populations is always equal to  $N$  (the total population density), so, once we have solved two of the three equations, we can take advantage of this  $N$ -fact to write the third solution.

Pair together two equations of your choice and try to solve them. Note that while you have a few options here (three equations can be paired in six ways if the order matters), there are some pairings that make solving easier and others that make it very hard or impossible.

24. Explore the possibilities and report all of your tries, even those that didn't quite work out. Try combining equations by adding, subtracting, multiplying, and dividing them. Remember that our goal is to find a solution in the simplest possible way. These are first-order equations, so let's strive for combining two equations to obtain a simple separable equation to be solved. Use  $R_0 = \beta/\gamma$  wherever you can.
25. Can you combine them to solve for  $S(t)$ ? Write a solution for  $S(t)$  in terms of  $R(t)$ .
26. Can you do the opposite? That is, write out a solution for  $R(t)$  in terms of  $S(t)$ ?
27. Note that each of these solutions should depend only on one undetermined constant. Take the expression you found for  $S(t)$  (in question 25) and set the value of the constant by applying the initial conditions  $S(0) = s_0$  and  $R(0) = r_0$ . Why do we need two conditions for one constant?
28. Now substitute the solution for  $S(t)$  found above in equation (A.3b) and solve, using  $R(0) = r_0$  and  $I(0) = i_0$ . You should find an expression for  $I(t)$  that depends on  $R(t)$  only (i.e., not on  $S(t)$ ).

29. Finally, derive the solution for  $R(t)$  based on the relationship between  $S$ ,  $I$ ,  $R$ , and  $N$  (as discussed earlier). You do not need to write this solution explicitly.

If you followed all the steps above, you should now have solutions to all three ODEs, where the susceptible model  $S(t)$  depends on  $R(t)$  only, the infectious model  $I(t)$  depends on  $R(t)$  only, and the recovered model  $R(t)$  is not written explicitly.

30. Compute the limit as  $t \rightarrow \infty$  for  $R(t)$ . What does this limit represent from an epidemiological point of view?
31. Assume that  $t \rightarrow \infty$  represents the *end* of the epidemic and that  $S(0) \neq 0$ . What does this limit imply with regard to the susceptible population?
32. Based on the conclusion above, how is *the end of an epidemic* defined? What is it caused by?
33. As we discussed earlier, the basic reproduction number  $R_0$  is very important in this model. Rewrite the second equation as

$$\frac{dI}{dt} = \left( \frac{R_0 S}{N} - 1 \right) \gamma I, \quad (\text{A.5})$$

and study the sign of the derivative (i.e., the sign of  $dI/dt$ ) in terms of  $R_0$ . Can you relate your conclusions to what you learned earlier about  $R_0$ ?

You have now read and understood the original SIR paper, be proud of yourself! This was no small feat! Make sure to take a few minutes to collect your thoughts and summarize the main concepts you learned in a bullet-point list.

### A.3 The Ebola Epidemic of 2014-2016 in West Africa

You should be ready to read and thoroughly understand a recent SIAM (Society of Industrial and Applied Mathematics<sup>9</sup>) article discussing the Ebola epidemic of 2014-2016 in West Africa.

- ★ Please read “Emerging Disease Dynamics – The Case of Ebola”, written by Sherry Towers, Oscar Patterson-Lomba, and Carlos Castillo-Chavez. This article appeared in *SIAM News* on November 3rd, 2014.<sup>10</sup> In the article there are a few technical terms. Take a look at section A.4 for some definitions.

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<sup>9</sup>The Society for Industrial and Applied Mathematics (SIAM) is an international community of over 14,000 individual members. Almost 500 academic, manufacturing, research and development, service and consulting organizations, government, and military organizations worldwide are institutional members. For more information about SIAM and to learn about student memberships, visit <https://www.siam.org/>.

<sup>10</sup><https://sinews.siam.org/Details-Page/emerging-disease-dynamics>.

Please answer the following questions:

34. What are the variables the article uses in the graphs? Which of the variables in the SIR model you are familiar with do these represent?
35. What is the chosen unit for time?
36. What type of model do they use to fit their data?
37. What do the authors do to validate their model?
38. Was there anything confusing for you in the article? Anything that you think is explained poorly?
39. Can you spot any weaknesses of the SIR model after reading this? What does the basic SIR model not take into account that is discussed in the article?

#### A.4 Some Vocabulary Related to the SIAM Article

**Optimal Control Strategies.** A general term indicating the fact that having a limited number of resources means not all possible measures to prevent an outbreak can be implemented. The issue is then: which approaches should be taken to have the greatest impact in slowing the spread of the disease? Full quarantine? Medical treatment? Isolation? Something else? And how much money should be put towards each?

**Dimensionless Quantity.** Refers to quantities like  $R_0$  which determine the qualitative behavior of a model. Unsurprisingly, the word *dimensionless* specifically refers to the fact that  $R_0$  has no physical units.

