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On Asymptotic Stability of Yang-Mills' Gradient Flow

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1 Introduction.

The aim of this note is to study the existence and asymptotic stability of Yang-Mills' gradient flow. The Yang-Mills functional is given by the square integral of the curvature R^∇ associated to a metric connection ∇ on a Riemannian vector bundle E over a Riemannian manifold M:

$$\mathcal{YM}(\nabla) = \frac{1}{2}\int_M \langle R^\nabla, R^\nabla\rangle_x.$$

This functional is defined on the space $\mathcal{C}_E$ of all smooth metric connections with the range $[0,\infty]$. Here we do not assume compactness of M, so the range contains ∞.

A critical point of the functional, if it exists, is called the *Yang-Mills connection* (for the pricise definition, see §2). In other words, the Yang-Mills connection is a solution of

$$\text{grad}\ \mathcal{YM}(\nabla) = 0, \tag{1.1}$$

where $-$ grad $\mathcal{YM}(\cdot)$ is the Euler-Lagrangian operator of $\mathcal{YM}(\cdot)$.

It is also a stationary solution of the equation which gives the Yang-Mills' gradient flow:

$$\frac{d\nabla(t)}{dt} = - \text{ grad } \mathcal{YM}(\nabla(\text{t})). \tag{1.2}$$

Therefore it is important to investigate the structure of the gradient flow in studying the corresponding variational problem.

Now we want to analyze (1.1) and (1.2) in terms of differential equations. Let ∇_0 be a fixed base connection. For every connection ∇, the difference

$$A = \nabla - \nabla_0 \tag{1.3}$$

is a global cross section of $\Omega^1(\mathfrak{G}_E)$ (for the definition of $\Omega^1(\mathfrak{G}_E)$ etc., see §2). Using fundamental calculation (Propositions in §2, below), we find that (1.1) is written in a system of second-order partial differential equations (the Yang-Mills equation) of A with the principal term $\delta^{\nabla_0} d^{\nabla_0} A$, where d^{∇_0} is the covariant derivation operator of ∇_0 and δ^{∇_0} is its formal adjoint operator. The operator $-\ \delta^{\nabla_0} d^{\nabla_0}$, however, is not elliptic type. Hence the Yang-Mills equation itself is not in the frame work of elliptic partial differential equations.

Similarly (1.2) is an evolution equation of A, but not parabolic type in usual sense. To avoid this difficulty, we use the gauge invariancy of the functional (Proposition 2.2 (2)). We take

$$A = g^{-1} \circ \nabla \circ g - \nabla_0; \qquad g \in \mathcal{G} \ : \text{the gauge group} \tag{1.4}$$

instead of (1.3). If we choose a "good" g, then it recovers the ellipticity of the principal term. The "goodness" of g is written in a certain differential equation. Under this choice, (1.2) is reduced to a system of semi-linear parabolic equations. By virtue of the standard technique for the parabolic system, we shall show the asymptotic stability of some critical points of the Yang-Mills functional.

2 Formulation of Problem.

In this section we shall formulate our problem precisely. The author gives here an account of only a part of the Yang-Mills theory which is in need of our formulation. The readers can see more details of the thoery in [1, 7, 8].

Let (M, g) be a smooth n-dimensional Riemannian manifold, where $n \geq 2$. We denote by $(E, \langle,\rangle)$ a Riemannian vector bundle over (M, g) of rank m. $\mathcal{C}_E$ is a space of all smooth connections. For any $\nabla \in \mathcal{C}_E$, we can define a naturally induced connection on $\mathrm{Hom}(E, E) \simeq E^* \otimes E$ in a canonical way. The $\mathrm{Hom}(E, E)$-valued 2-form R^∇ is defined by

$$R^\nabla_{V,W} = \nabla_V \nabla_W - \nabla_W \nabla_V - \nabla_{[V,W]}$$

for any smooth vector fields V and W on M. This form is called the *curvature.*

Definition 2.1. The *Yang-Mills functional* $\mathcal{YM} : \mathcal{C}_E \to [0, \infty]$ is given by

$$\mathcal{YM}(\nabla) = \frac{1}{2} \int_M \langle R^\nabla, R^\nabla \rangle_x.$$

To calculate the Euler-Lagrangian operator corresponding this functional, we need define the gauge group.

