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## CONSTRUCTIONS FOR CYCLIC STEINER 2-DESIGNS

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### ABSTRACT

This paper surveys direct and recursive constructions for cyclic Steiner 2-designs. A new method is presented for cyclic designs with blocks having a prime number of elements. Several new constructions are given for designs with block size 4 which are based on perfect systems of difference sets and additive sequences of permutations.

### 1. Introduction

A *balanced incomplete block design* (briefly BIBD) with parameters  $(v, k, \lambda)$  is a pair  $(V, B)$  where  $V$  is a  $v$ -set and  $B$  is a collection of  $k$ -subsets of  $V$  (called *blocks*) such that every 2-subset of  $V$  is contained in exactly  $\lambda$  blocks. A *Steiner 2-design* is a  $(v, k, \lambda)$  BIBD with  $\lambda = 1$ . An *automorphism* of a BIBD  $(V, B)$  is a bijection  $\phi: V \rightarrow V$  such that the induced mapping  $\Phi: B \rightarrow B$  is also a bijection. The set of all such mappings forms a group under composition called the automorphism group of the design.

A  $(v, k, \lambda)$  BIBD is *cyclic* if it has an automorphism consisting of a single cycle of length  $v$ . Cyclic  $(v, k, \lambda)$  BIBD's will be denoted by  $C(v, k, \lambda)$ . A  $(v, k, \lambda)$  *difference family* (briefly DF) is a collection of  $k$ -subsets  $D_1, \dots, D_t$  of the integers  $Z_v$  modulo  $v$  such that for each nonzero  $x \in Z_v$ , the congruence  $d_i - d_j \equiv x \pmod{v}$  has exactly  $\lambda$  solution pairs  $(d_i, d_j)$  with  $d_i, d_j \in D_l$ , for some  $l$ . A  $(v, k, \lambda)$  DF is called *simple* if  $\lambda = 1$ . It is easily verified that a necessary condition for the existence of a  $(v, k, \lambda)$  DF is  $\lambda(v-1) \equiv 0 \pmod{k(k-1)}$ . In particular, if a simple DF exists then  $v \equiv 1 \pmod{k(k-1)}$ . A  $(v, k, \lambda)$  DF generates a cyclic BIBD  $C(v, k, \lambda)$  with  $V = Z_v$  and  $B = \{\sigma^i D_l \mid 0 \leq i < v, 1 \leq l \leq t\}$ , where  $\sigma: V \rightarrow V, \sigma(x) = x + 1 \pmod{v}$  and  $n = \lambda(v-1)/(k(k-1))$ . The  $t$  blocks  $D_1, \dots, D_t$  are called *starter* or *base blocks* of the design  $(V, B)$  (they are representatives of the orbits of  $B$  under  $\sigma$ ). An orbit analysis of a cyclic Steiner 2-design  $C(v, k)$  yields the following necessary existence condition:

$$v \equiv 1, k \pmod{k(k-1)}. \tag{1}$$

The case  $v = k(k-1)t + 1$  corresponds to a simple DF. If  $v = k(k-1)t + k$  then there are  $t + 1$  starter blocks  $D_0, D_1, \dots, D_t$ , where  $D_0 = \{0, m, 2m, \dots, (k-1)m\}$ ,  $m = (k-1)t + 1$

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generates a  $m$ -orbit and  $D_1, \dots, D_t$  generate  $t$   $v$ -orbits under  $\sigma$ . It is clear, that the differences in  $D_1, \dots, D_t$  cover the elements  $Z_v \setminus D_0$  exactly once.

Two difference families  $D = \{D_1, \dots, D_t\}$  and  $D' = \{D'_1, \dots, D'_t\}$  are said to be *equivalent* if for some integers  $r, s_1, \dots, s_t$

$$\{D'_1, \dots, D'_t\} = \{rD_1 + s_1, \dots, rD_t + s_t\} \pmod{v}. \quad (2)$$

If  $D$  is equivalent with itself, then the corresponding  $r$  is called a *multiplier* of  $D$  and  $\tau: x \rightarrow rx, x \in Z_v$  is an automorphism of the cyclic design.

