京都大学学術情報リポジトリ
KURENAI 紅
Kyoto University Research Information Repository

京都大学
KYOTO UNIVERSITY

Title	Counterexamples to Termination for the Direct Sum of Term Rewriting Systems
Author(s)	TOYAMA, Yoshihito
Citation	数理解析研究所講究録 (1987), 625: 242-246
Issue Date	1987-05
URL	http://hdl.handle.net/2433/99946
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

Kyoto University

数理解析研究所講究録
第 625 巻 1987 年 242-246

項書き換えシステムの直和の停止性

Counterexamples to Termination for the Direct Sum of Term Rewriting Systems

外山芳人 (NTT 通研)

Yoshihito TOYAMA
NTT Electrical Communications Laboratories
3-9-11 Midori-cho, Musashino-shi, Tokyo 180 Japan

Abstract

The direct sum of two term rewriting systems is the union of systems having disjoint sets of function symbols. It is shown that the direct sum of two term rewriting systems is not terminating, even if these systems are both terminating.

Keyword: Term rewriting system, termination

Introduction

A term rewriting system R is a set of rewriting rules $M \to N$, where M and N are terms [1,3,5]. The direct sum system $R_1 \oplus R_2$ is defined as the union of two term rewriting systems with disjoint function symbols [8]. It was proved [8] that for any term rewriting systems R_1 and R_2,

$R_1 \oplus R_2$ is confluent iff R_1 and R_2 are confluent.

By replacing *confluent* with *terminating* in the above proposition, the analogous conjecture for the terminating property has the form:

$R_1 \oplus R_2$ is terminating iff R_1 and R_2 are terminating.

However, the answer to this conjecture is negative against our expectation. We show the counterexamples to this conjecture and its modifications.

Counterexamples

A counterexample to the above conjecture is obtained by R_1 and R_2 having the following rewriting rules [8]:

$$R_1 \quad \left\{ F(0,1,x) \to F(x,x,x) \right.$$

$$R_2 \quad \left\{ \begin{array}{l} G(x,y) \to x \\ G(x,y) \to y \end{array} \right.$$

It is trivial that R_1 and R_2 are terminating. However, $R_1 \oplus R_2$ is not terminating, because $R_1 \oplus R_2$ has the infinite reduction sequence:

$$F(G(0,1),G(0,1),G(0,1)) \to F(0,G(0,1),G(0,1)) \to F(0,1,G(0,1))$$
$$\to F(G(0,1),G(0,1),G(0,1)) \to \cdots.$$

This counterexample also provides a negative answer to the same question for the direct sum of recursive program schemes suggested by Klop [6].

Dershowitz showed the following theorem [1,2,3] for terminating of the union system:

> *Theorem (Dershowitz 1981). Let R_1 and R_2 be two term rewriting systems. Suppose that R_1 is left linear, and R_2 is right linear, and there is no overlap between the left-hand sides of R_1 and the right-hand sides of R_2. Then, the union of the two systems is terminating iff both R_1 and R_2 are terminating.*

However, Dershowitz's Theorem [2,3] is not correct, because the above counterexample refutes his theorem[1].

In this counterexample, note that R_2 is not confluent. Hence, Toyama conjectured that under the assumption of confluence for R_1 and R_2, $R_1 \oplus R_2$ is terminating iff R_1 and R_2 are terminating [8]. Since the direct sum of two term rewriting

[1]The version of Dershowitz's Theorem in [1] is correct since the definition of overlap in [1] is different from it in [2,3]. However, the examples in [1] are wrong, since the definition of overlap is in [2,3]. This remark is the basis of the letters from Leo Bachmair (on October 24, 1986) and from Nachum Dershowitz (on November 11, 1986).

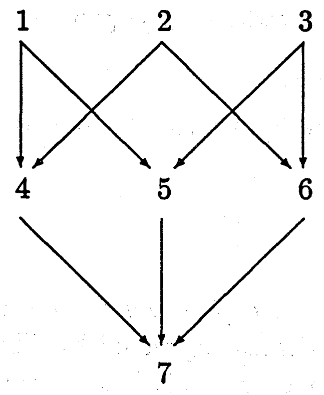
systems always preserves their confluence, this conjecture can be stated by the form:

$R_1 \oplus R_2$ is canonical iff R_1 and R_2 are canonical,

where canonical means confluent and terminating.