Definition 2.2. G_E and $\mathfrak{G}_E$ denote the bundles defined by

$$G_E = \{L \in \mathrm{Hom}(E, E)\ ;\ {}^tL = L^{-1}\},$$

$$\mathfrak{G}_E = \{L \in \mathrm{Hom}(E, E)\ ;\ {}^tL = -L\}.$$

$\mathcal{G}$ ans $\mathcal{Y}$ are spaces of all smooth sections of G_E and $\mathfrak{G}_E$ respectively. $\mathcal{G}$ is called a *gauge group.*

The gauge group $\mathcal{G}$ acts on $\mathcal{C}_E$ in the following way:

(2.1) $$g(\nabla) = g \circ \nabla \circ g^{-1} \ ; \ g \in \mathcal{G}, \ \nabla \in \mathcal{C}_E.$$

It is known that

Proposition 2.1 [7]. A difference $A = \nabla' - \nabla$ of two connections $\nabla', \nabla \in \mathcal{C}_E$ is a global cross section of $\Omega^1(\mathfrak{G}_E)$, and conversely $\nabla + A \in \mathcal{C}_E$ for any $\nabla \in \mathcal{C}_E$, $A \in \Omega^1(\mathfrak{G}_E)$. The curvature $R^{\nabla'}$ of $\nabla' = \nabla + A$ is expressed in the form

$$R^{\nabla'} = R^{\nabla} + d^{\nabla} A + [A, A].$$

Let $\nabla^{\varepsilon} = \nabla + \varepsilon A$ be a compactly supported variation of ∇ (*i.e.* $A \in \Omega^1_0(\mathfrak{G}_E)$: the subset of $\Omega^1(\mathfrak{G}_E)$ consisting of all element with compact support). Using Proposition 2.1, we find that if $\mathcal{YM}(\nabla) < \infty$, then

$$\frac{d}{d\varepsilon}\mathcal{YM}(\nabla^{\varepsilon})\Big|_{\varepsilon=0} = \int_M \langle R^{\nabla}, d^{\nabla} A\rangle_x = \int_M \langle \delta^{\nabla} R^{\nabla}, A\rangle_x.$$

Keeping this in mind, we define grad $\mathcal{YM}(\nabla)$ by

$$\text{grad } \mathcal{YM}(\nabla) = \delta^{\nabla} R^{\nabla}$$

even for ∇ with $\mathcal{YM}(\nabla) = \infty$.

Definition 2.3. A connection $\nabla \in \mathcal{C}_E$ is called the *Yang-Mills connection*, if

$$\delta^{\nabla} R^{\nabla} = 0$$

is satisfied.

According to the definition of grad $\mathcal{YM}(\cdot)$, the equation of the Yang-Mills' gradient flow (1.2) is writtten as

$$\frac{d\nabla(t)}{dt} = -\ \delta^{\nabla(t)} R^{\nabla(t)}. \tag{2.2}$$

We consider this equation around a fixed base connection ∇_0. As decsribing in §1, however, if we set $\nabla(t) = \nabla_0 + A(t)$, then (2.2) is an evolution equation of $A(t)$ but not parabolic type, and it is difficult to see the structure of flow. Hence we consider the flow under some gauge condition which recovers the parabolicity of (2.2).

The action (2.1) og $\mathcal{G}$ on $\mathcal{C}_E$ yields the following facts.

Proposition 2.2 [7]. (1) *The curvature* $R^{g(\nabla)}$ *of* $g(\nabla)$ *is*

$$R^{g(\nabla)} = g \circ R^{\nabla} \circ g^{-1}.$$

(2) $\langle R^{g(\nabla)}, R^{g(\nabla)}\rangle_x = \langle R^{\nabla}, R^{\nabla}\rangle_x$ *holds, and therefore the Yang-Mills functional is invariant under the gauge action:*

$$\mathcal{YM}(g(\nabla)) = \mathcal{YM}(\nabla).$$

Taking (2) into consideration, we set

$$\nabla(t) = g(t)(\nabla_0 + A(t)) = g(t) \circ (\nabla_0 + A(t)) \circ g^{-1}(t).$$

