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PERIODIC STATIONARITY OF A CHAOTIC MOTION IN THE SYSTEM GOVERNED BY A DUFFING'S EQUATION

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ABSTRACT

By means of a time series analysis developed by the authors a chaotic behavior of a system governed by a Duffing's equation with a periodic external force is demonstrated to have the properties of a periodic stationary random process whose probability distribution is invariant under periodic translations.

It was shown by one of the authors using a digital and analog computers that a system governed by a Duffing's equation with a periodic external force has a chaotic solution in a certain parameter range[1-4], and that the average component of the chaotic solution is a periodic function and the chaotic component has a continuous power spectrum. It was also shown that the attractor as well as the average and the spectrum is little dependent on the numerical error or the noise in the computer experiment. It is an interesting problem to see if such a solution process can be regarded as a sample function of a certain random process. It was conjectured [1,2] on some ground that the chaotic solution process of a Duffing's equation could be a sample function of a periodic stationary process (PSP) [5,6] with the same period with the enternal force. First we briefly describe the periodic stationary process to define its characteristics and then will give the method of time series analysis for such a process[7] in order to apply it to the chaotic solution process.

A random process $X(t)$ is called strictly PSP with period T if its probability distribution is invariant under the periodic translations

$$X(t) \rightarrow X(t + nT), \quad n = 0, \pm 1, \pm 2, \cdots \tag{1}$$

It is called PSP in the wide sense if the mean and the correlation function

$$M(t) = \langle X(t) \rangle \quad , \quad R(t,s) = \langle X(t)X(s) \rangle - M(t)M(s) \tag{2}$$

($\langle \ \rangle$ means the ensemble average) is invariant under periodic translations which means that $M(t)$ is a periodic function and $R(t,s)$ is periodic with respect to $(t + s)/2$ with period T. It was shown[6] that a wide-sense PSP has the spectral representation, which for the purpose of present time series analysis can be conveniently rewritten in the following form:

$$Y(t) = X(t) - M(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \, dZ(\lambda) \tag{3}$$

$$\langle \overline{dZ(\lambda)} dZ(\lambda') \rangle = \sum_{p=-\infty}^{\infty} \delta(\lambda - \lambda' - \frac{2\pi}{T}p) S_p(\lambda) d\lambda d\lambda' \tag{4}$$

$$\overline{S}_p(\lambda) = S_{-p}(-\lambda) = S_{-p}(\lambda - 2\pi p/T) \tag{5}$$

where — denotes the complex conjugate. Eq.(3) shows that the frequency component $dZ(\lambda)$, a random measure, has the nonvanishing covariance only when the frequency difference is $\lambda - \lambda' = 2\pi p/T$ and that $S_p(\lambda)$ gives the (complex) spectral density defined on the p-th diagonal shown in Fig.1 with the origin on the λ' axis. The case of a stationary process corresponds to the limit $T \rightarrow \infty$ so that we only have the spectral density with $p=0$.

Associated with the invariance of the probability measure under the periodic translations (1), we can define the shift $Z \rightarrow U^n Z$, $U^{m+n} = U^n U^m$, of a random variable generated by $X(t)$ or defined in the same probability space. We call ergodic the strict PSP $X(t)$ or the shift U^n if only a constant is invariant under U^n: then we have the law of large number[8]

$$\lim_{M\rightarrow\infty} \frac{1}{M} \sum_{n=0}^{M-1} U^n Z = \langle Z \rangle \tag{6}$$

For an ergodic PSP X(t) we have the ergodic theorem for the mean,

$$M(t) = \lim_{M\to\infty} \frac{1}{M}\sum_{n=0}^{M-1} X(t + nT) \tag{7}$$

and a set of ergodic theorems for the cross-correlation functions[7])

$$\gamma_p(\tau) = \lim_{A\to\infty} \frac{1}{A}\int_0^A Y(t)Y(t+\tau)e^{2\pi i p(t+\tau)/T}\,dt \tag{8}$$

$$= \frac{1}{T}\int_0^T \langle Y(t)Y(t+\tau)\rangle e^{2\pi i p(t+\tau)/T}dt = \int_0^\infty e^{i\lambda\tau}S_p(\lambda)d\lambda$$