**Ansatz.** A hypothesis, an educated guess, a particular form of a mathematical model. Their model is piecewise exponential. The word *ansatz* comes from German.<sup>11</sup>

**Time series.** Basically, a function that depends on time, typically used when referring to repeatedly sampled data over a span of time. Visually, a plot with time on the horizontal axis.

**95% Confidence Interval.** A statistical term representing the uncertainty in a prediction. Roughly this means that the authors were 95% confident that the true number of new Ebola cases would be somewhere in their predicted interval — though the strict statistical definition of “confidence interval” is more subtle than this.<sup>12</sup>

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<sup>11</sup>For more on the etymology of the word *ansatz* and its use in mathematics, see <https://en.wikipedia.org/wiki/Ansatz>.

<sup>12</sup>For a good explanation of the concept of *confidence interval* see, for example, the video <https://www.youtube.com/watch?v=tFWsu09f74o>.



## A.5 Next Steps

Let's recap what you have read and learned:

- Write a short summary of what you've learned. Note the key aspects of an epidemics and what you've learned about modeling it.
- Reflect on the Report and the *SIAM News* article. Draw any parallels you notice and make a list of main differences.
- Think about next steps, taking inspiration from your answers to questions 38 and 39. What is one key aspect of the SIR model and how it is applied that you think needs improvement? What improvements would be most important to you? Spending less money? Saving more lives? Shortening the epidemic? Distributing funds equitably? What else?

We hope you have enjoyed this guided exploration of the SIR model in an historical context. We will now continue working on applying the model to modern data sets of interest based on each student's preference.

# *Facing the Pandemic Together: Forming a Collaborative Research Group*

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**Keywords:** SIR model, epidemiological modeling, COVID-19, pandemic, research group  
Manuscript received on September 2, 2020; published on March 15, 2021.

**Abstract:** This is an account of how a reading and writing project in an introductory differential equations course was transitioned to a professor-student research group collaborative project, in response to the global COVID-19 pandemic. Adapting on the fly to the ever-evolving pandemic, we collected data, estimated parameters in our models, and computed numerical solutions to SIR-based systems of differential equations. This is a description of what we did and how we found comfort in the project in this time of great uncertainty. The collaboration yielded successes and more questions than we had answers for, but the situation provided an opportunity of a lifetime for my students to engage in a real-world developing situation.

## 1 Introduction

Nestled quietly in the northwest corner of New York State, my Spring 2020 introductory differential equations course began on January 22, 2020 like it had in so many previous spring semesters. We watched a *YouTube* clip of the movie *Contagion*, and I mentioned how my favorite part of *Contagion* and similar films is the cut to the scientific lab where we see a computer model predicting the spread of some deadly virus across the globe. Little were we to know that such images would saturate our lives not three months later. We discussed rates of change and the well-known SIR compartmental model. My course includes a group reading and writing project, and I encouraged students to choose their groups and begin thinking about topics of interest to them. Apart from a few first day jitters amongst the students, all was well.

In late January, the novel coronavirus was lurking in the headlines, and in the second week of the semester, I brought in a pre-print of a research article by Chen et al. [1], in which a detailed compartmental model is developed for the transmission of the novel coronavirus SARS-CoV-2, to share with the students as a way to further explain what I wanted them to do with their group projects. The deadline for group proposals came and went, and nobody chose a topic involving epidemiological models.

Fast forward six weeks to March 11. Many colleges and universities in the northeastern United States had announced plans to close or move education online, and my students

were told that their upcoming spring break had been extended by one week. We all know that things moved quickly after this point, and much of the next two weeks was spent with preparations for completing the semester in an online environment. I immediately thought about my students' group projects, and I took steps to offer an alternative option in order to complete the project. Five of my seven students took me up on the offer of forming a research group to explore variations on the standard SIR model. This is our story.

In putting together the pitch for this collaborative project, I found it necessary to temper the structure of my traditional project with the ever-evolving and constantly changing COVID-19 pandemic. There were some tasks that all group members completed, and there were clear ways in which certain parts could be delegated to individuals who would then report back their findings to the larger group. A regular Wednesday evening meeting served as a requirement that all research group members had to complete. My students embraced this opportunity to work together. We collected data, estimated parameters in our models, and computed numerical solutions to SIR-based systems of differential equations. In this time of great uncertainty, we found comfort in the project and sought ways to understand our models and their relationships to a real-life developing situation. While I hope that such a devastating scenario does not present itself again, the situation provided the opportunity of a lifetime for my students.

## 2 Our Story

The original reading and writing project required that the students create a five-page paper and a group presentation. While other educators adapted such group presentations to an online format, I felt it was necessary to abandon the existing structure so that the course could better align with a rapidly evolving current event. All too often, colleagues in other departments talk about how their students are reading about and learning about hot topics from the current news cycle. I fear that mathematics is sometimes left behind in this context and viewed by students and the general population as a rather stale discipline.

