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Author(s)	Kato, Mitsuyoshi; Namba, Makoto
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Branched Coverings of Complex Manifolds

Mitsuyoshi Kato and Makoto Namba

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Introduction. The theory of branched coverings is one of good examples of amalgamation of different branches of mathematics : topology, complex analysis and algebraic geometry. See, for example, Zariski [31], Fox [6], Kato [17], Hirzebruch [11], Höfer [13], Ishida [15], Fukui [7], Gaffney-Lazarsfeld [8], etc..

It need not to be mentioned that the theory of (Galois) branched coverings is a geometric counterpart of the Galois theory of function fields.

In this article, we present a theory of Galois and abelian branched coverings of complex manifolds, emphasizing existence theorems and examples mainly along the line of Namba [22]. In the last section, we discuss the equivalence problem of Kummer coverings after Kato [18].

Chapter 1. Galois Coverings.

1. Definition of Branched Coverings. First of all, we give a definition of branched coverings of complex manifolds. Since we treat infinite coverings as well as finite coverings, we define branched coverings as follows:

Definition 1. 1. Let M be an n-dimensional connected complex manifold. A branched covering of M is an irreducible normal complex space X together with a surjective holomorphic mapping $\pi : X \longrightarrow M$ satisfying the following 4 conditions:

i) Every fiber of π is discrete.

ii) $R_\pi = \{p \in X \mid \pi^* : \mathcal{O}_{M,\pi(p)} \longrightarrow \mathcal{O}_{X,p}$ is not isomorphic$\}$ and $B_\pi = \pi(R_\pi)$ are hypersurfaces (i.e., pure codimension 1) of X and M, respectively, called the ramification locus and the branch locus of π, respectively. (Here, $\mathcal{O}_{X,p}$ is the local ring of germs of holomorphic functions around p.)

iii) $\pi : X - \pi^{-1}(B_\pi) \longrightarrow M - B_\pi$ is a topological (i.e., unbranched) covering.

iv) For every point $q \in B_\pi$, there is an open neighborhood W of q in M such that, for every connected component U of

$\pi^{-1}(W)$, $\pi^{-1}(q) \cap U$ consists of one point and $\pi|_U : U \longrightarrow W$ is a surjective proper mapping (hence a finite mapping).

If R_π is empty, then $\pi : X \longrightarrow M$ should be called an unbranched covering. But we call such a covering also a branched covering by abuse of language. A branched covering is said to be <u>finite</u> if every fiber is a finite set. The mapping degree of of $\pi : X - \pi^{-1}(B_\pi) \longrightarrow M - B_\pi$ is called the <u>degree of</u> π. Using the <u>purity of branch loci</u> (see Fischer [4]), we have easily

<u>Proposition 1. 2.</u> An irreducible normal complex space X together with a surjective finite proper holomorphic mapping $\pi : X \longrightarrow M$ is a finite branched covering, and vice versa.

Let $\pi : X \longrightarrow M$ and $\pi' : X' \longrightarrow M$ be branched coverings of M. A <u>morphism of</u> π to π' is, by definition, a surjective holomorphic mapping $\phi : X \longrightarrow X'$ such that $\pi' \cdot \phi = \pi$. Thus we have the <u>category of branched coverings of</u> M. ϕ is an <u>isomorphism</u> if $\phi : X \longrightarrow X'$ is biholomorphic. In this case, we say that π and π' are <u>isomorphic</u>. In particular, if $X = X'$ and $\pi = \pi'$, then an isomorphism is called a <u>covering transformation of</u> π. The set G_π of all covering transformations of π forms a group under compositions, called the <u>covering transformation group</u>. G_π acts on every fiber of π. A branched covering $\pi : X \longrightarrow M$ is called a <u>Galois covering</u> if G_π acts transitively on every fiber. $\pi : X \longrightarrow M$ is called an <u>abelian</u> (resp. a <u>cyclic</u>) <u>covering</u> if π is a Galois covering and G_π is an abelian (resp. a cyclic) group.

We denote by Sing B_π the singular locus of the branch locus B_π. It can be shown that, for every point $q \in B_\pi - \mathrm{Sing}B_\pi$, every point $p \in \pi^{-1}(q)$ is a non-singualr point of both X and $\pi^{-1}(B_\pi)$. Moreover, for any sufficiently small open neighborhood W of q with a coordinate system $(w_1, \cdots, w_n)$ such that $q = (0, \cdots, 0)$ and $B_\pi \cap W = \{w_n = 0\}$, there is an open neighborhood U of p with a coordinate system $(z_1, \cdots, z_n)$ such that U is a connected component of $\pi^{-1}(W)$, $p = (0, \cdots, 0)$ and π is locally given by

$$\pi|U : (z_1,\cdots,z_n) \longrightarrow (w_1,\cdots,w_n) = (z_1,\cdots,z_{n-1},z_n^e),$$

where e is a positive integer, (see Roan [25] and Namba [22]). For an irreducible component C of $\pi^{-1}(B_\pi)$, the integer e is constant for points of $C - \pi^{-1}(\mathrm{Sing}B)$, and is called the <u>ramification index of</u> π <u>along</u> C. (For convenience, the ramification index of π along an irreducible hypersurface of X which is not contained in $\pi^{-1}(B_\pi)$ is defined to be 1.) If π is a Galois covering, then, for any irreducible component D_1 of B_π, the ramification index e of π along irreducible components of $\pi^{-1}(D_1)$ is constant. In this case, e is called the <u>ramification index of</u> π <u>along</u> D_1.

Let a hypersurface B of M be given. Suppose for simplicity that B has a finite number of irreducible components $D_1, \cdots, D_s$:

$$B = D_1 \cup \cdots \cup D_s.$$

Let $e_1, \cdots, e_s$ be positive integers greater than one, and

$$D = e_1 D_1 + \cdots + e_s D_s$$

be a positive divisor on M.

<u>Definition 1. 3.</u> A branched covering $\pi : X \longrightarrow M$ is

said to <u>branch along</u> D (resp. <u>at most along</u> D) if (i) $B_\pi = B$ (resp. $B_\pi \subset B$) and (ii) for every j $(1 \leqslant j \leqslant s)$ and for every irreducible component C of $\pi^{-1}(D_j)$, the ramification index of π along C is e_j (resp. divides e_j).

For branched coverings $\pi : X \longrightarrow M$ and $\pi' : X' \longrightarrow M$ of M, we denote $\pi \geqslant \pi'$ or $\pi' \leqslant \pi$ if there is a morphism of π to π'. If $\pi \geqslant \pi'$ and π branches at most along D, then π' branches at most along D. If π is a Galois covering, $\pi \geqslant \pi'$ and $\pi \leqslant \pi'$, then π and π' are isomorphic.

<u>Definition 1. 4.</u> A Galois covering $\pi : X \longrightarrow M$ is called a <u>D-universal covering</u> if (i) π branches along D and (ii) for any covering $\pi' : X' \longrightarrow M$ which branches at most along D, the relation $\pi \geqslant \pi'$ holds.

By the above remark, a D-universal covering is unique up to isomorphisms, if it exists. We denote it by

$$\tilde{\pi} : \tilde{M}(D) \longrightarrow M.$$

We now propose the following two problems:

<u>Problem 1.</u> When does a D-universal covering exist?

<u>Problem 2.</u> When does a finite Galois covering which branches along D exist?

As for a compact Riemann surface M, the problems were answered completely by Bundgaard-Nielsen [1] and Fox [5]:

<u>Theorem 1. 5.</u> Let M be a compact Riemann surface of genus g, $p_1, \cdots, p_s$ be points of M, $e_1, \cdots, e_s$ be positive integers greater than 1, and $D = e_1p_1 + \cdots + e_sp_s$ be a positive divisor on M. Then the following three conditions are equivalent:

(i) There <u>does</u> <u>not</u> exist a D-universal covering of M.

(ii) There <u>does</u> <u>not</u> exist a finite Galois covering $\pi : X \longrightarrow M$ which branches along D.

