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Author(s)	Kazamaki, N.; Sekiguchi, T.
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A NOTE ON BMO-MARTINGALES

By N.Kazamaki and T.Sekiguchi

(風卷 紀彦、関口 健)

Toyama University

(富山大・理)

Let $(\Omega, \mathcal{F}, P; (\mathcal{F}_t))$ be a probability system where the filtration (\mathcal{F}_t) satisfies the usual conditions. We now consider two important subclasses of BMO, namely the class L^∞ of all bounded martingales and the class H^∞ of all martingales with bounded quadratic variation. There is not always an inclusion relation between these classes. In fact, if $B=(B_t)$ is a one dimensional Brownian motion starting at 0, then $(B_{t \wedge 1}) \in H^\infty \setminus L^\infty$ clearly. On the other hand, the process B stopped at τ , where $\tau = \min\{t: |B_t|=1\}$, belongs to $L^\infty \setminus H^\infty$ (see [2]).

Our aim is to prove the following.

THEOREM. Suppose the sample continuity of any martingale adapted to (\mathcal{F}_t) . Then the following properties are equivalent:

- (a) $BMO = L^\infty$.
- (b) $BMO = H^\infty$.
- (c) $\mathcal{F}_t = \mathcal{F}_0$ for every t .

Furthermore, excepting such a trivial case, $H^\infty \cup L^\infty$ is not dense in BMO.

Recall that BMO is the class of those uniformly integrable martingales M which satisfy $\|M\|_{BMO} = \sup_T \|E[(M_\infty - M_{T-})^2 | \mathcal{F}_T]^{1/2}\|_\infty < \infty$

where the supremum is taken over all stopping times T . It is well-known that BMO is Banach space with the norm $\|M\|_{\text{BMO}}$.

PROOF. Firstly, we shall establish the implication (a) \rightarrow (c). In order to see (c), it suffices to prove that for any martingale M , almost all sample functions of M are constant, and by using the usual stopping argument we may assume $M \in \text{BMO}$. Suppose now $\text{BMO} = L^\infty$. Then the two norms $\|M\|_{\text{BMO}}$ and $\|M\|_\infty$ on BMO are equivalent by the Closed Graph Theorem. So there exists a constant $C > 0$, depending only on M , such that $\|K \cdot M\|_\infty < C$ for any predictable process $K = (K_t, \mathcal{F}_t)$ with $|K| \leq 1$. Here $K \cdot M$ denotes the stochastic integral of K relative to M . We show below that the negation of (c) causes a contradiction. If we deny (c), then there exist $t > 0$ and a partition $\Delta: 0 = t(0) < t(1) < \dots < t(n) = t$ of $[0, t]$ such that $P(A) > 0$ where $A = \{\sum_{j=1}^n |M_{t(j)} - M_{t(j-1)}| > 2C\}$. Let now

$$B_{j, \varepsilon(j)} = \begin{cases} \{M_{t(j)} - M_{t(j-1)} \geq 0\} & \text{if } \varepsilon(j) = 1, \\ \{M_{t(j)} - M_{t(j-1)} < 0\} & \text{if } \varepsilon(j) = -1 \end{cases}$$

for $j = 1, 2, \dots, n$. Since $A = \bigcup_{1 \leq j \leq n, \varepsilon(j) = \pm 1} A \cap B_{1, \varepsilon(1)} \cap \dots \cap B_{n, \varepsilon(n)}$, we have for some $\varepsilon^*(j)$ ($1 \leq j \leq n$)

$$P(A \cap B_{1, \varepsilon^*(1)} \cap \dots \cap B_{n, \varepsilon^*(n)}) > 0.$$

