数理解析研究所講究録
第 630 巻 1987 年 1-8



# Long Cycles through Specified Vertices in a Graph

*Akira Saito*
(斎藤 明)
Department of Electrical Communications
Tohoku University
Sendai, Miyagi 980
JAPAN

*ABSTRACT*

In this paper, we consider the length of the longest cycle through specified vertices. We show the following two results. (1) Let $G$ be a $k$-connected graph of order at least $2k$ and circumference $l$. Suppose $m < k$. Then for any $m$ vertices of $G$, $G$ has a cycle which contains all of them and has length at least $\frac{k-m}{k}l + 2m$. (2) Let $G$ be a 3-connected planar graph with circumference $l$. Then for any three vertices of $G$, there exists a cycle which contains all of them and has length at least $\frac{1}{4}l + 3$.

Here, we consider finite simple graphs. Let $G$ be a graph. By Dirac's theorem[3] $G$ has a cycle through specified $k$ vertices. In [2] Dirac also showed that a 2-connected graph of order $n$ and minimum degree at least $d$ has a cycle of length at least $\min\{n, 2d\}$. Locke[4] and Voss[7] generalized his result by showing that under the same conditions the graph has a cycle of length at least $\min\{n, 2d\}$ which contains specified two vertices.

These results lead us to the following question: Does a $k$-connected graph have a long cycle through specified $m$ vertices ($m \leq k$)? In this paper we investigate this question.

For basic graph-theoretic terminology, we refer the reader to [1]. Let $G$ be a graph. The *circumference* of $G$, denoted by $\mathrm{cir}(G)$, is the length of the longest cycle of $G$. We denote by $w(G)$ the number of components of $G$. For $k \geq 0$ and $S \subset V(G)$, we call $S$ a $k$-cutset if $w(G - S) \geq 2$ and $|S| = k$. We often identify a subgraph $H$ of $G$ with its vertex set $V(H)$. Especially, when $x$ is a vertex of $H$, we write $x \in H$ instead of $x \in V(H)$. Furthermore, we write $|H|$ instead of $|V(H)|$. When we consider a cycle, we always give it an orientation. Let $C^+$ be the orientation of a cycle $C$ and $C^-$ be its reverse orientation. Let $C^+ = x_0, x_1, \ldots, x_{n-1}, x_n$ be a cycle. For $x_i$, $x_j \in C$, we define a subpaths $C^+[x_i, x_j]$ and $C^-[x_i, x_j]$ of $C$ by

$$C^+[x_i, x_j] = x_i, x_{i+1}, \ldots, x_{j-1}, x_j,$$

and

$$C^-[x_i, x_j] = x_i, x_{i-1}, \dots x_{j+1}, x_j.$$

We also define $C^+(x_i, x_j)$ and $C^-(x_i, x_j)$ by

$$C^+(x_i, x_j) = C^+[x_i, x_j] - \{x_i, x_j\},$$

and

$$C^-(x_i, x_j) = C^-[x_i, x_j] - \{x_i, x_j\}.$$

Furthermore, $C^+[x_i, x_j) = C^+[x_i, x_j] - \{x_j\}$. Subpaths $C^-[x_i, x_j)$, $C^+(x_i, x_j]$, $C^-(x_i, x_j]$ are defined similarly. Let $x_1, x_2, \dots, x_s$ be a path. We denote by $\mathrm{end}(P)$ the set of endvertices of $P$; $\mathrm{end}(P) = \{x_1, x_s\}$. Let $P = x_1, x_2, \dots, x_s$ and $Q = y_1, y_2, \dots, y_t$ be paths such that $x_s = y_1$. We denote by $P \cdot Q$ the walk $x_1, x_2, \dots, x_s = y_1, y_2, \dots, y_t$.