(iii) Either (iii-1) $g = 0$ and $s = 1$ or (iii-2) $g = 0$, $s = 2$ and $e_1 \neq e_2$.

<u>Example 1. 6.</u> If M is a compact Reimann surface and $\tilde{\pi} : \tilde{M}(D) \longrightarrow M$ exists, then $\tilde{\pi}$ is an infinite covering, unless $M = \tilde{M}(D) = \mathbb{P}^1$, the complex projective line, and $\tilde{\pi}$ is isomorphic to one of the following rational functions, (see Klein [20], Hochstadt [12]):

(1) $w = z^m$ $(m = 1, 2, \cdots)$,

$D = m(\infty) + m(o)$, $\tilde{G} \simeq C_m$ (m-th cyclic group).

(2) $w = -(z^m - 1)^2/4z^m$,

$D = m(\infty) + 2(0) + 2(1)$, $\tilde{G} \simeq D_m$ (m-th dihedral group).

(3) $w = (z^4 + 2\sqrt{3}z^2 - 1)^3/(z^4 - 2\sqrt{3}z^2 - 1)^3$,

$D = 3(\infty) + 3(0) + 2(1)$, $\tilde{G} \simeq A_4$.

(4) $w = (z^8 + 14z^4 + 1)^3/108z^4(z^4 - 1)^4$,

$D = 4(\infty) + 3(0) + 2(1)$, $\tilde{G} \simeq S_4$.

(5) $$w = \frac{(z^{20} - 228z^{15} + 494z^{10} + 228z^5 + 1)^3}{-1728z^5(z^{10} + 11z^5 - 1)^5}$$

$D = 5(\infty) + 3(0) + 2(1)$, $\tilde{G} \simeq A_5$.

(Here (α) is the point divisor of $\alpha \in \mathbb{P}^1$, $\tilde{G} = G_{\tilde{\pi}}$ and A_n (resp. S_n) is the alternating (resp. symmetric) group of n letters.)

2. D-universal coverings. In this section, we give answers to the problems at the end of §1, using language of fundamental groups.

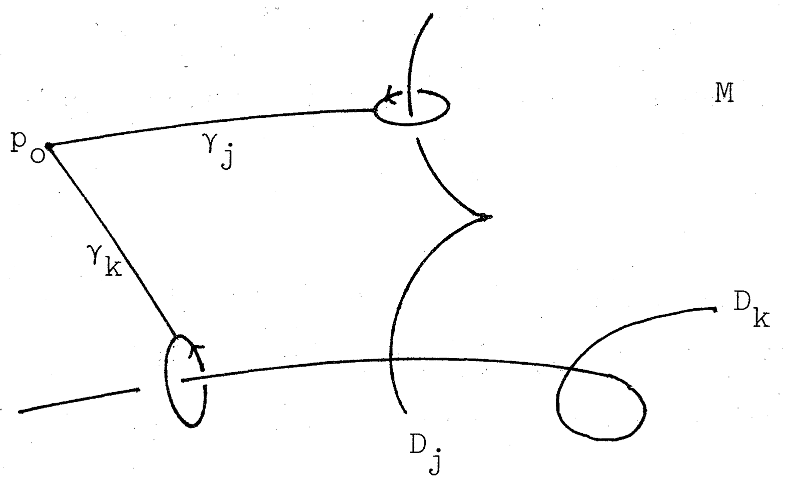

Figure 1

Take a point $p_o \in M - B$ and fix it once for all. Let γ_j be a loop in $M - B$ starting and terminating at p_o, encircling a point $p \in D_j - \mathrm{Sing}B$ in the positive sense as in Figure 1. γ_j is called a normal loop of D_j. We identify γ_j with its homotopy class in $\pi_1(M - B, p_o)$. Let

$$J = < \gamma_1^{e_1}, \cdots, \gamma_s^{e_s} >^{\pi_1}$$

be the smallest normal subgroup of $\pi_1(M - B, p_o)$ which contains $\gamma_1^{e_1}, \cdots, \gamma_s^{e_s}$.

Definition 2. 1. A subgroup K of $\pi_1(M - B, p_o)$ with $J \subset K$ is said to be D-faithful if the following condition is satisfied: If γ_j^d belongs to K, then $d \equiv 0 \pmod{e_j}$ for every j $(1 \leqslant j \leqslant s)$.

For every point $p \in \mathrm{Sing}B$, take a sufficiently small ball W (with respect to a metric on M) with the center p such that

$\pi_1(W - B) \simeq \pi_{1,loc,p}(M - B)$, (the local fundamental group at p).

Let

$$i_* : \pi_1(W - B) \longrightarrow \pi_1(M - B, p_o)$$

be the homomorphism induced by the inclusion mapping $i : W - B \hookrightarrow M - B$.

<u>Definition 2. 2.</u> A subgroup K of $\pi_1(M - B, p_o)$ with $J \subset K$ is said to be <u>locally cofinite</u> if $i_*^{-1}(K)$ is a subgroup of $\pi_1(W - B)$ of finite index for every point $p \in \mathrm{Sing}\, B$.

<u>Theorem 2. 3.</u> For any covering $\pi : X \longrightarrow M$ which branches at most along D, $K = \pi_*(\pi_1(X - \pi^{-1}(B)))$ contains J and is locally cofinite. Conversely, for any locally cofinite subgroup K ($\supset J$) of $\pi_1(M - B, p_o)$, there exists a unique (up to isomorphisms) covering $\pi : X \longrightarrow M$ which branches at most along D such that $\pi_*(\pi_1(X - \pi^{-1}(B))) = K$. In this case, π branches along D if and only if K is D-faithful.

For the proof of the converse, we construct a topological covering $\pi' : X' \longrightarrow M - B$ such that $K = \pi_*^{1}(\pi_1(X'))$, and then we extend π' to

$$\pi : X \longrightarrow M$$

using a theorem in Grauert-Remmert [9], (see also Grothendieck-Raynaud [10], p.340). Topologically, this is so called a <u>Fox completion</u>, (see Fox[6]). See Namba [22] for detail. By

Theorem 2.3,

<u>Theorem 2. 4.</u> There exists a finite Galois covering $\pi : X \longrightarrow M$ which branches along D if and only if there exists a normal subgroup K of $\pi_1(M - B, p_o)$ of finite index which contains J and is D-faithful. The correspondence $\pi \to K = \pi_*(\pi_1(X - \pi^{-1}(B)))$ between (isomorphism classes of) such π's and such K's is one-to-one. In this case, G_π is isomorphic to $\pi_1(M - B, p_o)/K$.

In fact, for such a normal subgroup K, we have

$$\frac{\pi_1(W-B)}{i_*^{-1}(K)} \simeq \frac{i_*(\pi_1(W-B))}{K \cap i_*(\pi_1(W-B))} \simeq \frac{K\cdot i_*(\pi_1(W-B))}{K} \subset \frac{\pi_1(M-B,p_o)}{K}$$

under the above notation. Hence K is necessarily locally cofinite.

Now, put

$$\tilde{K} = \cap K,$$

where the intersection $\cap$ runs over all subgroups K of $\pi_1(M - B, p_o)$ which contain J and are locally cofinite. $\tilde{K}$ is then a normal subgroup of $\pi_1(M - B, p_o)$ which contains J.

<u>Theorem 2. 5.</u> A D-universal covering $\tilde{\pi} : \tilde{M}(D) \longrightarrow M$ exists if and only if $\tilde{K}$ is locally cofinite and D-faithful. In this case, $\tilde{K} = \tilde{\pi}_*(\pi_1(\tilde{M}(D) - \tilde{\pi}^{-1}(B)))$ and $G_{\tilde{\pi}} \simeq \pi_1(M - B, p_o)/\tilde{K}$. Moreover, $\tilde{M}(D)$ is simply connected.

It is easy to see that $\tilde{M}(D)$ is simply connected. In fact, if $\mu : \tilde{X} \longrightarrow \tilde{M}(D)$ is a (topological) universal covering of $\tilde{M}(D)$, then $\tilde{\pi}\cdot\mu : \tilde{X} \longrightarrow M$ is a covering which branches along D such that $\tilde{\pi}\cdot\mu \geqslant \tilde{\pi}$. By the D-universality of $\tilde{\pi}$, we have $\tilde{\pi}\cdot\mu \leqslant \tilde{\pi}$. Hence μ is an isomorphism.

