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Discrete approximations for stochastic differential equations

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1 Introduction

Among discrete approximations for deterministic differential equations (DDEs), Runge-Kutta method (RK method) is well known. The aim of the present paper is to describe stochastic version of RK method and to consider of its applicability for stochastic differential equations (SDEs).

We consider stochastic initial value problem (SIVP) for scalar autonomous stochastic differential equations given by

$$\begin{cases} dX(t) = f(X)dt + g(X)dW(t), & t \in [t_0, T], \\ X(t_0) = X_0, \end{cases} \tag{1}$$

where $W(t)$ represents the standard Wiener processes, namely Gaussian stochastic variables, characterized by their mean and covariance as

$$\mathbf{E}(W(t)) = 0, \tag{2}$$

$$\mathbf{C}_W(t,s) = \mathbf{E}(W(t)W(s)) = \min(t,s). \tag{3}$$

SIVP(1) is equivalent to stochastic integral equation (SIE) for all t in some interval $[t_0, T]$:

$$X(t) = X(t_0) + \int_{t_0}^{t} f(X(s))ds + \int_{t_0}^{t} g(X(s))dW(s). \tag{4}$$

Here, the second integral, called as stochastic integral (SI), is defined by

$$\int_{t_0}^{t} g(X(s))dW(s) = \lim_{h\to 0} \sum_{k=0}^{n-1} g(\lambda X(t_{k+1}) + (1-\lambda)X(t_k))\Delta W_k \tag{5}$$

where $0 \le \lambda \le 1$, $\Delta W_k = W(t_{k+1}) - W(t_k)$, $h = \max(t_{k+1} - t_k)$ and the mode of convergence is in mean square. In particular if $\lambda = 0$, (5) is called Ito SI and if $\lambda = 1/2$, Stratonovich SI. Also corresponding to $\lambda = 0$ and $\lambda = 1/2$, eqn.(1) is called the Ito SDE and the Stratonovich SDE, respectively.

Hereafter we use the following notations in the Stratonovich case:

$$\oint \quad , \quad d_s.$$

2 Preliminaries

First we introduce the central tool of calculus ; Ito's formula:
Assume that the functions f and g satisfy the condition guaranteeing the existence and uniqueness of solution to SIVP(1), that is

i) The functions $f(x)$ and $g(x)$ are measurable with respect to x, for $x \in \mathbf{R}$.

ii) There exists a constant K satisfying for $x, y \in \mathbf{R}$

(a) Lipshitz condition

$$|f(x) - f(y)| + |g(x) - g(y)| \leq K|x - y|,$$

(b) linear growth condition

$$|f(x)|^2 + |g(x)|^2 \leq K^2(1 + |x|^2).$$

iii) X_0 is independent on $W(t)$ for $t > 0$, and $\mathbf{E}X_0^2 < \infty$.

If the real-valued function $F(x)$ has continuous derivatives F', F'' for $x \in \mathbf{R}$ and $X(t)$ is a solution of SDE (1), then $F(X(t))$ has the stochastic differential

$$dF(X(t)) = [fF' + \frac{1}{2}g^2F''](X(t))dt + [gF'](X(t))dW(t). \qquad (6)$$

For simplicity by introducing the following operators,

$$L_f = f\frac{d}{dx} + \frac{1}{2}g^2\frac{d^2}{dx^2},$$

$$L_g = g\frac{d}{dx},$$

(6) is expressed as

$$dF(X(t)) = [L_fF](X(t))dt + [L_gF](X(t))dW(t) \qquad (7)$$

and its integral version is given for $t \in [t_0, T]$ by

$$F(X(t)) = F(X(t_0)) + \int_{t_0}^{t} [L_fF](X(s))ds + \int_{t_0}^{t} [L_gF](X(s))dW(s). \quad (8)$$

Between the Ito SI and Stratonovich SI holds the following relationship:

$$\oint_a^b g(X(s))dW(s) = \int_a^b g(X(s))dW(s) + \frac{1}{2}\int_a^b [g'g](X(s))ds. \qquad (9)$$