To write down (2.2) in terms of $A(t)$ and $g(t)$, we need

Proposition 2.3 [7]. *We have*

$$\delta^{\nabla + A} S = \delta^{\nabla} S - [A, S] \quad \text{for} \quad S \in \Omega^2(\mathfrak{G}_E),$$

$$\delta^{g(\nabla)} R^{g(\nabla)} = g \circ \delta^{\nabla} R^{\nabla} \circ g^{-1}.$$

Using Propositions 2.1 - 2.3, we find that the explicit form of (2.2) is

$$\begin{aligned}\frac{dA(t)}{dt} &= -\delta^{\nabla_0}d^{\nabla_0}A(t) - \delta^{\nabla_0}R^{\nabla_0} - \delta^{\nabla_0}[A(t),A(t)] \\ &\quad + [\nabla_0 + A(t), Y(t)] + [A(t), R^{\nabla_0}] \\ &\quad + [A(t), d^{\nabla_0}A(t)] + [A(t),[A(t),A(t)]],\end{aligned} \tag{2.3}$$

where

$$Y(t) = g^{-1}\frac{dg(t)}{dt}. \tag{2.4}$$

The operator $-\delta^{\nabla_0}d^{\nabla_0}$ in the principal term of the right-hand side of (2.3) is not elliptic, but $-(\delta^{\nabla_0}d^{\nabla_0}+d^{\nabla_0}\delta^{\nabla_0})$ is elliptic. Therefore we impose some condition on $A(t)$ or $g(t)$ so that the term $-d^{\nabla_0}\delta^{\nabla_0}A(t)$ appears in the right-hand side.

Noting $Y(t) \in \Omega^0(\mathfrak{G}_E)$ because of $g(t) \in \mathcal{G}$, Yokotani imposed the condition

$$g^{-1}(t)\frac{dg(t)}{dt} = Y(t) = -\delta^{\nabla_0}A(t), \quad g(0) = \text{identity} \tag{2.5}$$

on $g(t)$. This makes $-d^{\nabla_0}\delta^{\nabla_0}A(t)$ of the term $[\nabla_0, Y(t)]$. He constructed under this method a local solution in [11].

On the other hand, if $A(t)$ is a Coulomb gauge, *i.e.*,

$$\delta^{\nabla_0}A(t) = 0 \tag{2.6}$$

is satisfied for all $t > 0$, then $-\delta^{\nabla_0}d^{\nabla_0}A(t) = -(\delta^{\nabla_0}d^{\nabla_0}+d^{\nabla_0}\delta^{\nabla_0})A(t)$ holds for all $t > 0$. Kozono, Maeda and Naito investigated the stability of the Yang-Mills' gradient flow (2.3) under (2.6) in [5, 10].

Hence our problem is to study the global existence of flow and the stability problem under Yokotani's method.

In what follows, we restrict ourselves to the case where M is the Euslidian space $\mathbf{R}^n$ or a bounded domain $\Omega \subset \mathbf{R}^n$ with smooth boundary, where $n \geq 2$. Suppose that E is the trivial Riemannian vector bundle over (M, g_0) of rank m, where g_0 is the standard metric on $\mathbf{R}^n$. We denote by ∇_0 a cannonical flat connection determnined by the trivialization of the bundle. Trivially ∇_0 is a Yang-Mills connection. We shall concern with the asymptotic stability of flow arround ∇_0. The following sections are summary of the papers [4, 9], one of which is a joint work with Kono (cf. [3]).

3 Construction of Flow.

Noting (2.5), we have the expression of $g(t)$ by the Peano-Baker series:

(3.1) $$g(t) = \sum_{m=0}^{\infty} \Phi_m(t),$$

where

$$\begin{cases} \Phi_0(t) = \text{identity}, \\ \Phi_{m+1}(t) = -\int_0^t \Phi_m(\tau)\delta^{\nabla_0}A(\tau)d\tau, \quad m = 0,\ 1,\ 2,\ \cdots. \end{cases}$$

Hence we want to solve (2.3) with $Y(t) = -\ \delta^{\nabla_0}A(t)$.