Let $z \in V(G)$ and $S \subset V(G) - \{z\}$. A subgraph $F$ of $G$ is called a $(z, S)$-*fan* if $F$ has the following decomposition $F = \cup_{i=1}^{k} P_i$, where
(1) each $P_i$ is a path between $z$ and $a_i \in S$, and
(2) $P_i \cap S = \{a_i\}$, and $P_i \cap P_j = \{z\}$ if $i \neq j$.
We call $k$ the *size* of the fan $F$. The vertices $a_1, \dots, a_k$ are called endvertices of $F$ and the set of its endvertices is denoted by $\mathrm{end}(F)$. Since $F$ is a tree, for any two vertices $x$, $y \in F$ the path in $F$ which joins $x$ and $y$ is unique. We denote this path by $F[x, y]$. We define $F[x, y)$ by $F[x, y) = F[x, y] - \{y\}$. Paths $F(x, y]$ and $F(x, y)$ are defined similarly.

The following theorem is well-known, called the generalized Menger's theorem.

**THEOREM A** ([1, Theorem 6.7]). *Let $G$ be a $k$-connected graph, $z \in V(G)$, and $S \subset V(G) - \{z\}$. Then $G$ has a $(z, S)$-fan of size* $\min\{|S|, k\}$. ■

The following theorem was proved by Perfect[5].

**THEOREM B** (Perfect[5]). *Let $G$ be a graph, $z \in V(G)$, and $S \subset V(G) - \{z\}$. Suppose $G$ has two $(z, S)$-fans $F_1$ and $F_2$ of size $k_1$ and $k_2$, respectively. If $k_1 \leq k_2$, then $G$ has a $(z, S)$-fan $F'$ of size $k_2$ such that* $\mathrm{end}(F_1) \subset \mathrm{end}(F')$. ■

We use these two theorems in the proofs our results.

First, we show that the existence of long cycles through specified $m$ vertices in a $k$-connected graph is assured if $m < k$. Note that a $k$-connected graph is hamiltonian if its order is at most $2k$, by Dirac's theorem.

**Theorem** 1. *Let $k \geq 2$, $0 \leq m \leq k$ and $G$ be a $k$-connected graph of order at least $2k$. For any $m$ vertices $x_1, \dots, x_m$ of $G$, there exists a cycle such that*
*(1) $x_1, \dots, x_m \in V(C)$, and*
*(2) $|C| \geq \frac{k-m}{k}\operatorname{cir}(G) + 2m$.*

Recently, Seymour and Truemper sent me a proof which is simpler than the original one. We show their proof.

**Proof** (due to Seymour and Truemper). The proof is by induction on $m$. For $m = 1$, let $x \in V(G)$, and let $C$ be a longest cycle in $G$. Since $|C| \geq 2k$,

$$\frac{k-1}{k}\operatorname{cir}(G) + 2 = |C| - \frac{|C|}{k} + 2 \leq |C|.$$

So we may assume $x \notin V(C)$. Now $G$ has an $(x, C)$-fan of size $k$. The endvertices of $F$ divide $C$ into $k$ paths, and any shortest one $P$ of these paths, say $P = C^+[u, v]$ has length at most $\frac{1}{k}\operatorname{cir}(G)$. So $C^+[v, u] \cdot F[u, v]$ is a cycle which contains $x$ and has length at least

$$|C| - \frac{\operatorname{cir}(G)}{k} + 2 = \frac{k-1}{k}\operatorname{cir}(G) + 2$$

as desired.

Suppose $m > 1$, and let $C$ be a longest cycle containing at least $m - 1$ members of $S$. By the induction hypothesis,

$$\begin{aligned} |C| &\geq \frac{k-m+1}{k}\operatorname{cir}(G) + 2(m-1) \\ &= \frac{k-m}{k}\operatorname{cir}(G) + 2m + \frac{\operatorname{cir}(G)}{k} - 2 \\ &\geq \frac{k-m}{k}\operatorname{cir}(G) + 2m. \qquad (*) \end{aligned}$$