Then the process K defined by $K_s = \sum_{j=1}^n \varepsilon^*(j) I_{]t(j-1), t(j)]}(s)$ is a predictable process with $|K| \leq 1$, so that $\|K \cdot M\|_\infty \leq C$ must follow. On the contrary, we find

$$(K \cdot M)_t = \sum_{j=1}^n |M_{t(j)} - M_{t(j-1)}| > 2C$$

on the set $A \cap B_{1,\varepsilon^*(1)} \cap \dots \cap B_{n,\varepsilon^*(n)}$. Thus (a) implies (c). The implication (c) \rightarrow (b) is trivial. Finally, to prove the implication (b) \rightarrow (a), let us suppose $BMO \neq L^\infty$. Then by the result of Dellacherie, Meyer and Yor L^∞ is not dense in BMO, and further Kazamaki and Shiota have recently shown in [2] that the BMO-closure of L^∞ contains H^∞ . Therefore, combining these results, we have $BMO \neq H^\infty$. That is, the contraposition of the implication (b) \rightarrow (a) is established. The latter half of the theorem follows at the same time. This completes the proof.

We now exemplify that H^∞ as well as L^∞ is not always closed in BMO under the same assumption as in the theorem. For that, consider the identity mapping S of R_+ onto R_+ . Let μ be the probability measure on R_+ defined by $\mu(S \in dx) = \sqrt{2/\pi} e^{-x^2/2} dx$ and \mathcal{G}_t be the μ -completion of the Borel field generated by $S \wedge t$. Then $(R_+, \mathcal{G}, \mu; (\mathcal{G}_t))$, where $\mathcal{G} = \bigvee_{t \geq 0} \mathcal{G}_t$, is a probability system. Clearly, S is a stopping time over (\mathcal{G}_t) . We next consider in the usual way a probability system $(\Omega, \mathcal{F}, P; (\mathcal{F}_t))$ by taking the product of the system $(R_+, \mathcal{G}, \mu; (\mathcal{G}_t))$ with another system $(\Omega', \mathcal{F}', P'; (\mathcal{F}'_t))$ which carries a one dimensional Brownian motion $B = (B_t)$ starting at 0. Then the filtration (\mathcal{F}_t) satisfies the usual conditions and S is also a stopping time over this filtration. Let M denote the process B stopped at S . It is a continuous martingale over (\mathcal{F}_t) such that $\langle M \rangle_t = t \wedge S$ where $\langle M \rangle$ denotes the continuous increasing process associated with M . We first verify $M \in BMO$. Since $\{S > t\}$ is an \mathcal{F}_t -atom, we have

$$E[\langle M \rangle_\infty - \langle M \rangle_t | \mathcal{F}_t] = E[S-t | \mathcal{F}_t] \mathbb{1}_{\{t < S\}}$$

$$\leq \left(\int_t^\infty e^{-x^2/2} dx \right)^{-1} \left(\int_t^\infty (x-t) e^{-x^2/2} dx \right),$$

which converges to 0 as $t \rightarrow \infty$. That is, there is a constant $C > 0$ such that $E[\langle M \rangle_\infty - \langle M \rangle_t | \mathcal{F}_t] \leq C$ for every t . In our setting, this yields that $E[\langle M \rangle_\infty - \langle M \rangle_T | \mathcal{F}_T] \leq C$ for every stopping time T . Thus $M \in \text{BMO}$. As a matter of course, we have $M \in H^\infty$. Secondly, let $M^{(n)} = B^{n \wedge S}$ ($n=1, 2, \dots$). Then $M^{(n)} \in H^\infty$. Since $\langle M^{(n)} \rangle_t - \langle M \rangle_t = t \wedge S - t \wedge n \wedge S$, we find

$$E[\langle M^{(n)} \rangle_\infty - \langle M^{(n)} \rangle_t | \mathcal{F}_t]$$

$$\leq \left(\int_{t \vee n}^\infty e^{-x^2/2} dx \right)^{-1} \left(\int_{t \vee n}^\infty (x - t \vee n) e^{-x^2/2} dx \right),$$

from which $M^{(n)}$ converges in BMO to M as $n \rightarrow \infty$. Consequently, $M \in \overline{H^\infty} \setminus H^\infty$.

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