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ON KNESER TYPE THEOREM FOR FUNCTIONAL DIFFERENTIAL EQUATIONS  
WITH THE PHASE SPACE  $\mathcal{E}_\gamma$  IN BANACH SPACES

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§ 1. Introduction.

Let  $R = (-\infty, \infty)$  and  $E$  be an infinite dimensional Banach space with norm  $\|\cdot\|_E$ . Let  $X = E$  or  $R$ . Denote by  $\mathcal{E}_\gamma^X$ ,  $\gamma \in R$ , the space of continuous functions  $\psi : (-\infty, 0] \rightarrow X$  having the limit  $\lim_{\theta \rightarrow -\infty} e^{\gamma\theta} \psi(\theta)$  with the norm

$$\|\psi\|_{\mathcal{E}_\gamma^X} = \sup_{-\infty < \theta \leq 0} e^{\gamma\theta} \|\psi(\theta)\|_X.$$

If  $x : (-\infty, \sigma+a) \rightarrow X$ ,  $0 < a \leq \infty$ , then for any  $t \in (-\infty, \sigma+a)$  we define  $x_t : (-\infty, 0] \rightarrow X$  by  $x_t(\theta) = x(t+\theta)$ ,  $-\infty < \theta \leq 0$ .

The purpose of this paper is to give Kneser type theorem on the set of solutions for the Cauchy problem of the functional differential equation (FDE) with infinite delay in a Banach space (for brevity, CP(1.1)),

$$\frac{dx}{dt} = f(t, x_t), \quad x_\sigma = \varphi \in \mathcal{E}_\gamma, \quad (1.1)$$

under the condition that  $f : [\sigma, \sigma+a] \times \mathcal{E}_\gamma^E(\varphi, r) \rightarrow E$ ,  $\mathcal{E}_\gamma^E(\varphi, r) := \{\psi \in \mathcal{E}_\gamma^E \mid |\varphi - \psi|_{\mathcal{E}_\gamma^E} \leq r\}$ , is a uniformly continuous mapping. The

argument in the proof of the main theorem (Theorem 3.4) is based on the idea in [9] and on properties of  $\mathcal{E}_\gamma^E$ . Our result extends the one obtained in [10] and is closely related to the one due to Kubiacyk [2] for ordinary differential equations (ODE's).

## § 2. Some Lemmas.

In this section, we shall show a differential inequality and a comparison theorem. For a continuous function  $w : (a, b) \rightarrow \mathbb{R}$  and for  $t \in (a, b)$ ,  $(D_+ w)(t)$ ,  $(\underline{D}_- w)(t)$  and  $(\overline{D}_+ w)(t)$  denote the right hand derivative, the left hand lower derivative and the right hand upper derivative, respectively.

Lemma 2.1. Let  $w : [\sigma, \sigma+a) \rightarrow \mathbb{R}$  be a continuous function such that  $(D_+ w)(t)$  exists for all  $t \in [\sigma, \sigma+a)$ . Then the following inequalities hold :

1)

$$\overline{D}_+ \sup_{\sigma \leq s \leq t} w(s) \leq |(D_+ w)(t)|.$$

2) If  $w(t) \geq 0$ , then

$$\bar{D}_+ \sup_{\sigma \leq s \leq t} e^{\gamma(s-t)} w(s) \leq \begin{cases} |(D_+ w)(t)| & \text{if } \gamma \geq 0 \\ |(D_+ w)(t)| - \gamma \sup_{\sigma \leq s \leq t} e^{\gamma(s-t)} w(s) & \text{if } \gamma < 0. \end{cases}$$

