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THE GEOMETRIC STRUCTURE OF THE SOLUTION MAP OF LINEAR DELAY EQUATIONS

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1 INTRODUCTION

In our monograph we have studied linear delay equations through the Laplace transform. This method applied within the right framework turned out to be very useful. This paper intents to present the main ideas. For complete proofs and extensions we refer to our monograph [21] and our forthcoming paper [22].

We start to rewrite a retarded functional differential equation as a renewal equation and then use the renewal equation to derive an analytic continuation for the Laplace transform of the solution. It turns out that this approach yields an easy proof of the large time behaviour of the solutions. Moreover, careful analysis yields necessary and sufficient conditions for completeness [20] and convergence results for series of spectral projections [22].

In section 2 we obtain a more natural interpretation of the structural operators which will be defined in a way which differs slightly from the one of Delfour and Manitius [6, 7], see Diekmann [8, 9]. In section 3 we discuss the types of completeness one can consider within this framework and we present necessary and sufficient conditions for both completeness and F-completeness. In addition, we present some tools to verify pointwise completeness. Finally, section 4 contains the case $n = 2$ which we will work out completely.

Consider a linear autonomous retarded functional differential equation (RFDE)

$$\dot{x}(t) = \int_0^h d\zeta(\theta)x(t-\theta) \tag{1.1}$$

satisfying the initial condition

$$x(t) = \varphi(t) \quad \text{for } -h \le t \le 0, \tag{1.2}$$

where $\varphi \in \mathcal{C} = C[-h,0]$ and the matrix-valued function ζ belongs to NBV$[0,h]$, that is, each element ζ_{ij} is of bounded variation, satisfies $\zeta_{ij}(0) = 0$ and is continuous from the left.

In the study of the behaviour of the solution of the above RFDE it turns out to be useful to rewrite the problem as a Volterra convolution integral equation (or, as it is frequently called, a renewel equation).

We split up the integral to separate the part involving the known φ from the part involving the unknown x:

$$\begin{aligned} \dot{x}(t) &= \int_0^t d\zeta(\theta)x(t-\theta) + \int_t^h d\zeta(\theta)\varphi(t-\theta) \\ &= -\int_0^t d_\theta\zeta(t-\theta)x(\theta) - \int_{-h}^0 d_\theta\zeta(t-\theta)\varphi(\theta) \end{aligned}$$

(recall that ζ is defined to be constant on $[h,\infty)$).
Next we integrate from 0 to t and obtain

$$x(t) - \varphi(0) = -\int_0^t\int_0^\sigma d_\theta\zeta(\sigma-\theta)x(\theta)d\sigma - \int_0^t\int_{-h}^0 d_\theta\zeta(\sigma-\theta)\varphi(\theta)d\sigma.$$

So, because of [21; 2.7]

$$\begin{aligned} x(t) - \varphi(0) &= -\int_0^t d_\theta \int_\theta^t \zeta(\sigma-\theta)d\sigma x(\theta) - \int_{-h}^0 d_\theta \int_0^t \zeta(\sigma-\theta)d\sigma\varphi(\theta) \\ &= -\int_0^t \zeta(t-\theta)x(\theta)d\theta + \int_{-h}^0 (\zeta(t-\theta)-\zeta(-\theta))\varphi(\theta)d\theta. \end{aligned}$$

We summarize the end result of our manipulations as follows. The solution x of (2.1) satisfies the renewal equation

$$x - \zeta * x = f, \tag{1.3}$$

where $\zeta * x = \int_0^t \zeta(\theta)x(t-\theta)$ and

$$f = F\varphi = \varphi(0) + \int_{-h}^0 (\zeta(t-\theta)-\zeta(-\theta))\varphi(\theta)d\theta. \tag{1.4}$$

REMARKS 1.1.

(i) The so-called forcing function $F\varphi$ defined by (1.4) is constant for $t \ge h$ and absolutely continuous.

(ii) The formula (1.4) makes perfect sense if $\varphi(0)$ is given as an element of $\mathbf{R}^n$ while $\varphi(\theta)$ for $-h \leq \theta \leq 0$ is given as an integrable function. Moreover, Delfour and Manitius [6, 7] proved that $F\varphi$ is still absolutely continuous, although there is no explicit formula for $\dot{F}\varphi$ anymore. Thus, the existence of the operator F makes it possible to extend the state space of (1.1) to $M_p = \mathbf{R}^n \times L^p[-h,0]$. Following Delfour and Manitius we call F which differs slightly from the one they studied, a structural operator.

(iii) Partial integration shows that the derivative of the solution of the linear autonomous RFDE (1.1) also satisfies a renewal equation of the form

$$\dot{x} - \zeta * \dot{x} = h,$$

where h is defined on $[0,\infty)$ and is constant on the interval $[h,\infty)$. Diekmann [8, 9, 10] extended his idea to associate a renewal equation with (1.1) into a complete frame-work which shows the natural connection between different choices for the state space of (1.1).

Next we derive the analytic continuation for the Laplace transform of the solution $x(\cdot\,;f)$ of equation (1.1). Since $e^{\cdot\gamma}x(\cdot\,;f) \in L^1$, we can Laplace transform the equation to obtain for $\Re(z) > \gamma$

$$L\{x\}(z) = \Delta^{-1}(z)\big(f(h) + z\int_0^h e^{-zt}(f(t) - f(h))dt\big), \tag{1.5}$$

where $\Delta(z)$ denotes the characteristic matrix

$$\Delta(z) = zI - \int_0^h e^{-zt}d\zeta(t). \tag{1.6}$$

The expression at the right hand side of (1.5) yields the analytic continuation of $L\{x\}$ to the whole complex plane. We denote this analytic continuation by $H_f(z)$. In order to be able to apply the Laplace inversion formula we first have to analyse the inverse $\Delta^{-1}(z)$ of the characteristic matrix (1.6). See [21] for the details.