This implies that the solution for the Stratonovich SDE

$$d_sX = f(X)dt + g(X)dW(t)$$

is also the solution of the Ito SDE

$$dX = [f + \frac{1}{2}g'g](X)dt + g(X)dW(t).$$

Conversely, the solution X(t) of the Ito SDE

$$dX = f(X)dt + g(X)dW(t)$$

solves the Stratonovich SDE

$$d_sX = [f - \frac{1}{2}g'g](X)dt + g(X)dW(t).$$

3 Runge-Kutta approximations
Stochastic version of RK method; m-stage RK method has the form

$$\bar{X}_n = \bar{X}_{n-1} + \sum_{i=1}^{m} p_i F_i h + \sum_{i=1}^{m} q_i G_i \Delta W_n, \tag{10}$$

where

$$\begin{aligned}
\bar{X}_0 &= X_0, \\
F_1 &= f(\bar{X}_{n-1}), \\
G_1 &= g(\bar{X}_{n-1}), \\
F_2 &= f(\bar{X}_{n-1} + \beta_{21}F_1 h + \gamma_{21}G_1\Delta W_n), \\
G_2 &= g(\bar{X}_{n-1} + \beta_{21}F_1 h + \gamma_{21}G_1\Delta W_n), \\
&\vdots \\
F_m &= f(\bar{X}_{n-1} + \textstyle\sum_{j=1}^{m-1}\beta_{mj}F_j h + \sum_{j=1}^{m-1}\gamma_{mj}G_j\Delta W_n), \\
G_m &= g(\bar{X}_{n-1} + \textstyle\sum_{j=1}^{m-1}\beta_{mj}F_j h + \sum_{j=1}^{m-1}\gamma_{mj}G_j\Delta W_n),
\end{aligned} \tag{11}$$

with

$$h = \Delta t_n = t_n - t_{n-1},$$
$$\Delta W_n = W(t_n) - W(t_{n-1}).$$

RK method (10) yields sequences which approximate the sample paths of the solution $X(t)$ of SIVP(1). That is, the numerical approximation is generated iteratively from the difference equation with increments

$$\Delta t_n = t_n - t_{n-1}$$

corresponding to the chosen interval partition

$$t_0 < t_1 < \cdots < t_n < \cdots < t_N = T,$$

and the Wiener increments

$$\Delta W_n = W(t_n) - W(t_{n-1}),$$

which are obtained as sample values of normal random variables of mean zero and variance Δt_n:

$$\Delta W_n = (\Delta t_n)^{\frac{1}{2}}\xi, \qquad \xi \in N(0,1).$$

It is convienient to work with equally spaced partitions. Therefore we now use the following notation

$$h = \Delta t_n = \frac{T - t_0}{N}, \qquad \Delta W = \Delta W_n = h^{\frac{1}{2}}\xi.$$

The continuous parameter process corresponding to RK method (10) is given by

$$\bar{X}_n = \bar{X}_{n-1} + (t - t_{n-1})\sum_{i=1}^{m} p_i F_i + [W(t) - W(t_{n-1})]\sum_{i=1}^{m} q_i G_i, \tag{12}$$

$$t \in (t_{n-1}, t_n]$$

and (11).

Rümelin (1982) has established the following convergence result for RK method (10).

Theorem 1 *(Rümelin [5])*

Suppose f, f', g, g', g'' are bounded. Then the corresponding continuous parameter process (12) defined by the m-stage RK method (10) converges uniformly on $[t_0, T]$ in the quadratic mean sense to the Ito solution of

$$dX = [f + \lambda g'g](X)dt + g(X)dW(t).$$

Here the correction factor is $\lambda = 0$ for $m = 1$ and

$$\lambda = \sum_{i=2}^{m} q_i \sum_{j=1}^{i-1} \gamma_{ij} \qquad \textit{for } m \geq 2. \tag{13}$$

Remark Let

$$d_i = \sum_{j=1}^{i-1} \gamma_{ij},$$

then expression (13) is rewritten as

$$\lambda = \sum_{i=2}^{m} q_i d_i.$$

Thus if RK method having the order larger than or equal to 2 as the quadrature for the second integral in (4), then we have

$$\lambda = \sum_{i=2}^{m} q_i d_i = \frac{1}{2}.$$

Therefore if this method is applied to Ito SDE, the numerical solution converges to Stratonovich solution. That is, to obtain Ito solution using RK method (10), one requires the following transformation:

$$f \rightarrow f - \frac{1}{2} g' g.$$

If $X(t)$ and $\bar{X}_n$ denote the exact solution and numerical solution of SIVP (1), respectively, the local error from $t = t_{n-1}$ to $t = t_n$ is defined by the following:

$$\mathbf{E}(|X(t_n) - \bar{X}_n|^2 | X(t_{n-1}) = \bar{X}_{n-1} = \bar{x}_{n-1})$$

where $\bar{x}_{n-1}$ is an arbitary real value.

Definition 1 *The numericl scheme $\bar{X}_n$ is of order γ iff*

$$\mathbf{E}(|X(t_n) - \bar{X}_n|^2 | X(t_{n-1}) = \bar{X}_{n-1} = \bar{x}_{n-1}) = O(h^{\gamma+1}) \qquad (h \downarrow 0).$$

4 RK schemes of lower order

First of all we give two known RK schemes for SDE.