Proof. For a proof of the assertion 1) refer to [1,6].  
Set  $u(t) = \sup(w(s) | \sigma \leq s \leq t)$ ,  $z(t) = \sup(e^{\gamma s} w(s) | \sigma \leq s \leq t)$  and  $I = [\sigma, \sigma+a)$ . Clearly,  $z(t)$  is nondecreasing in  $t \in I$ . Let any  $\tau \in I$  be a fixed number and  $\gamma \in (-\infty, 0)$ . Then we have, for  $h > 0$ ,

$$\begin{aligned} z(\tau+h) - z(\tau) &= e^{\gamma t_0} w(t_0) - z(\tau) \quad \text{for some } t_0 \in [\tau, \tau+h] \\ &\leq e^{\gamma \tau} \sup_{\sigma \leq s \leq \tau+h} w(s) - e^{\gamma \tau} \sup_{\sigma \leq s \leq \tau} w(s) \\ &= e^{\gamma \tau} (u(\tau+h) - u(\tau)), \end{aligned}$$

from which it follows that

$$\bar{D}_+ \sup_{\sigma \leq s \leq t} e^{\gamma s} w(s) \leq e^{\gamma t} \bar{D}_+ \sup_{\sigma \leq s \leq t} w(s).$$

It is easy to prove the assertion 2) in case where  $\gamma$  is a negative number. Let  $\gamma \geq 0$ . Then by the assertion 1) we have

$$\begin{aligned} \bar{D}_+ \sup_{\sigma \leq s \leq t} e^{\gamma(s-t)} w(s) &\leq -\gamma e^{-\gamma t} \sup_{\sigma \leq s \leq t} e^{\gamma s} w(s) + e^{-\gamma t} \bar{D}_+ \sup_{\sigma \leq s \leq t} e^{\gamma s} w(s) \\ &\leq -\gamma e^{-\gamma t} \sup_{\sigma \leq s \leq t} e^{\gamma s} w(s) + e^{-\gamma t} |D_+(e^{\gamma t} w(t))| \end{aligned}$$

$$\leq |(D_+ w)(t)|$$

as required.

Lemma 2.2. Let  $\gamma \geq 0$  and  $U : [\sigma, \sigma+a] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a continuous function, where  $\mathbb{R}^+ = [0, \infty)$ . Assume that

(1)  $u^* : [\sigma, \sigma+a] \rightarrow \mathbb{R}^+$  is the maximal solution of the scalar differential equation

$$\frac{du}{dt} = U(t, u(t)), \quad u(\sigma) = u_0 \geq 0; \text{ and}$$

(2)  $m : (-\infty, \sigma+a] \rightarrow \mathbb{R}$  is a continuous function such that  $m_\sigma \in \mathcal{E}_\gamma^{\mathbb{R}}$  and  $m(t) \geq 0$  on  $[\sigma, \sigma+a]$ , and that, for every  $t_1 \in [\sigma, \sigma+a]$  such that  $|m_{t_1}|_{\mathcal{E}_\gamma^{\mathbb{R}}} = m(t_1)$ , the differential inequality

$$(D_- m)(t_1) \leq U(t_1, m(t_1))$$

is satisfied.

If  $|m_\sigma|_{\mathcal{E}_\gamma^{\mathbb{R}}} \leq u_0$ , then

$$m(t) \leq u^*(t) \quad \text{for } t \in [\sigma, \sigma+a].$$

Proof. For any  $\varepsilon > 0$  we denote by  $u(t, \varepsilon)$  any solution of the differential equation

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$$\frac{d}{dt} u(t) = U(t, u(t)) + \varepsilon, \quad u(\sigma) = u_0 + \varepsilon. \quad (2.1)$$

Then, by Lemma 1.3.1 in [4], we have

$$\lim_{\varepsilon \rightarrow 0^+} u(t, \varepsilon) = u^*(t)$$

uniformly on  $[\sigma, \sigma+a]$ . Thus it is sufficient to show that for every  $\varepsilon > 0$ , sufficiently small,

$$m(t) \leq u(t, \varepsilon) \text{ on } [\sigma, \sigma+a].$$

Suppose, on the contrary, that the set

$$Z = \{t \in [\sigma, \sigma+a] \mid m(t) > u(t, \varepsilon)\}$$

is nonempty and define  $t_1 = \inf Z$ . Then we have  $t_1 > \sigma$ , because  $\lim_{\varepsilon \rightarrow 0^+} m_\sigma \leq u_0 < u_0 + \varepsilon$ . Moreover, since  $m(t_1) = u(t_1, \varepsilon)$  and  $m(t) < u(t, \varepsilon)$  for  $t \in [\sigma, t_1)$ , it is easy to see that

$$\begin{aligned} \underline{D}_m(t_1) &\geq \liminf_{h \rightarrow 0^-} \frac{1}{h} (u(t_1+h, \varepsilon) - u(t_1, \varepsilon)) \\ &= U(t_1, m(t_1)) + \varepsilon \quad \text{by (2.1)}. \end{aligned}$$

