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**The initial-boundary value problem for
a nonlinear degenerate parabolic equation**

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1. Introduction and main results.

Let $a < b$ and $\lambda > 0$. We consider nonnegative solutions of the initial-boundary value problem

$$\begin{cases} u_t = uu_{xx} - \lambda |u_x|^2 & (a < x < b, t > 0) & (1.1) \end{cases}$$

$$\begin{cases} u(a, t) = u(b, t) = 0 & (t > 0) & (1.2) \end{cases}$$

$$\begin{cases} u(x, 0) = u_0(x) & (a < x < b) & (1.3) \end{cases}$$

where initial data u_0 satisfy

$$(H.1) \quad u_0 \in W^{1, \infty}(a, b) \quad \text{and} \quad u_0(x) \geq 0 \quad (a \leq x \leq b)$$

In order to construct a solution to the problem (1.1)-(1.3), it might be natural to employ the well-known viscosity method: Let $\varepsilon > 0$ and let $u_\varepsilon(x, t)$ be an unique classical solution of the initial-boundary value problem for the uniformly parabolic equation:

$$\begin{cases} u_{\varepsilon t} = (u_\varepsilon + \varepsilon) u_{\varepsilon xx} - \lambda |u_{\varepsilon x}|^2 & (a < x < b, t > 0) & (1.1)_\varepsilon \end{cases}$$

$$\begin{cases} u_\varepsilon(a, t) = u_\varepsilon(b, t) = 0 & (t > 0) & (1.2)_\varepsilon \end{cases}$$

$$\begin{cases} u_\varepsilon(x, 0) = u_0(x) & (a < x < b) & (1.3)_\varepsilon \end{cases}$$

We call u the viscosity solution of the problem (1.1)-(1.3)

$$\text{if } u(x, t) = \lim_{\varepsilon \rightarrow 0} u_{\varepsilon}(x, t).$$

Let us consider solutions with compact support and define the interface $\zeta_{\pm}(t)$ by

$$\zeta_{\pm}(t) = \pm \sup \{ \pm x : u(x, t) > 0 \} \quad \text{for } t > 0$$

Differentiating $u(\zeta_{\pm}(t), t) = 0$ with respect to t and using eq. (1.1), we easily see that the interface $\zeta_{\pm}(t)$ satisfies formally

$$\frac{d\zeta_{\pm}(t)}{dt} = \lambda u_x(\zeta_{\pm}(t), t), \quad (1.4)$$

provided $u_x(\zeta_{\pm}(t), t) \neq 0$. Thus we might expect that the support of solutions shrinks if $u_x(\zeta_{\pm}(t), t) \neq 0$. Indeed, for $\lambda > \frac{1}{2}$ we have a special weak solution of the form

$$u(x, t) = (T_0 - t)^{\frac{1}{2\lambda-1}} \left[C_0^2 - \frac{1}{2(2\lambda-1)} x^2 (T_0 - t)^{\frac{2\lambda}{2\lambda-1}} \right]_+ \quad (1.5)$$

where T_0 and C_0 are positive constants such that

$$\left(-\sqrt{2(2\lambda-1)} C_0 T_0^{\frac{\lambda}{2\lambda-1}}, \sqrt{2(2\lambda-1)} C_0 T_0^{\frac{\lambda}{2\lambda-1}} \right) \subset [a, b]$$

and $[\cdot]_+ = \max(\cdot, 0)$.

Apparently its support shrinks to one point. But this conjecture is not true for viscosity solutions. In [1], Bertch, Dal Passo and Ughi show that every viscosity solution of the Cauchy problem for (1.1) has a property that

$$\text{supp } u(t) = \text{supp } u_0 \quad \text{for } t > 0. \quad (1.6)$$

It is a striking result. If $\lambda < 0$, equation (1.1) is called the pressure equation, related to the porous medium equation and

the support of solutions spreads out as time goes , as is suggested by the interface equation (1.4).

Another curious property of eq. (1.1) is the nonuniqueness phenomenon which was discovered by Dal Passo and Luckhaus [2] ($\lambda = 0$), Ughi [5] ($\lambda = 0$) and Bertch, Dal Passo and Ughi [1] ($\lambda \geq 0$). The existence of our special weak solution u also suggests the nonuniqueness phenomenon.

