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ON CYCLIC SEMI-REGULAR GROUP DIVISIBLE DESIGNS

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Abstract

This paper develops ceratin immediately applicable conditions for the existence of cyclic SRGD designs. The case when the block size equals 3 has been completely explored. Special attention has been given to the situation $\lambda_2 = \lambda_1+1$ which is important from statistical considerations. Some results have also been derived for the situation when short orbits are allowed.

1. Introduction

The group divisible (GD) designs constitute the largest, simplest and perhaps most important type of 2-associate partially balanced incomplete block (PBIB) designs. A GD design is an arrangement of $v(= mn)$ treatments in b blocks such that each block contains k $(< v)$ distinct treatments; each treatment is replicated r times; and the set of treatments can be partitioned into m (≥ 2) equivalence classes of n (≥ 2) treatments each , any two distinct treatments occurring together in λ_1 blocks if they belong to the same equivalence class, and in λ_2 blocks if they belong to different equivalence classes. It may be remarked that in the literature the commonly used terminology for these equivalence classes of treatments is 'groups', but here we deliberately prefer to use the phrase 'equivalence classes' in order to avoid a notational confusion with groups in a group-theoretic sense which we shall be considering shortly in this paper. GD designs may again be of three types: (a) *singular*, if $r = \lambda_1$; (b) *semi-regular*(SR) if $r>\lambda_1$ and $rk = \lambda_2 v$; (c) *regular*(R) if $r>\lambda_1$ and $rk>\lambda_2 v$.

If the automorphism group of a GD design contains a cyclic group of order

v, then the GD design is said to be *cyclic*. For a cyclic GD design, without loss of generality, we may represent the set of v treatments by V = {0,1,..., v-1} and in this case the automorphism of order v is $x \to x+1 \pmod v$. In the sequel, we shall use this notation to represent the treatments in a cyclic GD design. The following definitions will also be helpful. For a block $B = \{b_0, b_1,\ldots,b_{k-1}\}$ and any $i \in V$, define $B+i = \{b_0+i, b_1+i,\ldots,b_{k-1}+i\}$, addition being reduced mod v. The collection of blocks $\{B+i \mid i \in V\}$ is called the *full orbit* containing B. Let i_0 be the smallest positive integer such that $B+i_0 = B$. If $i_0 < v$, then the collection of blocks $\{B+i \mid 0 \leq i \leq i_0-1\}$ is called a *short orbit* containing B.

A large number of methods of constructing GD designs are available in the literature (cf. Clatworthy (1973), Raghavarao (1971)). However, most of the designs produced by them are not cyclic. Cyclic GD designs can be conveniently obtained by the method of differences of Bose (1939). Their flexibility, ease of representation and conduct of experimentation make them worthy of attention in their own right, as David and Wolock (1965) pointed out. In cyclic designs, no plan of experimental layout is needed since the initial block or blocks suffice (i.e., concise representation). This is readily implemented on a computer. Cyclic designs also permit an automatic two-way elimination of heterogeneity and a fairly straightforward and general method of analysis and come into serious consideration as a means of augmenting the experimenter's choice of designs.

The basic development of the constructions of GD designs was done by Bose, Shrikhande and Bhattacharya (1953). Freeman (1976) and Dey and Nigam (1985) gave some methods of constructing cyclic RGD designs. Huang, Lin and Clatworthy (1982) searched cyclic, symmetric PBIB designs systematically. But they do not discuss cyclic SRGD designs. Recently, Jimbo and Vanstone (1984) considered

cyclic GD designs with $\lambda_1 = 0$ and $\lambda_2 = 1$ to construct other block designs. From a point of view of various usefulness of cyclic designs, we here discuss the existence problem (with construction procedures and non-existence) of cyclic SRGD designs. A case $\lambda_2 = \lambda_1+1$, which has strong statistical significance in terms of optimality, is also treated. Within the scope of practical range of parameters in Clatworthy (1973), who tabulates practical plans with solutions not cyclic except three, we also produce cyclic solutions for SRGD designs which may be more convenient to be stored in a computer and be non-isomorphic to the previously published solutions. As a by-product, it is shown that a cyclic BIB(vk^2,k,1) design does not contain any cyclic BIB(vk,k,1) subdesign for $k \geq 3$. The case where short orbits are allowed has also been discussed.

For definitions of other designs treated in this paper refer to Raghavarao (1971).