<u>Theorem 2. 6.</u> Let $\pi : X \longrightarrow M$ be a Galois covering which branches along D. Suppose that X is non-singular and simply connected. Then π is D-universal. In this case, $\tilde{K} = J$ and $G_\pi = \pi_1(M - B, p_o)/J$.

In this theorem, the condition of the non-singularity of X can not be dropped, as the following example shows:

<u>Example 2. 7.</u> Put $M = \mathbb{C}^2$ and let (u, v) be the coordinate system on $\mathbb{C}^2$. Put $D_1 = \{u = 0\}$, $D_2 = \{v = 0\}$ and $D = 2D_1 + 2D_2$. Put $X = \{(u, v, w) \in \mathbb{C}^3 \mid w^2 = uv\}$ and

$$\pi : (u, v. w) \in X \longmapsto (u, v) \in \mathbb{C}^2.$$

Then π is a cyclic covering of degree 2 which branches along D. X is simply connected, for X is a cone. But π is not D-universal. In fact, putting $Y = \mathbb{C}^2$ and

$$\mu : (x, y) \in Y \longmapsto (u, v, w) = (x^2, y^2, xy) \in X,$$

the composition $\pi\cdot\mu : Y \longrightarrow \mathbb{C}^2$ is a covering of degree 4 which branches along D and $\pi\cdot\mu \geqslant \pi$. (By Theorem 2. 6, $\pi\cdot\mu$ is D-universal.)

<u>For the rest of this section, we suppose that B is simple normally crossing.</u>

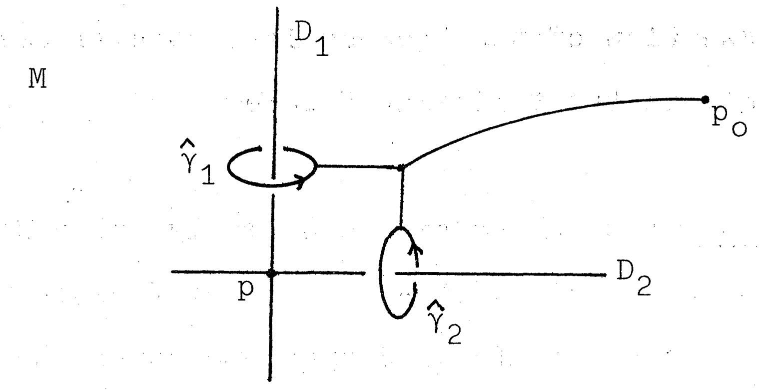

Figure 2

For any point $p \in \mathrm{Sing}B$, let $(w_1, \cdots, w_n)$ be a local coordinate system around p such that $p = (0, \cdots, 0)$ and

$$B = \{(w_1, \cdots, w_n) \mid w_k = \cdots = w_n = 0\}.$$

locally. Let

$$\{w_j = 0\} = D_j \quad (k \leqslant j \leqslant n),$$

locally, say. Let $\hat{\gamma}_j$ be a loop in $M-B$ starting and terminating at p_o, encircling a point of $D_j - \mathrm{Sing}B$ near p in the positive sense as in Figure 2. Then $\hat{\gamma}_j$ is conjugate to γ_j in $\pi_1(M - B, p_o)$. $\hat{\gamma}_k, \cdots, \hat{\gamma}_n$ are mutually commutative. For a sufficiently small ball W with the center p, we have

$$\pi_1(W - B) = (\hat{\gamma}_k)^{\mathbb{Z}} \cdots (\hat{\gamma}_n)^{\mathbb{Z}}$$

and

$$(\hat{\gamma}_k{}^{e_k})^{\mathbb{Z}} \cdots (\hat{\gamma}_n{}^{e_n})^{\mathbb{Z}} \subset i_*^{-1}(J) \subset \pi_1(W - B).$$

Hence J is locally cofinite, so that $\tilde{K} = J$. Thus

Theorem 2. 8. If B is simple normally crossing, then a D-universal covering $\tilde{\pi} : \tilde{M}(D) \longrightarrow M$ exists if and only if J is D-faithful. In this case, $J = \tilde{K}$ and $G_{\tilde{\pi}} \cong \pi_1(M - B, p_o)/J$. Moreover, if (under the above notation), $(\hat{\gamma}_k{}^{e_k})^{\mathbb{Z}} \cdots (\hat{\gamma}_n{}^{e_n})^{\mathbb{Z}} = i_*^{-1}(J)$ for every point $p \in \mathrm{Sing}B$, then $\tilde{M}(D)$ is non-singular.

The last assertion of the theorem is a special case of Kato [17], as well as the following theorem.

Theorem 2. 9. Let B be simple normally crossing. Let K be a normal subgroup of $\pi_1(M - B, p_o)$ of finite index which contains J and is D-faithful. Suppose moreover that, for any point $p \in \text{Sing}B$, K satisfies the following condition : (under the above notation)

$$\text{if } \hat{\gamma}_k^{d_k} \cdots \hat{\gamma}_n^{d_n} \in K, \text{ then } d_k \equiv 0 (\text{mod } e_k), \cdots, d_n \equiv 0 (\text{mod } e_n).$$

Then the irreducible normal complex space X is non-singular, where $\pi : X \longrightarrow M$ is the finite Galois covering which branches along D and corresponds to K under Theorem 2. 4.

3. Examples. It is not easy in general to apply the results of §2 to concrete examples. (Even the calculation of $\pi_1(M - B, p_o)$ is not easy.) In this section, we discuss two examples.

Case 1. Put $M = \mathbb{C}^2$, $B = D_1 = \{(x,y) \in \mathbb{C}^2 \mid x^3 = y^2\}$, ℓ : a positive integer greater than 1, and $D = \ell D_1$.

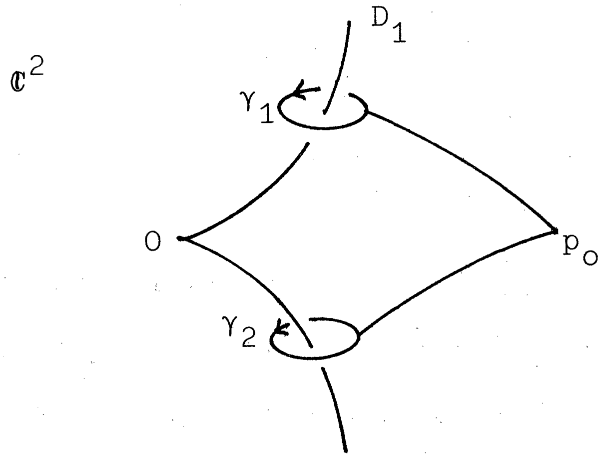

Figure 3

As is well known, $\pi_1(\mathbb{C}^2 - B, p_o)$ is isomorphic to 3rd braid group B_3 ; taking the loops γ_1 and γ_2 as in Figure 3, we have

$$\pi_1(\mathbb{C}^2 - B, p_o) = \langle \gamma_1, \gamma_2 \mid \gamma_1\gamma_2\gamma_1 = \gamma_2\gamma_1\gamma_2 \rangle .$$

Here the right hand side means the group generated by γ_1 and γ_2 with the generating relation $\gamma_1\gamma_2\gamma_1 = \gamma_2\gamma_1\gamma_2$. Since $\gamma_2 = (\gamma_2\gamma_1)^{-1}\gamma_1(\gamma_2\gamma_1)$, γ_2 is conjugate to γ_1. Let J be the smallest normal subgroup of $\pi_1(\mathbb{C}^2 - B, p_o)$ containing γ_1^{ℓ} (and so γ_2^{ℓ}). Then

$$\pi_1(\mathbb{C}^2 - B, p_o)/J \cong G_\ell = \langle a, b \mid a^\ell = b^\ell = 1, aba = bab \rangle$$

by the correspondence: $\gamma_1 \longmapsto a$, $\gamma_2 \longmapsto b$. We identify these groups through the isomorphism.