We now define weak solutions of the problem (1.1)-(1.3) as follows:

Definition 1. A nonnegative function $u \in L^\infty([0, \infty) : W^{1, \infty}[a, b])$ is called a weak solution of (1.1)-(1.3) if for any $T > 0$

$$u_t \in L^2([a, b] \times [0, T])$$

and for all $t \geq 0$

$$\int_a^b u(x, t) \psi(x, t) dx = \int_a^b u_0(x) \psi(x, 0) dx$$

$$+ \int_0^t \int_a^b \{u(x, s) \psi_t(x, s) - u(x, s) u_x(x, s) \psi_x(x, s) - (\lambda+1) |u_x(x, s)|^2 \psi(x, s)\} dx ds$$

for any function $\psi \in C^{2,1}([a, b] \times [0, \infty))$ with compact support in (a, b) .

Note that $u \in L^\infty([0, \infty) : W^{1, \infty}([a, b]))$ with $u_t \in L^2([a, b] \times [0, T])$ for any $T > 0$ implies that u is continuous in x and t .

In this paper we establish the global existence of (weak) solutions of (1.1)-(1.3) and investigate the uniqueness of solutions. We propose a new uniqueness class of solutions which is different from [1], [2] and [5].

As to the existence theorem, we have

Theorem 1. Let u_0 satisfy (H1). Then the problem (1.1)-(1.3) has at least one weak solution.

Theorem 2. Let $\lambda > \frac{1}{2}$. Assume that u_0 satisfies (H1) and

$$(H2) \quad \lim_{x \downarrow a} \frac{u_0(x)}{(x-a)^2} < \infty \quad \text{and} \quad \lim_{x \uparrow b} \frac{u_0(x)}{(b-x)^2} < \infty .$$

Then u satisfies

$$|u_{xx}(x, t)| \leq \frac{1}{t} \quad (1.8)$$

and, in particular, $u \in L^\infty([\delta, \infty); W^{2, \infty}([a, b]))$ as well as

$u_t \in L^\infty([\delta, \infty); L^\infty([a, b]))$ for any $\delta > 0$. Moreover, if we assume

that u_0 is semiconcave, that is,

$$u_{0xx} \leq C \quad \text{in} \quad \mathcal{D}'$$

for some constant C , then u is also semiconcave almost everywhere,

that is,

$$u_{xx}(x, t) \leq C \quad \text{for a.e. } (x, t) \in [a, b] \times (0, \infty)$$

where C is also a positive constant.

Remark 1. In theorem 2 the hypotheses (H1) can be weakened as follows:

$$(H1)_w \quad u_0 \in L^\infty([a, b]), \quad u_0(x) \geq 0 \text{ a.e.}$$

Corollary 1. Under the assumption $(H1)_w$ and (H2), the problem (1.1)-(1.3) has at least one weak solution which has properties in Theorem 2.

Concerning the uniqueness and continuous-dependence-on-data of solutions, we have

Theorem 3. Let u and v be two weak solutions corresponding to the initial data u_0 and v_0 , respectively. Assume that u and v are semiconcave almost everywhere. Then the inequality

$$\int_a^b |u(x, t) - v(x, t)| dx \leq e^{ct} \int_a^b |u_0(x) - v_0(x)| dx$$

holds valid for any $t > 0$ and a positive constant c .

Corollary 2. Let u_0 satisfy $(H1)_w$, (H2) and be semiconcave. Then the problem (1.1)-(1.3) has an unique weak solution u which is also semiconcave and depends on initial data continuously in $L^1(a, b)$.

Remark 2. Our special solution (1.5) is not semiconcave. Uniqueness theorem does not hold valid for the problem (1.1)-(1.3) with initial data

$$u_0(x) = (T_0)^{\frac{1}{2\lambda-1}} \left[C_0^2 - \frac{1}{2(2\lambda-1)} x^2 (T_0)^{\frac{2\lambda}{2\lambda-1}} \right]_+,$$

which does not satisfy (H2).

2. Proof of Theorem 1.

Before proving Theorem 1, we shall obtain a priori estimates of u_ε .

Lemma 1. Let u_0 satisfy (H1). Then

$$\|u_\varepsilon\|_{L^\infty([0, \infty); W^{1, \infty}([a, b]))} \leq C \quad (2.1)$$

and

$$\int_0^\infty \int_a^b (u_\varepsilon(x, t) + \varepsilon) |u_{\varepsilon X}(x, t)|^{p-1} u_{\varepsilon XX}^2(x, t) dx \leq C \quad (2.2)$$

for any $p \geq 1$, where and in the sequel C denotes various positive constants independent of ε .