2. Some existence theorems

Considering a cyclic GD design, we have the following three lemmas; the proofs of the first two lemmas are simple and hence omitted.

Lemma 2.1. If $a_1, a_2 \in V$ are first associates of each other (i.e., belong to the same equivalence class), then for any $i \in V$, $a_1+i \pmod v$ and $a_2+i \pmod v$ are also first associates of each other.

Lemma 2.2. If S is an equivalence class of a cyclic GD design, then for any $i \in V$, $S+i = \{s+i \pmod v \mid s \in S\}$ is also an equivalence class.

Lemma 2.3. If S_0 is an equivalence class of a cyclic GD design and if S_0 contains the identity element 0, then S_0 is a subgroup of the cyclic group $V = Z_v$, i.e., $S_0 = \{0,m,2m,\ldots,(n-1)m\}$, and all other equivalence classes are cosets of S_0.

Proof. If $a \in S_0$, then 0 and a are first associates since $0 \in S_0$. Hence -a

and 0 are first associates by Lemma 2.1, which means that the inverse element $-a$ is contained in S_0. If $a_1, a_2 \in S_0$, then a_1 and a_2 are first associates, hence $-a_1$ and a_2 are first associates and, again by Lemma 2.1, 0 and a_1+a_2 are first associates. Thus $a_1+a_2 \in S_0$. Hence S_0 is a subgroup of Z_v and, by Lemma 2.2, all other equivalence classes are cosets of S_0.

In view of Lemma 2.3, the equivalence classes of treatments in a cyclic GD design are of the form $S_i = \{i, m+i, \ldots, (n-1)m+i\}$ $(0 \le i \le m-1)$ and for any i,j $(0 \le i,j \le m-1)$,

$$\{a_1 - a_2 \pmod v \mid a_1 \in S_i,\ a_2 \in S_j\} = S_{i-j}\ , \tag{2.1}$$

the suffix $i-j$ in the right-hand side of (2.1) being reduced mod m.

Consider now cyclic SRGD designs in particular. It is well-known (cf. Raghavarao (1971)) that in an SRGD design k is an integral multiple of m and each block contains exactly k/m (= c, say) treatments from each equivalence class. The following theorem gives a characterization of cyclic SRGD designs without short orbits.

Theorem 2.1. For the existence of a cyclic SRGD design, without short orbits, it is necessary and sufficient that

(a) b is an integral multiple of v *and*

(b) if $b/v = \alpha$, then there exist integers f_u^{ij} $(0 \le i \le m-1,\ 1 \le j \le c,\ 1 \le u \le \alpha)$ such that defining $L = \{0,1,\ldots,n-1\}$, $f_u^{ij} \in L$ for all i,j,u, and

(i) in the set $\{f_u^{ij} - f_u^{it} \mid 0 \le i \le m-1,\ 1 \le j,t \le c\ (j \ne t),\ 1 \le u \le \alpha\}$, where the differences are reduced mod n, each non-zero member of L is repeated λ_1 times,

(ii) in each of the sets

$$\{f_u^{1j} - f_u^{0t},\ f_u^{2j} - f_u^{1t}, \ldots,\ f_u^{\overline{m-1}j} - f_u^{\overline{m-2}t},\ f_u^{0j} - f_u^{\overline{m-1}t} - 1 \mid 1 \le j,t \le c,\ 1 \le u \le \alpha\},$$

$$\{f_u^{2j} - f_u^{0t},\ f_u^{3j} - f_u^{1t}, \ldots,\ f_u^{\overline{m-1}j} - f_u^{\overline{m-3}t},\ f_u^{0j} - f_u^{\overline{m-2}t} - 1,\ f_u^{1j} - f_u^{\overline{m-1}t} - 1 \mid 1 \le j,t \le c,\ 1 \le u \le \alpha\},$$

$$\vdots$$

$$\{f_u^{\overline{m-1}j}-f_u^{0t},\ f_u^{0j}-f_u^{\overline{1}t}-1,\ldots,\ f_u^{\overline{m-2}j}-f_u^{\overline{m-1}t}-1 \mid 1\le j,t\le c,\ 1\le u\le\alpha\},$$

where the differences are reduced mod n, each member of L is repeated λ_2 times.