$$p = 0,1,2,\cdots \tag{9}$$

which can be shown by means of (6) with Z replaced by X(t) or by the integrand of (8). As a matter of fact $\gamma_p(\tau)$ gives the cross-correlation function of Y(t) and $Y(t)e^{2\pi ipt/T}$ in the sense of generalized harmonic analysis and its cross-power spectrum, by (9), agrees with the spectral density $S_p(\lambda)$ defined by (4). Note that we only have a single ergodic theorem corresponding to p = 0 in the stationary case. Further it can be shown that under a fairly general condition we have

$$\gamma_p(\tau) \equiv 0, \quad p \neq \text{integer} \tag{10}$$

by making the argument similar to Ref.8 (Chap.X, Sec.2 and 6).

For the data of finite length the cross-power spectrum $S_p(\lambda)$ can be estimated for instance by means of the smoothed periodogram: Putting

$$Y_A(\lambda) = \frac{1}{\sqrt{2\pi}}\int_0^A Y(t)e^{-i\lambda t}\,dt\,, \quad -\infty < \lambda < \infty \tag{11}$$

we have the cross-periodogram of the data of length A:

$$S_p^A(\lambda) = \frac{1}{A}\,\bar{\hat{Y}}_A(\lambda)\hat{Y}_A(\lambda - 2\pi p/T) \tag{12}$$

which can be smoothed by a suitable frequency window $W(\lambda)$,

$$\tilde{S}_p(\lambda) = \int_{-\infty}^{\infty} W(\lambda - \lambda')S_p^A(\lambda')d\lambda' \tag{13}$$

When a time series is observed it is important to know if it could be regarded as a sample function of a stationary process, or a periodic stationary process or otherwise (nonstationary). A powerful method to test this is to make use of the relation (10); that is, $\gamma_p(\tau)$ or $S_p(\lambda)$ vanishes unless p = integer if X(t) is periodic stationary, vanishes unless p = 0 if stationary, and otherwise if generally nonstationary. To see this we can estimate a test function, for instance,

$$\sigma(p) = \int |\tilde{S}_p(\lambda)| d\lambda \ , \quad -\infty < p < \infty \tag{14}$$

If $\sigma(p)$ shows a sharp peak only at p = 0 the observed time series can be considered as stationary, if it has peaks at p = integer the time series is stationary; and otherwise it is nonstationary. Although $\sigma(p)$ may have some background noise because of finite data length, we can check the periodic stationarity by means of the relative height of the possible peaks at p = integer.

We apply the above method to the solution processes of the three kinds of Duffing's equations driven by periodic external force:

i) $$\ddot{X}(t) + k\dot{X}(t) + X(t)^3 = B \cos t \tag{15}$$

ii) $$\ddot{X}(t) + k\dot{X}(t) + X(t)^3 = B_o + B \cos t \tag{16}$$

iii) $$\ddot{X}(t) + k\dot{X}(t) + X(t)^3 - X(t) = B \cos t \tag{17}$$

where $\dot{X}(t)$ and $\ddot{X}(t)$ represent the first and the second derivative with respect to t, respectively. The chaotic solution process was generated by by a digital computer with Runge-Kutta method over the length of M = 2,048 period ($T = 2\pi$). The mean function (7) which is to be a periodic function expressible in the form

$$M(t) = \sum_{p=-\infty}^{\infty} C_p \, e^{i\frac{2\pi}{T}pt} \tag{18}$$

was calculated approximately as the arithmetic mean over M periods,and the rapid convergence of the right-hand side of (7) with increasing number of the sum was checked numerically by computing its variance; thus verifying experimentally the ergodic theorem for the mean (7).

For smoothing a rectangular window with the width $|\lambda| \leqq 1/64$ was employed. We show here one example for each Duffing's equation i)-iii) in Figs.2 - 4, respectively. The parameters k, B and B_o are specified in the figures and corresponding strange attractors for Figs.2 and 3 are to be found in Refs.1 - 4; the attractor for Fig.4 is shown in Fig.5. The test function in Figs.2-4A shows peaks at p = integer clearly demonstrating the periodic stationarity. Note that in the cases of i) and iii) the peaks appearing at p = even-integer mean that Y(t) is a wide-sense PSP with apparent period T/2 ; X(t), however, has period T since M(t) has the same period as shown by nonvanishing $|C_1|^2$ in Figs.2-4B. The amplitude of spectral density $S_p(\lambda)$ estimated by the smoothed periodogram is shown in Figs.2-4B, where $S_o(\lambda)$ agrees with the ordinary power spectrum of Y(t) already obtained[1-4]. In view of the above time series analysis we conclude that a chaotic solution process of i)-iii) may be regarded as a sample function of an ergodic PSP or at least a wide-sense PSP with the period of external force.

