Title	A Word Problem In Coxeter Semigroups(Combinatorial Theory and Related Topics)
Author(s)	Enomoto, Hikoe; Mizutani, Hajime; Tokushige, Norihide
Citation	数理解析研究所講究録 (1990), 735: 28-42
Issue Date	1990-12
URL	http://hdl.handle.net/2433/102027
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
735巻 1990年 28-42

A Word Problem In Coxeter Semigroups

by

Hikoe Enomoto (榎本　彦衛) *)
Hajime Mizutani (水谷　一) **)
Norihide Tokushige (徳重　典英) ***)

Abstract.
The braid group, with generators $1, 2, \ldots, n$, has the presentation

$$\begin{cases} i\,j\,i = j\,i\,j & \text{if } |i-j| = 1 \\ i\,j = j\,i & \text{if } |i-j| > 1. \end{cases}$$

In this paper, we prove that the two products $P = 1 \cdot 3 \cdot 5 \cdot \ldots$ and $Q = 2 \cdot 4 \cdot 6 \cdot \ldots$ satisfy $P\,Q\,P\,Q \ldots = Q\,P\,Q\,P \ldots$ with $n+1$ letters on each side of the equals sign. This is an affirmative answer to a problem of Coxeter.

---

*) Department of Mathematics
Keio University
3-14-1 Hiyoshi, Kohokuku, Yokohama
223 Japan

**) 4-118 Hiecho, Minamiku, Yokohama
232 Japan

***) Department of Computer Science
Meiji University
1-1-1 Higashimita, Tamaku, Kawasaki
214 Japan

1. Introduction

In [2], Coxeter posed the following problem:

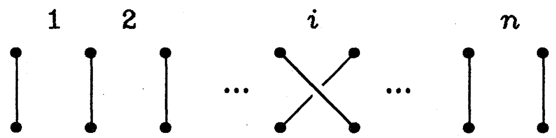
Problem. The braid group, with generators $1, 2, \ldots, n$, has the presentation

$$\begin{cases} i\,j\,i = j\,i\,j & \text{if } |i-j| = 1 \\ i\,j = j\,i & \text{if } |i-j| > 1. \end{cases}$$

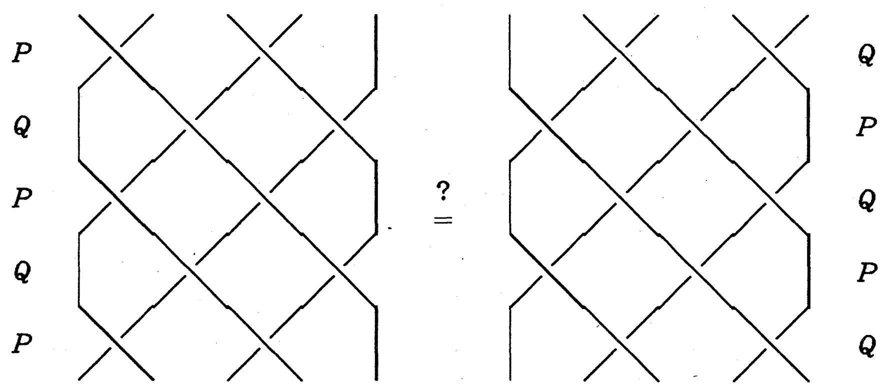
Prove or disprove that the two products $P = 1 \cdot 3 \cdot 5 \cdot \ldots$ and $Q = 2 \cdot 4 \cdot 6 \cdot \ldots$ satisfy $P\,Q\,P\,Q \ldots = Q\,P\,Q\,P \ldots$ with $n+1$ letters on each side of the equals sign. ■

In this paper, we give an affirmative answer to the problem.

Here, we consider a topological feature of this problem. A generator i of the braid gruop corresponds to the following state of braids:



For example, the problem for $n = 4$ corresponds to the following state of braids:



All proofs in this paper are combinatorial, but these topological images will be helpful to see some of the proofs.

2. Definitions and notation

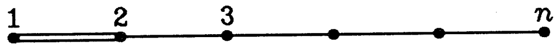
Let $\Gamma = (V,E)$ be a multigraph. We define the semigroup S corresponding to Γ as the following. S is generated by the elements of V subject to the relations

$$i\,j\,i\,j\,\ldots = j\,i\,j\,i\,\ldots \quad \text{for } i, j \in V$$

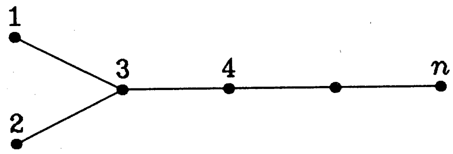
with $e(i,j) + 2$ letters on each side of the equals sign, where $e(i,j)$ denotes the number of edges which join i and j. Now we define Coxeter semigroups by the following graphs:

A_n

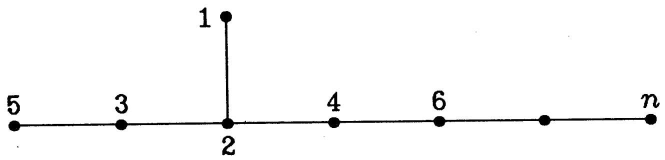
B_n



D_n

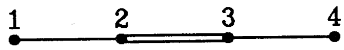


E_n



($n = 6, 7, 8$)

F_4



H_n

($n = 3, 4$)

Let X_n be a Coxeter semigroup with generators $1, 2, \ldots, n$. We use the following notation for elements of X_n.