However, this conjecture is also not true. Klop and Barendregt showed a counterexample [7] by extending Toyama's counterexample. Consider R_1 and R_2 having the following rewriting rules:

$$R_1 \begin{cases} F(4,5,6,x) \to F(x,x,x,x) \\ F(x,y,z,w) \to 7 \end{cases}$$



$$R_2 \begin{cases} G(x,x,y) \to x \\ G(x,y,x) \to x \\ G(y,x,x) \to x \end{cases}$$

Then, R_1 is confluent, because any term can be reduced into 7. R_1 is also terminating; no term can be reduced into 4, 5, and 6, hence, the first rule cannot be applied infinitely. Thus, R_1 is canonical. Clearly, R_2 is canonical.

However, $R_1 \oplus R_2$ is not canonical, since $F(t,t,t,t)$ with $t \equiv G(1,2,3)$ reduces to itself:

$$F(t,t,t,t) \to \cdots \to F(G(4,4,3), G(5,2,5), G(1,6,6), t) \to \cdots$$

$$\to F(4,5,6,t) \to F(t,t,t,t) \to \cdots.$$

We say R is irreducible if for any rule $M \to N$ in R, M and N are normal forms in $R - \{M \to N\}$. In Klop and Barendregt's counterexample, R_1 is not irreducible, since the left-hand side $F(4,5,6,x)$ and the right-hand side $F(x,x,x,x)$ of the first rule can be reduced by using other rules. Hence, Hsiang conjectured [4] that for irreducible term rewriting systems R_1 and R_2, $R_1 \oplus R_2$ is canonical iff R_1 and R_2 are canonical. Clearly, the direct sum of two systems always preserves their irreduciblility. Hence, Hsiang's conjecture can be shown in the form:

$R_1 \oplus R_2$ is canonical and irreducible iff R_1 and R_2 are canonical and irreducible.

However, Hsiang's conjecture is also not true. We can find the following counterexample to his conjecture by extending Klop and Barendregt's counterexample. Let R_1 and R_2 have the following rewriting rules:

$$R_1 \left\{ \begin{array}{l} F(f_4(x,x), f_5(x,x), f_6(x,x), y, x) \to F(y,y,y,y,x) \\ F(x,y,z,u,0) \to 1 \\ f_1(0,x) \to f_4(0,x) \\ f_1(x,0) \to f_5(x,0) \\ f_2(0,x) \to f_4(0,x) \\ f_2(x,0) \to f_6(x,0) \\ f_3(0,x) \to f_5(0,x) \\ f_3(x,0) \to f_6(x,0) \\ f_4(0,0) \to 1 \\ f_5(0,0) \to 1 \\ f_6(0,0) \to 1 \end{array} \right.$$

$$R_2 \left\{ \begin{array}{l} G(x,x,y) \to x \\ G(x,y,x) \to x \\ G(y,x,x) \to x \end{array} \right.$$

Then, we can show that R_1 and R_2 are canonical and irreducible. However, $R_1 \oplus R_2$ is not canonical, since $F(t,t,t,t,0)$ with $t \equiv G(f_1(0,0), f_2(0,0), f_3(0,0))$ reduces to itself:

$$F(t,t,t,t,0) \to \cdots \to$$

$$F(G(f_4(0,0), f_4(0,0), f_3(0,0)), G(f_5(0,0), f_2(0,0), f_5(0,0)), \\ G(f_1(0,0), f_6(0,0), f_6(0,0)), t, 0) \to \cdots$$

$$\to F(f_4(0,0), f_5(0,0), f_6(0,0), t, 0) \to F(t,t,t,t,0) \to \cdots.$$

Acknowledgment

The author would like to thank Jan Willem Klop and Henk Pieter Barendregt for the counterexample to Toyama's conjecture, Jieh Hsiang for pointing out a mistake in the original counterexample proposed by Klop and Barendregt and for Hsiang's conjecture, and Leo Bachmair for informing that Dershowitz's theorem is refuted by Toyama's counterexample.

References

[1] L. Bachmair and N. Dershowitz. Commutation, transformation, and termination, Lecture Notes in Comput. Sci. 230 (Springer-Verlag, 1986) 5-20.

[2] N. Dershowitz. Termination of linear rewriting systems: Preliminary version, Lecture Notes in Comput. Sci. 115 (Springer-Verlag, 1981) 448-458.

[3] N. Dershowitz. Termination, Lecture Notes in Comput. Sci. 220 (Springer-Verlag, 1985) 180-224.

[4] J. Hsiang. Private communication (July 28, 1986).

[5] G. Huet and D. C. Oppen. Equations and rewrite rules: a survey, in: R. Book, ed., Formal languages: perspectives and open problems (Academic Press, 1980) 349-393.

[6] J. W. Klop. Combinatory reduction systems, Dissertation, Univ. of Utrect (1980).

[7] J. W. Klop and H. P. Barendregt, Private communication (January 19, 1986).

[8] Y. Toyama. On the Church-Rosser property for the direct sum of term rewriting systems, to appear in J. ACM.