The cyclic covering

$$\pi_\ell : X_\ell \longrightarrow M = \mathbb{C}^2,$$

corresponding to the kernel of the homomorphism

$$f_\ell : G_\ell \longrightarrow \mathbb{Z}/\ell\mathbb{Z}, \ (f_\ell(a) = f_\ell(b) = 1)$$

is given by

$$\pi_\ell : X_\ell = \{(x,y,z) \in \mathbb{C}^3 \mid z^\ell = x^3 - y^2\} \longrightarrow M = \mathbb{C}^2$$
$$(x,y,z) \longmapsto (x, y)$$

and branches along $D = \ell D_1$. But π_ℓ is <u>not</u> D-universal.

The following argument on the structure of G_ℓ was informed by Mr. Mizutani. See also Coxeter [2]. First of all,

<u>Lemma 3. 1.</u> $c = (aba)^2$ is an element of the center $Z(G_\ell)$ of G_ℓ.

Next, consider the <u>Schwarz' triangular group</u>

$$G(2, 3, \ell) = < S, T \mid S^2 = T^3 = (ST)^{\ell} = 1 >,$$

and the homomorphism,

$$g : G_{\ell} \longrightarrow G(2, 3, \ell)$$

defined by $g(a) = ST$ and $g(b) = TS$.

<u>Proposition 3. 2.</u> The following sequence is exact:

$$1 \longrightarrow <c> \longrightarrow G_{\ell} \xrightarrow{g} G(2, 3, \ell) \longrightarrow 1$$

From this proposition, we have the following table:

ℓ	ord(c)	$G(2, 3, \ell)$	G_{ℓ}	ord G_{ℓ}
2	1	S_3	S_3	6
3	2	A_4	$SL(2, \mathbb{Z}/3\mathbb{Z})$	24
4	4	S_4	$G_4/Z(G_4) \simeq S_4$, $Z(G_4) \simeq \mathbb{Z}/4\mathbb{Z}$	96
5	10	A_5	$SL(2, \mathbb{Z}/5Z)\times(\mathbb{Z}/5\mathbb{Z})$	600
6	∞	infinite group	infinite solvable group	∞
$\geqslant 7$	∞	infinite group	infinite unsolvable group	∞

<u>If ℓ satisfies $2 \leqslant \ell \leqslant 5$,</u> then (under the notations of §2) $\tilde{K} = J$ and there exists a D-universal covering

$$\tilde{\pi} : \tilde{M}(D) \longrightarrow M = \mathbb{C}^2.$$

In this case, $\tilde{\pi}$ is a finite Galois covering such that $G_{\tilde{\pi}} \simeq G_{\ell}$. Moreover, we have $\tilde{M}(D) = \mathbb{C}^2$ and $\tilde{\pi}$ is the composition

$$\tilde{M}(D) = \mathbb{C}^2 \xrightarrow{\mu} X_{\ell} \xrightarrow{\pi_{\ell}} M = \mathbb{C}^2,$$

where μ is the projection

$$\mu : \tilde{M}(D) = \mathbb{C}^2 \longrightarrow X_\ell = \mathbb{C}^2/H,$$

where H is a finite subgroup of $GL(2, \mathbb{C})$. The origin of X_ℓ in this case is called the <u>Klein singularity</u>, (see Pinkham [24]).

<u>If $\ell = 6$</u>, then we have

<u>Proposition 3. 3.</u> The kernel of $f_6 : G_6 \longrightarrow \mathbb{Z}/6\mathbb{Z}$ $(f_6(a) = f_6(b) = 1)$ is given by $\langle a^{-1}b, ab^{-1}\rangle$ and is isomorphic to $N = \left\{ \begin{pmatrix} 1 & i & j \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \middle| \, i, j, k \in \mathbb{Z} \right\}$.

The isomorphism is given by

$$a^{-1}b \longmapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad ab^{-1} \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

We identify $\ker(f_6)$ with N through the isomorphism. For any positive odd integer r,

$$N(r) = \left\{ \begin{pmatrix} 1 & i & j \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \in N \,\middle|\, i \equiv j \equiv k \equiv 0 \pmod r \right\}$$

is a normal subgroup of G_6 of index $6r^3$ and is D-faithful. Hence

<u>Proposition 3. 4.</u> For any positive odd integer r, there is a Galois covering $\nu_r : Y_r \longrightarrow M = \mathbb{C}^2$ of degree $6r^3$ branching along $6D_1$. $\nu_r \leq \nu_{r'}$ if and only if $r|r'$.

Since
$$\bigcap_{r:\text{odd}} N(r) = \{1\},$$
We have

Proposition 3. 5. If $D = 6D_1$, then there does not exist a D-universal covering of $M = \mathbb{C}^2$.

Case 2. Put $M = \mathbb{P}^2$ (the complex projective plane), $B = D_1 \cup D_2$, D_1 = the closure in $\mathbb{P}^2$ of the affine curve $\{(x, y) \in \mathbb{C}^2 \mid x^3 - y^2 = 0\}$, $D_2 = L_\infty$ (the line at infinity), ℓ, m : positive integers greater than 1, and $D = \ell D_1 + mD_2$.

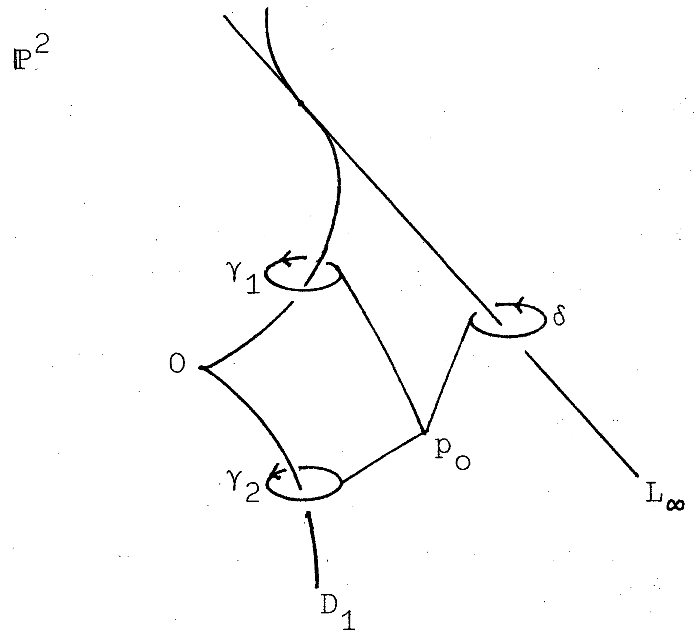

Figure 4

Taking the loops γ_1, γ_2 and δ as in Figure 4, we have
$$\pi_1(M - B, p_o) = \langle \gamma_1, \gamma_2, \delta \mid \gamma_1\gamma_2\gamma_1 = \gamma_2\gamma_1\gamma_2 = \delta^{-1} \rangle .$$
Let J be the smallest normal subgroup of $\pi_1(M - B, p_o)$ which contains γ_1^{ℓ}, γ_2^{ℓ} and δ^m. Then
$$\pi_1(\mathbb{P}^2 - B, p_o)/J \simeq G_{\ell,m} ,$$

where

$$G_{\ell,m} = < \alpha, \beta, \delta \mid \alpha^{\ell} = \beta^{\ell} = \delta^{m} = 1, \alpha\beta\alpha = \beta\alpha\beta = \delta^{-1} >,$$

$$(\gamma_1 \longmapsto \alpha, \gamma_2 \longmapsto \beta, \delta \longmapsto \delta).$$

Let G_ℓ be the group in Case 1. There is a surjective homomorphism

$$h : G_\ell \longrightarrow G_{\ell,m}$$

defined by $h(a) = \alpha$ and $h(b) = \beta$.