LEMMA 1.2. *The determinant of the characteristic matrix $\Delta(z)$, can be written as follows*

$$\det \Delta(z) = z^n - \sum_{j=1}^{n} \int_0^{jh} e^{-zt}d\eta_j(t)z^{n-j}. \tag{1.7}$$

LEMMA 1.3. *There exist constants $C_1, C_2 > 0$ such that*

$$\big|\det \Delta(z)\big| \geq C_2|z|^n$$

for $|z| \geq C_1|e^{-hz}|$.

COROLLARY 1.4. *The entire function* $\det \Delta(z)$ *has no zeros in the domain*

$$\{z : |z| > C_1 e^{-h\Re(z)}\}$$

for C_1 *sufficiently large. Consequently, there are only finitely many zeros in each strip*

$$-\infty < \gamma_1 < \Re(z) < \gamma_2 < \infty$$

and $\det \Delta(z)$ *has a zero free right half plane* $\Re(z) > \gamma$.

Now we turn to a representation for $\Delta^{-1}(z)$. Rewrite

$$\Delta^{-1}(z) = \frac{\text{adj}\Delta(z)}{\det \Delta(z)}, \tag{1.8}$$

where $\text{adj}\Delta(z)$ denotes the matrix of cofactors of $\Delta(z)$, i.e. the coefficients of $\text{adj}\Delta(z)$ are the $(n-1) \times (n-1)$ subdeterminants of $\Delta(z)$. Because of the exponential type calculus presented in Chapter 4 of [21] we have the following representation for the cofactors:

$$(\text{adj}\Delta(z))_{ij} = \delta_{ij} z^{n-1} + \sum_{k=1}^{n-1} \int_0^{kh} e^{-zt} d\eta_{ijk}(t) z^{n-1-k}, \tag{1.9}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Rewrite equation (1.5) as follows

$$L\{x\}(z) = \frac{\text{adj}\Delta(z)}{\det \Delta(z)} \big(f(0) + \int_0^h e^{-zt} df(t)\big). \tag{1.10}$$

Using Corollary 1.4 we can choose γ such that $\det \Delta(z)$ has no zeros in the right half plane $\Re(z) > \gamma$. Hence, the Laplace transform $L\{x\}$ is analytic in this half plane. So, from the Laplace inversion theorem, we obtain the following representation for the solution $x = x(\cdot\,; f)$ of the renewal equation (1.3)

$$x(t) = \frac{1}{2\pi i} \int_{L(\gamma)} e^{zt} \frac{\text{adj}\Delta(z)}{\det \Delta(z)} \big(f(0) + \int_0^h e^{-zt} df(t)\big) dz \quad \text{for } t > 0. \tag{1.11}$$

Next we analyze the singularities of

$$H(z,t) = e^{zt} \frac{\text{adj}\Delta(z)}{\det \Delta(z)} \big(f(0) + \int_0^h e^{-zt} df(t)\big). \tag{1.12}$$

Clearly the only singularities are poles of finite order, given by the zeros of $\det \Delta(z)$.

LEMMA 1.5. *If λ_j is a zero of $\det\Delta(z)$ of order m_{λ_j}, then the residue of $H(z,t)$ for $z=\lambda_j$ equals*

$$\operatorname*{Res}_{z=\lambda_j} H(z,t) = p_j(t)e^{\lambda_j t}, \tag{1.13}$$

where p_j is a polynomial in t of degree less than or equal to $(m_{\lambda_j}-1)$.

Denote the zeros of $\det\Delta(z)$ by $\lambda_1, \lambda_2, \ldots$. Using Corollary 1.4 we can define a sequence $\{\gamma_l\}$ such that the number of zeros of $\det\Delta(z)$ with real part strictly between γ_l and γ equals l. Define $\Gamma(\gamma,\gamma_l)$ to be the closed contour in the complex plane, which is composed of four straight lines and connects the points $\gamma_l - iN, \gamma - iN, \gamma + iN$, and $\gamma_l + iN$, where N is larger than $\max_{1\le j\le l}|\Im(\lambda_j)|$.

From the above lemma and the Cauchy theorem of residues we obtain

$$\frac{1}{2\pi i}\int_{\Gamma(\gamma,\gamma_l)} H(z,t)dz = \sum_{j=1}^{l} p_j(t)e^{\lambda_j t}. \tag{1.14}$$

In order to be able to shift the contour $L(\gamma)$ to $L(\gamma_l)$ we have to derive estimates for

$$|H(\sigma+i\omega;t)| \quad \text{for large values of } |\omega|.$$

LEMMA 1.6. *If $-\infty < \gamma_l < \gamma < \infty$. Then*

$$\lim_{|z|\to\infty} |H(z,t)| = 0$$

uniformly in $\gamma_l \le \Re(z) \le \gamma$.

From equation (1.14) and the above lemma we obtain by taking the limit $N\to\infty$ in (1.14) that

$$x(t) = \sum_{j=1}^{l} p_j(t)e^{\lambda_j t} + \frac{1}{2\pi i}\int_{L(\gamma_l)} H(z,t)dz. \tag{1.15}$$

So it remains to prove estimates for the remainder integral and we derive the following result [21; 6.12].