So we may assume that exactly one member $x$ of $S$ does not lie on $C$. Since $\operatorname{cir}(G) \geq 2k$, $|C| \geq 2k$. So $G$ has an $(x, C)$-fan of size $k$. The endvertices of $F$ divide $C$ into $k$ paths. We call such a path *bad* if it contains some member of $S$ internally, and we call it *good* if it is not bad. Let $b$ represent the number of bad paths, and let $L$ be the sum of lengths of the bad paths. Then some good path $P = C^+[u, v]$ has length at most

$$\frac{|C| - L}{k - b}$$

(, where $|C| \geq 2k$ and $k \geq m - 1$). Keeping $|C|$ and $k$ fixed, and under the conditions $L \geq 2b$ and $b \leq m - 1$, this is maximized when $L = 2b$ and $b = m - 1$. Hence,

$$|P| \leq \frac{|C| - 2(m-1)}{k - m + 1}.$$

A cycle $C^+[v,u] \cdot F[u,v]$ contains $S$, and from (*) it has length at least

$$|C| - \frac{|C| - 2(m-1)}{k-m+1} + 2 \geq \frac{k-m}{k}\mathrm{cir}(G) + 2m$$

as desired. ■

Theorem 1 is sharp. Let, $k \geq 2$, $s \geq 1$, and $0 \leq m \leq k$. Let $H_0, H_1, \ldots, H_k$ and $H_0'$ be graphs such that $H_1 \simeq \cdots \simeq H_k \simeq K_s$, $H_0 \simeq \overline{K_m}$ and $H_0' \simeq \overline{K_k}$. Suppose vertex sets $V(H_0), \ldots, V(H_k)$ and $V(H_0')$ are disjoint. Define $G(k,m,s)$ by $G(k,m,s) = (H_1 \cup \cdots \cup H_k \cup H_0) + H_0'$. Then $G(k,m,s)$ is $k$-connected, $|G(k,m,s)| = ks + k + m \geq 2k$, and $\mathrm{cir}(G(k,m,s)) = ks + k$. On the other hand, the length of the longest cycle through $V(H_0)$ is $(k-m)s + k + m$. The above example shows that large circumference does not assure the existence of long cycles through specified $k$ vertices in $k$-connected graphs.

Next, we confine ourselves to planar graphs. Even if we consider only planar graphs, the length of the longest cycle through specified two vertices in a 2-connected graph is independent of its circumference. Let $C = x_0, x_1, \ldots, x_m = x_0$ be a cycle of length $m$ ($m \geq 4$). Add a new vertex $y$ and join $yx_1$ and $yx_{m-1}$. Then this graph has circumference $m$, but the unique cycle through $y$ and $x_0$ has length four. On the other hand, by Tutte's theorem[6] 4-connected planar graphs are hamiltonian, and hence the length of the longest cycle through four specified vertices in a 4-connected planar graph is equal to its circumference. On a planar graph of connectivity three, we show the following theorem.

**THEOREM 2.** *Let $G$ be a 3-connected planar graph. Then any three vertices of $G$ lie on a cycle of length at least $\frac{1}{4}\mathrm{cir}(G) + 3$.*

The proof of Theorem 2 is given by the following two lemmas.

**LEMMA 1.** *Let $G$ be a 3-connected planar graph. Then for any two vertices $x$, $y$, there exists a cycle $C$ such that*
*(1) $x, y \in V(C)$.*
*(2) $|C| \geq \frac{1}{2}\mathrm{cir}(G) + 2$.*

**LEMMA 2.** *Let $G$ be a 3-connected planar graph, $x, y, z \in V(G)$ and $C$ be a cycle of $G$ such that $x, y \in V(C)$. Then there exists a cycle $C'$ such that*
*(1) $x, y, z \in V(C')$.*
*(2) $|C'| \geq \frac{1}{2}|C| + 2$.*

**Proof of Lemma 1.** If $G$ is hamiltonian, then the lemma clearly holds. So we may assume that $G$ is not hamiltonian, which implies $|G| \geq 7$ and $\mathrm{cir}(G) \geq 6$. Let $C$ be a longest cycle of $G$. We consider three cases.

*Case 1.* $\{x, y\} \subset V(C)$.

This case is trivial.

*Case 2.* $|\{x, y\} \cap V(C)| = 1$.

We may assume that $x \in V(C)$ and $y \notin V(C)$. Consider a $(y, C)$-fan $F$ of size three. Let $\mathrm{end}(F) = \{y_1, y_2, y_3\}$. If $x \in \{y_1, y_2, y_3\}$, say $x = y_1$, then we have two cycles $C^+[x, y_2] \cdot F[y_2, x]$ and $C^-[x, y_2] \cdot F[y_2, x]$, one of which has length at least $\frac{1}{2}|C| + 2 = \frac{1}{2}\mathrm{cir}(G) + 2$ and contains both $x$ and $y$. Next, assume $x \notin \{y_1, y_2, y_3\}$. We may assume $x \in C^+(y_3, y_1)$. Then one of the two cycles $C^+[y_3, y_2] \cdot F[y_2, y_3]$ and $C^-[y_1, y_2] \cdot F[y_2, y_1]$ has the desired properties.