For simplicity, we assume that m is a positive <u>even</u> integer. Then the following sequence is exact:

$$1 \longrightarrow <c^{m/2}> \longrightarrow G_\ell \xrightarrow{h} G_{\ell,m} \longrightarrow 1.$$

In particular, if the pair (ℓ, m) is one of the following table:

ℓ	2	3	4	5
m	2	4	8	20

then $G_{\ell,m} \cong G_\ell$ and there exists a D-universal covering

$$\tilde{\pi} : \tilde{M}(D) \longrightarrow M = \mathbb{P}^2 \quad (D = \ell D_1 + mD_2).$$

In this case, $\tilde{\pi}$ is a finite Galois covering such that $G_{\tilde{\pi}} \cong G_{\ell,m} \cong G_\ell$.

<u>If $\ell = 6$,</u> then we have by Proposition 3. 3,

<u>Proposition 3. 6.</u> For any positive integer m such that $m \equiv 2 \pmod 4$, there is a Galois covering $\phi_m : Z_m \longrightarrow \mathbb{P}^2$ of degree $6(m/2)^3$ branching along $D = 6D_1 + mL_\infty$. $\phi_m \leqslant \phi_{m'}$ if and only if $m|m'$.

On the other hand, since the sequence

$$1 \longrightarrow \langle c \rangle / \langle c^{m/2} \rangle \longrightarrow G_{\ell,m} \longrightarrow G(2,3,\ell) \longrightarrow 1$$

is exact, we have in particular (putting $m = 2$),

$$G_{\ell,2} \cong G(2, 3, \ell).$$

Putting $\ell = 2$, we identify $G_{6,2}$ with $G(2, 3, 6)$ through the isomorphism. It is well known that $G(2, 3, 6)$ has the normal subgroup L such that

$$G(2, 3, 6)/L \cong \mathbb{Z}/6\mathbb{Z},$$
$$L \cong \mathbb{Z} \oplus \mathbb{Z} \quad \text{(the direct sum).}$$

Identifying L with $\mathbb{Z} + \mathbb{Z}$ through the isomorphism, consider, for any positive integer q, the normal subgroup

$$L_q = \{(j, k) \in \mathbb{Z} \oplus \mathbb{Z} \mid j \equiv k \equiv 0 \pmod{q}\}$$

of index $6q^2$ of $G(2, 3, 6)$. Since

$$\bigcap_q L_q = \{1\} ,$$

we have

<u>Proposition 3. 7.</u> If $D = 6D_1 + 2L_\infty$, then there <u>does</u> <u>not</u> exist a D-universal covering of $\mathbb{P}^2$.

By another method (see Namba [23]), we can show

<u>Proposition 3. 8.</u> For any positive integer k, there exists a finite Galois covering $\pi : X \longrightarrow \mathbb{P}^2$ branching along $D = 6kD_1 + 2kL_\infty$.

4. Existence of Finite Galois Coverings. As for Problem 2 in §1, it is desirable to give (sufficient) conditions for the existence without using language of fundamental groups. Theorem 1.5 is such a theorem. In this section, we give such theorems.

Let $L_1, \cdots, L_s$ be distinct lines on $\mathbb{P}^2$ and put $B = L_1 \cup \cdots \cup L_s$. Put

$$\Delta = \{p \in B \mid m_p(B) \geqslant 3\},$$

where $m_p(B)$ is the multiplicity at p of the curve B. Δ is a finite point set.

Theorem 4. 1. Suppose that $L_j \cap \Delta$ is non-empty for every j $(1 \leqslant j \leqslant s)$. Then, for any postive integers $e_1, \cdots, e_s$ greater than 1, there exists a finite Galois covering $\pi : X \longrightarrow \mathbb{P}^2$ branching along $D = e_1L_1 + \cdots + e_sL_s$.

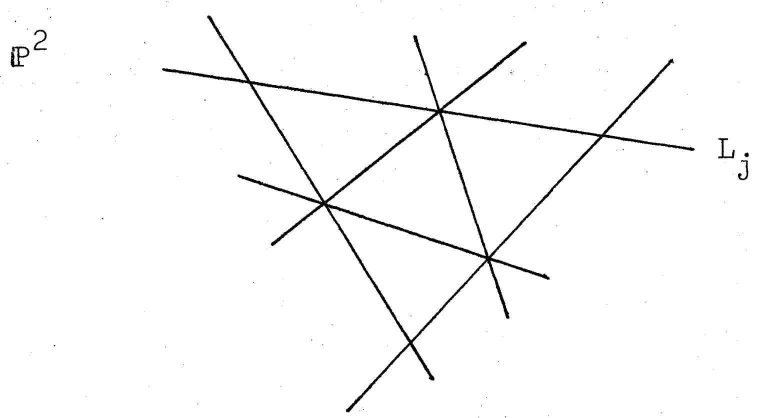

Figure 5

See Kato [16] for the proof of the theorem. We generalize the theorem as follows:

Theorem 4. 2. Let M be an n $(\geqslant 2)$ dimensional projective manifold and $D_1, \cdots, D_s$ be distinct irreducible hypersurfaces of M. Suppose that there are fixed component free linear pencils $\Lambda_1, \cdots, \Lambda_t$ on M such that (i) every D_j is a member of some Λ_k and (ii) every Λ_k contains at least three D_j's as its members. Then, for any positive integers $e_1, \cdots, e_s$ greater than 1, there exists a finite Galois covering $\pi : X \to M$ branching along $D = e_1D_1 + \cdots + e_sD_s$.

Note that Theorem 4. 1 follows from Theorem 4. 2, putting $M = \mathbb{P}^2$, $D_j = L_j$ $(1 \leq j \leq s)$ and Λ_k = the linear pencil given by the projection with the center point $p_k \in \Delta$. See Namba [22] for the proofs of Theorem 4. 2 and the following theorem:

Theorem 4. 3. Let $C_1, \cdots, C_s$ be distinct irreducible conics on $\mathbb{P}^2$ such that, for any C_j, there is a C_k which is tangent to C_j at two distinct points. Then, for any positive integers $e_1, \cdots, e_s$ greater than 1, there exists a finite Galois covering $\pi : X \longrightarrow \mathbb{P}^2$ branching along $D = e_1D_1 + \cdots + e_sD_s$.

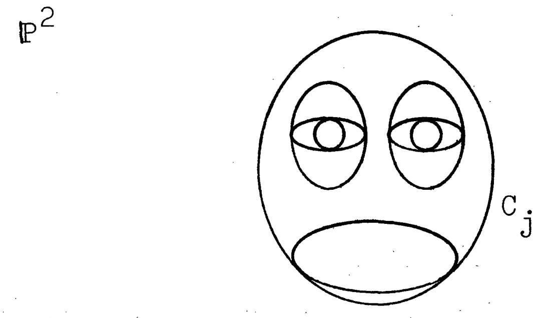

Figure 6

Chapter 2. Abelian Coverings.

5. Abelian D-universal coverings. Let M be an n-diemensional connected complex manifold. Let $B = D_1 \cup \cdots \cup D_s$, $D = e_1 D_1 + \cdots + e_s D_s$, $p_0 \in M - B$, γ_j $(1 \leqslant j \leqslant s)$ and $J = \langle \gamma_1^{e_1}, \cdots, \gamma_s^{e_s} \rangle_{\pi_1}$ be as in §1 and §2. Put

$$\hat{J} = J \cdot [\pi_1(M-B, p_o), \pi_1(M-B, p_o)],$$

where $[G, G]$ is the commutator subgroup of G. Then we can easily prove the following lemma.

Lemma 5. 1. $\pi_1(M - B, p_o)/\hat{J} \cong$

$$H_1(M - B; \mathbb{Z})/(\mathbb{Z}(e_1\gamma_1) + \cdots + \mathbb{Z}(e_s\gamma_s)).$$

Here $H_1(M - B; \mathbb{Z})$ is the first homology group of $M-B$ and $\mathbb{Z}(e_1\gamma_1) + \cdots + \mathbb{Z}(e_s\gamma_s)$ is the subgroup of $H_1(M - B ; \mathbb{Z})$ generated by $e_1\gamma_1, \cdots, e_s\gamma_s$, which are regarded as elements of $H_1(M - B; \mathbb{Z})$.