THEOREM 1.7. *Fix some $\gamma\in\mathbb{R}$ such that $\det\Delta(z)\neq 0$ on $L(\gamma)$. Then we have the following asymptotic expansion for the solution x of the renewal equation (1.3)*

$$x(t) = \sum_{\Re(\lambda_j)>\gamma} p_j(t)e^{\lambda_j t} + o(e^{\gamma t}) \quad \text{as } t\to\infty. \tag{1.16}$$

2 The structural operators F and G

In a natural way, through translation along the solution we can associate strongly continuous semigroups with (1.1) and (1.3). Define $T(t) : \mathcal{C} \to \mathcal{C}$ by

$$T(t)\varphi = x_t(\cdot\,;\varphi), \tag{2.1}$$

where $x_t(\theta) = x(t+\theta)$ for $-h \le \theta \le 0$. Suggested by Remark 1.1(i) we define the forcing function space for (1.3) to be $\mathcal{F}$, the supremum normed Banach space of continuous functions on $[0,\infty)$ which are constant on $[h,\infty)$. Define $S(t) : \mathcal{F} \to \mathcal{F}$ by

$$S(t)f = x^t - \zeta * x^t, \tag{2.2}$$

where $x^t(s) = x(t+s)$ for $0 \le s < \infty$.

In addition to $F : \mathcal{C} \to \mathcal{F}$ we can also define a structural operator $G : \mathcal{F} \to \mathcal{C}$ from the renewal equation (1.3) into the RFDE (1.1) by

$$Gf = x(\cdot + h; f), \tag{2.3}$$

i.e. G translates the solution of (1.3) corresponding to f backwards over a distance h. It is easy to verify that G is injective and onto. We formulate the result in a proposition to show the interplay between the structural operators and the semigroups.

Proposition 2.1.

(i) $FT(t) = S(t)G$;
(ii) $FG = S(h)$;
(iii) $GF = T(h)$;
(iii) $T(t) = GS(t)G^{-1}$.

The infinitesimal generator for $T(t)$ is given by $A\varphi = \dot\varphi$ defined on

$$\mathcal{D}(A) = \{\varphi \in \mathcal{C} : \dot\varphi \in \mathcal{C} \text{ and } \dot\varphi(0) = \int_0^h d\zeta(\theta)\varphi(-\theta)\ \}. \tag{2.4}$$

There are important duality relations between the introduced $\mathcal{C}_0$-semigroups and the structural operators. We describe the main result see Delfour and Manitius [6, 7], Diekmann [8, 9]. Let $S^T(t)$ denote the $\mathcal{C}_0$-semigroup associated with the transposed renewal equation

$$x - \zeta * x = f,$$

where $f \in \mathrm{NBV}[0,h]$ and constant on $[h,\infty)$ and $T^T(t)$ the $\mathcal{C}_0$- semigroup associated with the transposed delay equation

$$\dot x(t) = \int_0^h d\zeta^T(\theta)x(t-\theta).$$

Similar as above we can introduce structural operators F^T and G^T.

PROPOSITION 2.2.
(i) $F^T = F^*$;
(ii) $G^T = G^*$.

Furthermore, the following duality principle holds

THEOREM 2.3. *The C_0-semigroups $T^*(t)$ and $S^T(t)$ are equal.*

Next we describe the spectrum of A. Let

$$\Delta(z) = zI - \int_0^h e^{-zt} d\zeta(t) \tag{2.5}$$

denote the characteristic matrix associated with the RFDE (1.1) and let

$$R(z, A) : \mathcal{C} \to \mathcal{D}(A)$$

denote the resolvent

$$R(z, A) = (zI - A)^{-1} \tag{2.6}$$

of A. The following theorem yields an explicit formula for the resolvent of A.

THEOREM 2.4. *If $\varphi \in \mathcal{C}$ and if $\lambda \in \mathcal{C}$ is such that* $\det \Delta(\lambda) \neq 0$. *Then $\lambda \in \rho(A)$ and $R(\lambda, A)\varphi$ is given explicitly by*

$$R(\lambda, A)\varphi = e^{\lambda t}\{\Delta^{-1}(\lambda)K(\varphi) - \int_0^t e^{-\lambda s}\varphi(s)ds\}, \tag{2.7}$$

where

$$K(\varphi) = \lambda \int_0^\infty e^{-\lambda t} F\varphi(t)dt. \tag{2.8}$$

PROOF. Let $\psi = R(\lambda, A)\varphi$. From the definition of A it follows that

$$(\lambda I - A)\psi = \varphi$$

if and only if ψ satisfies the conditions

(i) $\lambda\psi - \dot{\psi} = \varphi$;

(ii) $\lambda\psi(0) - \int_0^h d\zeta(\theta)\psi(-\theta) = \varphi(0)$;

(iii) $\dot{\psi} \in \mathcal{C}$.