- For $X = x_1 x_2 \ldots x_k \in X_n$, ${}^tX := x_k x_{k-1} \ldots x_1$.
- $[i,j] := \begin{cases} i \cdot i+1 \cdot \ldots \cdot j & \text{if } i \leq j \\ {}^t[j,i] & \text{otherwise.} \end{cases}$
- $[i,j,k] := \begin{cases} [i,j][j-1,k] & \text{if } j > k \\ [i,j][j+1,k] & \text{if } j < k \\ [i,j] & \text{if } j = k. \end{cases}$
- $\overline{x_1 x_2 \ldots x_k} := \overline{x_1}\,\overline{x_2} \ldots \overline{x_k}$. ($\overline{x_i}$ will be defined later.)
- $S_n := \{ x_1 x_2 \ldots x_n \in X_n : \{ x_1, x_2, \ldots, x_n \} = \{ 1, 2, \ldots, n \} \}$.
- $k(X_n) := \min\{ k \in N : A^k = [1,n]^k \text{ for all } A \in S_n \subset X_n \}$.
- $\Delta_{\{1, 2, \ldots, n\}} = \Delta_n := [1,n][1,n-1]\ldots[1,2][1,1]$.
- For $A, B \in X_n$, we denote $A \equiv B$ if A and B are identical, letter

by letter.

- The Coxeter group corresponding to X_n is denoted by $\widetilde{X}_n$, i.e.,
$$\widetilde{X}_n := \left\langle 1, 2, \ldots, n \;\middle|\; \begin{array}{l} \text{relations in } X_n \\ i^2 = \text{identity} \quad \text{for} \quad 1 \leqq i \leqq n \end{array} \right\rangle .$$
- The Coxeter number of $\widetilde{X}_n$ is denoted by $h(\widetilde{X}_n)$, that is
$$h(\widetilde{X}_n) := \min\{\ h \mid X^h = \text{identity} \quad \text{for all} \quad X \in S_n \subset \widetilde{X}_n\ \}.$$

Remark. Note that $2k(X_n) \geqq h(\widetilde{X}_n)$. ■

3. The cancellation property of Coxeter semigroups

In this section, we show that the cancellation law holds in Coxeter semigroups. This property will be used in sections 4, 5.

Lemma 3.1 Let Γ be a multigraph with edge multiplicities $\leqq 3$, and contains no K_3 (the complete graph of order 3) nor . Let S be the semigroup corresponding to Γ. Suppose that $i\,X = j\,Y$ holds for $i, j \in V(\Gamma)$ and $X, Y \in S$. Then, it follows that

(i) if $i = j$, then $X = Y$,

(ii) if $i \neq j$ and $e(i,j) = 0$, then $X = j\,Z$, $Y = i\,Z$ for some Z,

(iii) if $e(i,j) = 1$, then $X = j\,i\,Z$, $Y = i\,j\,Z$ for some Z,

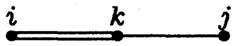
(iv) if $e(i,j) = 2$, then $X = j\,i\,j\,Z$, $Y = i\,j\,i\,Z$ for some Z.

(v) if $e(i,j) = 3$, then $X = j\,i\,j\,i\,Z$, $Y = i\,j\,i\,j\,Z$ for some Z.

Proof. The lemma for words X, Y of word-length s will be referred to as L_s. If $i\,X$ is transformed $j\,Y$ by a sequence of t single applications of the defining relations, then the whole transformation will be said to be chain-length t. We prove the lemma by induction on s. We assume (a) L_s is true for $0 \leqq s \leqq r$ for transformations of all chain-length, and (b) L_{r+1} is true for all chain-length $\leqq t$.

Let X, Y be of word-length $r+1$, and $i\,X = j\,Y$ through a transformation of chain-length $t+1$. Let the successive words of the transformation be
$$W_1 \equiv i\,X, \quad \ldots\ , \quad W_g \equiv k\,W, \quad \ldots\ , \quad W_{t+2} \equiv j\,Y,$$
with $2 \leqq g \leqq t+1$. The transformations $i\,X \to k\,W$, $k\,W \to j\,Y$ are each of chain-length $\leqq t$, and we can apply (b) to them.

We distinguish some cases according to the induced subgraph $\langle \{i, j, k\} \rangle_\Gamma$. The general pattern of the proof is, however, exactly the same for each case, and it will be sufficient here to deal with one case only, as a typical example of the common method of proof. See also [3].