1. $m = 1$, $\gamma = 1$.

   Euler-Maruyama scheme (Maruyama 1955):

$$\begin{cases} \bar{X}_0 &= X_0, \\ \bar{X}_n &= \bar{X}_{n-1} + f(\bar{X}_{n-1})h + g(\bar{X}_{n-1})\Delta W. \end{cases} \tag{14}$$

2. $m = 2$, $\gamma = 2$.

   Heun scheme (McShane 1974):

$$\begin{cases} \bar{X}_0 = X_0, \\ \bar{X}_n = \bar{X}_{n-1} + \frac{1}{2}[F_1 + F_2]h + \frac{1}{2}[G_1 + G_2]\Delta W. \end{cases} \tag{15}$$

where

$$\begin{aligned} F_1 &= F(\bar{X}_{n-1}), \\ G_1 &= g(\bar{X}_{n-1}), \\ F_2 &= F(\bar{X}_{n-1} + F_1 h + G_1 \Delta W), \\ G_2 &= g(\bar{X}_{n-1} + F_1 h + G_1 \Delta W), \end{aligned}$$

$$F = f - \frac{1}{2}g'g. \tag{16}$$

By virtue of the Remark for Theorem 1, (16) is required to give the solution of Ito SDE. Similarly, we can attempt to construct a 3-stage RK scheme; stochastic version of 3-stage Heun method:

$$\begin{cases} \bar{X}_0 = X_0, \\ \bar{X}_n = \bar{X}_{n-1} + \frac{1}{4}[F_1 + 3F_3]h + \frac{1}{4}[G_1 + 3G_3]\Delta W, \end{cases} \tag{17}$$

where

$$\begin{aligned} F_1 &= F(\bar{X}_{n-1}), \\ G_1 &= g(\bar{X}_{n-1}), \\ F_2 &= F(\bar{X}_{n-1} + \tfrac{1}{3}F_1 h + \tfrac{1}{3}G_1\Delta W), \\ G_2 &= g(\bar{X}_{n-1} + \tfrac{1}{3}F_1 h + \tfrac{1}{3}G_1\Delta W), \\ F_3 &= F(\bar{X}_{n-1} + \tfrac{2}{3}F_2 h + \tfrac{2}{3}G_2\Delta W), \\ G_3 &= g(\bar{X}_{n-1} + \tfrac{2}{3}F_2 h + \tfrac{2}{3}G_2\Delta W), \end{aligned}$$

$$F = f - \frac{1}{2}g'g.$$

But unfortunately this scheme has order 3 only if SDE (1) holds $fg' + \frac{1}{2}g^2g'' = f'g$. This result of Rümelin is described in the folowing theorem

Theorem 2 *(Rümelin [5])*

Suppose $f(x)$ and $g(x)$ have continuous and bounded derivatives up to the sixth order . Then if consistency condition

$$fg' + \frac{1}{2}g^2g'' = f'g,$$

namely

$$L_f g = L_g f \tag{18}$$

isn't satisfied, any RK method cannot attain order 3.

To verify above, one expands 3-stage RK scheme (17) at $(t_{n-1}, \bar{X}_{n-1})$ via Taylor series as follows

$$\begin{aligned} \bar{X}_n &= \bar{X}_{n-1} + [f - \tfrac{1}{2}g'g]_{n-1}h + g_{n-1}\Delta W \\ &\quad + \tfrac{1}{2}[g'g]_{n-1}(\Delta W)^2 \\ &\quad + \tfrac{1}{2}[f'g + g'f - g'^2 g - \tfrac{1}{2}g''g^2]_{n-1}h\Delta W \\ &\quad + \tfrac{1}{6}[g'^2 g + g''g^2]_{n-1}(\Delta W)^3 \\ &\quad + O(h^2) + O(h|\Delta W|^2) + O(|\Delta W|^4) \\ &= \bar{X}_{n-1} + [f - \tfrac{1}{2}L_g g]_{n-1}h + g_{n-1}\Delta W \\ &\quad + [\tfrac{1}{2}(L_f g + L_g f) - \tfrac{1}{2}L_g^2 g]_{n-1}h\Delta W \\ &\quad + \tfrac{1}{2}[L_g g]_{n-1}(\Delta W)^2 \\ &\quad + \tfrac{1}{6}[L_g^2 g]_{n-1}(\Delta W)^3 \\ &\quad + O(h^2) + O(h|\Delta W|^2) + O(|\Delta W|^4). \end{aligned} \tag{19}$$