Cyclic designs have a nice structure and interesting algebraic properties. Their concise representation makes them attractive in applications and for testing purposes. Cyclic BIBD's and difference systems have been studied by many authors [3], [7], [10], [13]. Results concerning cyclic Steiner 2-designs are surveyed in [5] which also contains a fairly extensive bibliography.

The present paper addresses the problem of existence of cyclic Steiner 2-designs  $C(v, k, 1)$ . In the next two sections we discuss direct and recursive constructions for general block sizes  $k$ . In addition to known techniques, several new constructions are presented for  $k = 4$  and  $5$ . We conclude with a list of open problems. The paper significantly extends the existence results given in [5] for cyclic Steiner 2-designs with block sizes  $k > 3$ .

## 2. Direct Constructions

The majority of direct methods for constructing cyclic designs are based on finite fields. In this section we survey those constructions which apply to Steiner 2-designs and apply them to generate some new designs with blocks of prime size.

We begin with two general constructions of Wilson for  $(v, k, 1)$  difference families [13].

**Theorem 1** Let  $p = k(k-1)t + 1$  be a prime and  $\alpha$  a primitive root of  $Z_p$ . Let  $H^m$  be the multiplicative subgroup of  $Z_p \setminus \{0\}$  generated by  $\alpha^m$  and let  $\omega = \alpha^{2mt}$ .

- (i) If  $k = 2m + 1$  is odd and  $\{\omega^{-1}, \omega^2 - 1, \dots, \omega^m - 1\}$  is a system of representatives for the cosets  $\alpha^i H^m, i = 0, 1, \dots, m-1$ , then the blocks  $D_{i+1} = \{\alpha^{mi}, \omega \alpha^{mi}, \dots, \omega^{2m} \alpha^{mi}\}, i = 0, 1, \dots, t-1$  form a  $(p, k, 1)$  DF.
- (ii) If  $k = 2m$  is even and  $\{1, \omega - 1, \dots, \omega^{m-1} - 1\}$  is a system of representatives for the cosets  $\alpha^i H^m, i = 0, 1, \dots, m-1$ , then the blocks  $D_{i+1} = \{0, \alpha^{mi}, \omega \alpha^{mi}, \dots, \omega^{2m-2} \alpha^{mi}\}, i = 0, 1, \dots, t-1$  form a  $(p, k, 1)$  DF in  $Z_p$ .

**Theorem 2** Let  $p = k(k-1)t + 1$  be a prime and  $\alpha$  a primitive root of  $Z_p$ . If there exists a set  $B = \{b_1, \dots, b_k\} \subset Z_p$  such that  $\{b_j - b_i \mid 1 \leq i < j \leq k\}$  is a system of representatives for the cosets  $\alpha^i H^m, i = 0, 1, \dots, m-1$ , where  $m = k(k-1)/2$  and  $H^m$  is the subgroup of  $Z_p \setminus \{0\}$  generated by  $\alpha^m$ , then  $D_{i+1} = \alpha^{2mi} B, i = 0, 1, \dots, t-1$  is a  $(p, k, 1)$  DF in  $Z_p$ .

Our next result concerns the case  $v \equiv k \pmod{k(k-1)}$ .

**Theorem 3** Let  $k = 2m + 1$  and  $p = 2mt + 1$ ,  $n \geq 2$  be two odd primes and let  $\alpha$  be a primitive root of  $Z_p$ . Define  $m - 1$  numbers  $r_i$  by the equations  $\alpha^{r_i} = \alpha^{t_i} - 1$ ,  $i = 1, \dots, m - 1$ . If there exists a  $\beta \in Z_k$  such that the  $2m$  elements  $\pm 1, \pm(\beta^{t_i} - 1)\beta^{-r_i}$ ,  $i = 1, \dots, m - 1$  are all distinct in  $Z_k$ , then the blocks

$$\begin{aligned} D_0 &= \{0_0, 0_1, \dots, 0_{2m}\} \\ D_{i+1} &= \{0_0, \alpha_{\beta^{t_i}}, \alpha_{\beta^{t_i}+1}, \dots, \alpha_{2mt-t+i}\beta^{t_i}\}, \quad i = 0, 1, \dots, t - 1 \end{aligned} \quad (3)$$

form a  $(kp, k, 1)$  DF in  $Z_{kp}$ .