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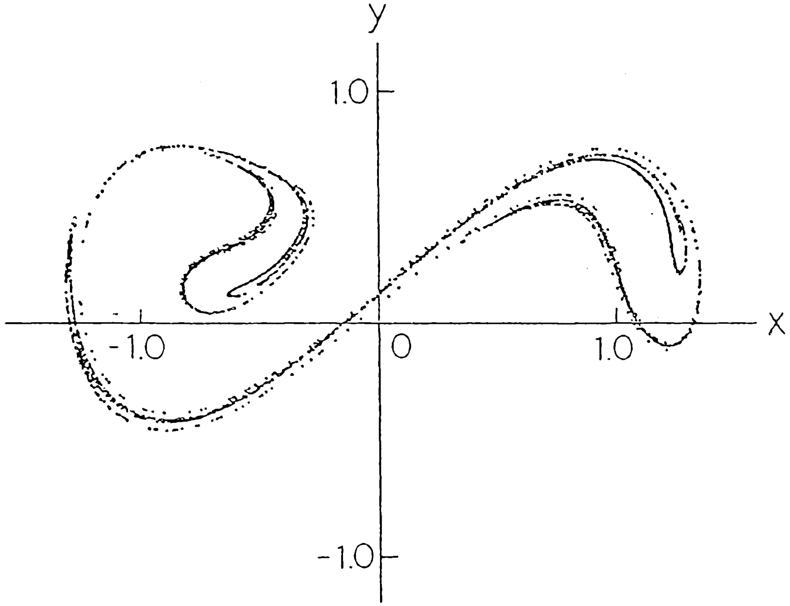


Fig.5

Strange attractor for iii) with k = 0.25, B = 0.3.

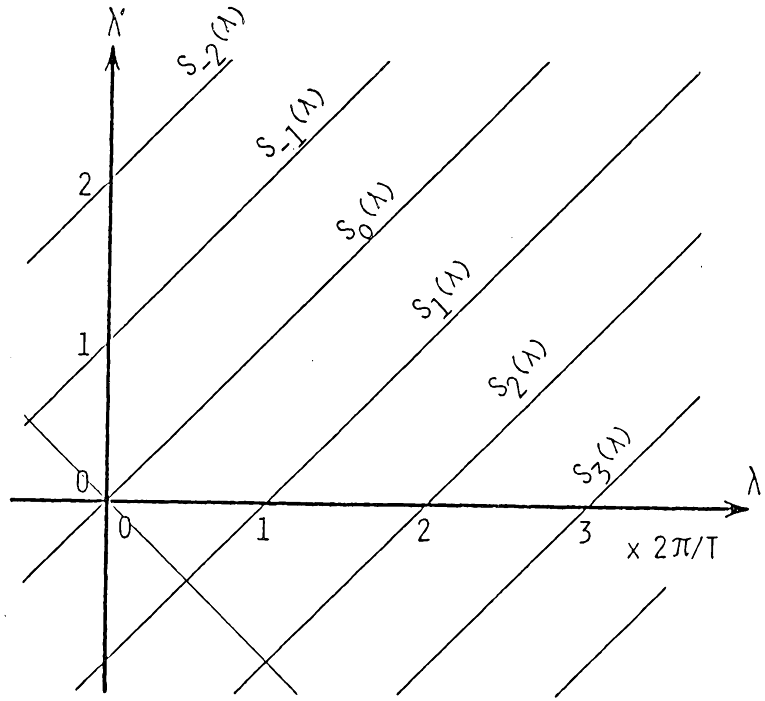
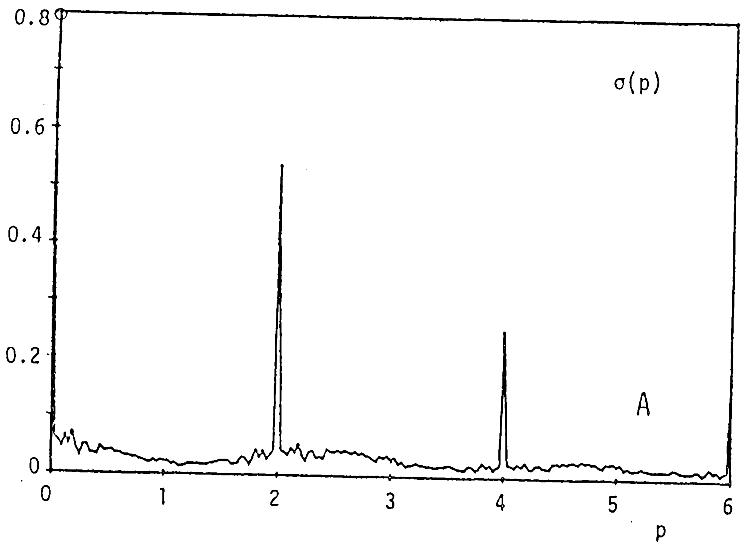