Define

$$\psi(t) = e^{\lambda t}\psi(0) + \int_t^0 e^{\lambda(t-s)}\varphi(s)ds, \tag{2.9}$$

where $-h \leq t \leq 0$. Then ψ satisfies the conditions (i) and (iii). Also, condition (ii) becomes

$$\Delta(\lambda)\psi(0) = K(\varphi). \tag{2.10}$$

Since $\det \Delta(\lambda) \neq 0$, we can solve

$$\psi(0) = \Delta^{-1}(\lambda)K(\varphi). \tag{2.11}$$

∎

COROLLARY 2.5. *The spectrum of A is all point spectrum and is given by*

$$\sigma(A) = \mathrm{P}\sigma(A) = \{\lambda \in \mathbb{C} : \det \Delta(\lambda) \neq 0\}. \tag{2.12}$$

PROOF. Because of the proof of Theorem 2.4 we have

$$\{\lambda \in \mathbb{C} : \det \Delta(\lambda) \neq 0\} \subset \rho(A).$$

To prove the reverse inclusion choose $\lambda \in \mathbb{C}$ such that $\det \Delta(\lambda) = 0$ and define

$$\varphi(t) = e^{\lambda t}\varphi^0 \quad \text{for } -h \leq t \leq 0,$$

where $\varphi^0 \neq 0$ is an element of the nullspace of $\Delta(\lambda)$. Then

$$A\varphi = \dot{\varphi} = \lambda\varphi.$$

Therefore, we conclude that $\lambda \in \mathrm{P}\sigma(A)$. ∎

COROLLARY 2.6.

$$\mathcal{N}\big((\lambda I - A)\big) = \big\{\varphi \in \mathcal{C} : \varphi(t) = e^{\lambda t}\varphi^0 \text{ and } \varphi^0 \in \mathcal{N}\big(\Delta(\lambda)\big)\big\}.$$

Let $\varphi \in \mathcal{C}$ be fixed and consider the function $R(z, A)\varphi$ as a function of z. By Theorem 2.4 we have that $R(z, A)\varphi$ is a meromorphic function with poles λ satisfying the equation

$$\det \Delta(z) = 0.$$

This property of $R(z, A)$ makes it possible to apply [19; V.10.1].

THEOREM 2.7. *If λ is a pole of $R(z, A)$ of order m. Then for some k with $1 \leq k \leq m$*

(i) $\mathcal{N}\big((\lambda I - A)^k\big) = \mathcal{N}\big((\lambda I - A)^{k+1}\big)$;

(ii) $\mathcal{R}\big((\lambda I - A)^k\big) = \mathcal{R}\big((\lambda I - A)^{k+1}\big)$;

(iii) $\mathcal{R}\big((\lambda I - A)^k\big)$ *is closed;*

(iv) $\mathcal{C} = \mathcal{N}\big((\lambda I - A)^k\big) \oplus \mathcal{R}\big((\lambda I - A)^k\big)$;

(v) *The spectral projection P_λ corresponding to the decomposition in (iv) can be represented by the contour integral*

$$P_\lambda\varphi = \frac{1}{2\pi i}\int_{\Gamma_\lambda} R(z, A)\varphi dz, \tag{2.13}$$

where Γ_λ is a circle enclosing λ but no other point of the discrete set $\sigma(A)$.

Let $\mathcal{M}_\lambda$ denote the generalized eigenspace $\mathcal{N}\big((\lambda I - A)^m\big)$ corresponding to an eigenvalue λ of A. By Theorem 2.4 and the definition of A we have that elements of $\mathcal{M}_\lambda$ involve combinations of

$$t^l e^{\lambda t} d_l, \tag{2.14}$$

where $l = 1, 2, \ldots, m$ and the constants $d_l \in \mathbb{R}^n$ satisfy a system of linear equations. So $\mathcal{M}_\lambda$ is finite dimensional and by using this system of linear equations one can construct an explicit base for $\mathcal{M}_\lambda$ that shows that the dimension of $\mathcal{M}_\lambda$ equals m_λ, the multiplicity of λ as zero of $\det \Delta(z)$, see [12].

Let Q_λ denote $\mathcal{R}\big((\lambda I - A)^k\big)$. Since the generator A and the $\mathcal{C}_0$-semigroup $T(t)$ commute, the linear subspaces $\mathcal{M}_\lambda$ and Q_λ are $T(t)$-invariant. Before we continue with the characterization of these $T(t)$-invariant subspaces, we first extend the equivalence between linear autonomous RFDEs and renewal equations (a similar result was proved by Banks and Manitius [1]). As a consequence of this extension we can translate the convergence results derived in [21] to results on spectral projection series for a state of the RFDE (1.1).

PROPOSITION 2.8. *The λ_j-th term of the Fourier type series expansion of $x(\cdot\,; f)$ of the renewal equation (1.3) equals the λ_j-th spectral projection of the corresponding state of the RFDE (1.1), i.e.*

$$P_{\lambda_j}(Gf)(t-h) = \operatorname*{Res}_{z=\lambda_j} \big\{e^{zt}\Delta^{-1}(z) z \int_0^\infty e^{-zt} f(t)dt\big\}. \tag{2.15}$$

As a result of the above proposition, residue calculus of the renewal equation and analysis of the spectrum of the resolvent $R(z, A)$ yield the same information. The only difference is that instead of the solution x we now analyze the state $x_t = x(t+\theta)$ as a function on the interval $[-h, 0]$. In Theorem 1.7 we derived an exponential estimate for the remainder term of $x(\,\cdot\,; f)$ and of course at the same time this yields an estimate for the state

$$T(t)Gf = x_t(\,\cdot\,; f).$$

Recall from Proposition 2.1 that the $\mathcal{C}_0$-semigroups are intertwined, i.e.

$$T(t) = GS(t)G^{-1}. \tag{2.16}$$

COROLLARY 2.9. *Let $\Lambda(\gamma)$ be the finite set of eigenvalues defined by*

$$\Lambda = \Lambda(\gamma) = \big\{\lambda \in \sigma(A) : \Re(\lambda) > \gamma\big\}.$$