<u>Case.</u> $\langle \{i, j, k\} \rangle_\Gamma$ = 

By (b) and

$$i\,X = k\,W = j\,Y,$$

we have

$$X = k\,i\,k\,P \tag{1}$$

$$W = i\,k\,i\,P = j\,k\,Q \tag{2}$$

$$Y = k\,j\,Q \tag{3}$$

for some P, Q. By (a) and (2),

$$k\,i\,P = j\,R \tag{4}$$

$$k\,Q = i\,R \tag{5}$$

for some R. By (4),

$$i\,P = j\,k\,S \tag{6}$$

$$R = k\,j\,S \tag{7}$$

for some S. By (6),

$$P = j\,T \tag{8}$$

$$k\,S = i\,T \tag{9}$$

for some T. By (9),

$$S = i\,k\,i\,U \tag{10}$$

$$T = k\,i\,k\,U \tag{11}$$

for some U. On the other hand, by (5) and (7),

$$Q = i\,k\,i\,M \tag{12}$$

$$R = k\,i\,k\,M = k\,j\,S \tag{13}$$

for some M. By (a) and (13),

$$i\,k\,M = j\,S$$

and so

$$k\,M = j\,N \tag{14}$$

$$S = i\,N \tag{15}$$

for some N. By (14),

$$M = j\,k\,O \tag{16}$$

$$N = k\,j\,O \tag{17}$$

for some O. By (10) and (15),

$$S = i\,k\,i\,U = i\,N. \tag{18}$$

By (17) and (18),

$$N = k\,j\,O = k\,i\,U,$$

and so

$$i\,U = j\,O. \tag{19}$$

By (19),

$$U = j\,A \tag{20}$$

$$O = i\,A \tag{21}$$

for some A. By (1), (8), (11), and (20),

$$\begin{aligned} X &= k\,i\,k\,P \\ &= k\,i\,k\,j\,T \\ &= k\,i\,k\,j\,k\,i\,k\,U \\ &= k\,i\,k\,j\,k\,i\,k\,j\,A \\ &= k\,i\,j\,k\,j\,i\,k\,j\,A \\ &= k\,j\,i\,k\,i\,j\,k\,j\,A \\ &= k\,j\,i\,k\,i\,k\,j\,k\,A \\ &= k\,j\,k\,i\,k\,i\,j\,k\,A \\ &= j\,k\,j\,i\,k\,i\,j\,k\,A \\ &= j\,k\,i\,j\,k\,j\,i\,k\,A \\ &= j \cdot k\,i\,k\,j\,k\,i\,k\,A. \end{aligned} \tag{22}$$

By (3), (12), (16), and (21),

$$\begin{aligned} Y &= k\,j\,Q \\ &= k\,j\,i\,k\,i\,M \\ &= k\,j\,i\,k\,i\,j\,k\,O \\ &= k\,j\,i\,k\,i\,j\,k\,i\,A \\ &= k\,i\,j\,k\,j\,i\,k\,i\,A \\ &= k\,i\,k\,j\,k\,i\,k\,i\,A \\ &= k\,i\,k\,j\,i\,k\,i\,k\,A \\ &= k\,i\,k\,i\,j\,k\,i\,k\,A \\ &= i \cdot k\,i\,k\,j\,k\,i\,k\,A. \end{aligned} \tag{23}$$

By (22) and (23), we have $X = j\,Z$, $Y = i\,Z$ for $Z := k\,i\,k\,j\,k\,i\,k\,A$. ■

Lemma 3.2 In Coxeter semigroups, it follows that

(i) if $i\,X = i\,Y$, then $X = Y$,

(ii) if $X\,i = Y\,i$, then $X = Y$.

Proof. Follows from Lemma 3.1 (i) and the fact that $A = B$ holds iff ${}^tA = {}^tB$. ■

4. Type A_n

In A_n, we define $\bar{i}^{\langle n\rangle} = \bar{i} := n+1-i$. Note that this operator is considered as a graph isomorphism of A_n. In this section, we prove the following theorem which gives an affirmative answer to the problem of Coxeter.

Theorem 4.1 In A_n, the following hold.

(1) For $A \in A_n$, $A^{n+1} = \Delta_n^2$ holds iff $A \in S_n$.

(2) Suppose that $AB \in A_n$, $n = 2m$. Then, $(AB)^m A = \Delta_n$ holds iff $AB \in S_n$ and $B = \bar{A}$.

(3) Suppose that $A \in A_n$, $n = 2m+1$. Then, $A^{m+1} = \Delta_n$ holds iff $A \in S_n$ and $\bar{A} = A$.

Remark. We verify that Theorem 4.1 answers to the question of Coxeter. In A_n, define $P := 1 \cdot 3 \cdot 5 \cdot \ldots$, $Q := 2 \cdot 4 \cdot 6 \cdot \ldots$.

(1) If $n = 2m$, then $\bar{Q} = P$, $\bar{P} = Q$, and so,

$$\begin{aligned} P\,Q\,P\,Q \ldots (n+1 \text{ letters}) &= (P\,Q)^m P \\ &= \Delta_n \\ &= (Q\,P)^m Q \\ &= Q\,P\,Q\,P \ldots (n+1 \text{ letters}). \end{aligned}$$

(2) If $n = 2m+1$, then $\overline{P\,Q} = P\,Q$, $\overline{Q\,P} = Q\,P$, and so,

$$\begin{aligned} P\,Q\,P\,Q \ldots (n+1 \text{ letters}) &= (P\,Q)^{m+1} \\ &= \Delta_n \\ &= (Q\,P)^{m+1} \\ &= Q\,P\,Q\,P \ldots (n+1 \text{ letters}). \end{aligned}$$ ■

To prove the theorem, we need several lemmas.