Hence, we have

$$\underline{D}_m(t_1) > U(t_1, m(t_1)). \quad (2.2)$$

On the other hand, since  $U(t,s) \geq 0$  and  $u(t,\varepsilon)$  is nondecreasing in  $t$ , we have

$$\begin{aligned}
 |m_{t_1}|_{\mathcal{E}_\gamma^R} &= \sup_{\theta \leq 0} e^{\gamma\theta} |m(t_1+\theta)| \\
 &= \max\left\{ \sup_{\theta \leq \sigma-t_1} e^{\gamma\theta} |m(t_1+\theta)|, \sup_{\sigma-t_1 \leq \theta \leq 0} e^{\gamma\theta} |m(t_1+\theta)| \right\} \\
 &= \max\left\{ \sup_{s \leq 0} e^{\gamma(s-t_1+\sigma)} |m(\sigma+s)|, m(t_1) \right\} \\
 &= \max\left\{ e^{\gamma(\sigma-t_1)} |m_\sigma|_{\mathcal{E}_\gamma^R}, m(t_1) \right\} \\
 &= m(t_1).
 \end{aligned}$$

Thus, from the assumption 2) we are led to the inequality

$$\underline{D}_m(t_1) \leq U(t_1, m(t_1)),$$

which is incompatible with (2.2). This implies that the set  $Z$  is empty. Therefore the proof is completed.

A function  $\eta : (\sigma, \sigma+a] \times [0, 2r] \rightarrow \mathbb{R}$  is said to be a Kamke-type function if the following conditions hold :

( $\eta_1$ )  $\eta = \eta(t,s)$  is a real-valued function, defined on

$(\sigma, \sigma+a] \times [0, 2r]$ , which is Lebesgue measurable in  $t$  for each fixed  $s \in [0, 2r]$  and is continuous in  $s$  for a.a.  $t \in [\sigma, \sigma+a]$ :

( $\eta_2$ ) There exists a function  $\alpha$ , defined on  $(\sigma, \sigma+a]$  and locally integrable there, such that  $|\eta(t, s)| \leq \alpha(t)$  for a.a.  $t \in (\sigma, \sigma+a]$  and all  $s \in [0, 2r]$ .

The following result is a modification of the one given by [8, Lemma 3.1]. The proof is obvious.

Lemma 2.3. Let  $\eta(t, s) : (\sigma, \sigma+a] \times [0, 2r] \rightarrow \mathbb{R}$  be a Kamke-type function and let  $\{w^n\}$  and  $\{z^n\}$  converge pointwise to functions  $w^0$  and  $z^0$  on  $[\sigma, \sigma+a]$  as  $n \rightarrow \infty$ , respectively.

Assume that

1) there are a constant  $H > 0$  such that

$|w^n(t) - w^n(s)| \leq H|t-s|$  for all  $t, s \in [\sigma, \sigma+a]$  and all  $n \in \mathbb{N}$ ; and

2)  $w^n$  and  $z^n$  are related to each other as

$$\frac{d}{dt} w^n(t) \leq \eta(t, z^n(t)) + \sigma_n \quad \text{for a.a. } t \in (\sigma, \sigma+a),$$

where  $\sigma_n \geq 0$  and  $\sigma_n \rightarrow 0$  as  $n \rightarrow \infty$ .