**Proof** We note, that since in the family of blocks  $\mathbf{B} = \{B_1, \dots, B_t\}$ ,  $B_{i+1} = \{0, \alpha^i, \dots, \alpha^{2mt-t+i}\}$  each nonzero difference appears exactly  $k - 2m + 1$  times,  $\mathbf{B}$  forms a  $(p, k, k)$  DF in  $Z_p$ . To complete the proof, it suffices to show that for any fixed difference in  $\mathbf{B}$  the corresponding subscript differences cover every non-zero element of  $Z_k$  exactly once. Since for each  $i$ ,  $(\alpha^{t_i} - 1)\alpha^{-r_i} = 1$  this is equivalent to the assumption that  $\pm 1, \pm(\beta^{t_i} - 1)\beta^{-r_i}$ ,  $i = 1, \dots, m - 1$  are distinct in  $Z_k$ . Finally, since  $k$  and  $p$  are distinct primes the design is cyclic in  $Z_{kp}$ .  $\square$

We will apply Theorem 3 to blocksize  $k = 7$ . Then  $m = 3$  and  $p$  is a prime of the form  $p = 6t + 1$ ,  $t \geq 2$ . If  $\alpha$  is a primitive root of  $Z_p$ , then  $\alpha^{3t} = -1$  and since

$$(\alpha^t + 1)\alpha^{2t} = \alpha^{2t} - 1 = (\alpha^t + 1)(\alpha^t - 1)$$

we have  $\alpha^t - 1 = \alpha^{2t}$ . Let  $r$  be the solution of  $\alpha^r = \alpha^{2t} - 1$ . We require that for some  $\beta \in Z_7$  the 6 numbers

$$\pm\beta^{2t}, \pm(\beta^t - 1), \pm\beta^{2t-r}(\beta^{2t} - 1) \quad (4)$$

cover the non-zero elements of  $Z_7$ . Since  $\beta^{2t}$  cannot be congruent to 1 modulo 7, we see that  $t \equiv 1$  or  $2 \pmod{3}$ . If  $t \equiv 1 \pmod{3}$ , then (4) are distinct if either  $\beta = 2$  and  $r \equiv 0 \pmod{3}$ , or  $\beta = 4$  and  $r \equiv 2 \pmod{3}$ . If  $t \equiv 2 \pmod{3}$ , then we need either  $\beta = 2$  and  $r \equiv 1 \pmod{3}$ , or  $\beta = 4$  and  $r \equiv 0 \pmod{3}$ . Combining all these conditions we obtain the following result.

**Corollary 4** Let  $p = 6t + 1$  be a prime,  $t \geq 2$ ,  $t \not\equiv 0 \pmod{3}$ , and let  $\alpha$  be a primitive root in  $Z_p$ . Then the blocks (3) form a  $(7p, 7, 1)$  DF for some  $\beta \in Z_7$  if and only if  $t \not\equiv r \pmod{3}$ , where  $r$  satisfies  $\alpha^r = \alpha^{2t} - 1$ .

We note, that for some values of  $t$  we obtain two non-isomorphic cyclic designs. If  $t \equiv 4 \pmod{6}$ , then (4) are distinct also if either  $\beta = 3$  and  $r \equiv 2 \pmod{3}$ , or  $\beta = 5$  and  $r \equiv 0 \pmod{3}$ . If  $t \equiv 2 \pmod{6}$ , then (4) are distinct also if either  $\beta = 3$  and  $r \equiv 0 \pmod{3}$ , or  $\beta = 5$  and  $r \equiv 1 \pmod{3}$ .