This moment, a global pandemic, was our time to shine. I felt that I would be remiss in not addressing epidemiological models in my course. I clearly remember thinking that, if I did not somehow change my current plans, I would regret the decision in the future. I asked myself the question "How could I adapt the course project in a way to harness our limited preliminary knowledge of ODEs to explore dire questions of disease spread?" I dove into the rapidly expanding literature. It seemed that multitudes were drafting research articles aimed at understanding the transmission and control of COVID-19. Many were seeking to predict its course. I thought to myself "Is there any reason why my students and I cannot be involved with this process?" Fearlessly and humbly, I decided that the answer was "No." I set to work formulating a plan for how we could gather data, process the data, and share in the pandemic experience as a research group. While I had supervised groups of students in course projects and as research assistants, I had never led a research group with students whose mathematical preparation consisted, in some cases, of only a two-semester sequence in calculus. Based on the vast amounts of freely available COVID-19 data being reported daily, I knew that the revised project should include a data

collection component. After finding De Castro [2], I also knew that I wanted to have the students perform parameter estimation, a topic that I had never included in my first-level ODE course. Now, I am a staunch believer in adhering to the syllabus as an agreement between my students and myself, and I only very rarely make adjustments to those items in the syllabus that were essentially agreed upon by all at the beginning of the term. As such, I decided a fair alternative would be to give my students an option for their project completion. Classes were slated to resume remotely after the extended spring break on Monday, March 30. Nearly a week beforehand, I emailed my class. I invited all course students to a video conference in order to discuss course changes, and I gave students a choice:

- Project Option #1: Continue the group project from before without the group presentation.
- Project Option #2: Engage in collaborative work concerning an epidemiological model for COVID-19.

We discussed my ideas for Project Option #2. Student groups were to notify me their preference the next week. All but one group selected Project Option #2. We were all set to go!

In my search of the growing literature, I selected a handful of articles and pre-prints including

- Chen et al. [1], “A Mathematical Model for Simulating the Phase-based Transmissibility of a Novel Coronavirus,” *Infection Diseases of Poverty*, (2020) 9:24.
- De Castro [2], “SIR Model for COVID-19 Calibrated with Existing Data and Projected for Colombia,” pre-print accessed from arXiv:2003.11230 [q-bio.PE] 29 March 2020.
- Sameni [3], “Mathematical Modeling of Epidemic Diseases; A Case Study of the COVID-19 Coronavirus,” pre-print accessed from arXiv:2003.1137v1 [q-bio.PR] 25 March 2020.

The paper by Chen et al. is a slightly revised version of the pre-print that I had shared with the entire class in January during my explanation of the course project. Deciding that all students in the class, not just the five of seven in our research group, could benefit from a more in-depth analysis of SIR and infectious disease models, we all discussed parts of the work by Sameni during the first week of remote instruction after the extended spring break.

Sameni’s article presents an interesting and applicable modification of the SIR model to include an exposed, but not symptomatic, group. Dubbed SEIR in the paper, the model allows for the exposed compartment to increase in size from interactions between susceptible individuals and either symptomatic or asymptomatic individuals. Moreover, a passed-away group is included, and perhaps we might refer to the model as SEIRP. Figure 1 is a schematic for the SEIRP model showing the flow between compartments. As we will see in the details that follow, one reason why I decided to share SEIRP with the entire class is because it gave a description for how a model parameter could be

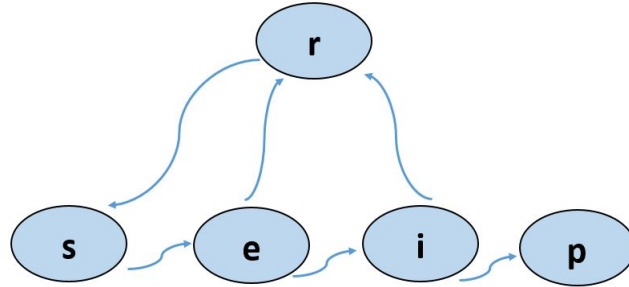


Figure 1: Schematic for SEIRP model adapted from Sameni [3, Figure 5].

estimated using the fourteen-day quarantine that we had all been hearing about in our news feeds. In particular, if one only considers the flow from the exposed compartment to the infected compartment, the differential equation for the exposed compartment is simply the exponential equation

$$\frac{de}{dt} = -\kappa e$$

with solution  $e(t) = e(0)e^{-\kappa t}$ . One can verify that after approximately  $6\kappa^{-1}$  time units, the size of the exposed population decreases by 99.75%. With a 14-day quarantine period, we might then argue that  $14 = 6\kappa^{-1}$  which gives  $\kappa \approx 0.43$  (inverse days).