Then

$$\frac{d}{dt} w^0(t) \leq \eta(t, z^0(t)) \quad \text{for a.a. } t \in (\sigma, \sigma+a).$$



## § 3. Main results.

For a bounded set  $\Omega$  of  $E$ , the  $\alpha$ -measure of  $\Omega$  is defined as follows :

$$\alpha(\Omega) = \inf\{d > 0 \mid \Omega \text{ has a finite cover of diameter } < d\}.$$

Let  $\mathcal{X}$  be the set of functions  $x$  on  $(-\infty, \sigma+a]$ ,  $0 < a < \infty$ , into  $E$  such that  $x$  is continuous on  $[\sigma, \sigma+a]$  and  $x_\sigma \in \mathcal{E}_\gamma$ . For a subset  $\mathcal{X} \subset \mathcal{X}$ , we will use the following notations :

$$\mathcal{X}(t) = \{x(t) \in E \mid x \in \mathcal{X}\}, \quad \mathcal{X}_t = \{x_t \mid x \in \mathcal{X}\} \quad \text{for } t \in [\sigma, \sigma+a]$$

and

$$\mathcal{X}[c,d] = \{x|_{[c,d]} \mid x \in \mathcal{X}\},$$

where  $c, d \in [\sigma, \sigma+a]$  and  $x|_{[c,d]}$  is the restriction of  $x$  to  $[c,d]$ . We denote by  $C([a,b], E)$  the set of all the continuous functions  $x : [a,b] \rightarrow E$  with supremum norm. For brevity, we denote  $\mathcal{E}_\gamma$  the phase space  $\mathcal{E}_\gamma^E$  when  $X = E$ . The following lemma is concerned with the phase space  $\mathcal{E}_\gamma$ .

Lemma 3.1 (Shin [7]). If  $\mathcal{X}_\sigma$  is relatively compact in  $\mathcal{E}_\gamma$  and if  $\mathcal{X}[\sigma, t]$  is a bounded and equicontinuous set in  $C([\sigma, t], E)$ , then

$$\alpha(x_t) = e^{-\gamma t} \sup_{\sigma \leq s \leq t} e^{\gamma s} \alpha(x(s)).$$

Lemma 3.2 (Shin [9]). Let  $\{S_n\}$  be a family of nonempty bounded subsets of a Banach space  $Y$  such that  $S_{n+1} \subset S_n$  for  $n \in \mathbb{N}$ . If  $S_n$  is connected for every  $n \in \mathbb{N}$  and if  $\alpha(S_n) \rightarrow 0$  as  $n \rightarrow \infty$ , then the set  $\bigcap_{n=1}^{\infty} \text{cl } S_n$  is nonempty, compact and connected, where  $\text{cl } A$  stands for the closure of  $A$ .

Assume that  $f : [\sigma, \sigma+a] \times \mathcal{E}_\gamma(\varphi, r) \rightarrow E$  is a uniformly continuous function such that  $\|f\|_E \leq M$ . Then a function  $u : (-\infty, \sigma+\xi] \rightarrow E$ ,  $0 < \xi \leq a$ , said to be an  $\frac{1}{n}$ -approximate solution for CP(1.1) if the following conditions hold :

- (1)  $u$  is continuous on  $J$ ,  $J = [\sigma, \sigma+\xi]$ , and  $u_\sigma = \varphi \in \mathcal{E}$  ;
- (2)  $u$  has the right hand derivative  $(D_+ u)(t)$  such that  $\|(D_+ u)(t)\|_E \leq M$  on  $[\sigma, \sigma+\xi)$ , and satisfies

$$u(t) = \varphi(0) + \int_{\sigma}^t (D_+ u)(s) ds \quad \text{for } t \in J ; \text{ and}$$

- (3)  $\|(D_+ u)(t) - f(t, u_t)\|_E \leq \frac{1}{n}$  for  $t \in [\sigma, \sigma+\xi)$ .

We denote by  $Q^n[d]$  the set of all the  $\frac{1}{n}$ -approximate solutions, defined on  $(-\infty, \sigma+d]$ , for CP(1.1). Then there is a  $\xi > 0$  and the set  $Q^n := Q^n[\xi]$  is nonempty (see [8, Lemma 2.1]).