Let $A = \sum A^\alpha dx_\alpha, A^\alpha = (u^\alpha_{\beta\gamma})$, and $\{u^i\}$ be a rearrangement of $\{u^\alpha_{\beta\gamma}\}$. In the case which is described in the end of the previous section, $-\ (d^{\nabla_0}\delta^{\nabla_0} + d^{\nabla_0}\delta^{\nabla_0})$ is the Laplace operator $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_n^2}$ (with the homogeneous Dirichlet condition if $\partial M \neq \emptyset$) and $R^{\nabla_0} = 0$. Therefore observing the non-linearity of (2.3), we have only to investigate the following system of semilinear heat equations:

$$
(3.2)\quad \begin{cases} u_t = \Delta u + F_1(u, \partial u) + F_2(u) \quad \text{on} \quad M \times (0, \infty), \\ u(0) = a, \\ u|_{\partial M} = 0 \quad \text{if} \quad \partial M \neq \emptyset, \\ u = (u^1. \cdots, u^N), \end{cases}
$$

where

$$
(3.3)\quad \begin{cases} F_1(u, \partial u) = \sum_{i,j,k} a_{ijk} u^i \partial_j u^k, \\ F_2(u) = \sum_{i.j.k} b_{ijk} u^i u^j u^k, \end{cases}
$$

and

$$
\partial_j = \frac{\partial}{\partial x_j}.
$$

The coefficient a_{ijk}, b_{ijk} are bounded together with their derivatives ∂a_{ijk}, ∂b_{ijk}.

It is well-known that (3.2) is converted into

$$
(3.4)\quad u = e^{t\Delta} a + \int_0^t e^{(t-\tau)\Delta} \{F_1(u(\tau), \partial u(\tau)) + F_2(u(\tau))\} d\tau,
$$

where $\{e^{t\Delta}\}_{t \geq 0}$ is a strongly continuous semigroup generated by Δ with its domain $\mathcal{D}(\Delta) = \overset{\circ}{W}{}^{1,p}(M) \cap W^{2,p}(M)$. This semigroup satisfies the L^p-L^q-estimates:

(3.5)

$$
\begin{cases} \|e^{t\Delta} a\|_p \leq C(p, q, n) t^{-(n/q - n/p)/2} \|a\|_q \qquad (1 < p \leq q < \infty), \\ \|\partial e^{t\Delta} a\|_p \leq C(p, q, n) t^{-(1 + n/q - n/p)/2} \|a\|_q \qquad (1 < p \leq q < \infty), \end{cases}
$$

where $\|\cdot\|_p$ is the $L^p(M)$-norm.

Using these estimates, we can construct a unique global solution u to (3.4) provided $\|a\|_n$ is sufficiently small by a successive approximation (cf. [2]). This solution u also satisfies (3.2).

Theorem 3.1. *Let $a \in L^n(M)$. Then there exists a positive constant λ such that if $\|a\|_n < \lambda$ then there exists a unique global solution $u(t) \in \overset{\circ}{W}{}^{1,n}(M) \cap W^{2,n}(M)$ for $t > 0$ to (3.2) satisfying the following properties:*

$$\begin{cases} t^{(1-n/p)/2}u(t) \in BC([0,\infty)\,;\, L^p(M)) & \text{for} \quad n \le p \le \infty, \\ t^{1-n/(2q)}\partial u(t) \in BC([0,\infty)\,;\, L^q(M)) & \text{for} \quad n \le q < \infty. \end{cases}$$

with values

$$\begin{cases} t^{(1-n/p)/2}u(t)\Big|_{t=0} = \begin{cases} a & \text{for} \quad p = n, \\ 0 & \text{for} \quad n < p \le \infty, \end{cases} \\ t^{1-n/(2q)}\partial u(t)\Big|_{t=0} = 0 \quad \text{for} \quad n \le q < \infty. \end{cases}$$

Moreover $u(t)$ belongs to $C^0([0,\infty)\,;\, L^n(M)) \cap C^1((0,\infty)\,;\, L^n(M))$.

This theorem gives us the global existence of a unique flow for (2.3) with $Y(t) = -\,\delta^{\nabla_0}A(t)$ and the asymptotic stability under the smallness in the $L^n(M)$-norm of $A(0)$. We, however, need the $L^\infty(M)$-estimate of $\delta^{\nabla_0}A(t)$ (*i.e.* $\partial u(t)$) to show the convergence of (3.1). Therefore Theorem 3.1 is not sufficient for the original problem.