Then the state space $\mathcal{C}$ can be decomposed into two closed $T(t)$-invariant subspaces $\mathcal{M}_\Lambda$ and Q_Λ

$$\mathcal{C} = \mathcal{M}_\Lambda \oplus Q_\Lambda, \tag{2.17}$$

where

$$\mathcal{M}_\Lambda = \bigoplus_{\lambda\in\Lambda} \mathcal{M}_\lambda$$

and

$$Q_\Lambda = \bigoplus_{\lambda\in\Lambda} Q_\lambda.$$

The spectral projection P_Λ on $\mathcal{M}_\Lambda$ is given by

$$P_\Lambda = \sum_{\lambda\in\Lambda} P_\lambda.$$

Besides, if

$$\varphi = P_\Lambda\varphi + (I - P_\Lambda)\varphi,$$

according to the above decomposition. Then

$$\|T(t)(I - P_\Lambda)\varphi\| \leq Ke^{\gamma t}\|(I - P_\Lambda)\varphi\| \tag{2.18}$$

for some positive constant K and $t \geq 0$.

Assume that all roots have negative real part, then we can choose $\gamma < 0$ in Corollary 2.9 and we derive exponential asymptotic stability: for all $\varphi \in \mathcal{C}$

$$\|T(t)\varphi\| \leq Ke^{\gamma t}\|\varphi\| \tag{2.19}$$

for some positive constant K and negative γ.

Let $\mathcal{M}_\mathcal{C}$ denote the linear subspace generated by $\mathcal{M}_\lambda$, i.e.

$$\mathcal{M}_\mathcal{C} = \bigoplus_{\lambda\in\sigma(A)} \mathcal{M}_\lambda. \tag{2.20}$$

This linear subspace is called the generalized eigenspace of A.

DEFINITION 2.10. The generalized eigenspace $\mathcal{M}_\mathcal{C}$ is called complete if and only if $\mathcal{M}_\mathcal{C}$ is dense in $\mathcal{C}$, that is $\overline{\mathcal{M}_\mathcal{C}} = \mathcal{C}$.

DEFINITION 2.11. A small solution x of (1.1) is a solution x such that

$$\lim_{t\to\infty} e^{kt}x(t) = 0$$

for all $k \in \mathbf{R}$.

We can now characterize the subspace $\mathcal{N}\big(T(\alpha)\big)$.

THEOREM 2.12.

$$\mathcal{N}\big(T(\alpha)\big) = \big\{\varphi \in \mathcal{C} : z \mapsto R(z, A)\varphi \ \text{ is entire }\big\}.$$

PROOF. From Theorem 2.4 it follows that only the fact

$$\varphi \in \mathcal{N}\big(T(\alpha)\big) \text{ if and only if } x(\,\cdot\,; F\varphi) \text{ is a small solution}$$

remains to be proved. But this is clear from the definitions of F and α. ∎

From the exponential estimates derived in Corollary 2.9 we can also characterize the closed subspace

$$\bigcap_{\lambda \in \sigma(A)} Q_\lambda. \tag{2.21}$$

COROLLARY 2.13.

$$\bigcap_{\lambda \in \sigma(A)} Q_\lambda = \mathcal{N}\big(T(\alpha)\big).$$

PROOF. Let $\varphi \in \mathcal{N}\big(T(\alpha)\big)$. From Theorem 2.12 and the representation (2.15) we derive for all $\lambda \in \sigma(A)$

$$P_\lambda \varphi = 0.$$

Hence

$$\varphi \in \bigcap_{\lambda \in \sigma(A)} Q_\lambda.$$

On the other hand if $\varphi \in \bigcap_{\lambda \in \sigma(A)} Q_\lambda$, then we derive from Corollary 2.9 the exponential estimate

$$\|T(t)\varphi\| \leq K e^{\gamma t} \quad \text{for } t \geq 0 \tag{2.22}$$

for every $\gamma \in \mathbb{R}$ and some positive constant K. Therefore, $x(\,\cdot\,;\varphi)$ is a small solution and Henry's theorem on small solutions yields $\varphi \in \mathcal{N}\big(T(\alpha)\big)$. ∎

3 TYPES OF COMPLETENESS

DEFINITION 3.1. An entire function $F : \mathbb{C} \to \mathbb{C}$ is of order 1 if and only if

$$\limsup_{r \to \infty} \frac{\log \log M(r)}{\log r} = 1, \tag{3.1}$$

where

$$M(r) = \max_{0 \leq \theta \leq 2\pi} \left\{ |F(re^{i\theta})| \right\}. \tag{3.2}$$

An entire function of order 1 is of exponential type if and only if

$$\limsup_{r \to \infty} \frac{\log M(r)}{r} = \mathrm{E}(F), \tag{3.3}$$

where $0 \leq \mathrm{E}(F) < \infty$. In that case $\mathrm{E}(F)$ is called the exponential type of F. A vector-valued function $F = (F_1, \ldots, F_n) : \mathbb{C} \to \mathbb{C}^n$ will be called an entire function of exponential type if and only if the components F_j of F are entire functions of order 1 that are of exponential type. Furthermore, the exponential type will be defined by

$$\mathrm{E}(F) = \max_{1 \leq j \leq n} \mathrm{E}(F_j). \tag{3.4}$$

Next we shall give a characterization of the smallest possible time t_0 such that all small solutions vanish for $t \geq t_0$. This characterization of t_0 is needed in order to establish the results concerning completeness of the system of generalized eigenfunctions which we present shortly.