Lemma 3.3 (Shin [9]). Let  $f : [\sigma, \sigma+a] \times \mathcal{E}_\gamma(\varphi, r) \rightarrow E$  be

uniformly continuous and  $\|f\|_E \leq M$  on  $[\sigma, \sigma+a] \times \mathcal{E}_\gamma(\varphi, r)$ . Then  $Q^n$  is nonempty and  $Q^n \setminus J$  is connected in  $C(J, E)$  for every  $n \in \mathbb{N}$ .

Now, we state the main result in this paper, which is related to the result due to Kubiacyk[2] for ODE's.

Theorem 3.4. Assume that  $f : [\sigma, \sigma+a] \times \mathcal{E}_\gamma(\varphi, r) \rightarrow E$  is a uniformly continuous function such that  $\|f\|_E \leq M$  on  $[\sigma, \sigma+a] \times \mathcal{E}_\gamma(\varphi, r)$ , and that there exists a Kamke-type function  $\omega(t, s) : (\sigma, \sigma+a] \times [0, 2r] \rightarrow \mathbb{R}^+$  such that

1)  $\omega(t, s)$  is nondecreasing in  $s$  ;

2)  $\omega(t, z(t)) \rightarrow 0$  as  $t \rightarrow \sigma+0$  and  $\int_\sigma^t \omega(s, z(s)) ds < \infty$

whenever  $z : [\sigma, \sigma+a] \rightarrow [0, 2r]$  is an absolutely continuous function satisfying the condition  $(D^+z)(\sigma) = z(\sigma) = 0$ , where

$$(D^+z)(\sigma) := \lim_{t \rightarrow \sigma+} \frac{z(t)}{t-\sigma} ;$$

3)  $z \equiv 0$  is the unique absolutely continuous function, mapping  $[\sigma, \sigma+a]$  into  $\mathbb{R}^+$ , which satisfies the initial condition  $(D^+z)(\sigma) = z(\sigma) = 0$  and the scalar differential equation

$$\frac{dz}{dt} = \begin{cases} \omega(t, z(t)) & \gamma \geq 0 \\ \omega(t, z(t)) - \gamma z(t) & \gamma < 0 \end{cases} \quad \text{for a.a. } t \in (\sigma, \sigma+a) ; \text{ and}$$

4)

$$\underline{D}_- \alpha(A(t)) := \liminf_{h \rightarrow 0^-} \frac{1}{h} [\alpha(A(t)) - \alpha(\{x(t) - hf(t, x_t) : x \in A\})]$$

$$\leq \omega(t, \alpha(A_t))$$

for a.a.  $t \in (\sigma, \sigma+a]$  and for any subset  $A \subset \mathcal{I}$  such that  $A|[\sigma, \sigma+a]$  is equicontinuous and that  $A_t \subset \mathcal{E}_\gamma(\varphi, r)$  for all  $t \in [\sigma, \sigma+a]$ .

Then the set of all the solutions for CP(1.1) defined on  $J$  ( $=[\sigma, \sigma+\xi]$ ) is nonempty, compact and connected in  $C(J, E)$ .

Proof. From Lemma 3.2 and Lemma 3.3 it is sufficient to see that  $\alpha(Q^n|J) \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $Q^n|J$  is an equicontinuous subset of  $C(J, E)$ , we have  $\alpha(Q^n|J) \leq \sup\{\alpha(Q_t^n) | t \in J\}$  by Theorem 2.1 in [5]. Thus we must prove that  $\alpha(Q_t^n) \rightarrow 0$  uniformly on  $J$  as  $n \rightarrow \infty$ . From the properties of the  $\alpha$ -measure of noncompactness, we have, for  $t \in (\sigma, \sigma+\gamma]$  and  $h > 0$ ,

$$\begin{aligned} & \frac{1}{h} (\alpha(Q^n(t)) - \alpha(Q^n(t-h))) \\ & \leq \frac{1}{h} (\alpha(Q^n(t)) - \alpha(\{x(t) - hf(t, x_t) | x \in Q^n\})) \\ & \quad + \frac{1}{h} \alpha(\{x(t) - x(t-h) - hf(t, x_t) | x \in Q^n\}). \end{aligned} \quad (3.1)$$