Fig.1

Support for the spectral covariance on (λ,λ') plane. The Spectral density $S_p(\lambda)$ with its origin on λ'-axis is defined on the p-th diagonal and is symmetric with respect to the line $\lambda = -\lambda'$ crossing the diagonals in the figure.



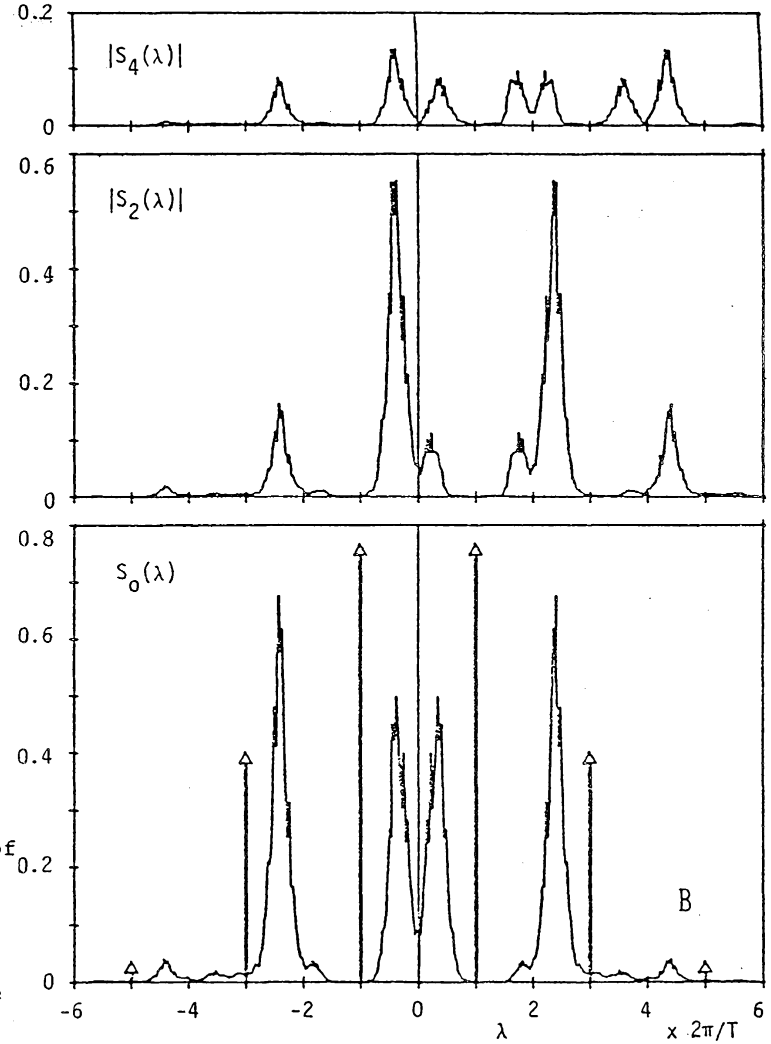


Fig.2

A. Test function of the periodic stationarity for i) with k = 0.1, B = 12.0. A small circle at p = 0 indicates the height of the peak.