For  $k = 7$  solutions exist when  $t = 2^*, 5, 7, 13, 16^*, 26^*, 35, 37, 38^*, 40^*, 46^*, 47$ , etc. The base blocks for  $t = 2^*, 5$  and  $7$  are

|       |        |        |        |        |        |        |  |       |        |        |        |        |        |        |
|-------|--------|--------|--------|--------|--------|--------|--|-------|--------|--------|--------|--------|--------|--------|
| $0_0$ | $1_1$  | $4_4$  | $3_2$  | $12_1$ | $9_4$  | $10_2$ |  | $0_0$ | $1_1$  | $4_4$  | $3_2$  | $12_1$ | $9_4$  | $10_2$ |
| $0_0$ | $2_2$  | $8_1$  | $6_4$  | $11_2$ | $5_1$  | $7_4$  |  | $0_0$ | $2_5$  | $8_6$  | $6_3$  | $11_5$ | $5_6$  | $7_3$  |
|       |        |        |        |        |        |        |  |       |        |        |        |        |        |        |
| $0_0$ | $1_1$  | $26_4$ | $25_2$ | $30_1$ | $5_4$  | $6_2$  |  | $0_0$ | $1_1$  | $37_2$ | $36_4$ | $42_1$ | $6_2$  | $7_4$  |
| $0_0$ | $3_2$  | $16_1$ | $13_4$ | $28_2$ | $15_1$ | $18_4$ |  | $0_0$ | $3_2$  | $25_4$ | $22_1$ | $40_2$ | $18_4$ | $21_1$ |
| $0_0$ | $9_4$  | $17_2$ | $8_1$  | $22_4$ | $14_2$ | $23_1$ |  | $0_0$ | $9_4$  | $32_1$ | $23_2$ | $34_4$ | $11_1$ | $20_2$ |
| $0_0$ | $27_1$ | $20_4$ | $24_2$ | $4_1$  | $11_4$ | $7_2$  |  | $0_0$ | $27_1$ | $10_2$ | $26_4$ | $16_1$ | $33_2$ | $17_4$ |
| $0_0$ | $19_2$ | $29_1$ | $10_4$ | $12_2$ | $2_1$  | $21_4$ |  | $0_0$ | $38_2$ | $30_4$ | $35_1$ | $5_2$  | $13_4$ | $8_1$  |
|       |        |        |        |        |        |        |  | $0_0$ | $28_4$ | $4_1$  | $19_2$ | $15_4$ | $39_1$ | $24_2$ |
|       |        |        |        |        |        |        |  | $0_0$ | $41_1$ | $12_2$ | $14_4$ | $2_1$  | $31_2$ | $29_4$ |

The solutions for  $t = 5$  and  $7$  are first examples of BIBD's with the parameters  $(217,7,1)$  and  $(301,7,1)$ , respectively. For  $k = 11$  solutions exist when  $t = 33, 54^*, 57, 91, 94^*$ , etc. and for  $k = 13, t = 13, 19, 59$ , etc. (\* indicates 2 solutions).

We conclude this section with a well-known result in finite geometries [6].

**Theorem 5** Let  $q$  be a prime power. Then the lines in the projective geometry  $PG(n, q)$ ,  $n \geq 2$  form a cyclic design with parameters  $((q^{n+1} - 1)/(q - 1), q + 1, 1)$ .

### 3. Recursive Constructions

Given two difference families it is sometimes possible to combine them to construct a new one. Several such constructions are known for general cyclic BIBD's [4] [8] [14]. To apply them, various conditions on the block sizes are usually required.

We begin with a construction by C.J. Colbourn and M.J. Colbourn [4].

**Theorem 6** Let  $A^i = \{0, a^i_1, \dots, a^i_{k-1}\}$ ,  $i = 1, \dots, t$  be a  $(v, k, 1)$  DF in  $Z_v$  and let  $B^j = \{0, b^j_1, \dots, b^j_{k-1}\}$ ,  $j = 1, \dots, s$  be a  $(w, k, 1)$  DF in  $Z_w$ .

- (i) If  $v = k(k - 1)t + 1$  and  $w$  is relatively prime to  $(k - 1)!$ , then for  $i = 1, \dots, t$ ,  $j = 1, \dots, s$  and  $l = 0, 1, \dots, w - 1$

$$\left. \begin{array}{l} \{0, a^i_1 + lv, a^i_2 + 2lv, \dots, a^i_{k-1} + (k-1)lv\} \\ \{0, vb^j_1, vb^j_2, \dots, vb^j_{k-1}\} \end{array} \right\} \quad (5)$$

is a  $(vw, k, 1)$  DF in  $Z_{vw}$ .