Proof. The maximum principle gives

$$0 \leq u_\varepsilon(x, t) \leq \max_{a \leq x \leq b} u_0(x). \quad (2.3)$$

Multiplying (1.1)_ε by $\frac{1}{p} (|u_{\varepsilon X}(x, t)|^{p-1} u_{\varepsilon X})_X$ and integrating by parts on $[a, b]$, we have

$$\begin{aligned} & \frac{1}{p(p+1)} \frac{d}{dt} \int_a^b |u_{\varepsilon X}|^{p+1} dx + \int_a^b (u_{\varepsilon} + \varepsilon) |u_{\varepsilon X}|^{p-1} u_{\varepsilon XX}^2 dx \\ & + \frac{\lambda}{p+1} |u_{\varepsilon X}(a, t)|^p u_{\varepsilon X}(a, t) - \frac{\lambda}{p+1} |u_{\varepsilon X}(b, t)|^p u_{\varepsilon X}(b, t) = 0 \quad (2.4) \end{aligned}$$

Here and from now on we abbreviate x and t variables in the integrand. Since u_{ε} is nonnegative, we easily see that

$$u_{\varepsilon X}(a, t) \geq 0 \quad \text{and} \quad u_{\varepsilon X}(b, t) \leq 0.$$

Hence integrating (2.4) from 0 to t , we obtain that, any $p \geq 1$

$$\begin{aligned} & \frac{1}{p(p+1)} \int_a^b |u_{\varepsilon X}|^{p+1} dx + \int_0^t \int_a^b (u_{\varepsilon} + \varepsilon) |u_{\varepsilon X}|^{p-1} u_{\varepsilon XX}^2 dx dt \\ & \leq \frac{1}{p(p+1)} \int_a^b |u_{0X}|^{p+1} dx \end{aligned}$$

from which it follows that

$$\|u_{\varepsilon X}(t)\|_{L^{p+1}(a, b)}^{p+1} \leq \|u_{0X}\|_{L^{p+1}(a, b)}^{p+1} \quad \text{for any } t > 0 \quad (2.5)$$

and

$$\int_0^t \int_a^b (u_{\varepsilon} + \varepsilon) |u_{\varepsilon X}|^{p-1} u_{\varepsilon XX}^2 dx dt \leq C \|u_{0X}\|_{L^{p+1}(a, b)}^{p+1} \quad (2.6)$$

From (2.5) we easily have (2.1).

Lemma 2. Let $0 < \varepsilon < \varepsilon_0$ where ε_0 is a fixed number. For any $T > 0$,

$$\|u_{\varepsilon t}\|_{L^2([a,b] \times [0,T])} \leq C \quad (2.7)$$

where C is a positive constant independent of ε .

Proof. Using (1.1) _{ε} and integrating by parts, we get

$$\begin{aligned} \int_0^T \int_a^b u_{\varepsilon t}^2 dx dt &= \int_0^T \int_a^b (u_{\varepsilon} + \varepsilon)^2 u_{\varepsilon XX}^2 dx dt - \frac{2}{3} \varepsilon \lambda \int_0^T \{u_X(b,t)^2 - u_X(a,t)^2\} dt \\ &\quad + \left(\frac{2}{3} \lambda + \lambda^2\right) \int_0^T \int_a^b u_{\varepsilon X}^4 dx dt \\ &\leq (\|u_{\varepsilon}\|_{L^\infty([a,b] \times [0,T])} + \varepsilon_0)^2 \int_0^T \int_a^b (u_{\varepsilon} + \varepsilon) u_{\varepsilon XX}^2 dx dt \\ &\quad + \frac{4}{3} \varepsilon \lambda T \|u_{\varepsilon X}\|_{L^\infty([a,b] \times [0,\infty))}^3 + \left(\frac{2}{3} \lambda + \lambda^2\right) (b-a) T \|u_{\varepsilon X}\|_{L^\infty([a,b] \times [0,\infty))}^4 \end{aligned}$$

From (2.1) and (2.2) with $p = 1$, we can easily obtain (2.7).