Our newly formed research group would eventually work with SEIRP and consider its many other parameters, but we needed to back up a bit first. Students had read the details of Option #2, and those who had decided to join the group had some preliminary decisions to make. In particular, Figure 2 is a portion of the document that I had distributed to the entire class. There were tasks for every member to complete, and there was also a division of labor for other parts of the work.

A shared Excel file *COVID\_19\_model* would become the main working spreadsheet for our group. At this point, it mainly consisted of data that I had started to collect for Niagara County, where Niagara University is located, and the spreadsheet listed the number of positive cases as reported via the New York State’s Department of Health COVID-19 Tracker (see Appendix A.1). I also included a file that served as an example for how to implement Euler’s method to obtain numerical solutions to an SIR model. This file had been shared with the entire class earlier in the semester when we had previously considered Euler’s method.

In preparation for our first research group meeting, students were to each select one location and time period. This part of the project was tailored to the fact that the group consisted of six total individuals, the five students and myself. In an email to my research group, I wrote “The distinction between early and late dates will be something for us to decide as a group based on when we think the early and late stages of the particular outbreak took place. Also, we should decide which US location we wish to focus on. Should we focus on all of NY? Should we focus on Niagara County? Should we focus on all of NY except NYC? Should we focus on NYC? There are lots of options. Please let me know your preference.”

**Tasks for Everyone:**

1. Read the article Chen et al. [1], "A Mathematical Model for Simulating the Phase-based Transmissibility of a Novel Coronavirus" available on Canvas.
2. Read the article De Castro [2], "SIR Model for COVID-19 Calibrated with Existing Data and Projected for Colombia" available on Canvas.
3. Download Excel files *COVID\_19\_model* and *numericalSolsEx1* for personal exploration.
4. Participate in Wednesday group meetings.

**Individual Tasks - Choose 1:**

1. Compile data for *Confirmed*, *Deaths*, and *Recovered* from selected location in China during appropriate time periods.
  - (a) Early dates
  - (b) Dates near the peak of the epidemic - present
2. Compile data for *Confirmed*, *Deaths*, and *Recovered* from selected location in Italy during appropriate time periods.
  - (a) Early dates
  - (b) Dates near the peak of the pandemic - present
3. Compile data for *Confirmed*, *Deaths*, and *Recovered* from selected location in the United States during the appropriate time periods.
  - (a) Early dates
  - (b) Dates near the peak of the pandemic - present

Figure 2: Portion of document describing Project Option #2.

### 3 The First Meeting

Our first meeting took place on Wednesday, April 8 via Zoom. It was comforting to see each other, and many students expressed their eagerness to engage in the work with the group. One of the first questions that arose was how to specifically define the early dates and the dates near the peak of the pandemic for the various geographical regions. We also discussed some of the limitations of the basic SIR model, including the fact that it assumes a fixed population size, recovery from the illness, and subsequent immunity from re-infection. To better address the fixed population size, I asked participants to refine their geographical location provided that the relevant data was available. Students chose Hubei in China, New York City (NYC) in the US, and the entirety of Italy. Our main source for data was the "COVID-19 Data Repository by the Center for Systems Science and Engineering (CSSE) at Johns Hopkins University" (see Appendix A.2). For brevity, we occasionally refer to this data as the "JHU CSSE COVID-19 Data."

The discussion about time periods was interesting, and students had much to offer. One student suggested looking at a site (see Appendix A.3) that had charts like Figure 3. Some of the suggested ideas for time period cut-offs included finding dates at which a certain percentage of the peak number of infected individuals had occurred and just an *ad hoc* selection of a cut-off date based on when a certain number of infected individuals had been reached. Since the peak number of infected individuals might not have been

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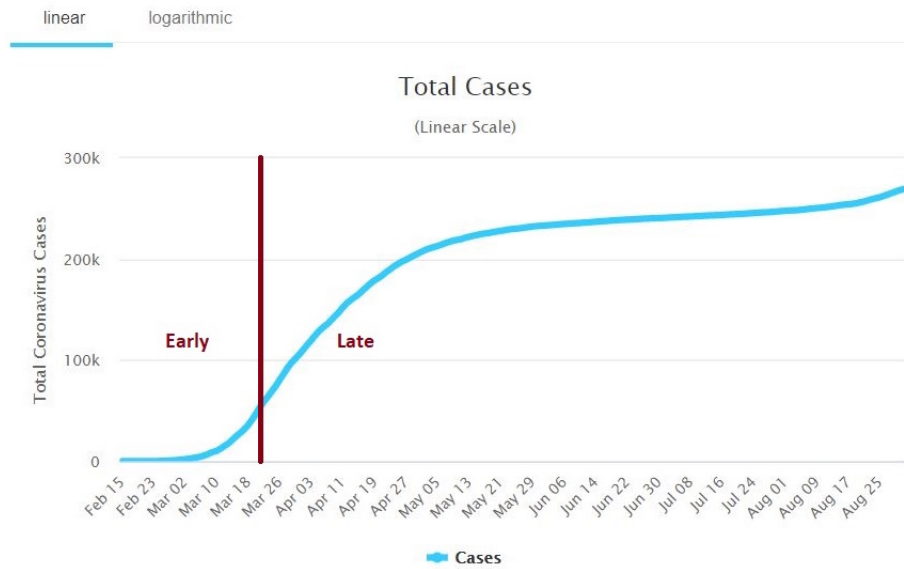