Proof. The necessity of (a) follows immediately counting the number of full orbits in a cyclic SRGD design. To prove the necessity of (b), let for such a design, $b/v = \alpha$, the number of full orbits. Then by our preceding discussion (and also the fact that in an SRGD design each block contains exactly $k/m = c$, say, treatments from each equivalence class), it follows that in a cyclic SRGD design, without short orbits, the α initial blocks must be of the form

$$\{mf_u^{0l},\ldots,mf_u^{0c},mf_u^{1l}+1,\ldots,mf_u^{1c}+1,\ldots,mf_u^{\overline{m-1}l}+(m-1),\ldots,mf_u^{\overline{m-1}c}+(m-1)\},$$

$1\le u\le\alpha$, where $0\le f_u^{ij}\le n-1$ $(0\le i\le m-1,\ 1\le j\le c,\ 1\le u\le\alpha)$.

By (2.1), among the ordered differences (mod v) arising out of the distinct elements in the α initial blocks, each non-zero element of S_0 is repeated λ_1 times and each element of $V-S_0$ is repeated λ_2 times. Therefore, considering in particular the occurrence of the elements of S_1 among these ordered differences, it follows that among

$$\{m(f_u^{1j}-f_u^{0t})+1,\ m(f_u^{2j}-f_u^{1t})+1,\ldots,m(f_u^{\overline{m-1}j}-f_u^{\overline{m-2}t})+1,\ m(f_u^{0j}-f_u^{\overline{m-1}t}-1)+1 \mid 1\le j,t\le c,\ 1\le u\le\alpha\},$$

all reduced mod v, each element of S_1 is repeated λ_2 times. Hence the necessity of our assertion regarding the first set in ((b),(ii)) is immediate. The necessity of our assertions regarding the other sets in ((b),(ii)) and the set in ((b),(i)) follow in a similar manner. This proves the necessity part of the theorem. The sufficiency part follows by retracing the above steps.

Although apparently Theorem 2.1 looks somewhat involved, computational experience shows that (see Section 3) in proving existence or non-existence of cyclic SRGD designs, without short orbits, using computers, application of Theorem 2.1 can tremendously reduce the computational time. Moreover, one can

obtain simpler necessary conditions starting from Theorem 2.1 as stated below.

Theorem 2.2. For the existence of a cyclic SRGD design, without short orbits, it is necessary that b is an integral multiple of v and at least one of the following holds:

(i) n odd and λ_2 is an integral multiple of m,

(ii) n even, λ_1 even, λ_2 even and λ_2 is an integral multiple of m,

(iii) n even, λ_1 even, λ_2 odd and m=2.

Proof. Summing (mod n) the elements in the set in Theorem 2.1((b),(i)) in two ways, one obtains the necessary condition

(2.2) $\quad n(n-1)\lambda_1/2 \equiv 0 \pmod{n}$.

Similarly, considering the sets in Theorem 2.1((b),(ii)) one can derive the necessary conditions

(2.3) $\quad n(n-1)\lambda_2/2 \equiv -ic^2\alpha \pmod{n}$ $(1 \le i \le m-1)$.

Considering separately the cases of odd and even n and making use of the identity $c^2\alpha/n = \lambda_2/m$, it is possible to complete the proof from (2.2) and (2.3).

The above theorem is a very powerful tool in identifying the SRGD designs for which a cyclic construction may be possible and also in proving non-existence results concerning cyclic designs (i.e., a cyclic SRGD design cannot be constructed unless its parameters are as stipulated by Theorem 2.2). Theorem 2.3 and Example 2.1 below illustrate the ideas. In particular, Theorem 2.3 completely exhausts the situation k = 3.

Theorem 2.3. An SRGD design with k = 3 can be cyclic (without short orbits) if and only if its parameters are of the form

(a) $v=3n$, $m=3$, n, $b=3n^2t$, $r=3nt$, $k=3$, $\lambda_1=0$, $\lambda_2=3t$ (n odd; $t \ge 1$),

or (b) $v=3n$, $m=3$, n, $b=6n^2t$, $r=6nt$, $k=3$, $\lambda_1=0$, $\lambda_2=6t$ (n even; $t \ge 1$).

Proof. For an SRGD design with k = 3, clearly m = 3 and $\lambda_1 = 0$. Now, if such a design is cyclic, considering separately the cases of odd and even n

and applying the conditions (i) and (ii) in Theorem 2.2, one obtains respectively the forms (a) and (b) stated above. This proves the 'only if' part of the theorem. Note that the condition (iii) in Theorem 2.2 cannot arise in this situation.