Proof of Theorem 1. From (2.1), (2.3) and (2.7), we see that there exists a nonnegative function $u \in L^\infty([0,\infty) : C([a,b]) \cap W^{1,\infty}[a,b])$ with $u_t \in L^2([a,b] \times [0,T])$ (for any $T > 0$) and we can extract a subsequence of $\{u_\varepsilon\}$, which is denoted by $\{u_{\varepsilon_i}\}$, such that, as $\varepsilon_i \rightarrow 0$,

$$u_{\varepsilon_i} \longrightarrow u \quad \text{strongly in } C([a, b] \times [0, T])$$

$$u_{\varepsilon_i X} \longrightarrow u_X \quad \text{weakly star in } L^\infty([a, b] \times [0, \infty))$$

and

$$u_{\varepsilon_i t} \longrightarrow u_t \quad \text{weakly in } L^2([a, b] \times [0, T]) .$$

In order to show that u is a weak solution of (1.1)-(1.3), it suffices to show that, for any $T > 0$

$$|u_{\varepsilon_i X}|^2 \longrightarrow |u_X|^2 \quad \text{in } L^1([a, b] \times [0, T]) ,$$

and this implies

$$u_{\varepsilon_i X} \longrightarrow u_X \quad \text{strongly in } L^2([a, b] \times [0, T]) .$$

From (2.1) and (2.2), we have

$$\| (u_\varepsilon^2)_{XX} \|_{L^2([a, b] \times [0, T])} \leq 2 \| u_\varepsilon u_{\varepsilon XX} + u_{\varepsilon X}^2 \|_{L^2([a, b] \times [0, T])} \leq C .$$

We also have

$$\| (u_\varepsilon^2)_{xt} \|_{L^2(0, T; H^{-1}(a, b))} \leq C \| (u_\varepsilon^2)_t \|_{L^2([a, b] \times [0, T])} \leq C .$$

By virtue of Aubin's compactness theorem (see J.L.Lions [4]), we may assume that

$$(u_{\varepsilon_i}^2)_X = 2u_{\varepsilon_i} u_{\varepsilon_i X} \longrightarrow 2uu_X = (u^2)_X \quad \text{strongly in } L^2([a, b] \times [0, T]) .$$

Hence we may also assume that

$$u_{\varepsilon_i} u_{\varepsilon_i X} \longrightarrow uu_X \quad \text{a.e. in } [a, b] \times [0, \infty)$$

from which it follows that

$$u_{\varepsilon_i X} \longrightarrow u_X \quad \text{a.e. in } [a, b] \times [0, \infty) \quad (2.8)$$

since $\frac{\partial u}{\partial x} = 0$ a.e. in $E = \{x \in [a, b]; u=0\}$ (see Kinderlehrer-Stampacchia [3], p53) and

$$u_{\varepsilon_i x} \longrightarrow u_x \quad \text{a.e. in } {}^c E = \{x \in [a, b]; u > 0\}.$$

In view of Lebesgue's bounded convergence theorem we can easily obtain

$$\lim_{m \rightarrow \infty} \int_0^T \int_a^b |u_{mx}^2| \, dx dt = \int_0^T \int_a^b |u_x^2| \, dx dt. \quad (2.9)$$

On the other hand, from (2.1) we may assume that u_{mx} converges to u_x weakly in $L^2([a, b] \times [0, T])$. Hence

$$u_{mx} \longrightarrow u_x \quad \text{strongly in } L^2([a, b] \times [0, T]) \quad (2.10)$$

This completes the proof of Theorem 1.

3. Proof of Theorem 2.

Lemma 3. Let u_0 satisfy (H1)_w and (H2). Then, for any $t > 0$

$$|u_\varepsilon(a, t)| \leq \sqrt{\varepsilon} C \quad (3.1)$$

and

$$|u_\varepsilon(b, t)| \leq \sqrt{\varepsilon} C \quad (3.2)$$

Proof. We only show that (3.1) hold valid. From (H2) we see that for some $\delta > 0$ and $C_1 > 0$

$$0 \leq u_0(x) \leq C_1 \{(x-a)^2 + \sqrt{\varepsilon} (x-a)\} \quad \text{for any } x \in (a, a+\delta) \quad (3.3)$$

Let $T > 0$ be fixed. For any $(x, t) \in [a, a+\delta] \times [0, T]$ set

$$\bar{u}(x, t) = A \{(x-a)^2 + \sqrt{\varepsilon} (x-a)\} \quad (3.4)$$

where A is chosen so large that

$$A \geq C_1 \quad (3.5)$$

and

$$A(\delta^2 + \sqrt{\varepsilon} \delta) \geq \max_{\substack{a \leq x \leq a+\delta \\ 0 \leq t \leq T}} u_\varepsilon(x, t) \quad (3.6)$$