Lemma 4.2 For all $A \in S_n$, $G\,A = [1,n]\,G$ holds for some $G \in A_{n-1}$.

Proof. We prove this lemma by induction on n. We can write A in the form as $A = X\,n$, or $A = n\,X$. If $A = n\,X$, we have $X\,A = X\,n\,X$. In both cases, we have $X\,A = Y\,n\,X$ for some $X \in A_{n-1}$, $Y \in S_{n-1}$. By induction hypothesis, we have $Z\,Y = [1,n-1]\,Z$ for some $Z \in A_{n-2}$. So,

$$Z\,X\,A = Z\,Y\,n\,X = [1,n-1]\,Z\,n\,X = [1,n]\,Z\,X.$$ ■

Formulae 4.3 In A_n, the following hold.

(i) $k[1,n] = [1,n]\,k-1$ for $2 \leqq k \leqq n$,

(ii) $1[1,n]^2 = [1,n]^2 n$,

(iii) $k[1,n]^{n+1} = [1,n]^{n+1}k$ for $1 \leqq k \leqq n$,

(iv) $A^{n+1} = [1,n]^{n+1}$ for $A \in S_n$,

(v) $k\Delta_n = \Delta_n \bar{k}$ for $1 \leqq k \leqq n$,

(vi) $\overline{\Delta_n} = {}^t\Delta_n = \Delta_n$.

Proof. (i)-(iii) follow from the definition of A_n. We prove (iv). Suppose that $A \in S_n$. By Lemma 4.2, there exists $G \in A_n$ such that $G\,A = [1,n]G$. So, we have

$$G\,A^{n+1} = [1,n]^{n+1}G = G[1,n]^{n+1}.$$

This and Lemma 3.2 give $A^{n+1} = [1,n]^{n+1}$. For the proof of (v) and (vi), see [3]. ■

Lemma 4.4 Suppose that $A\bar{A} \in S_n$, $n = 2m$.

(1) If $A = 1\,B$, then there exist $x_1, \dots, x_m, y_1, \dots, y_m$ such that $A\bar{A} = x_1 \dots x_m y_1 \dots y_m$, $x_1 < \dots < x_m$, $y_1 > \dots > y_m$, $x_1 = 1$, $y_1 = n$.

(2) If $A = B\,1$, then there exist $x_1, \dots, x_m, y_1, \dots, y_m$ such that $A\bar{A} = x_1 \dots x_m y_1 \dots y_m$, $x_1 < \dots < x_m$, $y_1 > \dots > y_m$, $y_1 = n$, $y_m = 1$.

Proof. This lemma is easily proved by induction on m. ■

We say that $A = a_1 a_2 \dots a_k \in A_n$ is <u>increasing</u> (resp. <u>decreasing</u>) if $a_1 < \dots < a_k$ (resp. $a_1 > \dots > a_k$).

Lemma 4.5 Suppose that $D\,1\,U \in S_{n-1}$, D decreasing, and U increasing. Then, it follows that $[1,n]D\,1\,U = U\,n\,D[1,n]$.

Proof. Induction on n.

<u>Case 1.</u> $D = D'2$

$$\begin{aligned}[1,n]D\,1\,U &= 1[2,n]D'\,2\,1\,U \\ &= 1([2,n]D'\,2\,U)1 \\ &= 1(U\,n\,D'[2,n])1 \quad \text{(by induction hypothesis)} \\ &= U\,n\,D'\,1\,2\,1[3,n] \\ &= U\,n\,D'\,2\,1\,2[3,n] \\ &= U\,n\,D[1,n].\end{aligned}$$

<u>Case 2.</u> $U = 2\,U'$

$$[1,n]D\,1\,U = [1,n]D\,1\,2\,U'$$

$= 1\ 2\ 1[3,n]D\ 2\ U'$
$= 2\ 1([2,n]D\ 2\ U')$
$= 2\ 1(U'\,n\ D[2,n])$ (by induction hypothesis)
$= 2\ U'\,n\ D\ 1[2,n]$
$= U\ n\ D[1,n].$ ■

Lemma 4.6 Suppose that $1\,X\,n\,Y \in S_n$, $n = 2m$, $X \equiv x_1 \dots x_{m-1}$ increasing, $Y \equiv y_1 \dots y_{m-1}$ decreasing. Define $U_0 := [1,n-1]$, $D_0 := [n,2]$, and for $1 \leqq j \leqq m-1$, define

$U_j :=$ the word obtained by deleting y_j from U_{j-1},
$D_j :=$ the word obtained by deleting x_j from D_{j-1}.

Then, it follows that $U_j D_j 1\,X\,n\,Y = X\,Y\,U_{j-1} D_{j-1}$.

Proof. Define $[i,j]_Z := [i,j] \cap Z$.