*Case 3.* $\{x, y\} \cap V(C) = \emptyset$.

First, we show the following claims.

*Claim 1. Suppose there exists a path $P$ in $G$ such that*
*(1) $P$ joins two distinct vertices of $C$ and $P$ intersects $C$ only at its endvertices.*
*(2) $x, y \in V(P)$.*
*Then the Lemma follows.*

*Proof.* Let $a$ and $b$ be endvertices of $P$. Then one of the two cycles $P[a, b] \cdot C^+[b, a]$ and $P[a, b] \cdot C^-[b, a]$ satisfies the desired properties.

*Claim 2. Suppose there exist two paths $P$ and $Q$ such that*
*(1) $V(P) \cap V(Q) = \emptyset$.*
*(2) Both $P$ and $Q$ join two vertices of $C$.*
*(3) $V(P) \cap V(C) = \mathrm{end}(P)$ and $V(Q) \cap V(C) = \mathrm{end}(Q)$.*
*(4) Vertices of $\mathrm{end}(P)$ and vertices of $\mathrm{end}(Q)$ appear alternately around $C^+$.*
*(5) $x \in V(P)$ and $y \in V(Q)$.*
*Then the lemma follows.*

*Proof.* Let $\mathrm{end}(P) = \{x_1, x_2\}$ and $\mathrm{end}(Q) = \{y_1, y_2\}$. We may assume $x_1$, $y_1$, $x_2$ and $y_2$ appear in this order around $C^+$. Then one of the two cycles

$$C^+[x_1, y_1] \cdot Q[y_1, y_2] \cdot C^-[y_2, x_2] \cdot P[x_2, x_1]$$

and

$$C^-[x_1, y_2] \cdot Q[y_2, y_1] \cdot C^+[y_1, x_2] \cdot P[x_2, x_1]$$

has the desired properties.

Let $\mathrm{end}(F_1) = \{x_1, x_2, x_3\}$. We may assume that $x_1, x_2, x_3$ appear in this order around $C^+$. If $y \in V(F_1)$, then the theorem follows by Claim 1. Suppose $y \notin V(F_1)$. Let $D = C \cup F_1$. Let $F_2$ be a $(y, D)$-fan of size three. Let $\mathrm{end}(F_2) = \{y_1, y_2, y_3\}$. If $\mathrm{end}(F_2) \cap (F_1 - \{x_1, x_2, x_3\}) \neq \emptyset$, then the lemma follows by Claim 1. So we may assume $\mathrm{end}(F_2) \subset V(C)$.

*Claim 3. If $\{y_1, y_2, y_3\} \subset C^+[x_i, x_{i+1}]$ (If $i = 3$, we consider $x_4 = x_1$), then the lemma follows.*

*Proof.* We may assume $y_1, y_2, y_3 \in C^+[x_1, x_2]$ and $y_1, y_2$ and $y_3$ appear in this order around $C^+$. Then

$$C^+[x_3, y_1] \cdot F_2[y_1, y_2] \cdot C^+[y_2, x_2] \cdot F_1[x_2, x_3]$$

or

$$C^+[x_1, y_2] \cdot F_2[y_2, y_3] \cdot C^+[y_3, x_3] \cdot F_1[x_3, x_1]$$

has the desired properties.