Moreover, we can prove:

Proposition 5. 2. $\hat{J}$ is a normal subgroup of $\pi_1(M - B, p_o)$ which contains J and is locally cofinite.

The covering $\pi_o : X_o \longrightarrow M$ which branches at most along D, corresponding to $\hat{J}$ by Theorem 2. 1 is an abelian covering. Moreover, for any abelian covering $\pi : \mathfrak{X} \longrightarrow M$ which branches at most along D, the relation $\pi_o \geqslant \pi$ holds.

Definition 5. 3. An abelian covering $\tilde{\pi}_{ab} : \tilde{M}_{ab}(D) \longrightarrow M$ is called an abelian D-universal covering if (i) $\tilde{\pi}_{ab}$ branches along D and (ii) for any abelian covering $\pi : X \longrightarrow M$ which branches at most along D, the relation $\tilde{\pi}_{ab} \geqslant \pi$ holds.

By the above consideration, if an abelian D-universal covering $\tilde{\pi}_{ab} : \tilde{M}_{ab}(D) \longrightarrow M$ exists, then it must be isomorphic to $\pi_o : X_o \longrightarrow M$. Conversely, if $\pi_o : X_o \longrightarrow M$ branches along D, then it is an abelian D-universal covering. Thus

<u>Theorem 5. 4.</u> There exists an abelian D-universal covering $\tilde{\pi}_{ab} : \tilde{M}_{ab}(D) \longrightarrow M$ if and only if the following condition is satisfied: if $d\gamma_j \in \mathbb{Z}(e_1\gamma_1) + \cdots + \mathbb{Z}(e_s\gamma_s)$, then $d \equiv 0 \pmod{e_j}$ for every $1 \leqslant j \leqslant s$. In this case, the covering transformation group of $\tilde{\pi}_{ab}$ is isomorphic to $\tilde{G}_{ab} = H_1(M - B;\ \mathbb{Z})/(\mathbb{Z}(e_1\gamma_1) + \cdots + \mathbb{Z}(e_s\gamma_s))$.

For example, let $M = \mathbb{C}^2$, $B = D_1$ and $D = \ell D_1$ be as in Case 1 of §3. Then we have $H_1(\mathbb{C}^2 - B\ ;\ \mathbb{Z}) = \mathbb{Z}\gamma_1$ and the condition in Theorem 5. 4 is clearly satisfied. In this case, $\tilde{\pi}_{ab} : \widetilde{\mathbb{C}^2}_{ab}(D) \longrightarrow \mathbb{C}^2$ is nothing but the cyclic covering

$$\pi_\ell : X_\ell = \{(x,\ y,\ z) \in \mathbb{C}^3 \mid z^\ell = x^3 - y^2\} \longrightarrow \mathbb{C}^2$$
$$(x,\ y,\ z) \longmapsto (x,\ y)$$

considered in Case 1 of §3.

Let $M = \mathbb{P}^2$, $B = D_1 \cup L_\infty$ and $D = \ell D_1 + mL_\infty$ be as in Case 2 of §3. Then we have

$$H_1(\mathbb{P}^2 - B\ ;\ \mathbb{Z}) = (\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2)/\mathbb{Z}(3\gamma_1 + \gamma_2).$$

Hence the condition in Theorem 5. 4 is equivalent in this case to the condition: $\ell/(3,\ \ell) = m$, where $(3,\ \ell)$ is the GCD of 3 and ℓ. If this is the case, $\tilde{\pi}_{ab} : \widetilde{\mathbb{P}^2}_{ab}(D) \to \mathbb{P}^2$ is a finite covering.

In general, if $M = \mathbb{P}^n$, D_j is an irreducible hypersurface of degree d_j $(1 \leqslant j \leqslant s)$ and $B = D_1 \cup \cdots \cup D_s$, then we have

Thus
$$H_1(\mathbb{P}^n - B ; \mathbb{Z}) = (\mathbb{Z}\gamma_1 + \cdots + \mathbb{Z}\gamma_s)/(\mathbb{Z}(d_1\gamma_1 + \cdots + d_s\gamma_s)).$$

<u>Theorem 5. 5.</u> Let D_j be distinct irreducible hypersurfaces of degree d_j $(1 \leqslant j \leqslant s)$ of the complex projective space $\mathbb{P}^n$. Put $D = e_1D_1 + \cdots + e_sD_s$. Then there exists an abelian D-universal covering $\tilde{\pi}_{ab} : \widetilde{\mathbb{P}^n}_{ab}(D) \longrightarrow \mathbb{P}^n$ if and only if $e_j/(d_j, e_j)$ divides

$$\langle e_1/(d_1,e_1), \cdots, e_{j-1}/(d_{j-1},e_{j-1}), e_{j+1}/(d_{j+1},e_{j+1}), \cdots, e_s/(d_s,e_s)\rangle$$

for every j $(1 \leqslant j \leqslant s)$, where $(\cdots)$ and $\langle\cdots\rangle$ denote the GCD and LCM of the components, respectively. In this case, $\tilde{\pi}_{ab}$ is a finite covering.

As for a compact Riemann surface M, Theorem 5. 4 can be rewritten as

<u>Theorem 5. 6.</u> Let p_j $(1 \leqslant j \leqslant s)$ be distinct points on a compact Riemann surface M of genus g. Put $D = e_1p_1 + \cdots + e_sp_s$. Then there exists an abelian D-universal covering $\tilde{\pi}_{ab} : \tilde{M}_{ab}(D) \longrightarrow M$ if and only if e_j divides.

$$\langle e_1, \cdots, e_{j-1}, e_{j+1}, \cdots, e_s\rangle$$

for every j $(1 \leqslant j \leqslant s)$. In this case, $\tilde{\pi}_{ab}$ is an infinite covering if $g \geqslant 1$.

Finally, as for finite abelian coverings of a complex manifold M, we have

Theorem 5. 7. Let M be a connected complex manifold, $B = D_1 \cup \cdots \cup D_s$, $D = e_1 D_1 + \cdots + e_s D_s$ and γ_j $(1 \leqslant j \leqslant s)$ be as before. Then there exists a one-to-one correspondence $\pi \longrightarrow K = K(\pi)$ between isomorphism classes of finite abelian coverings $\pi : X \longrightarrow M$ which branches at most along D, and subgroups K of finite index of

$$\tilde{G}_{ab} = H_1(M - B; \mathbb{Z})/(\mathbb{Z}(e_1\gamma_1) + \cdots + \mathbb{Z}(e_s\gamma_s)).$$

The correspondence satisfies (1) $G_\pi \cong \tilde{G}_{ab}/K(\pi)$, (2) $\pi_1 \leqslant \pi_2$ if and only if $K(\pi_1) \supset K(\pi_2)$ and (3) π branches along D if and only if $K(\pi)$ satisfies the following condition: if $d\gamma_j \in K(\pi)$, then $d \equiv 0 \pmod{e_j}$ for $1 \leqslant j \leqslant s$.

6. Finite Abelian Coverings of Projective Manifolds. In this section, we suppose that M is a projective manifold. We discuss finite abelian coverings of M. Here are two typical examples of abelian coverings.

Example 6. 1. Let $\hat{\pi} : L \longrightarrow M$ be a holomorphic line bundle on M and $\xi = \{\xi_\alpha\}$ be a holomorphic section of $L^{\otimes e}$ (the e-times tensor product of L for a postive integer e greater than 1), where ξ_α is a holomorphic function on an open set U_α on which L is trivial. Suppose that the zero divisor (ξ) of ξ has no multiple component:

$$(\xi) = D_1 + \cdots + D_s,$$

where D_j are distinct prime divisors. Put

$$D = e(\xi) = eD_1 + \cdots + eD_s.$$

Put

$$X = \bigcup_{\alpha}\{(p, z_\alpha) \in U_\alpha \times \mathbb{C} \mid z_\alpha{}^e = \xi_\alpha(p)\}.$$

Then X can be considered as an irreducible normal hypersurface of the bundle space L. Put

$$\pi = \hat{\pi}|_X : X \longrightarrow M.$$

Then π is a cyclic covering which branches along D.