- (ii) If  $v = k\alpha$ ,  $w = k\beta$  and  $\beta$  is relatively prime to  $(k-1)!$ , then for  $i = 1, \dots, t$ ,  $j = 1, \dots, s$  and  $l = 0, 1, \dots, w-1$

$$\left. \begin{array}{l} \{0, a^i_1 + lv, a^i_2 + 2lv, \dots, a^i_{k-1} + (k-1)lv\} \\ \{0, \alpha b^j_1, \alpha b^j_2, \dots, \alpha b^j_{k-1}\} \\ \{0, \alpha\beta, 2\alpha\beta, \dots, (k-1)\alpha\beta\} \end{array} \right\} \quad (6)$$

is a  $(k\alpha\beta, k, 1)$  DF in  $Z_{k\alpha\beta}$ . Here  $\alpha = (k-1)t + 1$ ,  $\beta = (k-1)s + 1$ , and only full orbit base blocks  $A^i_i, B^s_j$  are considered.

We note that the construction can be used if either  $w$  or  $\beta$  are prime. Then the existence of a  $(w, k, 1)$  DF implies the existence of a  $(w^n, k, 1)$  DF for every  $n \geq 1$ . Similarly, from a  $(k\beta, k, 1)$  DF we obtain a  $(k\beta^n, k, 1)$  DF. Also, if a  $(v, k, 1)$  DF exists with  $v \equiv 1 \pmod{k(k-1)}$  and prime  $k$  then there exists a  $(vk, k, 1)$  DF.

In [8] M. Jimbo and S. Kuriki have introduced a more general construction for cyclic BIBD's which is based on orthogonal arrays. Applying it to Steiner 2-designs we obtain the following typical result.

**Theorem 7** Suppose there exists a  $C(v, k, 1)$  and a  $C(w, k, 1)$ , where  $v \equiv 1 \pmod{k(k-1)}$  and  $k$  is an odd prime. Then there exists a  $C(vw, k, 1)$ . If, in addition,  $w \equiv 1 \pmod{k(k-1)}$ , then the conclusion holds for  $k$  a prime power.

So, for example, if  $k$  is an odd prime not dividing  $v$ , then the existence of a  $C(v, k, 1)$  implies the existence of both  $C(v^n, k, 1)$  and  $C(kv^n, k, 1)$  for any  $n \geq 1$ .

The next construction employs cyclic pairwise balanced designs. A *pairwise balanced design* (briefly PBD) is a pair  $(V, \mathcal{B})$  where  $V$  is a  $v$ -set and  $\mathcal{B}$  is a collection of subsets of  $V$  (blocks) such that every 2-subset of  $V$  is contained in exactly one block. A PBD will be denoted by  $(v, K, 1)$ , where  $K = \{k_1, \dots, k_n\}$  is the set of block sizes.

**Theorem 8** Suppose there exists a cyclic  $(v, K, 1)$  PBD with  $K = \{k_1, \dots, k_n\}$  and that for each  $k_i$  there exists a  $(k_i, k, 1)$  Steiner 2-design. Then there exists a  $C(v, k, 1)$ .

**Proof** Replace each base block in the PBD by the blocks of the corresponding Steiner 2-design to obtain the base blocks of the final  $C(v, k, 1)$ .  $\square$

In the next section we shall give some other recursive constructions for cyclic designs with blocks of size 4 and 5 which are based on the concepts of perfect systems of difference sets and additive sequences of permutations.

#### 4. Special Constructions

The existence question for cyclic Steiner triple systems has been completely settled by Peltesohn [10], who constructed  $C(v,3,1)$  for all  $v \equiv 1,3 \pmod{6}$ ,  $v \neq 9$ .

For block sizes  $k > 3$  the existence problem for  $C(v,k,1)$  remains unsolved. The state of affairs is most promising for the cases  $k = 4$  and  $5$ .

In order to present additional recursive constructions we require a few more definitions.