By Claims 1, 2, 3, the only possible case in which the lemma would not hold is $\{x_1, x_2, x_3\} = \{y_1, y_2, y_3\}$. We may assume $x_i = y_i$ $(i = 1, 2, 3)$. Let $D' = D \cup F_2$. Since $C$ is a longest cycle, $C^+(x_1, x_2) \neq \emptyset$. Since $G$ is 3-connected, there exists a path $P$ joining $C^+(x_1, x_2)$ and $D' - C^+[x_1, x_2]$ in $G - \{x_1, x_2\}$. Let $\mathrm{end}(P) = \{u, v\}$, $u \in C^+(x_1, x_2)$ and $v \in D' - C^+[x_1, x_2]$. If $v \in V(F_1) \cup V(F_2)$, then the lemma follows by Claim 2. So we may assume $v \in C^+(x_2, x_3]$. Then $F_1$, $F_2$, $C^+[x_1, x_2]$ and $P[u, v] \cdot C^+[v, x_3]$ form a subdivision of $K_{3,3}$. This contradicts the planarity of $G$. Therefore, the lemma follows. ■

**Proof of Lemma 2.** Let $C_0$ be a longest cycle which contains $x$ and $y$. Then $|C_0| \geq |C|$. If $G$ is hamiltonian, then $C_0$ is a hamiltonian cycle, and $|C_0| \geq 4$. Hence the result follows. Threfore, we may assume $G$ is not hamiltonian, and $|G| \geq 7$. By Lemma 1, $|C_0| \geq \frac{1}{2} \cdot 7 + 2 \geq 5$. So $|C_0| \geq \frac{1}{2}|C_0| + 2 \geq \frac{1}{2}|C| + 2$. Hence we may assume $z \notin C_0$. Consider a $(z, C_0)$-fan $F_1$. Let $\mathrm{end}(F_1) = \{z_1, z_2, z_3\}$. We may assume that $z_1, z_2, z_3$ appear in this order around $C^+$. We consider three cases.

*Case 1.* $\mathrm{end}(F_1) \subset C_0^+[x, y]$ *or* $\mathrm{end}(F_1) \subset C_0^+[y, x]$.

We may assume $\{z_1, z_2, z_3\} \subset C_0^+[x, y]$. Then one of the two cycles $C_0^+[z_2, z_1] \cdot F_1[z_1, z_2]$ and $C_0^+[z_3, z_2] \cdot F_1[z_2, z_3]$ has the desired properties.

*Case 2. One of* $\text{end}(F_1)$ *lies on* $C_0^+(y,x)$ *and the other two lie on* $C_0^+(x,y)$.

We may assume $z_1, z_2 \in C_0^+(x,y)$ and $z_3 \in C_0^+(y,x)$. Let $C_1 = C_0^+[z_2,z_1] \cdot F_1[z_1,z_2]$. Then $C_0 - C_1 = C_0^+(z_1,z_2)$. Let $D = C_0 \cup F_1$. By Theorem B, there exists an $(x, D - C_0^+(z_3,z_1))$-fan $F_2$ of size three, such that $z_1$, $z_3 \in \text{end}(F_2)$. Let $\text{end}(F_2) = \{z_1, z_3, a\}$. If $a \in F_1[z,z_1)$ or $a \in F_1[z,z_2)$, let

$$C_2 = C_0^+[z_1,z_3] \cdot F_1[z_3,a] \cdot F_2[a,z_1].$$

If $a \in F_1[z,z_3)$, let

$$C_2 = C_0^+[z_1,z_3] \cdot F_2[z_3,a] \cdot F_1[a,z_1].$$

If $a \in C_0^+(z_2,y]$, let

$$C_2 = C_0^+[a,z_3] \cdot F_1[z_3,z_2] \cdot C_0^-[z_2,z_1] \cdot F_2[z_1,a].$$

If $a \in C_0^+(y,z_3)$, let

$$C_2 = C_0^-[a,z_1] \cdot F_1[z_1,z_3] \cdot F_2[z_3,a].$$

Then in either case, $C_0^+(z_1,z_2) \subset C_2$ and either $C_1$ and $C_2$ satisfies the desired properties. So the only remaining case is $a \in C_0^+(z_1,z_2]$. Let $D' = D \cup F_2$.