Example 6. 2. Let L be a holomorphic line bundle on M and $\xi_1, \cdots, \xi_s$ be holomorphic sections of L. Suppose that $D_1 = (\xi_1), \cdots, D_s = (\xi_s)$ are distinct prime divisors such that $D_1 \cap \cdots \cap D_s = \phi$. For a positive integer e greater than 1, put

$$B = D_1 \cup \cdots \cup D_s,$$

$$D = eD_1 + \cdots + eD_s.$$

Consider the Kummer extension

$$F = \mathbb{C}(M)((\xi_1/\xi_s)^{1/e}, \cdots, (\xi_{s-1}/\xi_s)^{1/e})$$

of the field $\mathbb{C}(M)$ of meromorphic functions on M. Let

$$\pi : X \longrightarrow M$$

be the F-normalization of M, (see Iitaka [14]). Then π is a finite abelian covering of M which branches along D such that $G_\pi \cong (\mathbb{Z}/e\,\mathbb{Z})^{s-1}$. The covering $\pi : X \longrightarrow M$ is called a Kummer covering. In this case, we can prove that, if B is simple normally crossing, then X is non-singular.

Now, let $B = D_1 \cup \cdots \cup D_s$ and $D = e_1D_1 + \cdots + e_sD_s$ be as in §1. We rewrite Theorem 5. 7 using language of rational divisors. A rational D-divisor is a rational divisor $\hat{E}$ on M

of the following type:

$$\hat{E} = (a_1/e_1)D_1 + \cdots + (a_s/e_s)D_s + E,$$

where a_j $(1 \leqslant j \leqslant s)$ are integers and E is an integral divisor. Rational D-divisors form an additive group $\mathrm{Div}^{\mathbb{Q}}(M, D)$. Let $\mathrm{Div}_o{}^{\mathbb{Q}}(M, D)$ be the subgroup of $\mathrm{Div}^{\mathbb{Q}}(M, D)$ consisting of all $\hat{E}$ such that

$$c_{\mathbb{Q}}(\hat{E}) = (a_1/e_1)c_{\mathbb{Q}}([D_1]) + \cdots + (a_s/e_s)c_{\mathbb{Q}}([D_s]) + c_{\mathbb{Q}}(E)$$
$$= 0 \in H^2(M ; \mathbb{Q}),$$

where $[D_j]$ is the line bundle determined by D_j and $c_Q : \mathrm{Pic}(M) \longrightarrow H^2(M ; \mathbb{Q})$ is the homomorphism of rational Chern class.

Two rational D-divisors $\hat{E}$ and $\hat{E}'$ are said to be <u>linearly equivalent</u>, $\hat{E} \sim \hat{E}'$, if $\hat{E} - \hat{E}'$ is an integral and principal divisor. Consider the additive group

$$\mathrm{Pic}_o{}^{\mathbb{Q}}(M, D) = \mathrm{Div}_o{}^{\mathbb{Q}}(M, D)/\sim .$$

<u>Theorem 6. 3.</u> There exists a one-to-one correspondence $\pi \longrightarrow S = S(\pi)$ between isomorphism classes of finite abelian coverings $\pi : X \longrightarrow M$ which branches at most along D, and subgroups S of finite index of $\mathrm{Pic}_o{}^{\mathbb{Q}}(M, D)$. The correspondence satisfies (1) $G_\pi \cong S(\pi)$ and (2) $\pi_1 \leqslant \pi_2$ if and only if $S(\pi_1) \subset S(\pi_2)$.

<u>Theorem 6. 4.</u> There exists a finite abelian covering $\pi : X \longrightarrow M$ which branches along D if and only if there is a subgroup S of finite index of $\mathrm{Pic}_o{}^{\mathbb{Q}}(M, D)$ with the following condition: for every j $(1 \leqslant j \leqslant s)$, there is an element $\hat{E}(j)/\sim \in S$ such that $(a_j, e_j) = 1$, where

$\hat{E}(j) = (a_1/e_1)D_1 + \cdots + (a_j/e_j)D_j + \cdots + (a_s/e_s)D_s + E$,

(E : an integral divisor).

For the proofs of the above theorems, we make use of the theory of harmonic integrals by de Rham-Kodaira [3].

For example, the cyclic covering $\pi : X \longrightarrow M$ in Example 6.1 corresponds to

$$S = \{(a/e)(D_1 + \cdots + D_s) - aE \mid 0 \leqslant a \leqslant e - 1\}/\sim,$$

where E is an integral divisor on M such that $[E] = L$.

The Kummer covering $\pi : X \longrightarrow M$ in Example 6. 2 corresponds to

$$S = S_{12} + S_{23} + \cdots + S_{n-1,n} + S_{n,1},$$

where

$$S_{12} = \{(a/e)D_1 - (a/e)D_2 \mid 0 \leqslant a \leqslant e - 1\} / \sim, \text{ etc..}$$

As applications of Theorem 6. 4,

<u>Theorem 6. 5.</u> Let $D_1, \cdots, D_s$ $(s \geqslant 2)$ be linearly equivalent distinct prime divisors on a projective manifold M. Suppose that, for every j $(1 \leqslant j \leqslant s)$, e_j divides

$$< e_1, \cdots, e_{j-1}, e_{j+1}, \cdots, e_s >.$$

Then there exists a finite abelian covering $\pi : X \longrightarrow M$ which branches along $D = e_1D_1 + \cdots + e_sD_s$.

<u>Theorem 6. 6.</u> Let $p_1, \cdots, p_s$ be distinct points of a compact Riemann surface M. Put $D = e_1p_1 + \cdots + e_sp_s$, $(e_j \geqslant 2)$. Then there exists a finite abelian covering $\pi : X \longrightarrow M$ which branches along D if and only if, for every j $(1 \leqslant j \leqslant s)$, e_j divides

$$\langle e_1, \cdots, e_{j-1}, e_{j+1}, \cdots, e_s\rangle .$$

Finally, we move D and consider various $\mathrm{Pic}_o^{\mathbb{Q}}(M, D)$'s. Let $\mathrm{Div}^{\mathbb{Q}}(M)$ be the additive group of all rational divisors on M, and $\mathrm{Div}_o^{\mathbb{Q}}(M)$ be the subgroup of $\mathrm{Div}^{\mathbb{Q}}(M)$ consisting of all rational divisors whose rational Chern classes vanish. Two rational divisors $\hat{E}$ and $\hat{E}'$ are said to be <u>linearly equivalent</u>, $\hat{E} \sim \hat{E}'$, if $\hat{E} - \hat{E}'$ is integral and principal. Consider the additive group

$$\mathrm{Pic}_o^{\mathbb{Q}}(M) = \mathrm{Div}_o^{\mathbb{Q}}(M)/\sim.$$

Let $\mathbb{C}(M)$ be the field of meromorphic functions on M. Note that isomorphism classes of finite Galois (resp. abelian) branched coverings $\pi : X \longrightarrow M$ and (isomorphism classes of) finite Galois (resp. abelian) extensions $F/\mathbb{C}(M)$ of $\mathbb{C}(M)$ are in one-to-one correspondence under

$$\pi \longrightarrow F = \mathbb{C}(X),$$
$$F \longrightarrow F\text{-normalization of } M.$$

Then, by Theorem 6. 3, we have

<u>Theorem 6. 7.</u> For a projective manifold M, there exists a one-to-one correspondence $F \longrightarrow S = S(F)$ between the set of all (isomorphism classes of) finite abelian extensions $F/\mathbb{C}(M)$ and the set of all finite subgroups S of $\mathrm{Pic}_o^{\mathbb{Q}}(M)$. The correspondence satisfies (1) $S(F) \cong \mathrm{Gal}(F/\mathbb{C}(M))$ and (2) $F_1 \subset F_2$ if and only if $S(F_1) \subset S(F_2)$.

Note that the class field theory for fields of algebraic functions (of one variable) asserts the dual version of this theorem, using the generalized Jacobian variety, (see Serre [27]).