To prove the 'if' part, observe that for odd n, a design with parameters as in (a) may be constructed cyclically from the initial blocks {0, 3j+1, 6j+2} ($0 \le j \le n-1$), each repeated t times. Similarly, for even n, a cyclic construction of a design with parameters given by (b) is possible using the initial blocks {0, 3j+1, 6j+2}, {0, 3j+1, 6j+5} ($0 \le j \le n-1$), each repeated t times. The elements in these initial blocks are, of course, reduced mod v. It may be remarked that the choice of initial blocks as above is motivated essentially by the idea of row difference schemes considered by Jimbo and Kuriki (1983).

Example 2.1. We examine the situations under which an SRGD design with parameters of the form

(2.4) $\quad v=mn,\ m,\ n,\ b=n^2,\ r=n,\ k=m,\ \lambda_1=0,\ \lambda_2=1$

($m,n \ge 2$) can be cyclic (without short orbits). If the design is cyclic, then b/v and hence n/m (= α, say) is an integer. Evidently, for such a design the conditions (i) or (ii) of Theorem 2.2 cannot hold. The condition (iii) holds provided m=2. Then the parameters of the design become $v=4\alpha$, m=2, $n=2\alpha$, $b=4\alpha^2$, $r=2\alpha$, k=2, $\lambda_1=0$, $\lambda_2=1$, and a cyclic construction is always possible starting from the initial blocks {0,1}, {0,3},..., {0,2α-1}. Thus an application of Theorem 2.2 shows that an SRGD design with parameters as in (2.4) can be cyclic if and only if m=2 and n even.

In the situations considered in Theorem 2.3 and Example 2.1, the necessary conditions stated in Theorem 2.2 turn out to be sufficient as well. In general, however, this is not true and some examples in this regard will be

presented in the next section. Anyway, by a complete enumeration of all possibilities, we get the satisfying observation that at least over the practicable range $r,k \leq 10$, considered in Clatworthy (1973), the conditions stated in Theorem 2.2 are not only necessary but also sufficient. Table 2.1 presents a complete list of cyclic SRGD designs, together with their initial blocks, over the range $r,k \leq 10$. Some of these cyclic solutions may be non-isomorphic to previously published solutions. Also, note that the complement of a cyclic SRGD design is again a cyclic SRGD design.

Table 2.1. Cyclic SRGD designs for $r,k \leq 10$

No.	Design parameters								Serial no. in Clatworthy's tables	Initial blocks
	v	b	r	k	m	n	λ_1	λ_2		
1	4	4	2	2	2	2	0	1	SR 1	{0,1}
2	4	8	4	2	2	2	0	2	SR 2	Take two copies of No. 1
3	4	12	6	2	2	2	0	3	SR 3	Take three copies of No. 1
4	4	16	8	2	2	2	0	4	SR 4	Take four copies of No. 1
5	4	20	10	2	2	2	0	5	SR 5	Take five copies of No. 1
6	6	18	6	2	2	3	0	2	SR 7	{0,1},{0,3},{0,5}
7	8	16	4	2	2	4	0	1	SR 9	{0,1},{0,3}
8	8	16	8	4	4	2	0	4	SR 39	{0,1,2,3},{0,1,6,3}
9	8	32	8	2	2	4	0	2	SR 10	Take two copies of No. 7
10	9	27	9	3	3	3	0	3	SR 25	{0,1,2},{0,4,8}, {0,5,7}
11	10	50	10	2	2	5	0	2	SR 12	{0,1},{0,3},{0,5}, {0,7},{0,9}
12	12	36	6	2	2	6	0	1	SR 13	{0,1},{0,3},{0,5}
13	16	64	8	2	2	8	0	1	SR 15	{0,1},{0,3},{0,5}, {0,7}
14	20	100	10	2	2	10	0	1	SR 17	{0,1},{0,3},{0,5}, {0,7},{0,9}

The following result shows an interesting application of Theorem 2.2 in a slightly different context relating to inner structure of Steiner systems.

Corollary 2.2.1. A cyclic BIB($v'k^2$,k,1) design does not contain a cyclic BIB(v'k,k,1) subdesign for $k \geq 3$.

Proof. If a cyclic BIB($v'k^2$,k,1) design contains a cyclic BIB(v'k,k,1) subdesign, then by deleting all orbits which contain blocks of this subdesign we obtain a cyclic SRGD design with parameters

(2.5) $v=v'k^2$, $m=k$, $n=v'k$, $b=v'^2k^2$, $r=v'k$, k, $\lambda_1=0$, $\lambda_2=1$.