Figure 3: Total coronavirus cases in Italy as reported by Worldometers.info [4].

reached at the time of this first meeting, we abandoned the idea of selecting the cut-off date based on the timing of the peak. Collectively, we agreed to use the rule that the early time period ends when the total number of cases in a given location surpasses 50,000 individuals. This value seemed reasonable to use as a cut-off based on our review of the reported numbers of cases in our regions. As an illustration, the early Italy time period is February 15 through March 20 since the total number of cases on March 20 was 47,044, and the total number of cases on March 21 was 53,598. The other time periods are shown in Table 1, together with the starting population size values,  $N_0$ , as described in Section 4.

Table 1: Time Periods and Population Sizes

Region	Time Period	$N_0$
Early China	(01/22/20 - 02/11/20)	57,237,740
Late China	(02/12/20 - 04/13/20)	57,236,672
Early Italy	(02/15/20 - 03/20/20)	60,461,826
Late Italy	(03/21/20 - 05/03/20)	60,457,794
Early US	(03/02/20 - 03/22/20*)	8,550,971
Late US	(03/23/20 - 04/13/20)	8,550,748

Note that the total number of cases in the US had not yet exceeded 50,000 by the time of our discussion about cut-off dates. As such, we used 10,000 as the cut-off value to obtain the distinction between early and late US time periods. The end dates for the late time periods vary depending on when each student last collected data for their region. While



there are arguably better ways to define the time periods of interest, my students and I could get started with our data collection using this procedure. With so much uncertainty surrounding pandemic modeling, we decided to see what conclusions we would be led to with this decision. We could review our time period definition later if necessary to see if the data seemed to warrant a change. The main task for the group for the next week was individual data collection. Specifically, student learning objectives for the data collection phase of the project were to locate relevant data for their location and time period from the JHU CSSE COVID-19 Data and to compile the cleaned data in a spreadsheet with daily values of confirmed cases, deaths, and recovered cases.

## 4 The Second Meeting

Our second group Zoom conference took place in the evening on Wednesday, April 15. Students had collected data and sent me their spreadsheets beforehand. As an example, let us look at some data that was collected for the late Italy time period. Figure 4 is an image of part of the Excel file created by one of the students.

	A	B	C	D	E
1	Date	Confirmed	Deaths	Recovered	Active
2	3/21	53578	4825	6072	
3	3/22	59138	5476	7024	46638
4	3/23	63927	6077	7432	50418
5	3/24	69176	6820	8326	54030
6	3/25	74386	7503	9362	57521
7	3/26	80589	8215	10361	62013
8	3/27	86498	9134	10950	66414
9	3/28	92472	10023	12384	70065
10	3/29	97689	10779	13030	73880
11	3/30	101739	11591	14620	75528
12	3/31	105792	12428	15729	77635

Figure 4: Portion of initial spreadsheet with cumulative totals for late Italy time period.

The Confirmed, Deaths, and Recovered columns were filled in by hand from the JHU CSSE COVID-19 Data. During our second meeting, we looked at the data. Thinking it wise to start with a previously presented model, we discussed the De Castro [2] paper in which a traditional SIR model is employed. With only the Infected class  $I$  (from the Active data) and the Recovered class  $R$  (which included both the Deaths and Recovered columns) data available at first, we all quickly noticed the need to create a corresponding Susceptible class,  $S$ , data for each date. Students were tasked with this process for the next meeting. For the students working with the early time periods, a starting population size  $N$  was required. We decided that students would find this value via an internet search while noting their source. Using the fixed population size assumption  $S + I + R = N$ , students could then compute an  $S$  value for each day with  $S = N - I - R$ . Students charged with the late time periods decided to find their starting  $N$  value as the final  $S$  value from the corresponding early group. A more realistic approach is to select the starting  $N$  value for the late time periods by calculating the difference between the early time period  $N$  value and the total number of deaths during the early time period. The starting population size,  $N_0$ , values reported for the late time periods in Table 1 are computed this way.