By the uniform continuity of  $f$ , for any  $\varepsilon > 0$  there exists a  $\delta = \delta(\varepsilon) > 0$  such that  $|f(s, \varphi_1) - f(t, \varphi_2)|_E \leq \varepsilon/2$  if  $|t-s| < \delta$  and

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$|\varphi_1 - \varphi_2| < \delta$ . Since  $\{x_t^n \mid x^n \in Q^n\}$  is uniformly equicontinuous on  $[\sigma, \sigma + \xi]$ , we have, for any  $x \in Q^n$  and  $h \in (0, \delta)$ ,

$$\begin{aligned} & |x(t) - x(t-h) - hf(t, x_t)| \\ & \leq \left| \int_{t-h}^t [D_+ x(s) - f(s, x_s)] ds \right| + \left| \int_{t-h}^t [f(s, x_s) - f(t, x_t)] ds \right| \\ & \leq \frac{h}{n} + \frac{\varepsilon}{2} h. \end{aligned} \quad (3.2)$$

Set  $w^n(t) = \alpha(Q^n(t))$  and  $z^n(t) = \alpha(Q_t^n)$  for all  $t \in J$ .

Clearly, we have, for  $t, s \in J$  and any  $n \in \mathbb{N}$ ,

$$w^{n+1}(t) \leq w^n(t), \quad |w^n(t) - w^n(s)| \leq 2M|t-s|$$

and, Lemma 3.1,

$$z^n(t) = \sup_{\sigma \leq s \leq t} e^{\gamma(s-t)} w^n(s). \quad (3.3)$$

Hence we get

$$z^{n+1}(t) \leq z^n(t) \quad \text{and} \quad |z^n(t) - z^n(s)| \leq 2M \sup_{-a \leq \theta \leq 0} e^{\gamma\theta} |t-s|.$$

These imply that the sequences  $\{w^n(t)\}$  and  $\{z^n(t)\}$  converges to functions  $w^0(t)$  and  $z^0(t)$  uniformly on  $J$ , respectively.

Now, we shall show that  $z^0(t) \equiv 0$  on  $J$ . From (3.1), (3.2) and the assumption 4) it follows that

$$\frac{dw^n(t)}{dt} \leq \omega(t, z^n(t)) + \frac{2}{n} + \varepsilon \quad \text{for a.a. } t \in (\sigma, \sigma + \xi).$$

Using Lemma 2.3 and the relation (3.3), we can obtain

$$\frac{dw^0(t)}{dt} \leq \omega(t, z^0(t)) + \varepsilon \quad \text{for a.a. } t \in (\sigma, \sigma + \xi) \quad (3.4)$$

and

$$z^0(t) = \sup_{\sigma \leq s \leq t} e^{\gamma(s-t)} w^0(s). \quad (3.5)$$

Moreover, it is easy to see that  $(D_+ z^0)(\sigma) = z^0(\sigma) = 0$ . Using the assumption 2), we can put

$$u(t) = \int_{\sigma}^t \omega(s, z^0(s)) ds + \varepsilon(t - \sigma),$$

from which it follows that  $w^0(t) \leq u(t)$  for  $t \in J$ . Therefore we can obtain

$$0 \leq \frac{du}{dt} = \omega(t, z^0(t)) + \varepsilon \quad \text{for a.a. } t \in (\sigma, \sigma + \xi).$$

Put  $v(t) = \sup_{\sigma \leq s \leq t} e^{\gamma(s-t)} u(s)$ . Then, by Lemma 2.1 we can see that

$$\frac{dv}{dt} \leq \begin{cases} \omega(t, z^0(t)) + \varepsilon \\ \omega(t, z^0(t)) - \gamma z^0(t) + \varepsilon \end{cases} \quad \text{for a.a. } t \in (\sigma, \sigma + \xi), \quad (3.6)$$