The content of this section can be generalized to finite Galois coverings of a projective manifold, using language of unitary flat generalized vector bundles, along the line of Weil [30]. See Namba [22] for detail.

7. Equivalence Problem and Autonomorphism Groups of Kummer Coverings. Let $\pi : X \longrightarrow M$ be a Galois covering of M branching along $D = e_1D_1 + \cdots + e_sD_s$ with the covering transformation group $G = G_\pi$. In this case, we also write in this section

$$\pi : (G, X) \longrightarrow (M : D).$$

For a second Galois covering $\pi' : (G', X') \longrightarrow (M' : D')$, a biholomorphism $h : X \longrightarrow X'$ is refered to as an equivalence, written $h : \pi \approx \pi'$, if there is a biholomorphism $\bar{h} : M \longrightarrow M'$ making a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{\bar{h}} & M'. \end{array}$$

For a biholomorphism $h : X \longrightarrow X'$, if we put

$$G'^h = \{h^{-1}g'h \mid g' \in G'\}.$$

Then we have

$$h : \pi \approx \pi' \Longleftrightarrow G = G'^h.$$

Let $E(\pi)$ be the subgroup of $\mathrm{Aut}(X)$ consisting of equivalences of π onto itself. We have an obvious short exact sequence:

$$\{1\} \longrightarrow G \longrightarrow E(\pi) \longrightarrow \mathrm{Aut}(M, D),$$

where $Aut(M, D) = \{f \in Aut(M) \mid f^*D = D\}$.

Definition 7. 1. A Galois covering $\pi : X \longrightarrow M$ is said to be rigid, if $E(\pi) = Aut(X)$.

Equivalence Problem. Are which kinds of Galois coverings rigid?

This problem in the case of cyclic branched coverings of $\mathbb{P}^1$ was proposed by H. Shiga in Wakabayashi's problem session, Wakabayashi [29].

The second named author, Namba [21] showed that cyclic branched coverings of $\mathbb{P}^1$ are rigid under some conditions. Moreover, by making use of a theorem of Matsumura-Monsky, he proved that an m-fold cyclic covering $\pi : X \longrightarrow \mathbb{P}^n$ branching along a non-singular hypersurface of degree m in $\mathbb{P}^n$ is rigid, provided that

(i) $m \geqslant 4$, if $n = 1$,

(ii) $m \geqslant 3$, if $n \geqslant 2$, and

(iii) $(m, n) \neq (4, 2)$.

T. Kato [19] improved the results of Namba in the case of cyclic branched coverings of $\mathbb{P}^1$.

Let $L = L_1 + \cdots + L_s$ be a reduced divisor of $\mathbb{P}^n$ consisting of s distinct hyperplanes $L_1, \cdots, L_s$, which will be refered to as a hyperplane configuration of $\mathbb{P}^n$.

A Kummer covering

$$\pi : (G, X) \longrightarrow (\mathbb{P}^n : mL)$$

of $\mathbb{P}^n$ branching along mL is nothing but a branched covering

obtained as the Fox completion of a covering spread
$X_o \longrightarrow \mathbb{P}^n - L \subset \mathbb{P}^n$ associated with a $\mathbb{Z}/m\mathbb{Z}$-Hurewicz homomorphism

$$\pi_1(\mathbb{P}^n - L, *) \longrightarrow H_1(\mathbb{P}^n - L ; \mathbb{Z}) \longrightarrow H_1(\mathbb{P}^n - L ; \mathbb{Z}/m\mathbb{Z}).$$

Thus

$$G \simeq H_1(\mathbb{P}^n - L ; \mathbb{Z}/m\mathbb{Z}) = (\mathbb{Z}/m\mathbb{Z})^{s-1}$$

and G is generated by covering transformations $g_1, \cdots, g_s$ corresponding to the normal loops $\gamma_1, \cdots, \gamma_s$ of $L_1, \cdots, L_s$, respectively.

We are interested in the case where $n = 2$.

Let q be an r-ple point of L ; $q = L_{i_1} \cap \cdots \cap L_{i_r}$.

Let

$$\phi : B_q(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$$

be the blowing up of $\mathbb{P}^2$ at q. Then $\phi^{-1}(q) = E$ is a non-singular rational curve and we have a reduced divisor

$$p_1 + \cdots + p_r$$

on E, where

$$p_k = \overline{(\phi^* L_k - E)} \cap E$$

for $k = 1, \cdots, r$.

<u>Definition 7. 2.</u> If a Kummer covering of E branching along $m(p_1 + \cdots + p_r)$ is rigid, then $(\mathbb{P}^2 : mL)$ is said to be <u>rigid at</u> q. We shall say that $(\mathbb{P}^2 : mL)$ is <u>locally rigid</u>, if for each r-ple $(r \geqslant 4)$ point q of L, $(\mathbb{P}^2 : mL)$ is rigid at q.

In M. Kato [18], the first named author proved essentially

<u>Theorem 7. 3.</u> Let $\pi : (G, X) \longrightarrow (\mathbb{P}^2 : mL)$ and $\pi' : (G', X') \longrightarrow (\mathbb{P}^2 : mL')$ be kummer coverings of $\mathbb{P}^2$ such that L and L' are line configurations of $\mathbb{P}^2$. Suppose that

(1) $m \geqslant 6$,

(2) each L_j contains at least three singular points of L and

(3) $(\mathbb{P}^2 : mL)$ is locally rigid.

If a biholomorphism $h : X \longrightarrow X'$ exists, then $h : \pi \approx \pi'$. In particular, $\pi : X \longrightarrow \mathbb{P}^2$ is rigid.

Since the Kummer covering $\pi : (G, X) \longrightarrow (\mathbb{P}^n : mL)$ is an abelian mL-universal covering, it follows that a natural homomorphism

$$E(\pi) \longrightarrow \mathrm{Aut}(\mathbb{P}^n, L) \; (= \mathrm{Aut}(\mathbb{P}^n, mL), \; (m > 0))$$

is surjective. Thus we have

<u>Corollary 7. 4.</u> Under the assumption of Theorem 7. 3, we have a short exact sequence:

$$\{1\} \longrightarrow G(\simeq (\mathbb{Z}/m\mathbb{Z})^{s-1}) \longrightarrow \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(\mathbb{P}^2, L) \longrightarrow \{1\}.$$

The following results about rigidity of a Kummer covering $\pi : X \longrightarrow \mathbb{P}^1$ branching along $m(p_1 + \cdots + p_s)$ are known:

<u>Theorem 7. 5.</u> (1) if $\chi(X) \geqslant 0$, i.e., either $s = 2$ or $s = 3$ and $m \leqslant 3$, then π is not rigid.

(2) if $s = 3$ and $m \geqslant 4$, then π is rigid (see Namba [21]).

(3) if $s \geqslant 4$ and $m \geqslant 5(s - 1)$, then π is rigid (see M. Kato [18]).

Therem 7. 6. (Sakurai-Suzuki [26]). Suppose that $\chi(X) < 0$, $s \geqslant 4$ and that for any subset P' of $\{p_1, \cdots, p_s\}$ with $\#P' \geqslant 4$, $\mathrm{Aut}(\mathbb{P}^1, p') = \{1\}$. Then π is rigid.

Remark 7. 7. Recently, Sakurai is improving the result above extensively. He has announced in February, 1987, that π is rigid, if $\chi(X) < 0$ and $m \geqslant 11$. It is plausible that π is rigid, if $\chi(X) < 0$, i. e., Aut(X) is finite.

The proof of Theorem 7. 3 is based on the following facts:

(I) If X is a surface of general type, then Aut(X) is finite.

(II) The covering transformation group G is generated by 'complex reflections' $g_1, \cdots, g_s$ of the surface X.

(III) If a finite unitary reflection group of $\mathbb{C}^2$ contains a unitary reflection of order $\geqslant 6$, then if is abelian, refer to Shephard-Todd [28].

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Department of Mathematics
Kyushu University, Fukuoka, Japan

and

Mathematical Institute,
Tohoku University, Sendai, Japan