Note that $u \in C^{2,1}((a, b) \times [0, T])$. Direct calculation gives

$$\begin{aligned} & \bar{u}_t - (\bar{u} + \varepsilon) \bar{u}_{xx} + \lambda (\bar{u}_x)^2 \\ &= 2(2\lambda - 1)A^2(x-a)^2 + 2(2\lambda - 1)A^2\sqrt{\varepsilon}(x-a) + 2\varepsilon A(\lambda A - 2) \\ &\geq 0 \quad \text{in } (a, a+\delta) \times (0, T) \end{aligned} \quad (3.7)$$

provided that $\lambda \geq \frac{1}{2}$ and that A is so large that

$$\lambda A - 2 > 0.$$

By virtue of (3.3)-(3.7) we apply the maximum principle to obtain

$$0 \leq u_\varepsilon(x, t) \leq \bar{u}(x, t) \quad \text{in } [a, a+\delta] \times [0, T].$$

Hence

$$0 \leq u_\varepsilon(a, t) = \lim_{h \downarrow 0} \frac{u_\varepsilon(a+h, t) - u_\varepsilon(a, t)}{h} \leq \lim_{h \downarrow 0} \frac{u(a+h)}{h} = A\sqrt{\varepsilon}.$$

Thus we have (3.1).

Lemma 4. Under the same assumption

$$|u_{\varepsilon XX}| \leq \frac{C}{t} \quad \text{for all } t > 0. \quad (3.8)$$

Moreover, if $u_{0XX} \leq C_2$ then

$$u_{\varepsilon XX} \leq C_3 \quad (3.9)$$

where C_3 is a constant.

Proof. Putting $p = \frac{u_{\varepsilon t}}{u_{\varepsilon} + \varepsilon}$, we have

$$p_t = (u_{\varepsilon} + \varepsilon) p_{XX} + 2(1-\lambda) u_{\varepsilon X} p_X + p^2 \quad (x, t) \in (a, b) \times (0, \infty)$$

$$p(a, t) = p(b, t) = 0 \quad t \in (0, \infty)$$

$$p(x, 0) = u_{0XX} - \frac{\lambda u_0^2}{u_0 + \varepsilon} \quad x \in (a, b)$$

The standard comparison theorem yields that

$$p \geq -\frac{1}{t}$$

Using (1.1) _{ε} , we easily see that

$$u_{\varepsilon XX} \geq -\frac{1}{t} \quad (3.10)$$

We put $q = u_{\varepsilon XX}$ to obtain that

$$q_t = (u_{\varepsilon} + \varepsilon) q_{XX} + 2(1-\lambda) u_{\varepsilon X} q_X + (1-2\lambda) q^2 \quad (3.11)$$

As for the boundary conditions, we utilize (1.1) _{ε} to get

$$q(a, t) = \frac{\lambda}{\varepsilon} |u_{\varepsilon X}(a, t)|^2, \quad q(b, t) = \frac{\lambda}{\varepsilon} |u_{\varepsilon X}(b, t)|^2 \quad (3.12)$$

for any $t > 0$. In view of Lemma 3, we see that

$$0 \leq q(a, t) \leq \lambda C^2, \quad 0 \leq q(b, t) \leq \lambda C^2 \quad (3.13)$$

Hence the comparison theorem yields that, if $\lambda > \frac{1}{2}$

$$q(x, t) = u_{\varepsilon XX}(x, t) \leq \frac{C}{(2\lambda - 1)}$$

for some constant $C > 0$.

If $u_{0XX} \leq C_2$, $\lambda \geq \frac{1}{2}$ and (3.11)-(3.13) yield that

$$u_{\varepsilon XX}(x, t) \leq C_3 \quad (3.14)$$

where $C_3 = \max(\lambda C^2, C_2)$ is independent of ε .

Proof of Theorem 2. Because of Lemma 4, we see that $\{u_{\varepsilon XX}\}$ is bounded in $L^\infty([a, b] \times [\delta, \infty))$ for every $\delta > 0$. Hence we can assume that

$$u_{\varepsilon_i XX} \longrightarrow u_{XX} \quad \text{weakly star in } L^\infty([a, b] \times [\delta, \infty))$$

and

$$|u_{XX}(x, t)| \leq \frac{C}{t} \quad \text{for any } (x, t) \in [a, b] \times [\delta, \infty).$$

If $u_{0XX} \leq C$, from (3.14) we have

$$u_{XX}(x, t) \leq C \quad \text{for any } (x, t) \in [a, b] \times [0, \infty).$$

This completes the proof of Theorem 2.