To establish the $L^\infty(M)$-estimate of $\delta^{\nabla_0}A(t)$ $(\partial u(t))$, we assume $a = u(0) \in \overset{\circ}{W}{}^{1,n}(M)$. When $M = \mathbf{R}^n$, the derivation ∂ commutes with $e^{t\Delta}$, and when $M = \Omega$, the fractional power $(-\Delta)^\alpha$ $(0 < \alpha < 1)$ commutes with $e^{t\Delta}$. Hence u satisfies

(3.6)

$$\begin{cases} \partial u = e^{t\Delta}\partial a \\ \qquad + \int_0^t e^{(t-\tau)\Delta}\partial\{F_1(u(\tau),\partial u(\tau)) + F_2(u(\tau))\}d\tau \\ \qquad \text{for} \quad M = \mathbf{R}^n, \\ (-\Delta)^{\alpha+1/2}u = (-\Delta)^{\alpha}e^{t\Delta}(-\Delta)^{1/2}a \\ \qquad + \int_0^t (-\Delta)^{\alpha+1/2}e^{(t-\tau)\Delta} \\ \qquad \{F_1(u(\tau),\partial u(\tau)) + F_2(u(\tau))\}d\tau \\ \qquad \text{for} \quad M = \Omega \quad \left(0 \le \alpha < \frac{1}{2}\right). \end{cases}$$

The operator $(-\Delta)^{\alpha}$ satisfies

(3.7)
$$||a||_{2\alpha,p} \le C(\alpha,p,n)||(-\Delta)^{\alpha}a||_p,$$

where $||\cdot||_{k,p}$ is the norm of the Bessel potential space $\mathcal{L}^{k,p}(M)$.

Applications (3.5) and (3.7) to (3.6) yields the desired estimate.

Theorem 3.2. *We assume the hypothesis in Theorem 3.1 and $a \in \overset{\circ}{W}{}^{1,n}(M)$. Then the solution $u(t)$ constructed by Theorem 3.1 satisfies*

$$t^{\gamma}\partial u(t) \in BC([0,\infty)\ ;\ L^{\infty}(M)),$$

where

$$\gamma = \begin{cases} \frac{1}{2} & \textit{for} \quad M = \mathbf{R}^n, \\ \frac{1}{4} + \frac{\beta}{2} & \textit{for} \quad M = \Omega \quad \left(\frac{1}{2} < \beta < 1\right). \end{cases}$$

Moreover $u(t)$ *belongs to* $C^0([0,\infty)\ ;\ \overset{\circ}{W}{}^{1,n}(M)) \cap C^1((0,\infty)\ ;\ \overset{\circ}{W}{}^{1,n}(M))$.

By virtue of this theorem, we get

$$\|\delta^{\nabla_0} A(t)\|_\infty \le Ct^{-\gamma} \tag{3.8}$$

for some $\gamma \in (0,1)$. Hence we can show the convergence of (3.1) and

$$\|g(t)\|_\infty \le \exp\left\{\frac{Ct^{1-\gamma}}{1-\gamma}\right\}.$$

without difficulties. It is also easy to see

$$g(t) \in C^0([0,\infty)\ ;\ L^\infty(M)) \cap C^1((0,\infty)\ ;\ L^\infty(M)).$$

The formulation in §2 is all the C^∞ category. Hence we must discuss the regularity of $A(t)$ and $g(t)$. The regularty of $A(t)$ implies that of $g(t)$ via (3.1). Therefore it is sufficient to show the regularity result for the solution u to (3.2).

We use the notation

$$\|u\|_{p,q,T} = \left\{\int_0^T \|u(t)\|_p^q dt\right\}^{1/q}, \quad \|u\|_{p,T} = \|u\|_{p,p,T}, \quad T \in (0,\infty].$$

The (3.5) implies

$$\begin{cases} \|e^{t\Delta}a\|_{p_1,q,\infty} & \le\ C(p_1,q,s,n)\|a\|_s, \\ \|\partial e^{t\Delta}\|_{p_2,r,\infty} & \le\ C(p_2,r,s,n)\|a\|_s \end{cases} \tag{3.9}$$

for

$$\frac{1}{q} = \left(\frac{1}{s} - \frac{1}{p_1}\right)\frac{n}{2}, \quad \frac{1}{r} = \left(\frac{1}{n} + \frac{1}{s} - \frac{1}{p_2}\right)\frac{n}{2},$$

$$p_1,\ q,\ r > s > 1, \quad p_2 > \left(\frac{1}{n} + \frac{1}{s}\right)^{-1}.$$