$U_j\,D_j 1\,X\,n\,Y$
$= U_j\,[n,x_j{+}1]\,[x_j{-}1,2]_{D_j}\,1\,[2,x_j{-}1]_X\,[x_j,n{-}1]_X\,n\ \ [n{-}1,x_j{+}1]_Y\,[x_j{-}1,2]_Y$
$= U_j\,[x_j{-}1,2]_{D_j}\,1\,[2,x_j{-}1]_X\,[n,x_j]\,[x_j{+}1,n{-}1]_X\,n\,[n{-}1,x_j{+}1]_Y\,[x_j{-}1,2]_Y$
$= U_j\,[x_j{-}1,2]_{D_j}\,1\,[2,x_j{-}1]_X\,[n{-}1,x_j{+}1]_Y\,x_j\,[x_j{+}1,n{-}1]_X\,[n,x_j]\,[x_j{-}1,2]_Y$
(by Lemma 4.5)
$= U_j\,[n{-}1,x_j{+}1]_Y\,[x_j{-}1,2]_{D_j}\,1\,[2,x_j{-}1]_X\,[x_j,n{-}1]_X\,D_{j-1}$
$= U_j\,Y\,1\,X\,D_{j-1}$
$= [1,y_j{-}1]\,[y_j{+}1,n{-}1]_{U_j}\,[n{-}1,y_j{+}1]_Y\,[y_j,2]_Y\,1\,[2,y_j{-}1]_X\,[y_j{+}1,n{-}1]_X\,D_{j-1}$
$= [y_j{+}1,n{-}1]_X\,[n{-}1,y_j{+}1]_Y\,[1,y_j]\,[y_j{-}1,2]_Y\,1\,[2,y_j{-}1]_X\,[y_j{+}1,n{-}1]_X\,D_{j-1}$
$= [y_j{+}1,n{-}1]_X\,[n{-}1,y_j{+}1]_Y\,[2,y_j{-}1]_X\,y_j\,[y_j{-}1,2]_Y\,[1,y_j]\,[y_j{+}1,n{-}1]_X\,D_{j-1}$
(by Lemma 4.5)
$= [2,y_j{-}1]_X\,[y_j{+}1,n{-}1]_X\,[n{-}1,y_j{+}1]_Y\,[y_j,2]_Y\,U_{j-1}\,D_{j-1}$
$= X\,Y\,U_{j-1}\,D_{j-1}.$ ■

Lemma 4.7 Under the same assumption in the previous lemma, it follows that $(1\,X\,n\,Y)^m = (X\,Y)^{m-1}[1,n,2]$.

Proof. Using Lemma 4.6, we have

$$\begin{aligned}(X\,Y)^{m-1}[1,n,2] &= (X\,Y)^{m-1}U_0 D_0\\ &= (X\,Y)^{m-2}U_1 D_1\,1\,X\,n\,Y\\ &= \dots\\ &= U_{m-1}D_{m-1}(1\,X\,n\,Y)^{m-1}\end{aligned}$$

$$= (1\,X\,n\,Y)^m. \qquad \blacksquare$$

Proposition 4.8 If $A\bar{A} \in S_n$, $n = 2m$, then $(A\bar{A})^m A = \Delta_n$.

Proof. Induction on m.

Case 1. $A = 1\,B$.

By Lemma 4.4, $A\,\bar{A} = 1\,X\,n\,Y$ holds for some X increasing, Y decreasing, with $|X| = |Y|$.

$$\begin{aligned}(A\,\bar{A})^m A &= (1\,X\,n\,Y)^m A \\ &= (X\,Y)^{m-1}[1,n,2]1\,B \text{ (by Lemma 4.7)} \\ &= (B\,\bar{B})^{m-1}[1,n,1]B \\ &= (B\,\bar{B})^{m-1}B[1,n,1] \\ &= \Delta_{\{2,\,\dots\,,n-1\}}[1,n,1] \text{ (by induction hypothesis)} \\ &= [1,n][n-1,1][2,n-1][2,n-2]\,\dots\,[2,2] \\ &= \Delta_n.\end{aligned}$$

Case 2. $A = B\,1$.

$$\begin{aligned}(A\,\bar{A})^m A \cdot \bar{A} &= (B\,1\,\bar{B}\,n)^{m+1} \\ &= B \cdot (1\,\bar{B}\,n\,B)^m 1\,\bar{B} \cdot n \\ &= B\,\Delta_n n \text{ (by case 1)} \\ &= \Delta_n \bar{B}\,n = \Delta_n \bar{A}.\end{aligned}$$

By Lemma 3.2, we have $(A\,\bar{A})^m A = \Delta_n$.

Case 3. $1 \notin A$.

By cases 1, 2, we have $(\bar{A}\,A)^m \bar{A} = \Delta_n$. Hence,

$$(A\,\bar{A})^m A = \overline{\Delta_n} = \Delta_n. \qquad \blacksquare$$

Lemma 4.9 The following two conditions are equivalent.

(1) $A = \bar{A} \in S_n$, $n = 2m + 1$.

(2) There exist $P \equiv p_1 \dots p_m$ increasing, $Q \equiv q_1 \dots q_m$ decreasing such that $A = P\,n\,Q$ and $Q = \overline{P}^{\langle n-1 \rangle}$, i.e., $p_j + q_j = n$ for $1 \leqq j \leqq m$.