The function $\det \Delta(z)$ is an entire function of exponential type less than or equal to nh. Define ϵ by

$$\mathrm{E}(\det \Delta(z)) = nh - \epsilon. \tag{3.5}$$

Let $\mathrm{adj}\Delta(z)$ denote the matrix of cofactors of $\Delta(z)$. Since the cofactors $C_{ij}(z)$ are $(n-1) \times (n-1)$-subdeterminants of $\Delta(z)$, it follows that the exponential type of the cofactors is less than or equal to $(n-1)h$. Define σ by

$$\max_{1 \leq i,j \leq n} \mathrm{E}(C_{ij}) = (n-1)h - \sigma. \tag{3.6}$$

LEMMA 3.2. *If $\epsilon > 0$, then $\sigma < \epsilon$.*

PROOF. Suppose $\sigma = \epsilon$. We shall calculate $\mathrm{E}\big(\det \mathrm{adj}\Delta(z)\big)$ in two different ways. Since $\sigma = \epsilon$ we have

$$\begin{aligned} \mathrm{E}\big(\det \mathrm{adj}\Delta(z)\big) &\leq n\big((n-1)h - \epsilon\big) \\ &= (n-1)(nh - \epsilon) - \epsilon. \end{aligned} \tag{3.7}$$

Using the exponential type calculus [21]

$$\begin{aligned} \mathrm{E}\big(\det \mathrm{adj}\Delta(z)\big) &= \mathrm{E}\big((\det \Delta(z))^{n-1}\big) \\ &= (n-1)(nh - \epsilon). \end{aligned} \tag{3.8}$$

Hence

$$(n-1)(nh - \epsilon) \leq (n-1)(nh - \epsilon) - \epsilon, \tag{3.9}$$

which is a contradiction if $\epsilon > 0$. ∎

We can now state and prove a sharp version of Henry's theorem on small solutions [20] for the RFDE (1.1). See also [21; 10.11].

THEOREM 3.3. *The ascent α of the $\mathcal{C}_0$-semigroup $\{T(t)\}$ associated with the RFDE (1.1) is finite and is given by*

$$\alpha = \epsilon - \sigma. \tag{3.10}$$

Using Lemma 3.2 we have the following corollary.

COROLLARY 3.4. *The $\mathcal{C}_0$-semigroup $\{T(t)\}$ associated with the RFDE (1.1) is injective if and only if* $\mathrm{E}(\det \Delta(z)) = nh$.

COROLLARY 3.5. *The ascent α of $T(t)$ and the ascent δ of $T^*(t)$ are equal.*

PROOF. Using the duality principle Theorem 2.4 the corollary is clear since

$$\operatorname{adj}\Delta(z;\zeta) = \operatorname{adj}\Delta(z;\zeta^T)$$

and hence $\epsilon^T = \epsilon$ and $\sigma^T = \sigma$. ∎

The following corollary answers a question of Delfour and Manitius and extends a result of Bartosiewicz [2].

COROLLARY 3.6. *$\mathcal{N}(F) = \{0\}$ if and only if $\mathcal{N}(F^*) = \{0\}$.*

The following corollary yields an easy to verify necessary and sufficient condition for completeness. The proof is a combination of Henry's result $\overline{\mathcal{M}_C} = \overline{\mathcal{R}(T(\delta))}$ and the above corollaries.

COROLLARY 3.7. *The following statements are equivalent:*
(i) *The generalized eigenspace $\mathcal{M}_C$ is complete;*
(ii) *The ascent $\alpha = 0$;*
(iii) $\mathcal{N}(F) = \{0\}$;
(iv) $\mathcal{N}(F^*) = \{0\}$;
(iv) $\mathrm{E}(\det \Delta(z)) = nh$.

REMARK 3.8. Note, that Theorem 3.3 proves the existence of a small solution if $\mathrm{E}(\det \Delta(z)) < nh$.

Delfour and Manitius also introduced the concept of F-completeness. The generalized eigenspace $\mathcal{M}_C$ is called F-complete if

$$\overline{F\mathcal{M}_C} = \overline{\mathcal{R}(F)}. \tag{3.11}$$

They proved that F-completeness holds if and only if $\mathcal{N}(T^*(\delta)) = \mathcal{N}(F^*)$, that is, if and only if $\delta \le h$. So, we have

COROLLARY 3.9. *$\mathcal{M}_C$ is F-complete if and only if $\epsilon - \sigma \le h$.*

EXAMPLE 3.10. Consider the following system of differential-difference equations

$$\begin{aligned} \dot{x}_1(t) &= -x_2(t) + x_3(t-1) \\ \dot{x}_2(t) &= x_1(t-1) + x_3(t-\frac{1}{2}) \\ \dot{x}_3(t) &= x_3(t). \end{aligned} \tag{3.12}$$