On the other hand, Taylor scheme of order 3 proposed by Wagner and Platen (1978) (see [1] in detail) which is derived from Ito's formula (6) has the following form:

$$\begin{aligned}\bar{X}_0 &= X_0,\\ \bar{X}_n &= \bar{X}_{n-1} + [f - \tfrac{1}{2}L_g g]_{n-1}h + g_{n-1}\xi_1 h^{\frac{1}{2}}\\ &\quad+\tfrac{1}{2}[L_f g]_{n-1}(\xi_1 + \tfrac{1}{\sqrt{3}}\xi_2)h^{\frac{3}{2}}\\ &\quad+\tfrac{1}{2}[L_g f]_{n-1}(\xi_1 - \tfrac{1}{\sqrt{3}}\xi_2)h^{\frac{3}{2}}\\ &\quad-\tfrac{1}{2}[L_g^2 g]_{n-1}\xi_1 h^{\frac{3}{2}}\\ &\quad+\tfrac{1}{2}[L_g g]_{n-1}\xi_1^2 h\\ &\quad+\tfrac{1}{6}[L_g^2 g]_{n-1}\xi_1^3 h^{\frac{3}{2}},\end{aligned} \tag{20}$$

where ξ_1 and ξ_2 are independent of the random variables $N(0,1)$.

Replacing with

$$\Delta W = \xi_1 h^{\frac{1}{2}}, \qquad \Delta\tilde{W} = \xi_2 h^{\frac{1}{2}},$$

expresion (20) turns to

$$\begin{aligned}\bar{X}_0 &= X_0,\\ \bar{X}_n &= \bar{X}_{n-1} + [f - \tfrac{1}{2}L_g g]_{n-1}h + g_{n-1}\Delta W\\ &\quad+[\tfrac{1}{2}(L_f g + L_g f) - \tfrac{1}{2}L_g^2 g]_{n-1}h\Delta W\\ &\quad+\tfrac{1}{2}[L_g g]_{n-1}(\Delta W)^2\\ &\quad+\tfrac{1}{6}[L_g^2 g]_{n-1}(\Delta W)^3\\ &\quad+\tfrac{1}{2\sqrt{3}}[L_g f - L_f g]_{n-1}h\Delta\tilde{W}.\end{aligned} \tag{21}$$

Comparing (19) with (21), we establish an improved version of 3-stage RK scheme:

$$\begin{aligned}\bar{X}_0 &= X_0,\\ \bar{X}_n &= \bar{X}_{n-1} + \tfrac{1}{4}[F_1 + 3F_3]h + \tfrac{1}{4}[G_1 + 3G_3]\Delta W,\\ &\quad+\tfrac{1}{2\sqrt{3}}[L_g f - L_f g]_{n-1}h\Delta\tilde{W}\end{aligned} \tag{22}$$

where

$$\begin{aligned}F_1 &= F(\bar{X}_{n-1}),\\ G_1 &= g(\bar{X}_{n-1}),\\ F_2 &= F(\bar{X}_{n-1} + \tfrac{1}{3}F_1 h + \tfrac{1}{3}G_1\Delta W),\\ G_2 &= g(\bar{X}_{n-1} + \tfrac{1}{3}F_1 h + \tfrac{1}{3}G_1\Delta W),\\ F_3 &= F(\bar{X}_{n-1} + \tfrac{2}{3}F_2 h + \tfrac{2}{3}G_2\Delta W),\\ G_3 &= g(\bar{X}_{n-1} + \tfrac{2}{3}F_2 h + \tfrac{2}{3}G_2\Delta W),\end{aligned}$$

$$F = f - \frac{1}{2}g'g$$

with independent random variables ΔW and $\Delta\tilde{W}$ of normal distribution $N(0,h)$. Note that if consistency condition (18) $L_g f = L_f g$ is satisfied, the improved 3-stage RK scheme (22) coinsides with the 3-stage RK scheme (17).