Proof. (1) implies (2):

Let $A \equiv p_1 p_2 \dots p_k n\, q_1 q_2 \dots q_l$, $\{p_j\}$ increasing, $\{q_j\}$ decreasing. Suppose that $p_1 = 1$. Since

$$\bar{A} \equiv \overline{p_1} \dots \overline{p_k}\, 1\, \overline{q_1} \dots \overline{q_l} \equiv A,$$

we have $\overline{q_1} = 2$, i.e., $q_1 = n - 1$. Next, $q_2 = 2$ or $q_l = 2$, and say $q_l = 2$. Since

$$A \equiv 1\, p_2 \dots p_k\, n\, q_1 \dots p_{l-1}\, 2$$

$$= \overline{p_1} \dots \overline{p_k}\, 1\, 2\, \overline{q_2} \dots \overline{q_l} \equiv \overline{A},$$

we have $\overline{q_k} = 3$, i.e., $q_k = n - 2$. Continuing this way, we have (2).

(2) implies (1):

Suppose that $A \equiv p_1 \dots p_m\, n\, q_1 \dots q_m$ satisfies the condition of (2). Assume that $p_1 = 1$, then $q_1 = 2$ and so

$$\overline{A} \equiv \overline{p_1} \dots \overline{p_m}\, 1\, 2\, \overline{q_2} \dots \overline{q_m} = 1\, B$$

holds for some B. Next, $p_2 = 2$ or $q_m = 2$, and say $p_2 = 2$. Then, we have $q_2 = 3$ and so,

$$\overline{A} \equiv \overline{p_1} \dots \overline{p_m}\, 1\, 2\, 3\, \overline{q_3} \dots \overline{q_m} = 1\, 2\, C$$

holds for some C. Continuing this way, we have (1). ■

Lemma 4.10 Suppose that $n\, X \in S_n$, $X \equiv x_2 \dots x_n$, $x_2 > \dots > x_i < \dots < x_n$ for some i, $2 \leqq i \leqq n$. Define $x_1 := n$, $V_0 := [n,1][1,n]$, and for $V_{j-1} \equiv P\, x_j\, Q\, x_j\, R$ with $1 \leqq j \leqq n$, define $V_j :\equiv P\, x_j\, Q\, R$ if $1 \leqq j \leqq i$, $V_j :\equiv P\, Q\, x_j\, R$ if $i < j \leqq n$. Then, $X\, V_{j-1} = V_j\, n\, X$ holds.

Proof. Induction on n. Define $Z^{(k)} := Z \cap \{1, \dots, k\}$. If $j = 1$, then

$$X\, V_0 = X[n,1][1,n] = [n,1][1,n]X = [n,1][1,n-1]n\, X = V_1 n\, X.$$

So, we may assume $j > 1$.

<u>Case 1.</u> $x_n = n - 1$, $V_{j-1} \equiv n \cdot n - 1 \cdot V_{j-1}^{(n-2)} \cdot n - 1$, $X^{(n-2)} \neq V_{j-1}^{(n-2)}$.

$$\begin{aligned} X\, V_{j-1} &= X^{(n-2)} \cdot n - 1 \cdot n \cdot n - 1 \cdot V_{j-1}^{(n-2)} \cdot n - 1 \\ &= X^{(n-2)} \cdot n \cdot n - 1 \cdot n \cdot V_{j-1}^{(n-1)} \cdot n - 1 \\ &= n \cdot X^{(n-2)}(n - 1 \cdot V_{j-1}^{(n-2)})n \cdot n - 1 \\ &= n(n - 1 \cdot V_j^{(n-2)})\, (n - 1 \cdot X^{(n-2)})n \cdot n - 1 \\ &\qquad \text{(by induction hypothesis)} \\ &= V_j X^{(n-2)} \cdot n \cdot n - 1 \\ &= V_j \cdot n \cdot X^{(n-2)} \cdot n - 1 = V_j\, n\, X. \end{aligned}$$

<u>Case 2.</u> $x_n = n - 1$, $V_{j-1} \equiv n \cdot n - 1 \cdot V_{j-1}^{(n-2)} \cdot n - 1$, $X^{(n-2)} = V_{j-1}^{(n-2)}$.

$$\begin{aligned} X\, V_{j-1} &= X^{(n-2)} \cdot n - 1 \cdot n \cdot n - 1 \cdot V_{j-1}^{(n-2)} \cdot n - 1 \\ &= X^{(n-2)} \cdot n \cdot n - 1 \cdot n \cdot X \\ &= n \cdot X^{(n-2)} \cdot n - 1 \cdot n \cdot X = V_j\, n\, X. \end{aligned}$$

<u>Case 3.</u> $x_2 = n - 1$, $V_{j-1} \equiv n \cdot n - 1 \cdot V_{j-1}^{(n-2)} \cdot n - 1$.