Then the characteristic matrix becomes

$$\Delta(z) = \begin{pmatrix} z & 1 & -e^{-z} \\ -e^{-z} & z & -e^{-\frac{1}{2}z} \\ 0 & 0 & z+1 \end{pmatrix} \tag{3.13}$$

with determinant

$$\det \Delta(z) = (z+1)(z^2 + e^{-z}). \tag{3.14}$$

Therefore,

$$\epsilon = 2.$$

Since the cofactor

$$\begin{aligned} C_{23}(z) &= -\begin{vmatrix} z & -e^{-z} \\ -e^{-z} & -e^{-\frac{1}{2}z} \end{vmatrix} \\ &= ze^{-\frac{1}{2}z} + e^{-2z} \end{aligned} \tag{3.15}$$

has exponential type 2, we derive that $\sigma = 0$. Therefore, from Theorem 3.3, the ascent of the system (3.12) equals two. Thus there exists a (non trivial) small solution $x = x(\cdot\,; \varphi)$ such that

$$\mathrm{supp}(x) = [-1, 1].$$

EXAMPLE 3.11. Consider the following system of differential-difference equations

$$\begin{aligned} \dot{x}_1(t) &= -x_2(t) + x_3(t) \\ \dot{x}_2(t) &= x_1(t-1) + x_3(t - \frac{1}{2}) \\ \dot{x}_3(t) &= x_3(t). \end{aligned} \tag{3.16}$$

Then the characteristic matrix becomes

$$\Delta(z) = \begin{pmatrix} z & 1 & 1 \\ -e^{-z} & z & -e^{-\frac{1}{2}z} \\ 0 & 0 & z+1 \end{pmatrix} \tag{3.17}$$

with determinant

$$\det \Delta(z) = (z+1)(z^2 + e^{-z}). \tag{3.18}$$

Therefore,

$$\epsilon = 2.$$

Furthermore, in this case we derive $\sigma = 1$. Therefore, from Theorem 3.3, the ascent of the system (3.16) equals one. Thus all small solutions are trivial, in the sense that they are identical zero for $t \geq 0$.

From Corollary 3.7 it follows that completeness of the generalized eigenfunctions fails if and only if there are $c \in \mathbb{R}^n$ and $q(z) = \int_0^h e^{-zt}\psi(t)dt$ with $\psi \in L^2[0, h]$ such that

$$\Delta^{-1}(z)[c + q(z)] = \text{ an entire function.}$$

Hence, from the Paley-Wiener theorem

$$\Delta^{-1}(z)[c + q(z)] = \int_0^{\tau} e^{-zt}x(t)dt, \tag{3.19}$$

where x denotes the corresponding small solution with support $[0, \tau]$ and $\tau \leq \alpha$. If $\mathrm{E}(\det \Delta(z)) = nh$, an exponential type consideration shows that both c and q must be zero. Theorem 3.3 now states that as soon as $\mathrm{E}(\det\Delta(z)) < nh$ we can find a couple c and q. In the proof we did explicitly construct a pair.

Thus, the above results can be formulated as follows: If completeness of the generalized eigenfunctions fails we can construct a pair (c, q) so that (3.19) is satisfied. We still have F-completeness as long as $\tau \leq h$. If F-completeness fails too one might check point-wise completeness, that is, the solution set at time t : $\{x(t) : x_0 = \varphi\}$ fills $\mathbf{R}^n$. Using the representation of the solution through the fundamental solution, that is the matrix-valued solution U with initial data $U(0) = I$ and $U(t) = 0$ for $-h \leq t < 0$ one can prove Kappel [14] : Pointwise completeness holds if and only if there is a $0 \neq c \in \mathbf{R}^n$ such that

$$\Delta^{-1}(z)\, c = \text{ an entire function.} \tag{3.20}$$

In some special cases (for example point delays), there are necessary and sufficient conditions on the matrices defining the kernel ζ but these conditions are difficult to verify. We shall discuss a tool useful to verify pointwise completeness. First of all it is clear that $\alpha > h$ is needed in order to have an entire solution of (3.20). Given $\alpha > h$ one can analyze the chain of roots of both the numerator and the denominator by studying the Newton polygons. This may give chains of roots that can not be eliminated and hence (3.20) can not be entire.

To show the procedure we define a Newton polygon, see [21] for more information. Consider an exponential polynomial

$$H(z, w) = z^n + e^{l_1 z} z^{n-1} + \ldots + e^{l_n z}, \tag{3.21}$$

where $l_0 = 0$ and for $j = 1, \ldots, n$ the exponents l_j are nonnegative real numbers. Assign to every term of (3.21) a point A_j with coordinates $(l_j, n-j)$.

DEFINITION 3.12. The Newton polygon associated with H and denoted by $\mathrm{N}(H)$, is defined by the polygon determined by the upper convex envelope of the set of points $\{A_j : j = 0, 1, \ldots, n\}$. The upper convex property implies that the slopes of the line segments of the Newton polygon are negative and decrease.

Consider for Example 3.10 the first row of (3.20) that means we have to find $c \in \mathbf{R}^3$ such that

$$\frac{P(z)}{\det\Delta(z)} = \text{ an entire function,}$$

where

$$P(z) = c_1 z^2 + (c_1 - c_2 + c_3 e^{-z})z - c_3 e^{-z/2}$$

and

$$\det\Delta(z) = z^3 + z^2 + z e^{-z} + e^{-z}.$$

Therefore, the slope of the Newton polygon of $\det\Delta(z)$ equals $-1/2$ but the slope of the Newton polygon of P equals -1 unless $c_3 = 0$. If $c_3 = 0$ then P is a polynomial and cannot cancel the roots of $\det\Delta(z)$, hence we can assume the slope Newton polygon of

P to be -1, but if the polygons have different slopes then the characterization of the chains of roots of exponential polynomials, Bellman and Cooke [3; 12.10] shows that P cannot cancel the roots of $\det \Delta(z)$. (The chains of roots of an exponential polynomial are given by $|ze^{-\mu_r z}| = C$, where μ_r runs over the slopes of the line segments of the Newton polygon and C is a positive constant).