Since it is the unlikely case that  $N$  is fixed, we were led to discuss other limitations of DeCastro. De Castro [2] In particular, students pointed out that the traditional SIR model assumes full recovery and immunity for those who transition from the  $I$  to  $R$  compartments. Certainly the deceased data is different than recovered data, and immunity for COVID-19 is still an unknown question. Despite these drawbacks to a traditional SIR approach, the work by De Castro [2] gave our group a way to approximate the two parameters in the model, namely the transmission coefficient  $\psi$  and the recovery rate  $\delta$ . With this notation, the SIR system in De Castro [2] is written as

$$\begin{aligned}\frac{dS}{dt} &= -\psi SI \\ \frac{dI}{dt} &= \psi SI - \delta I \\ \frac{dR}{dt} &= \delta I.\end{aligned}$$

Taking the time step as  $\Delta t = 1$  day (consistent with our available data) and approximating the derivatives  $\frac{dS}{dt}$  and  $\frac{dI}{dt}$  with finite backward differences, coefficients  $\psi_n$  and  $\delta_n$  could be computed for each day via

$$\psi_n = -\frac{S_n - S_{n-1}}{S_n I_n} \quad \text{and} \quad \delta_n = \frac{\psi_n S_n I_n - (I_n - I_{n-1})}{I_n},$$

where the subscript  $n$  refers to the  $n$ th day in one's data set. Letting  $N_d$  be the total number of days,  $\psi$  and  $\delta$  can then be approximated as numerical means

$$\psi = \frac{\sum_{n=1}^{N_d} \psi_n}{N_d} \quad \text{and} \quad \delta = \frac{\sum_{n=1}^{N_d} \delta_n}{N_d}.$$

After demonstrating these computations to the group during our second meeting, students were asked to perform the computations with their own data for the next week. Students seemed to agree that these computations were straightforward to complete. We forged on. Equipped with  $\psi$  and  $\delta$ , we then used Euler's method to compute numerical solutions for the SIR model. Not surprisingly, as I assured the group, the numerical solution predictions were vastly different than the data. Since we were just getting started with the project, we did not stress about this too much. I followed-up with a numerical solution from Matlab's ODE45 solver that uses a Runge-Kutta method. While more accurate, the Matlab approximation for the number of infected individuals on the  $n$ th day was still quite different than  $I_n$ .

## 5 The Third Meeting

The students and I were largely unconcerned with the inaccuracies of our SIR model since we had plans to consider the SEIRP model which was anticipated to be more accurate. We met next on April 22. After receiving and reviewing the requested computations of  $S$ ,  $\psi$ , and  $\delta$  from each student, I decided that in our third meeting we would look

instead at the SEIRP model in Sameni [3]. As is standard in the analysis of an SIR model, we divide each of  $S, E, I, R,$  and  $P$  by the total population size  $N$  in order to form the proportions in each class. Lowercase letters are used to label the proportions, i.e.,  $s = S/N, e = E/N, i = I/N, r = R/N,$  and  $p = P/N$ . The SEIRP system from Sameni [3, Eq. (12)] is

$$\begin{aligned} s' &= -\alpha_e se - \alpha_i si + \gamma r \\ e' &= \alpha_e se + \alpha_i si - \kappa e - \rho e \\ i' &= \kappa e - \beta i - \mu i \\ r' &= \beta i + \rho e - \gamma r \\ p' &= \mu i. \end{aligned}$$

Students agreed that this model seemed more realistic as it contained the additional compartments  $E$  for exposed, asymptomatic individuals and  $P$  for those who had died. Based on what we had all been hearing about COVID-19, SEIRP seemed like a better choice as a model. News sources frequently discussed exposed, asymptomatic individuals, and everyone regularly heard updated death counts. While including more compartments seemed to be appropriate, the SEIRP model included more parameters which needed to be estimated. The group decided on a few different methods for the selection of the parameters. I summarized the plan in a document to the students, and it is copied in Figure 5 (on the next page) for convenience. Item 4 and Item 6 in Figure 5 serve as explicit student learning objectives for the modeling portion of the project. Parameter values for our regions are given in Table 2.

Table 2: Calculated Parameter Values

Region	Recovery Rate $\beta$	Mortality Rate $\mu$
Early China	0.01007	0.01054
Late China	0.08561	0.00198
Early Italy	0.02623	0.02103
Late Italy	0.01893	0.00693
Early US	0	0.00157
Late US	0	0.00604

For our Euler’s method numerical solution, group members noted that initial conditions were needed for all of the compartments. The glaringly difficult one to estimate was  $E_0$ , since  $E_0$  is impossible to know exactly for a given location and starting time. We discussed how we might instead use  $E_0$  as a variable in our computations. In other words, we could compute an estimate to  $I_n$  and check our accuracy for different values of  $E_0$ . In this way, the group decided that we might be able to discover some acceptable  $E_0$  values.

#### Parameter and Initial Condition Estimation/Exploration

Here are some things to do with our data in order to compute numerical solutions to the SEIRP model.