If this cyclic SRGD design involves short orbits, then from the facts $\lambda_1=0$, $\lambda_2=1$ it can be deduced that such short orbits must be composed of blocks of the form $\{i,n+i,2n+i,\ldots,(m-1)n+i\}$ $(0 \leq i \leq n-1)$ and, consequently, m and n must be relatively prime (cf. proof of Lemma 4.1) which is clearly not the case. Therefore, a cyclic SRGD design with parameters as in (2.5) cannot involve short orbits. Hence the non-existence of such a cyclic design for $k \geq 3$ follows in view of our findings in Example 2.1, completing the proof of the result.

3. The case $\lambda_2 = \lambda_1+1$

GD designs with $\lambda_2=\lambda_1\pm1$ have many interesting statistical optimality properties (see e.g., Takeuchi (1961), Cheng (1978)). For SRGD designs it is well-known that $\lambda_2>\lambda_1$. Considering, therefore, the special case of cyclic SRGD designs with $\lambda_2=\lambda_1+1$, we have the following result.

Theorem 3.1. For an SRGD design with $\lambda_2=\lambda_1+1$ to be cyclic (without short orbits), it is necessary that the parameters should be of one of the following forms:

(i) $v=mu^2$, m, $n=u^2$, $b=m\alpha u^2$, $r=mp\alpha u$, $k = mpu$, $\lambda_1 = mp^2\alpha-1$, $\lambda_2 = mp^2\alpha$,

where p,u are positive integers, $u(\geq 3)$ is odd and $\alpha = (u^2-1)/[mp(u-p)]$ is a positive integer;

(ii) $v=4\alpha$, $m=2$, $n=2\alpha$, $b=4\alpha^2$, $r=2\alpha$, $k=2$, $\lambda_1=0$, $\lambda_2=1$,

where α is a positive integer;

(iii) $v=4p^2\alpha$, $m=2$, $n=2p^2\alpha$, $b=4p^2\alpha^2$, $r=2p\alpha u$, $k=2pu$, $\lambda_1=u^2-1$, $\lambda_2=u^2$,

where p,u are positive integers, $u(\geq 3)$ is odd and $\alpha = (u^2-1)/[2p(u-p)]$ is a positive integer.

Proof. The following lemma which has been proved in Mukerjee, Kageyama and Bhagwandas (1986) will be helpful:

Lemma 3.1. Let $u', s(\geq 2)$ be fixed positive integers. Then the equation $u'u^2 - spuu' + sp^2 = 1$ does not have positive integral-valued solutions (u,p).

For an SRGD design one has

$$r(k-1) = \lambda_1(n-1) + \lambda_2(m-1)n, \tag{3.1}$$

$$rk = \lambda_2 v = \lambda_2 mn. \tag{3.2}$$

Consider now a cyclic SRGD design, without short orbits, having

$$\lambda_2 = \lambda_1 + 1. \tag{3.3}$$

For such a design, b is an integral multiple of v and the parameters must satisfy one of the necessary conditions (i),(ii),(iii) in Theorem 2.1. First suppose that the parameters satisfy the condition (i) in Theorem 2.1. Then n is odd and

$$\lambda_2 = mw, \tag{3.4}$$

w being a positive integer. By (3.1)-(3.4),

$$r = \lambda_2 + n - 1 = mw + n - 1, \tag{3.5}$$

and by (3.2),(3.4),(3.5),

$$k/m = \lambda_2 n/r = mnw/(mw+n-1), \tag{3.6}$$

which is a positive integer. Hence,

$$b/v = r/k = (mw+n-1)^2/(m^2nw), \tag{3.7}$$

which is again a positive integer. In particular, therefore, $(mw+n-1)^2/m^2$ is a positive integer, and consequently,

$$n = m\xi + 1, \tag{3.8}$$

for some positive integer ξ. Applying (3.8) in (3.6),(3.7), it follows that both $(m\xi+1)w/(w+\xi)$ and $(w+\xi)^2/[(m\xi+1)w]$ are positive integers. The integrality of $(w+\xi)^2/[(m\xi+1)w]$ implies that of ξ^2/w. Hence ξ and w are of the form

(3.9) $\xi = pq\alpha$, $w = p^2\alpha$,

where p,q,α are positive integers and p and q are relatively prime. Now,

$$(m\xi+1)w/(w+\xi) = (mpq\alpha+1)p/(p+q),$$

$$(w+\xi)^2/[(m\xi+1)w] = \alpha(p+q)^2/(mpq\alpha+1),$$

which are positive integers. Since p,q are relatively prime and so are $\alpha, mpq\alpha+1$, it follows that both $(mpq\alpha+1)/(p+q)$ and $(p+q)^2/(mpq\alpha+1)$ are positive integers. This means that there exist positive integers u,u' such that

(3.10) $mpq\alpha+1 = u'u^2$, $p+q = u'u$.