A collection of  $t$   $k$ -subsets  $D_i = \{d^i_0, d^i_1, \dots, d^i_{k-1}\}$ ,  $0 = d^i_0 < d^i_1 < \dots < d^i_{k-1}$ ,  $i = 1, \dots, t$  is said to be a *perfect difference family* (PDF) in  $Z_v$ ,  $v = k(k-1)t + 1$ , if the  $tk(k-1)/2$  differences  $d^i_l - d^j_j$ ,  $0 \leq j < l < k$  cover the set  $\{1, 2, \dots, tk(k-1)/2\}$ . PDF's are equivalent to regular perfect systems of difference sets starting with 1, which have been studied by many authors (see [1] for a recent survey). It has been shown [2] that PDF's can exist only when  $k$  is 3, 4 or 5. For  $k = 3$  the existence of a PDF is related to Skolem's partitioning problem [1].

Let  $X^1$  be the  $m$ -vector  $(-r, -r+1, \dots, -1, 0, 1, \dots, r-1, r)$ ,  $m = 2r + 1$  and let  $X^2, \dots, X^n$  be permutations of  $X^1$ . Then  $X^1, \dots, X^n$  is an *additive sequence of permutations* (ASP) of order  $m$  and length  $n$  if the vector sum of every subsequence of consecutive permutations is again a permutation of  $X^1$ . ASP's play an important role in recursive constructions for PDF's and vice versa [1] [11] [12].

#### Block size 4

We begin with two direct constructions.

**Theorem 9** let  $p = 12t + 1$ ,  $t \geq 1$  be a prime and let  $\alpha$  be a primitive root of  $Z_p$ .

(i) ([3] [13]) If  $p \neq x^2 + 36y^2$  for any integers  $x$  and  $y$  then

$$\{0, \alpha^{2i}, \alpha^{4i+2i}, \alpha^{8i+2i}\} \quad i = 0, 1, \dots, t-1 \quad (7)$$

is a  $(p, 4, 1)$  DF in  $Z_p$ .

(ii) ([5]) If  $\alpha \equiv 3 \pmod{4}$  (and such an  $\alpha$  always exists in  $Z_p$ ) then

$$\left. \begin{array}{l} \{0, \alpha^{4i}, \alpha^{4i+3}, \alpha^{4i+6}\} \quad i = 0, \dots, 3t-1 \\ \{0, \alpha^{4j+1}, \alpha^{4j+4}, \alpha^{8j+4j+1}\} \quad j = 0, \dots, t-1 \\ \{0, p, 2p, 3p\} \end{array} \right\} \quad (8)$$

form a  $(4p, 4, 1)$  DF in  $Z_{4p}$ .

The next two constructions will exhibit the relationship between PDF's and ASP's.

**Theorem 10** ([3] [13]) Let  $D_i = \{0, a_i, b_i, c_i\}$ ,  $i = 1, \dots, t$  be a PDF in  $Z_{12t+1}$  and let  $X^1, X^2, X^3$  be an ASP of order  $m = 2r + 1$ ,  $r \geq 2$  and length 3. Then

- (i) For  $i = 1, \dots, t$  and  $j = 1, \dots, m$  the  $6tm$  positive differences in the family

$$\Delta_{mi-m+j} = \{0, ma_i + \alpha_j, mb_i + \beta_j, mc_i + \gamma_j\} \quad (9)$$

cover the set  $\{r+1, r+2, \dots, r+6tm\}$ . Here  $\alpha, \beta$  and  $\gamma$  are the  $m$ -vectors  $X^1, X^1 + X^2, X^1 + X^2 + X^3$ , respectively.

- (ii) For  $i = 1, \dots, t$

$$\left. \begin{aligned} X_i^1 &= (-c, a-c, -b, b-c, a-b, -a, a, b-a, c-b, b, c-a, c)_i \\ X_i^2 &= (c-b, c, b-a, c-a, b-c, a-c, -b, a, b, -c, a-b, -a)_i \\ X_i^3 &= (b-a, -b, a-c, a, c, c-b, b-c, -c, -a, c-a, b, a-b)_i \end{aligned} \right\} \quad (10)$$

the  $(12t+1)$ -vectors  $X^j = (0, X_i^j, \dots, X_i^j)$ ,  $j = 1, 2, 3$  form an ASP of order  $12t + 1$  and length 3.

In order to utilize products of the form (9) for constructing new difference families we need to find additional base blocks with differences covering the set  $\{1, \dots, r\}$  and possibly  $\{r + 6tm + 1, \dots, 6x\}$  for some  $x \geq 1$ .