1. Perhaps use  $\kappa = 0.43$  as in Sameni's article.
2. Explore the use of different values for  $\alpha_i$  and  $\alpha_e$ . Note that we expect  $\alpha_e > \alpha_i$ .
3. Explore different  $\gamma$  values. Note that  $\gamma = 0$  corresponds to the case where the disease confers immunity.
4. Estimate the recovery rate  $\beta$  by performing computations with your data like we did for the SIR model in DeCastro's article. You should be able to compute a sequence of recovery rates  $\beta_n$  with the formula

$$\beta_n = \frac{r_{n+1} - r_n}{i_n},$$

where the subscript  $n$  refers to the  $n$ th day. Then define  $\beta$  as the average

$$\beta = \frac{\sum_{n=1}^{N_d} \beta_n}{N_d},$$

where  $N_d$  is the total number of days of data.

5. As in Sameni's work, make the parameter  $\rho$  of the same order as  $\beta$ . Explore the use of different  $\rho$  values.
6. Estimate the mortality rate  $\mu$  by performing computations with your data like we did for the SIR model in DeCastro's article. You should be able to compute a sequence of mortality rates  $\mu_n$  with the formula

$$\mu_n = \frac{p_{n+1} - p_n}{i_n},$$

where the subscript  $n$  refers to the  $n$ th day. Then define  $\mu$  as the average

$$\mu = \frac{\sum_{n=1}^{N_d} \mu_n}{N_d},$$

where  $N_d$  is the total number of days of data.

Figure 5: Portion of document describing parameter selection and computation.

## 6 The Fourth Meeting

We convened as a research group for the final time on Monday, May 4. I had previously compiled recent modifications to all of the student's individual spreadsheets. In this updated spreadsheet, I also provided formulas for an Euler's method solution to the SEIRP model using parameter values as in Sameni [3, Examples 3-5]. One such solution can be seen in Figure 6.

Students were also given a file containing additional tasks to complete in order to finish the project. In particular, Figure 7 is a portion of that document.

While I hoped that many students would complete the tasks, I realized that it might be necessary to relax my expectations for the project completion. Our fourth meeting would provide me with an opportunity to assess how far along we had come. We started our discussion by re-visiting Sameni [3]. Students had attempted to compute the mortality rate and recovery rates, but not everyone had succeeded. Additionally, attempts by the students to compute numerical solutions to the SEIRP model were largely unsuccessful. I decided that it was necessary to spend more time discussing the examples that I had provided to the group. During this final meeting, we spent time looking at the Euler's method formulas that I had implemented in our shared Excel file. This discussion seemed to clear up some of the misunderstandings that students had, and I encouraged them to pursue our shared goals from the list of additional tasks on their own time.

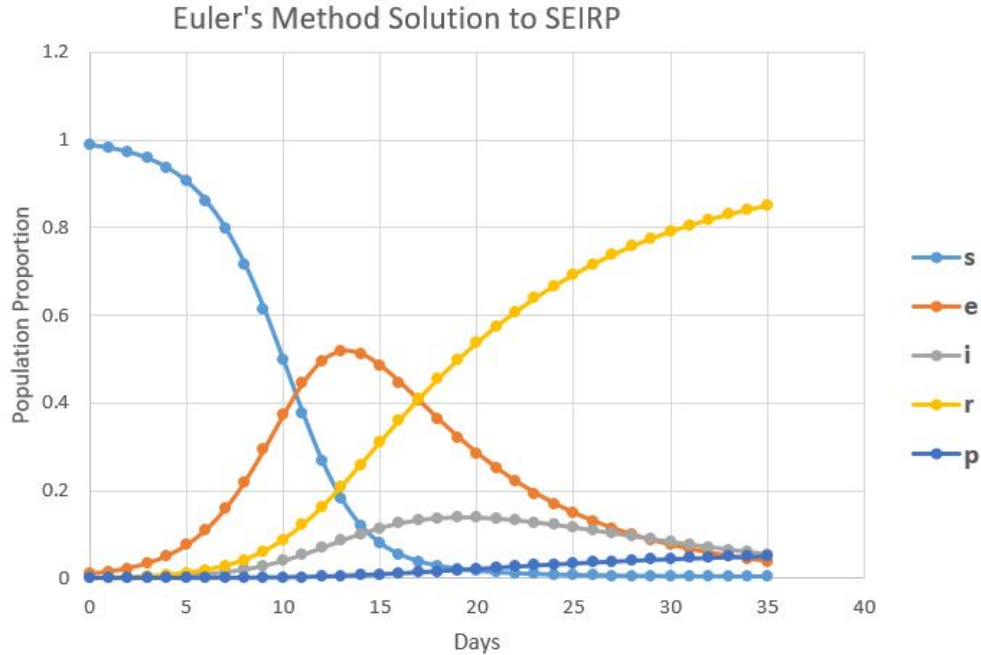


Figure 6: Numerical Solutions to SEIRP with parameter values as in Sameni [3, Ex. 3].