B. Amplitude of the spectral density $S_p(\lambda)$, p = 0,2,4. $S_p(\lambda)$ is symmetric with respect to the point $\lambda = \pi p/T$. Arrows in the bottom figure show $|C_p|^2$ for M(t) in (18).

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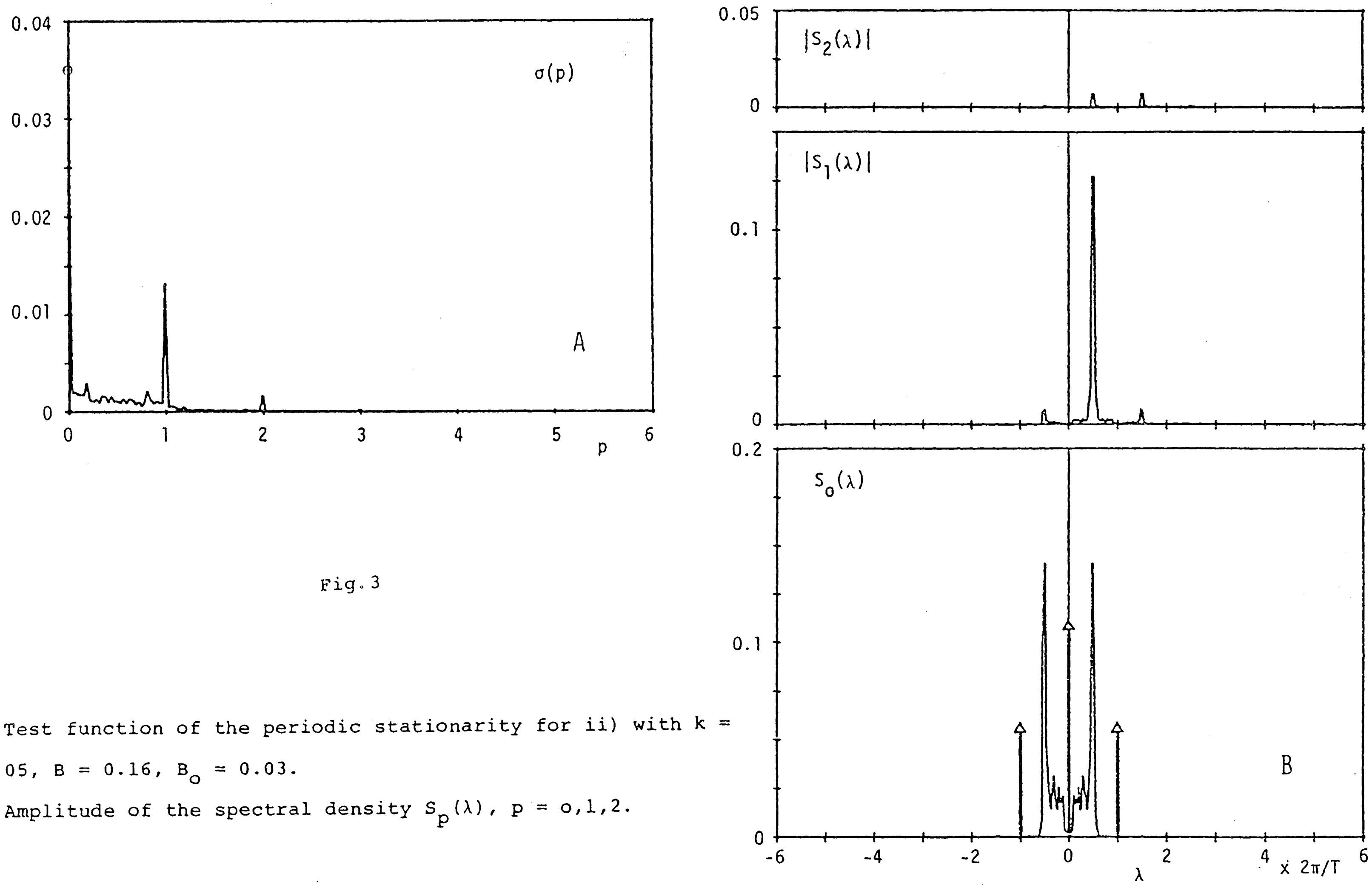
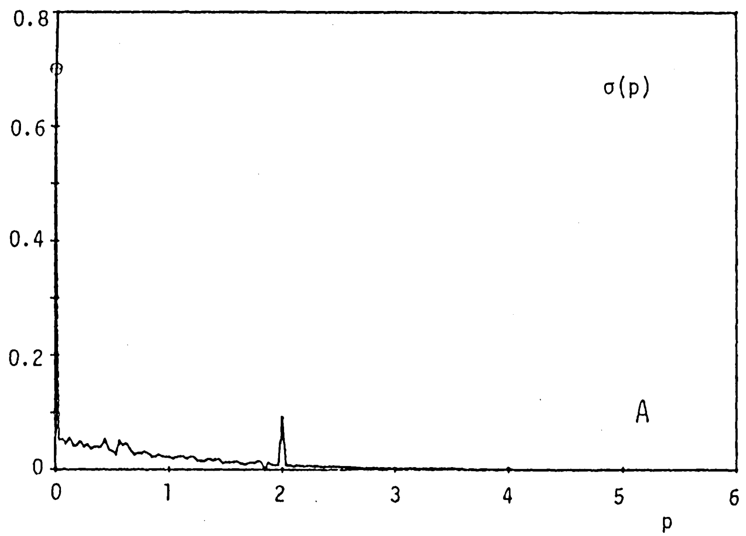