Next, consider a $(y, D' - C_0^+(z_2,z_3))$-fan $F_3$ such that $\{z_2,z_3\} \subset \text{end}(F_3)$. Let $\text{end}(F_3) = \{z_2,z_3,b\}$. If $b \in (F_1 - \text{end}(F_1)) \cup C_0^+(z_3,z_1)$, then the lemma follows by the same argument. If $b \in F_2(x,a) \cup F_2(x,z_1)$, let

$$C_3 = F_3[b,z_2] \cdot C_0^-[z_2,z_1] \cdot F_1[z_1,z_3] \cdot F_2[z_3,b].$$

If $b \in F_2(x,z_3)$, let

$$C_3 = F_3[b,z_3] \cdot F_1[z_3,z_2] \cdot C_0^-[z_2,z_1] \cdot F_2[z_1,b].$$

Then in either case $C_0^+(z_1,z_2) \subset C_3$ and hence either $C_1$ or $C_3$ satisfies the desired properties. So the lemma follows unless $b \in C_0^+[z_1,z_2)$. (Possibly $a = b$.)

Now we consider the case $a \in C_0^+(z_1,z_2)$ and $b \in C_0^+(z_1,z_2)$. If $z_1, b, a, z_2$ appear in this order around $C_0^+$, let

$$C_4 = F_3[z_3,b] \cdot C_0^+[b,z_2] \cdot F_1[z_2,z_1] \cdot C_0^-[z_1,z_3]$$

and

$$C_5 = F_2[z_3,a] \cdot C_0^-[a,z_1] \cdot F_1[z_1,z_2] \cdot C_0^+[z_2,z_3].$$

If $z_1, a, b, z_2$ appear in this order around $C^+$, let

$$C_4 = F_3[z_2, b] \cdot C_0^-[b, z_3] \cdot F_1[z_3, z_2]$$

and

$$C_5 = F_2[z_1, a] \cdot C_0^+[a, z_3] \cdot F_1[z_3, z_1].$$

Then in either case we have $\{x, y, z\} \subset C_4 \cap C_5$, $C_0 \subset C_4 \cup C_5$, and hence $|C_4| \geq \frac{1}{2}|C_0| + 2$ or $|C_5| \geq \frac{1}{2}|C_0| + 2$. So the lemma follows.

Now, we may assume that $a = z_2$ or $b = z_1$. If $a = z_2$, then $F_1$, $F_2$, $F_3$ and $C_0^-[b, z_1]$ form a subdivision of $K_{3,3}$. If $b = z_1$, then $F_1$, $F_2$, $F_3$ and $C_0^+[a, z_2]$ form a subdivision of $K_{3,3}$. Hence both contradicts the planarity of $G$. Therefore, the proof in this case is complete.

*Case 3.* $|\{x, y\} \cap \mathrm{end}(F_1)\}| = |C_0^+(x, y) \cap \mathrm{end}(F_1)| = |C_0^+(y, x) \cap \mathrm{end}(F_1)| = 1.$

We may assume $z_1 = x$, $z_2 \in C_0^+(x, y)$ and $z_3 \in C_0^+(y, x)$. Then either

$$C_6 = F_1[z_1, z_2] \cdot C_0^+[z_2, z_1], \quad \text{or}$$
$$C_7 = F_1[z_1, z_3] \cdot C_0^-[z_3, z_1]$$

satisfies the desired properties.

Therefore, in each case, $G$ has a cycle through $x$, $y$ and $z$ of length at least $\frac{1}{2}|C_0| + 2$. ■

## References


[1] M. Behzad, G. Chartrand and L. Lesniak-Foster, Graphs and Digraphs, Prindle, Weber & Shmidt, Boston MA (1979).

[2] G.A. Dirac, Some theorem on abstract graphs, *Proc. London Math. Soc.* **2** (1952) 69-81.

[3] G.A. Dirac, In abstrakten Graphen vorhandene vollstandige 4-Graphen und ihre unterteilungen, *Math. Nachr.* **22** (1960) 61-85.

[4] S.C. Locke, A generalization of Dirac's theorem, *Combinatorica* **5** (1985) 149-159.

[5] H. Perfect, Application of Menger's graph theorem, *J. Math. Annal. Appl.* **22** (1968) 96-111.

[6] W.T. Tutte, A theorem on planar graphs, *Trans. Amer. Math. Soc.* **82** (1956) 99-116.

[7] H.J. Voss, Bridges of longest circuits and of longest paths in graphs, *Beitrage zur Graphen-theorie und deren Anwendungen (Proceedings of the International Colloquim, Oberhog)* (1977) 275-286.