We list now the known recursive constructions for  $1 \leq m \leq 25$ .

**Theorem 11** Let  $D(t) = \{D_1, \dots, D_t\}$  be a PDF and let  $\Delta(mt) = \{\Delta_1, \dots, \Delta_{mt}\}$  be defined by (9), where  $m = 2r + 1$  and  $\alpha = (-r, -r+1, \dots, -1, 0, 1, \dots, r-1, r)$ . Then

1. For  $r = 2$

$$\beta = (-2, 0, 2, -1, 1), \quad \gamma = (0, -2, 1, -1, 2)$$

$$D(5t+1) = \Delta(5t) \cup \{0, 1, 30t+4, 30t+6\}$$

is a PDF in  $Z_{60t+13}$ .

2. For  $r = 3$

$$\beta = (-1, -2, -3, 3, 2, 1, 0), \quad \gamma = (-2, 1, -3, 0, 3, -1, 2)$$

$$D(7t+1) = \Delta(7t) \cup \{0, 2, 3, 42t+7\}$$

is a DF in  $Z_{84t+13}$ .

3. For  $r = 6$

$$\beta = (-4, -5, -1, -2, 3, -6, 6, 5, 1, -3, 0, 2, 4)$$

$$\gamma = (-1, -5, -6, 3, -3, -4, 4, 2, 5, -2, 6, 1, 0)$$

$$D(13t+1) = \Delta(13t) \cup \{0, 1, 4, 6\}$$

is a PDF in  $Z_{156t+13}$ .

4. For  $r = 9$

$$\beta = (-6, -7, 1, -2, -3, 5, 3, -9, -4, 7, -8, 0, -5, 9, -1, 6, 2, 4, 8)$$



$$\gamma = (-2, 1, -8, -9, 3, -1, -5, -6, 5, 2, -7, 7, -3, 8, -4, 6, 0, 9, 4)$$

$$D(19t+4) = \Delta(19t) \cup \{0, 1, 7, x+23\} \cup \{0, 2, x+14, x+19\} \\ \cup \{0, 3, x+13, x+21\} \cup \{0, 4, x+15, x+24\}, \quad x = 114t$$

is a PDF in  $Z_{228t+49}$ .

5. For  $r = 11$

$$\beta = (0, -2, 1, -7, 2, -6, -1, -5, 4, -10, -11, -9, 6, -4, -8, -3, 11, 8, 10, 3, 5, 7, 9)$$

$$\gamma = (9, 5, 4, -10, 0, -7, 10, -9, 8, -4, -3, 1, -5, -11, -8, 2, 6, -2, 11, -6, 7, -1, 3)$$

$$D(23t+5) = \Delta(23t) \cup \{0, 1, 8, x+28\} \cup \{0, 2, x+14, x+24\} \cup \\ \{0, 3, x+18, x+29\} \cup \{0, 4, x+17, x+23\} \cup \{0, 5, x+21, x+30\}, \quad x = 138t$$

is a PDF in  $Z_{276t+61}$ .

6. For  $r = 12$

using  $\alpha, \beta, \gamma$  and  $\Delta(5t)$  from 1 to obtain  $\Delta(25t)$

$$D(25t+5) = \Delta(25t) \cup \{0, 1, x+18, x+29\} \cup \{0, 4, x+20, x+26\} \\ \cup \{0, 3, 8, x+27\} \cup \{0, 7, x+21, x+30\} \cup \{0, 10, 12, x+25\}, \quad x = 150t$$

is a PDF in  $Z_{300t+61}$ , and

$$D(25t+6) = \Delta(25t) \cup \{0, 1, x+34, x+36\} \cup \{0, 3, x+18, x+29\} \\ \cup \{0, 4, x+20, x+28\} \cup \{0, 5, x+22, x+32\} \\ \cup \{0, 6, x+19, x+31\} \cup \{0, 7, x+21, x+30\}, \quad x = 150t$$

is a PDF in  $Z_{300t+79}$ .