Using this method it is easy to show that if $n = 2$ and $\mathrm{E}(\det \Delta(z)) > 0$ then pointwise completeness always holds for any RFDE, see also Popov [17] for the case of one point delay.

THEOREM 3.13. *Given a linear autonomous RFDE. If $n = 2$ and $\mathrm{E}(\det \Delta(z)) > 0$ then the system is pointwise complete.*

PROOF. In order to have

$$\Delta^{-1}(z)\, c = \text{ an entire function}$$

we clearly need $\mathrm{E}(\det \Delta(z)) < h$ (One can also use the fact that for $\mathrm{E}(\det \Delta(z)) \geq h$ completeness holds and hence pointwise complete holds too). Since for $n = 2$ the cofactors are just the coefficients there has to be one cofactor with exponential type h. Therefore, the numerator has positive type greater than $\mathrm{E}(\det \Delta(z))$, but since $\mathrm{E}(\det \Delta(z)) < h$ the slopes of the Newton polygons of $\det \Delta(z)$ and the numerator cannot be the same for any choice of c_1 and c_2. Thus pointwise completeness holds if $\mathrm{E}(\det \Delta(z)) > 0$. ∎

The case $\mathrm{E}(\det \Delta(z)) = 0$ is not completely clear, in the case of one point delay, pointwise completenss still holds if $\mathrm{E}(\det \Delta(z)) = 0$. So far one has to check if a $c \neq 0$ exists, but since in this case $\det \Delta(z)$ reduces to a polynomial there are only a finite number of possibilities to check. For higher dimensional systems the situation is even less clear. For $n = 3$ a careful analysis lead us to the conjecture that $\mathrm{E}(\det \Delta(z))$ must be zero in order for pointwise completeness to fail (compare the counter example by Popov), but this is by no means a general statement since one can make a 4-dimensional system out of Popov's example and a decoupled additional equation to get $\mathrm{E}(\det \Delta(z)) = h$ and pointwise completenss still fails.

We conclude this section with a convergence result for the spectral projection series [21]. Although there are convergence results for $t > 0$ we present here only a convergence result for $t > h$. From the application point of view this is quite natural. The initial condition φ is given and hence, there is no need to expand $T(t)\varphi$ for $t \leq h$ in a spectral projection series. Of course, if one would like to study the closure of the set of generalized eigenfuctions series expansions of $T(t)\varphi$ for $t > 0$ are needed. This is done in [21] where we proved a complete characterization of the closure of the generalized eigenspace $\overline{\mathcal{M}_C}$.

THEOREM 3.14. *If*

$$\mathrm{N}(z\, \mathrm{adj}\Delta(z)) \leq \mathrm{N}(\det \Delta(z)) \tag{3.22}$$

and if $\varphi \in C$ such that φ is locally of bounded variation. Then for every $\epsilon > 0$ the state $T(h + \epsilon)\varphi$ can be represented by a convergent spectral projection series.

4 An example: The case $n = 2$

As an illustration of the general theory we consider a two-dimensional system

$$\dot{x}(t) = Ax(t) + Bx(t-1). \tag{4.1}$$

The characteristic matrix of (4.1) becomes

$$\Delta(z) = \begin{pmatrix} z - a_{11} - b_{11}e^{-z} & -a_{12} - b_{12}e^{-z} \\ -a_{21} - b_{21}e^{-z} & z - a_{22} - b_{22}e^{-z} \end{pmatrix} \tag{4.2}$$

and the determinant $\det \Delta(z)$ satisfies

$$\det \Delta(z) = z^2 - (a_{11} + a_{22} + (b_{11} + b_{22})e^{-z})z - q(z), \tag{4.3}$$

where

$$q(z) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \left(\begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix} \right) e^{-z} + \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} e^{-2z} \tag{4.4}$$

and $|\cdot|$ denotes the determinant of the matrix. Therefore $\mathrm{E}(\det \Delta(z)) = 2$ if and only if $\det(B) \neq 0$ and hence

THEOREM 4.1. *The system of generalized eigenfunctions is complete if and only if* $\det(B) \neq 0$

Note that $\sigma = 0$ and so the ascent $\alpha = \epsilon = 2 - \mathrm{E}(\det \Delta(z))$. Furthermore, if $\det(B) = 0$ but $\mathrm{trace}(B) = b_{11} + b_{22} \neq 0$ then we still have F-completeness. In fact, for this special class of two dimensional systems with one point delay we conclude that F-completenss holds if and only if the characteristic equation has an infinite number of roots. This is special, it is easy to construct a two dimensional system with two time delays so that the characteristic equation has an infinite number of roots but F-completenss fails [21; 13.3].

To study the convergence of the spectral projection series we have to consider two cases: If completeness holds or F-completeness with $\mathrm{trace}(B) \neq 0$ then (3.22) holds and for φ locally of bounded variation $T(h+\epsilon)\varphi$ has a convergent spectral projection series for every $\epsilon > 0$. If $\mathrm{trace}(B) = 0$, then a more careful analysis is required which results in either more smoothness requirements of φ or replacing h be a larger number. The analysis uses the characterization of the closure of the generalized eigenspace [21]. In general, one can only say that for φ locally of bounded variation $T(2h+\epsilon)\varphi$ has a convergent spectral projection series for every $\epsilon > 0$.

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