$$X\, V_{j-1} = n - 1 \cdot X^{(n-2)}[n,1][1,n - 1]$$

$= n - 1 \cdot [n,1][1,n-1]X^{(n-2)}$

$= n \cdot n - 1 \cdot n[n-2,1][1,n-2]\, n - 1 \cdot X^{(n-2)}$

$= [n,1][1,n-2]n\, X = V_j\, n\, X.$

Case 4. $x_2 = n - 1,\ V_{j-1} \equiv n \cdot n - 1 \cdot V_{j-1}^{(n-2)},\ X^{(n-2)} \neq V_{j-1}^{(n-2)}.$

$X\, V_{j-1} = n - 1 \cdot X^{(n-2)} n \cdot n - 1 \cdot V_{j-1}^{(n-2)}$

$= n - 1 \cdot n \cdot X^{(n-2)}(n - 1 \cdot V_{j-1}^{(n-2)})$

$= n - 1 \cdot n\, (n - 1 \cdot V_j^{(n-2)})\, (n - 1 \cdot X^{(n-2)})$

(by induction hypothesis)

$= n \cdot n - 1 \cdot n \cdot V_j^{(n-2)}\, X = V_j\, n\, X.$

Case 5. $x_2 = n - 1,\ V_{j-1} \equiv n \cdot n - 1 \cdot V_{j-1}^{(n-2)},\ X^{(n-2)} = V_{j-1}^{(n-2)}.$

$X\, V_{j-1} = (n - 1 \cdot X^{(n-2)})\, n\, (n - 1 \cdot V_{j-1}^{(n-2)}) = V_j\, n\, X.$ ■

Proposition 4.11 If $A = \overline{A} \in S_n,\ n = 2m + 1,$ then $A^{m+1} = \Delta_n.$

Proof. We distinguish two cases.

Case 1. $A = n\, X.$

By Lemma 4.10, $X^m V_0 = V_m (n\, X)^m$, where V_i is defined in Lemma 4.10. By Lemma 4.9, we have $n\, X = P\, n\, Q$, where P, Q satisfy the conditions in Lemma 4.9 (2). So, we have

$$(P\, Q)^m [n,1][1,n] = [n,1]1\, \overline{Q}\, A^m.$$

Since $Q = \overline{P}^{\langle n-1 \rangle}$, using Proposition 4.8, we have

$$Q\, (P\, Q)^m [n,1][1,n] = \Delta_{n-1}[n,1][1,n] = [n,1]\Delta_n.$$

On the other hand,

$$Q[n,1]1\, \overline{Q}\, A^m = [n,1]\overline{P}\, 1\, \overline{Q}\, A^m = [n,1]\overline{A}\, A^m = [n,1]A^{m+1}.$$

Combining these identities, we have $A^{m+1} = \Delta_n.$

Case 2. $A = X\, n.$

By case 1, we have $({}^t A)^{m+1} = \Delta_n.$ So, $A^{m+1} = {}^t\Delta_n = \Delta_n.$ ■

Proposition 4.12 In A_n, the following hold.

(1) If $A\, B \in A_n,\ n = 2m,$ and $(A\, B)^m A = \Delta_n,$ then $B = \overline{A},\ A\, B \in S_n.$

(2) If $A \in A_n,\ n = 2m + 1,$ and $(A\, B)^{m+1} = \Delta_n,$ then $A = \overline{A} \in S_n.$

Proof. (1) Suppose that $(A\, B)^m A = \Delta_n,$ then $A\, B \in S_n.$ Suppose $Z\, \overline{Z} \in S_n,$ then by Proposition 4.8, $(Z\, \overline{Z})^m Z = \Delta_n.$ By Lemma 4.2, $G\, A\, B = Z\, \overline{Z}\, G$ holds for some $G \in A_n.$ Hence,

$$G\,\overline{B}\Delta_n = G\,\Delta_n B = G\,(A\,B)^{m+1} = (Z\,\overline{Z})^{m+1}G = Z\,\Delta_n G = Z\,\overline{G}\,\Delta_n,$$

which implies $G\,\overline{B} = Z\,\overline{G}$, or $\overline{G}\,B = \overline{Z}\,G$. So,

$$G\,A\,B = Z\,\overline{Z}\,G = Z\,\overline{G}\,B,$$

which implies $G\,A = Z\,\overline{G} = G\,\overline{B}$. This gives $B = \overline{A}$.

(2) Suppose that $A^{m+1} = \Delta_n$, then $A \in S_n$. Suppose $Z = \overline{Z} \in S_n$, then by Proposition 4.11, $Z^{m+1} = \Delta_n$. By Lemma 4.2, $G\,A = Z\,G$ holds for some $G \in A_n$. Hence,

$$G\,\Delta_n = G\,A^{m+1} = Z^{m+1}G = \Delta_n G = \overline{G}\,\Delta_n,$$

which implies $G = \overline{G}$. Therefore,

$$G\,\overline{A} = \overline{G\,A} = \overline{Z\,G} = Z\,G = G\,A.$$

This gives $A = \overline{A}$. ■

Proof of Theorem 4.1 Suppose that $A^{m+1} = \Delta_n$, then $A \in S_n$. This and Formulae 4.3 (iv) give (1), since $\Delta_n = [1,n]^{n+1}$. (2) and (3) follow from Propositions 4.8, 4.11, and 4.12. ■

Remark. Since $h(\widetilde{A}_n) = n + 1$, Theorem 4.1 gives $k(A_n) = n + 1$. ■

5. Types B_n and D_n

In this section, we deal with B_n and D_n.