Under the hypothesis in Theorem 3.2, we get the following estimate with helps of (3.9) and the Hardy-Littlewood-Sobolev inequality:

$$||u||_{s(n+2)/n,\infty} + ||\partial u||_{p,r,\infty} \leq C$$

for

$$s \geq n,$$
$$p > \max\left\{\frac{n+2}{n}, \left(\frac{1}{n}+\frac{1}{s}\right)^{-1}\right\},$$
$$r > \max\left\{\frac{n+2}{n+1}, s\right\}.$$

provided

$$\frac{1}{p} - \frac{1}{n} < \frac{1}{n+2}\left(2+\frac{n}{2}\right) < \frac{1}{p} + \frac{1}{n}.$$

It follows from this estimate that $F_1(u, \partial u) + F_2(u) \in L^{p_1}(M \times (0,T))$ for some $p_1 \in \left(\frac{n+2}{3}, \frac{n+2}{2}\right)$ and for any $T \in (0, \infty)$. The regularity result [6, VII, Theorem 10.4] yields $u \in W^{2,1,p_1}_{x,t}(M \times (0,T))$ and $\partial_t u \in L^{p_1}(M \times (0,t))$ provided $a \in W^{2-/p_1,p_1}(M)$. By virtue of [6, II, Lemma 3.3] we have $F_1(u, \partial u) + F_2(u) \in L^{p_2}(M \times (0,T))$ for some $p_2 > \frac{n+2}{2}$. Repeating a similar procedure and applying the Schauder estimate [6, VII, Theorem 10.1/10.2], we obtain

Theorem 3.3. *We assume that a_{ijk} and b_{ijk} are Hölder continuous in $\bar{M} \times [0, \infty)$. If $a \in \overset{\circ}{W}{}^{1,n}(M) \cap \bigcap_{s \geq n} W^{2-2/s,s}(M) \cap C^{2+\alpha}(\bar{M})$, and $||a||_n$ is small, then there exists a unique global classical solution u to (3.2).*

Using a standard bootstrap argument, we finally get

Theorem 3.4. *Assume the hypotheses in Theorem 3.3 and C^∞-smoothness of a_{ijk}, b_{ijk} and a. If the compatibility conditions of any* *order between initial and boundary data hold, then the solution is also* C^∞.

Remark 3.1. We have a interior regularity result in a similar manner, when a_{ijk}, b_{ijk} and a have only interior smoothness.

4 Conclusion and Remarks.

We restate the results in §3 in terms of the Yang-Mills functional as a main result.

Theorem 4.1. *Let a Riemannian manifold* (M, g_0) *and a Riemannian vector bundle* $(E, \langle , \rangle)$ *be as stated in the last paragraph of §2. For* $\varepsilon > 0$ *we denote a neighborhood*

$$\left\{\nabla \in \mathcal{C}_E \ ; \ \nabla - \nabla_0 \in \overset{\circ}{W}{}^{1,n}(M) \cap \bigcap_{s \geq n} W^{2-2/s,s}(M), \quad \|\nabla - \nabla_0\|_n < \varepsilon\right\},$$

of the flat connection $\nabla_0 \in \mathcal{C}_E$ *by* $U_\varepsilon(\nabla_0)$. *Then there exists a positive constant* ε *such that for any* $\nabla \in U_\varepsilon(\nabla_0)$ *there exist a* $\mathcal{C}_E$*-valued smooth function* $\nabla(t)$ *and a* $\mathcal{G}$*-valued smooth function* $g(t)$ *satisfying*

$$\begin{cases} \dfrac{d\nabla(t)}{dt} &= -\ \mathrm{grad}\ \mathcal{YM}(\nabla(t)) \qquad t \in (0, \infty), \\ \nabla(0) &= \nabla, \\ g(0) &= \textit{identity}\ , \end{cases}$$

and

$$\lim_{t \to \infty} g^{-1}(t) \circ \nabla(t) \circ g(t) = \nabla_0$$

in $L^p(M)$ for $n < p \leq \infty$ with decay rate $t^{-(1-n/p)/2}$.