Fig. 3

A. Test function of the periodic stationarity for ii) with k = 0.05, B = 0.16, B_o = 0.03.

B. Amplitude of the spectral density $S_p(\lambda)$, p = o,1,2.



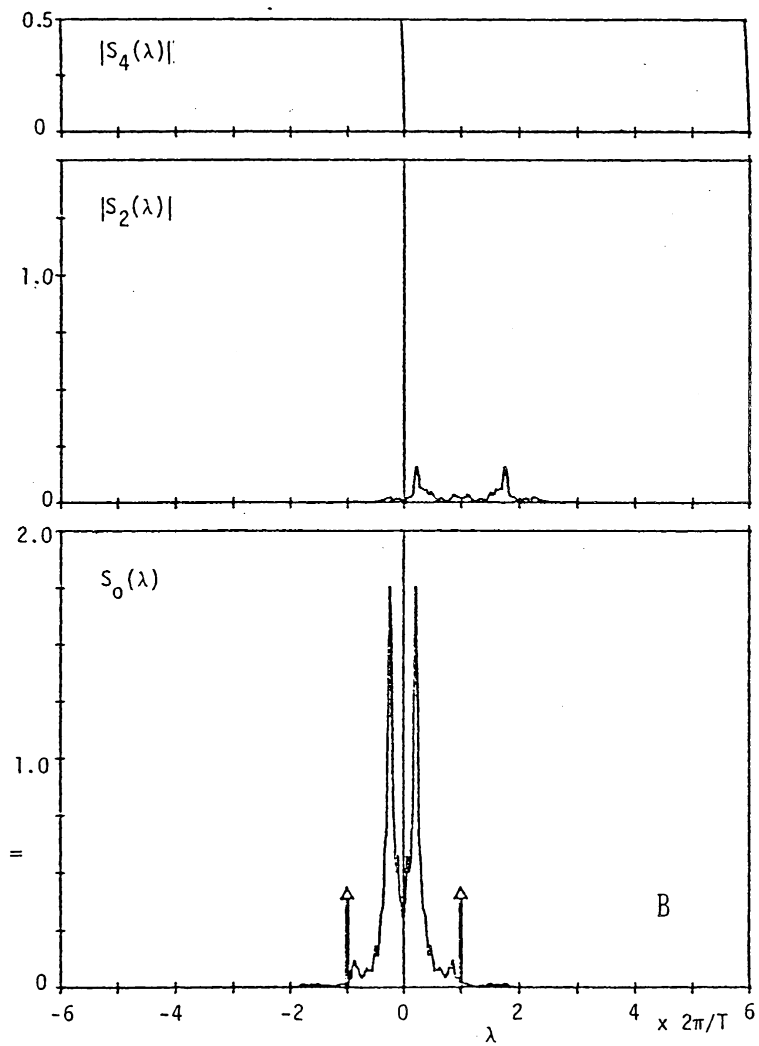


Fig.4

A. Test function of the periodic stationarity for iii) with k = 0.25, B = 0.3.

B. Amplitude of the spectral density $S_p(\lambda)$, p = 0,2,4

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