5 A numerical example

The schems presented in the previous section will now be demonstrated through a simple example, the stochastic Ginzburg-Landau equation (see [2])

$$d_s X = [\alpha X - X^3]dt + \sigma X dW, \tag{23}$$

$$dX = [(\alpha + \frac{1}{2}\sigma^2)X - X^3]dt + \sigma X dW, \tag{24}$$

with parameters α and σ. Note that this equation doesn't satisfy the consistency condition (18). We will determine the second moment at $t = 3$ with the starting value $X(0) = 1$ and parameters $\alpha = \sigma = 2$. The simulation was done with sample number N= $100,000$ and different time stepsizes. We used three numerical schemes: (i) the Euler-Maruyama scheme (14), (ii) 2-stage RK scheme (15) and (iii) improved 3-stage RK scheme (22). Also numerical solution was considered for the Ito solution. Namely the scheme (i) is applied to eqn.(24), while the schemes (ii) and (iii) without transformation (16) are applied to eqn.(23). The second moment of the exact solution has stationary value:

$$Y \equiv \mathrm{E}X^2 = \alpha = 2.$$

The results of three schemes are shown Table 1 and Fig. 1. In Table 1 no result of (i) with $h = 0.03$ means that a stochastic numerical instability arises. From the results we can conclude that the improved RK scheme is superior to other two schemes.

6 Future aspects

1. Derivation of high order RK scheme

So far we gave the concept of order in strong sense. It is however difficult to derive scheme of order 4 in this situation. In many purposes it is not necessary to consider this mode of convergence. We only require weak convergence; for example the convergence for the first two moments $\mathrm{E}\bar{X}_n$, $\mathrm{E}\bar{X}_n^2$. Thus we attempt to derive high order RK scheme with weak order. Also there may exist some problems when RK method is applied to n-dim SDE.

2. Weak-sense linear stability analysis

By applying the numerical scheme to the test equation (supermartingale eqn.)

$$\begin{cases} dX &= \lambda X dt + \mu X dW \quad (\lambda < 0, 2\lambda + \mu^2 < 0), \\ X(0) &= 1, \end{cases}$$

we will consider the numerical stability of the first two moments of the solution $X(t)$.

References

[1] T.C. Gard, *Introduction to Stochastic Differential Equations*, Marcel Dekker, New York, 1988.

[2] A. Greiner, W. Strittmatter and J. Honerkamp, *Numerical integration of stochastic differential equations*, J. Stat. Phys. **51**(1987) 95-108

[3] J.R. Klauder and W.P. Petersen, *Numerical integration of multiplicative-noise stochastic differential equations*, SIAM J. Numer. Anal. **22**(1985), 1153-1166

[4] P.E. Kloeden and E. Platen, *A survey of numerical methods for stochastic differential equations*, J.Stoch.Hydrol.Hydraulics **3**(1989),155-178.

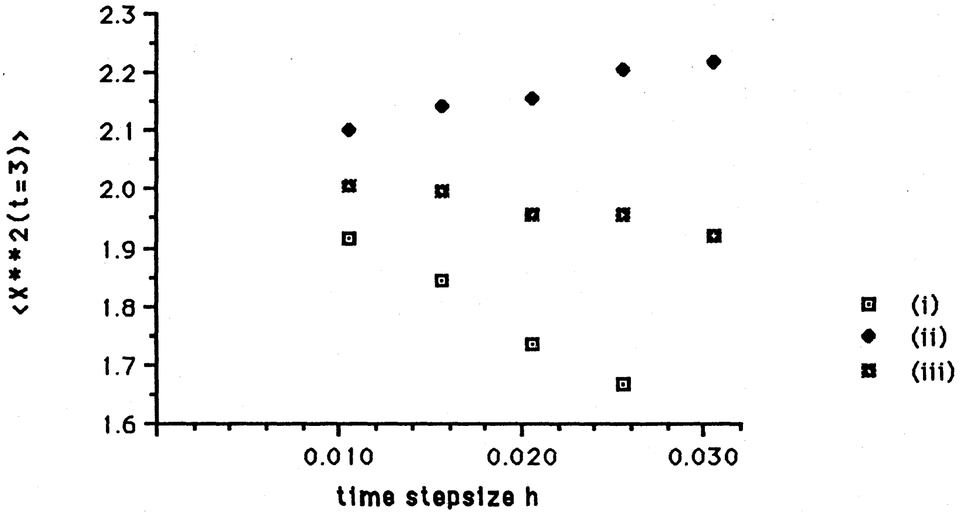
[5] W. Rümelin, *Numerical treatment of stochastic differential equations*, SIAM J.Numer.Anal. **19**(1982),604-613

Table 1

h	Y (t=3)		
	(i)	(ii)	(iii)
0.030	*.****	2.2015	1.9010
0.025	1.6500	2.1893	1.9387
0.020	1.7176	2.1382	1.9402
0.015	1.8259	2.1239	1.9778
0.010	1.9003	2.0813	1.9866

(FACOM M-780)

Fig. 1


2.3
2.2
2.1
2.0
1.9
1.8
1.7
1.6
<X**2(t=3)>
0.010
0.020
0.030
time stepsize h
(i)
(ii)
(iii)