On the other hand, since  $w^0(t) \leq u(t)$ , we have  $z^0(t) \leq v(t)$  by (3.5). Letting  $\varepsilon \rightarrow 0$  and using the assumption 1), we see that the relation (3.6) becomes

$$\frac{dv}{dt} \leq \begin{cases} \omega(t, v(t)) \\ \omega(t, v(t)) - \gamma v(t) \end{cases} \quad \text{for a.a. } t \in (\sigma, \sigma + \xi).$$

It is easy to see that  $(D_+ v)(\sigma) = v(\sigma) = 0$ . Thus, by Lemma 4.1 in [6] and the assumption 3), we have  $v(t) \equiv 0$  and so,  $z^0(t) \equiv 0$ . This implies  $\alpha(Q^n | J) \rightarrow 0$  as  $n \rightarrow \infty$ . Hence the proof is complete.

Corollary 3.5. The conditions 1) - 4) in Theorem 3.4 can be replaced as follows :

- 1) the condition 3) in Theorem 3.4 is satisfied ;and
- 2)

$$\alpha(f(t, B)) \leq \omega(t, \alpha(B)) \quad \text{for a.a. } t \in (\sigma, \sigma + a) \quad \text{and all } B \in \mathcal{B}_\gamma(\varphi, r).$$

Combining the argument in the proof of Theorem 3.4 and Lemma 2.2, we have the following result.

Proposition 3.6. Let  $\gamma \geq 0$ . Assume that  $f :$

$[\sigma, \sigma+a] \times \mathcal{B}_\gamma(\varphi, r) \rightarrow E$  is a uniformly continuous function such that  $\|f\|_E \leq M$  on  $[\sigma, \sigma+a] \times \mathcal{B}_\gamma(\varphi, r)$ , and that there exists a continuous function  $\omega(t, s) : [\sigma, \sigma+a] \times [0, 2r] \rightarrow \mathbb{R}^+$  such that

1) for every  $t_1 \in [\sigma, \sigma+a]$  such that  $\alpha(A_{t_1}) = \alpha(A(t_1))$ , where  $A$  is as in Theorem 3.4, the differential inequality

$$D_- \alpha(A(t_1)) \leq \omega(t, \alpha(A(t_1))),$$

is satisfied ; and

2)  $u(t) \equiv 0$  is the unique continuous function, mapping  $[\sigma, \sigma+a]$  into  $[0, 2r]$ , which satisfies the scalar differential equation

$$\frac{du(t)}{dt} = \omega(t, u(t)), \quad u(\sigma) = 0.$$

Then the conclusion of Theorem 3.4 remains valid.

#### References

- [1]. Y. Hino, S. Murakami and T. Naito, Functional Differential Equations with Infinite Delay, to appear.
- [2]. I. Kubiacyk, Kneser type theorems for ordinary differential equations in Banach spaces, J. Differential Equations, 45, 139-146 (1982).
- [3]. V. Lakshmikantham and S. Leela, Nonlinear



Differential Equations in Abstract Spaces. Pergamon Press, New York, 1981.

[4]. V. Lakshmikantham and S. Leela, Differential and Integral Inequalities. Vol. I, II, Academic Press, New York, 1969.

[5]. J. S. Shin, An existence theorem of functional differential equations with infinite delay in a Banach space, Funkcial. Ekvac., 30, 19-29 (1987).

[6]. J. S. Shin, On the uniqueness of solutions for functional differential equations with infinite delay, Funkcial. Ekvac., 30, 225-236 (1987).

[7]. J. S. Shin, Existence of solutions and Kamke's theorem for functional differential equations in Banach spaces, to appear in J. Differential Equations.

[8]. J. S. Shin, Remarks on the existence of unique solution for functional differential equations in a Banach space, to appear.

[9]. J. S. Shin, Kneser type theorems for functional differential equations in Banach spaces. to appear.

[10]. S. Szyfla, Structure of the solutions set of ordinary differential equations in Banach space, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys., 21, 141-144 (1972).