4. Proof of Theorem 3.

Let u and v be two weak solutions of (1.1)-(1.3) with initial data u_0 and v_0 , respectively. Let $T > 0$ be fixed and put $w(x, t) = u(x, t) - v(x, t)$ and $w_0(x) = u_0(x) - v_0(x)$.

Then we have

$$\int_a^b w(x, T) \psi(x, T) dx = \int_a^b w_0(x) \psi(x, 0) dx + \int_0^T \int_a^b \{w \psi_t - (uw_x - vv_x) \psi_x - (\lambda+1)(|u_x|^2 - |v_x|^2) \psi\} dx dt \quad (4.1)$$

for any $\psi \in C^{2,1}([a, b] \times [0, \infty))$ with compact support in (a, b) .

For each $n \in \mathcal{N}$ define

$$g_n(x) = \begin{cases} 1 & \text{if } \frac{1}{n} < s \\ ns & \text{if } |s| \leq \frac{1}{n} \\ -1 & \text{if } s < -\frac{1}{n} \end{cases}$$

and

$$\Psi = \{g_n((u^2 - v^2) \theta_{k,m} \star \rho_\nu \star \sigma_\mu) \star \rho_\nu \star \sigma_\mu\} \theta_{k,m}$$

where ρ_ν and σ_μ are the standard mollifiers with respect to x

and t , respectively; $\theta_k(\frac{x}{k})$ where $\theta \in C_0^\infty((a, b))$ with $0 \leq \theta \leq 1$

and $\theta(x) = 1$ in a neighborhood of 0 (we may assume $0 \in (a, b)$) and

$\theta_m(t) \in C_0^\infty((0, \infty))$ such that $0 \leq \theta_m \leq 1$ and $\theta_m(t)$ tends to the indicator function of $[s_1, s_2]$ ($0 < s_1 < s_2$) as $m \rightarrow \infty$. Then

$\Psi \in C_0^\infty((a, b) \times (0, \infty))$ and $\Psi(x, t) \geq 0$ for any $(x, t) \in (a, b) \times (0, \infty)$.

Substituting Ψ for a test function ψ in (3.1), we have

$$\int_0^T \int_a^b \{w\Psi_t - (uu_x - vv_x)\Psi_x - (\lambda+1)(|u_x|^2 - |v_x|^2)\Psi\} dxdt. \quad (4.2)$$

From $w_t \in L^2([a, b] \times [0, T])$ for any $T > 0$ and $\Psi \in C_0^\infty((a, b) \times (0, \infty))$ we get

$$\int_0^T \int_a^b w\Psi_t dxdt = - \int_0^T \int_a^b w_t\Psi dxdt.$$

Letting ν and μ tend to infinity, we can easily see that

$$I_1(k, m, n) - I_2(k, m, n) - I_3(k, m, n)$$

$$= \int_0^T \int_a^b w_t \theta_k \theta_m g_n((u^2 - v^2) \theta_k \theta_m) dxdt$$

$$- \left[- \int_0^T \int_a^b (uu_x - vv_x) \{g_n((u^2 - v^2) \theta_k \theta_m)\}_x dxdt \right]$$

$$- \left[-(\lambda+1) \int_0^T \int_a^b (|u_x|^2 - |v_x|^2) \theta_k \theta_m g_n((u^2 - v^2) \theta_k \theta_m) dxdt \right] = 0 \quad (4.3)$$

As n tends to infinity, we find $I_1(k, m, n)$ tends to

$$\tilde{I}_1(k, m) = \int_0^T \int_a^b w_t \theta_k \theta_m \operatorname{sgn}((u-v) \theta_k \theta_m) dx dt \quad (4.4)$$

since $\operatorname{sgn}((u^2 - v^2) \theta_k \theta_m) = \operatorname{sgn}((u-v) \theta_k \theta_m)$.