**Proof** Use (9) to check that the required sets are covered by all differences from the base blocks.  $\square$

We note that the constructions 2,5,6b are new and that 1,3,4,6a have been known [11]. The ASP with  $r = 11$  has been found by P.J. Laufer.

If we apply all methods listed in Sections 2, 3 and 4 and add the computer generated DF's from [5] we obtain the following results for  $1 \leq t \leq 50$ :

(12t+1,4,1) PDF

$$t = 1, 4, 8, 14, 21, 23, 26, 28, 30, 31, 36, 41$$

(12t+1,4,1) DF

$$t = 1, 3, 10, 14, 15, 19, 21, 23, 26, 28, 31, 34, 36, 38, 40, 41, 43, 45, 50$$

(12t+4,4,1) DF

$$t = 3-6, 12, 20, 24, 30, 32, 36, 43.$$

### Block size 5

As before, two direct constructions are known.

**Theorem 12** Let  $p = 20t + 1$ ,  $t \geq 1$  be a prime and let  $\alpha$  be a primitive root of  $Z_p$ .

(i) ([3] [13]) If  $p \neq x^2 + 100y^2$  for any integers  $x$  and  $y$  then

$$\{\alpha^{2i}, \alpha^{4t+2i}, \alpha^{8t+2i}, \alpha^{12t+2i}, \alpha^{16t+2i}\} \quad i = 0, 1, \dots, t-1 \quad (11)$$

is a  $(p, 5, 1)$  DF in  $Z_p$ .

(ii) ([5]) If  $\alpha^r + 1 = \alpha^s(\alpha^r - 1)$  for some odd integers  $r$  and  $s$  then

$$\{0, \alpha^{2i}, \alpha^{2i+r}, \alpha^{2t+2i}, \alpha^{2t+2i+r}\} \quad i = 0, 1, \dots, t-1 \quad (12)$$

$$\{0, p, 2p, 3p, 4p\}$$

form a  $(5p, 5, 1)$  DF in  $Z_{5p}$ .

Concerning PDF's with blocks of size 5 and ASP of length 4, results can be proved which are similar to those stated in Theorem 10 [1]. They can be used to derive the following construction.

**Theorem 13** Let  $D(t) = \{D_1, \dots, D_t\}$  be a PDF in  $C_{20t+1}$  and let  $D(s) = \{D_1, \dots, D_s\}$  be a DF in  $C_{20s+1}$ . Then a DF  $D(r)$  exists in  $C_{20r+1}$ ,  $r = 20st+s+t$  and  $D(r)$  is perfect whenever  $D(s)$  is perfect.

**Proof** Use  $D(t)$  to construct an ASP of length 4 and order  $m = 20t+1$  [1]. With help of this ASP construct the blocks  $\Delta(ms)$  in a similar way as in (9). Then  $D(r) = \Delta(ms) \cup D(s)$ .  $\square$

PDF's with  $k = 5$  can exist only if  $t$  is even and  $t \geq 6$  [1]. They have been enumerated for  $t = 6$  [9] and examples are known for  $t = 8, 10, 732, 974$ , etc.

Difference families are known for the following values of  $t$ ,  $1 \leq t \leq 50$ :

$(20t+1)$  PDF

$$t = 6, 8, 10$$

$(20t+1)$  DF

$$t = 1-3, 6, 8, 10, 12, 14, 21-22, 30, 32-33, 35, 41, 43-44$$

$(20t+5)$  DF

$$t = 3-5, 7, 9-10, 13, 15, 18, 22, 24-25, 27-28, 30, 34, 37, 39-40, 42-43, 45, 48-50.$$

**Open Problems**

1. Does there exist a  $(12t+1,4,1)$  DF for every  $t \geq 3$ ? Can all of these DF's be perfect if  $t \geq 4$ ?
2. Does there exist a  $C(v,4,1)$  for every  $v \neq 16, 25$  and  $28$ ?
3. Does there exist an ASP of length 3 for every order  $m \geq 5, m \neq 9,10$ ?
4. Do there exist  $C(v,5,1)$  for  $v = 81$  and  $85$ ?
5. Construct examples of PDF's  $D(t)$  for  $k = 5$  and even  $t \geq 12$ .
6. Construct examples of ASP of length 4 for orders  $m \geq 7$ .

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