In order to enforce the constraint  $s + e + i + r + p = 1$ , solve for  $s = 1 - e - i - r - p$  and substitute this expression into the  $e'$  equation in place of  $s$ . The resulting system has one fewer equation. We no longer need the equation for  $s'$  since we can determine  $s$  by knowing the other four variable values.

$$\begin{aligned} e' &= (1 - e - i - r - p)(\alpha_e e + \alpha_i i) - \kappa e - \rho e \\ i' &= \kappa e - \beta i - \mu i \\ r' &= \beta i + \rho e - \gamma r \\ p' &= \mu i \end{aligned}$$

Note that this reduced system is Eq. (27) in Sameni's article. The SEIRP tab in our *COVID\_19\_model* Excel spreadsheet is set up to use Euler's method to solve this reduced system for Examples 3, 4, and 5 of Sameni's article. Values for the parameters are given in those examples in such a way as to illustrate a number of different features of the (reduced) SEIRP model.

Our task is to use the data that we found in order to

- Estimate and explore relevant parameter values for COVID-19 in our different locations.
- With our parameter values, assess how well the SEIRP model fits the observed data.
- Use acceptable parameter values in order to predict when the peak of the infection will occur and how many individuals will be infected at that time.

Figure 7: Portion of document describing final project analysis.

The student working with the late Italy time period made good progress with implementing the Euler's method formulas to solve the SEIRP system. Table 3 lists the values that were used for the late Italy time period parameters. To set the initial conditions, the students computed  $e(0) = 10,000/N$ ,  $i(0) = i_0/N$ ,  $r(0) = r_0/N$ ,  $p(0) = p_0/N$ , where the subscript refers to the first actual data value (March 21) for the respective compartment.

The peak of the infection was found to occur on April 18, with 13,528,012 people infected. Comparing this with the data, we see good agreement with the actual date of the peak (April 19), but poor agreement with the actual number of infected individuals (108,257). A more accurate numerical solver will give better results, but it is likely that the parameter values in the model need to be adjusted for better agreement with the observations.

Table 3: Late Italy Parameter Values

$\beta$	$\mu$	$\alpha_e$	$\alpha_i$	$\kappa$	$\rho$	$\gamma$
0.01893	0.00693	0.6	0.4	0.43	1	0

## 7 Project Assessment

Assignment of project grades remained as a final task for me to complete. The course project accounted for 15% of the total course grade. Project proposals submitted before the pandemic interruption were still included in the project grade. For the remaining portion of the project grade, I decided to use a modified version of a presentation rubric that I had used in previous semesters. In the original rubric, three items were assessed on a five point scale (0-4): Time requirements are met at a comfortable pace, Voice is loud and clear, and Style and content complement each other. The modified rubric used the five point scale for three different items: Requested components were submitted in a timely fashion, Student actively participated in Wednesday meetings, and Reasonable mathematics was used within a student's spreadsheet submissions, respectively. Overall, my small research group performed well.

## 8 Conclusions

That is our story. I am happy to share it with you, and I hope that it reminded you of your own journey through these tumultuous times or that it has given you inspiration to forge your own path into real-time modeling with an undergraduate class. We learned a lot through the process. We had fun, we scratched our heads a lot, and we shared some of our feelings of anxiety and loss during an extremely challenging semester. Most importantly, my students obtained a real-life experience that led to a better appreciation for how mathematics is alive and relevant in the world today.

## Appendix A Links to Data and Files

A.1 This is the New York State’s Department of Health COVID-19 Tracker:

<https://covid19tracker.health.ny.gov/views/NYS-COVID19-Tracker/NYSDOHCOVID-19Tracker-Map?%3Aembed=yes&%3Atoolbar=no&%3Atabs=n>

A.2 Our main source for data was the “COVID-19 Data Repository by the Center for Systems Science and Engineering (CSSE) at Johns Hopkins University” (JHU CSSE)

<https://github.com/CSSEGISandData/COVID-19>.

A.3 Some charts like Figure 3 can be found at

<https://www.worldometers.info/coronavirus/#countries>.

## References

- [1] T. Chen, J. Rui, Q. Wang, Z. Zhao, J. Cui, and L. Yin. A mathematical model for simulating the phase-based transmissibility of a novel coronavirus. *Infectious Diseases of Poverty*, page 9:24, 2020.
- [2] C. De Castro. SIR model for COVID-19 calibrated with existing data and projected for colombia. 2020. Pre-print accessed from [arXiv:2003.11230\[q-bio.PE\]](https://arxiv.org/abs/2003.11230).
- [3] R. Sameni. Mathematical modeling of epidemic diseases; a case study of the COVID-19 coronavirus. 2020. Pre-print accessed from [arXiv:2003.1137v1\[q-bio.PR\]](https://arxiv.org/abs/2003.1137v1).
- [4] Worldometers.info. <http://worldometers.info>, 2020. Accessed: 2020-09-01.