Moreover, $\theta_m(t) = 0$ near 0 and T , then we have

$$\begin{aligned} \tilde{I}_1(k, m) &= \int_0^T \int_a^b (|w \theta_k \theta_m|)_t dx dt - \int_0^T \int_a^b |w \theta_k| (\theta_m)_t dx dt \\ &= - \int_0^T \int_a^b |w \theta_k| (\theta_m)_t dx dt \end{aligned} \quad (4.5)$$

As for $I_2(k, m, n)$, using chain rule, we get

$$\begin{aligned} I_2(k, m, n) &= -2 \int_0^T \int_a^b (uu_x - vv_x)^2 g'_n((u^2 - v^2) \theta_k \theta_m) \theta_k \theta_m dx dt \\ &\quad - \int_0^T \int_a^b (uu_x - vv_x) g'_n((u^2 - v^2) \theta_k \theta_m) (u^2 - v^2) \theta_m^2 \theta_k (\theta_k)_x dx dt \\ &\quad - \int_0^T \int_a^b (uu_x - vv_x) g_n((u^2 - v^2) \theta_k \theta_m) \theta_m^2 \theta_k (\theta_k)_x dx dt. \end{aligned}$$

Since the first term on the right hand side is nonpositive and

$$|(\theta_k)_x| \leq \frac{C}{k}, \text{ we have}$$

$$I_2(k, m, n) \leq \frac{C}{k} (\|u\|_{L^\infty}^3 + \|v\|_{L^\infty}^3 + \|u\|_{L^\infty} \|v\|_{L^\infty}) (\|u_x\|_{L^2} + \|v_x\|_{L^2})$$

where $L^p = L^p([a, b] \times [0, T])$ ($p=2, \infty$). Since $\|u\|_{L^\infty}$, $\|v\|_{L^\infty}$, $\|u_x\|_{L^2}$

and $\|v_x\|_{L^2}$ are bounded, we get

$$I_2(k, m, n) \leq \frac{C}{k} \quad (4.6)$$

where C depends on $\|u\|_{L^\infty}$, $\|v\|_{L^\infty}$, $\|u_x\|_{L^2}$ and $\|v_x\|_{L^2}$.

Since $\text{sgn}((u^2 - v^2)\theta_k \theta_m) = \text{sgn}(w\theta_k \theta_m)$, letting $n \rightarrow \infty$

we see that $I_3(k, m, n)$ tends to

$$\tilde{I}_3(k, m) = -(\lambda+1) \int_0^T \int_a^b (|u_x|^2 - |v_x|^2) \theta_k \theta_m \text{sgn}(w\theta_k \theta_m) dx dt.$$

Recalling that u_{xx} and v_{xx} are semiconcave, we have

$$\tilde{I}_3(k, m) = -(\lambda+1) \int_0^T \int_a^b (|w\theta_k \theta_m|)_x (u_x + v_x) dx dt$$

$$- (\lambda+1) \int_0^T \int_a^b (u-v)(u_x - v_x) (\theta_k)_x \theta_m \text{sgn}(w\theta_k \theta_m) dx dt$$

$$\begin{aligned}
&\leq (\lambda+1) \int_0^T \int_a^b |w \theta_k \theta_m| (u_{xx} + v_{xx}) dx dt \\
&\quad + (\lambda+1) \int_0^T \int_a^b (|u| + |v|) (|u_x| + |v_x|) |(\theta_k)_x| dx dt \\
&\leq C \int_0^T \int_a^b |w \theta_k \theta_m| dx dt + \frac{C}{k} . \tag{4.7}
\end{aligned}$$

Hence eq. (4.3) with (4.5), (4.6) and (4.7) implies that

$$\int_0^T \int_a^b |w \theta_k| (\theta_m)_t dx dt \leq C \int_0^T \int_a^b |w \theta_k \theta_m| dx dt + \frac{C}{k} \tag{4.8}$$

In (4.8) letting $k, m \rightarrow \infty$, we find that

$$\int_a^b |w(x, s_2)| dx - \int_a^b |w(x, s_1)| dx \leq C \int_{s_1}^{s_2} \int_a^b |w(x, s)| dx ds$$

for any s_1 and s_2 ($0 < s_1 < s_2$).

As $s_2 = t$ and s_1 tends to 0, we have

$$\int_a^b |w(x, t)| dx - \int_a^b |w_0(x)| dx \leq C \int_0^t \int_a^b |w(x, s)| dx ds$$

from which it follows that, for any $t \geq 0$

$$\int_a^b |w(x, t)| dx \leq e^{ct} \int_a^b |w_0(x)| dx \quad (4.9)$$

This completes the proof of Theorem 3. Corollary 2 is easily obtained from (4.9).

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