Proof. What we have not shown yet is the fact $A(t) \in \Omega^1(\mathfrak{G}_E)$ and $g(t) \in \mathcal{G}$ for $t > 0$.

We take transpose of both sides of (2.3) with $Y(t) = -\,\delta^{\nabla_0}A(t)$, $R^{\nabla_0} = 0$, and put $-{}^tA(t) = B(t)$. Then it is easy to see that $B(t)$ satisfies the same equation with the replacement $A(t)$ by $B(t)$. Since $B(0) = A(0)$ and since the solution of the equation is unique under the smallness condition $||A(0)||_n = ||B(0)||_n < \varepsilon$, we can conclude that $A(t)$ is skew-symmetric, *i.e.* $A(t) \in \Omega^1(\mathfrak{G}_E)$.

To see $g(t) \in \mathcal{G}$, we define the series

$$\tilde{g}(t) = \sum_{m=0}^{\infty} \Psi_m(t),$$

where

$$\begin{cases} \Psi_0(t) &= \text{identity}, \\ \Psi_{m+1}(t) &= \int_0^t \delta^{\nabla_0}A(\tau)\Psi(\tau)d\tau, \qquad m = 0,\ 1,\ \cdots. \end{cases}$$

It follows from (3.8) that $\tilde{g}(t)$ is well-defined, and satisfies

$$\frac{d\tilde{g}(t)}{dt} = \delta^{\nabla_0}A(t)\tilde{g}(t), \quad \tilde{g}(0) = \text{identity}.$$

Since $\frac{d}{dt}(g(t)\tilde{g}(t)) = 0$ and $g(0)\tilde{g}(0) = \text{identity}$, $\tilde{g}(t)$ is the inverse of $g(t)$. By virtue of $A(t) \in \Omega^1(\mathfrak{G}_E)$, ${}^t\tilde{g}(t)$ is a solution of (2.5), *i.e.*

$${}^t\tilde{g}(t)^{-1}\frac{d\,{}^t\tilde{g}(t)}{dt} = -\,\delta^{\nabla_0}A(t), \qquad {}^t\tilde{g}(0) = \text{identity}.$$

It is easy to show the uniqueness of solutions to (2.5). Thus $g(t) = {}^t\tilde{g}(t) =^t g(t)^{-1}$ holds. $\square$

Remark 4.1. It seems to the author that the equations of Yang-Mills' gradient flow (2.3) and either (2.5) or (2.6) are very similar to the equations of fluid mechanics. The motion of viscous fluid is described by the following system:

$$\text{(4.1)} \qquad \begin{cases} \dfrac{D\rho}{Dt} &= -\ \rho \ \mathrm{div}\ v, \\ \rho\dfrac{Dv}{Dt} &= Lv - \nabla p, \\ \left(\dfrac{D}{Dt}\right. &= \left.\dfrac{\partial}{\partial t} + (v \cdot \nabla)\right), \end{cases}$$

where ρ is a density, v is a velocity of fluid and p is a pressure. The operator L is elliptic of second-order.

When the fluid is incompressible (*i.e.* div $v = 0$), we put $\rho =$ const. and take (v, p) as unknowns. Then the system is reduced into

$$\frac{\partial v}{\partial v} = Lv - (v \cdot \nabla)v - \nabla p \qquad \text{and} \qquad \mathrm{div}\ v = 0,$$

which is remindful of (2.3) and (2.6). In fact Kozono, maeda and Naito [5, 10] employed an analysis analogous to the mathematical theory of incompressible viscous fluid (cf. [2]).

In compressible case, we assume that the pressure p is a function of ρ. This case looks like to (2.3) and Yokotani's condition (2.5), especially the first equation of (4.1) resembles us (2.5), *i.e.*

$$\frac{dg(t)}{dt} = -\ g(t)\delta^{\nabla_0}A(t).$$

Remark 4.2. In the proofs of theorems in §3, we do not use the properties of the Yang-Mills functional, but the non-linearity (3.3). Therefore Theorem 4.1 is only one application of the discussions in §3. The author poits out other applications in [9].

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