Lemma 5.1 For all $A \in S_n$, $G\,A = [1,n]\,G$ holds for some $G \in S_{n-1}$ in B_n and D_n.

Proof. Similar to the proof of Lemma 4.2. ■

Formulae 5.2 In B_n, the following hold.

(i) $k[1,n] = [1,n]\,k-1$ for $3 \leqq k \leqq n$,

(ii) $2[1,n]^2 = [1,n]^2 n$,

(iii) $1\,2[1,n] = 2[1,n]1$,

(iv) $k[1,n]^n = [1,n]^n k$ for $1 \leqq k \leqq n$.

Proof. It is not difficult to check these formulae using the defining relations of B_n. ■

Theorem 5.3 For $A \in B_n$, $A^n = [1,n]^n$ holds iff $A \in S_n$.

Proof. The proof of this theorem is similar to the proofs of Formulae

4.3 (iv) and Proposition 4.12 (2). ■

In D_n, we define $\bar{i} := i$ if $3 \leqq i \leqq n$, $\bar{\alpha} := \beta$ if $\{\alpha, \beta\} = \{1, 2\}$. Note that this is a graph isomorphism of D_n.

Formulae 5.4 In D_n, the following hold.

(i) $k[1,n] = [1,n]\, k - 1$ for $4 \leqq k \leqq n$,

(ii) $3[1,n]^2 = [1,n]^2 n$,

(iii) $\alpha[1,n]^2 = [1,n]^2 \bar{\alpha}$ for $\alpha = 1, 2$,

(iv) $k[1,n]^{n-1} = [1,n]^{n-1} \bar{k}$ for $1 \leqq k \leqq n$, n odd,

(v) $k[1,n]^{n-1} = [1,n]^{n-1} k$ for $1 \leqq k \leqq n$, n even.

Proof. It is not difficult to check these formulae using the defining relations of D_n. ■

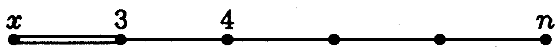
Theorem 5.5 In D_n, the following hold.

(1) For $A \in D_n$, $A^{2n-2} = [1,n]^{2n-2}$ iff $A \in S_n$.

(2) Suppose that $A \in D_n$, n odd. Then, $A^{n-1} = [1,n]^{n-1}$ holds iff $A \in S_n$ and $\bar{A} = A$.

(3) Suppose that $A \in D_n$, n even. Then, $A^{n-1} = [1,n]^{n-1}$ holds iff $A \in S_n$.

Proof. Using Formulae 5.4 (iv) and (v), one can prove (1) and (3) in the similar way of the proofs of Formulae 4.3 (iv) and Proposition 4.12 (2). We prove (2). The proof that $A^{n-1} = [1,n]^{n-1}$ implies $\bar{A} = A \in S_n$ is also similar to the proof of Proposition 4.12 (2). So, suppose that $\bar{A} = A \in S_n$. Since $\bar{A} = A$, A is represented as $A = P\, x\, Q$ where $x = \alpha\beta$, $\{\alpha, \beta\} = \{1, 2\}$. Note that $x, 3, 4, \ldots, n$ satisfy the relations of B_{n-1}:



Therefore, by Theorem 5.3,

$$A^{n-1} = (x \cdot 3 \cdot 4 \cdot \ldots \cdot n)^{n-1} = [1,n]^{n-1}.$$ ■

Remark. Since $h(\widetilde{B}_n) = 2n$ and $h(\widetilde{D}_n) = 2n - 2$, Theorems 5.3, 5.5 give $k(B_n) = n$, $k(D_{2m}) = 2m - 1$, and $k(D_{2m+1}) = 4m$. ■

6. Other types

Results in this section are obtained by calculations using a computer.

Theorem 6.1 In the exceptional types, we have
$k(E_6) = 12$, $k(E_7) = 9$, $k(E_8) = 15$, $k(F_4) = 6$, $k(H_3) = 5$, $k(H_4) = 15$.

In these six types, $2k(X_n) = h(\widetilde{X}_n)$ holds except E_6 which satisfies $k(E_6) = h(\widetilde{E}_6)$. In E_6, define $\overline{\alpha} := \beta$ if $\{\alpha, \beta\} = \{3, 4\}, \{5, 6\}$, and $\overline{i} := i$ otherwise. Note that this is a graph isomorphism of E_6.

Theorem 6.2 For $A \in E_6$, $A^6 = [1,6]^6$ holds iff $A \in S_6$ and $\overline{A} = A$.

References

[1] N. Bourbaki, Groupes et algèbres de Lie, Paris: Hermann 1968, 1975.

[2] H. S. M. Coxeter, Research Problems: Problem 100, Discrete Mathematics 73 (1989) 311.

[3] F. A. Garside, The braid group and other groups, Quart. J. Math. Oxford (2), 20 (1969), 235-254.