Writing $m\alpha = s$ and eliminating q, it follows from (3.10) that

$$u'u^2 - spuu' + sp^2 = 1.$$

Since $s = m\alpha (\geq 2)$, it is clear by Lemma 3.1 that $u' = 1$. Hence by (3.10),

(3.11) $mpq\alpha+1 = u^2$, $p+q = u$.

By (3.3)-(3.9) and (3.11), it may now be seen that the parameters of the design are of the form (i) in Theorem 3.1 where by (3.11), $\alpha = (u^2-1)/[mp(u-p)]$.

In a similar manner, the parametric forms (ii) and (iii) in Theorem 3.1 follow from the situation (iii) in Theorem 2.2. Observe that the situation (ii) in Theorem 2.2 cannot arise if $\lambda_2=\lambda_1+1$.

From Theorem 3.1, the following is evident:

Corollary 3.1.1. For a cyclic SRGD design (without short orbits) with $\lambda_2 = \lambda_1+1$:

(a) If n is odd, then n must be a perfect square and $n \equiv 1 \pmod m$;

(b) If n is even, then m=2 and λ_2 must be an odd perfect square.

From Example 2.1, it is clear that the designs of the form (ii) in Theorem 3.1 can always be constructed cyclically. The same, however, cannot be established for the designs of the forms (i) or (iii). Considering the range $r,k \leq 20$, we find that over this range there are only two designs of the form (i) in Theorem 3.1, namely,

D_1: v=18, m=2, n=9, b=36, r=12, k=6, λ_1=3, λ_2=4,

D_2: v=36, m=4, n=9, b=36, r=12, k=12, λ_1=3, λ_2=4,

and only one design of the form (iii) in Theorem 3.1, namely,

D_3: v=8, m=2, n=4, b=16, r=12, k=6, λ_1=8, λ_2=9.

The design D_3 is the complement of the design

D_3': v=8, m=2, n=4, b=16, r=4, k=2, λ_1=0, λ_2=1,

and from Example 2.1, it follows that D_3', and hence D_3, can be constructed cyclically. A computer search, however, reveals that a cyclic construction of the design D_1 or D_2 is impossible. This investigation completely explores the cyclic SRGD designs with $\lambda_2=\lambda_1+1$ over the range $r,k \leq 20$ and, incidentally, demonstrates that the necessary conditions in Theorem 2.2 are not sufficient in general although they are sufficient over the Clatworthy (1973) range.

It may be further remarked that the computer search for proving the non-existence of cyclic constructions for D_1 and D_2 was done over a microcomputer PC 9801/VM 2 (NEC). For D_1, application of first principles (based on a general program searching cyclic GD designs) established the non-existence in about 15.25 hours, while application of Theorem 2.1 did the same in only 43 seconds. As for D_2, from first principles the search could not be completed even in 70 hours, while applying Theorem 2.1 non-existence followed in about 19 hours. This shows that despite its cumbersome appearance, Theorem 2.1 is of considerable help in so far as computer enumerations are concerned.

4. Cyclic SRGD designs with short orbits

In general, various possibilities arise if short orbits are allowed in cyclic SRGD designs. In order to give some idea about the possible extensions of the results in Section 2 without going into too much of complexities, we treat here only the case when the block size k is prime. Clearly, then m=k and λ_1=0.

Lemma 4.1. If a cyclic SRGD design with prime k has short orbits, then m(= k) and n are relatively prime.

Proof. It is well-known (cf. Rao (1945)) that if a cyclic SRGD design with prime k has short orbits then such short orbits are necessarily generated by the initial block $\{0,n,2n,\ldots,(m-1)n\}$. Since $\lambda_1=0$, it follows that the treatments $0,n,2n,\ldots,(m-1)n$ must belong to different equivalence classes. Now if m and n are not relatively prime, let $h(>1)$ be their highest common factor and define $z = m/h$. Clearly then $1\leq z\leq m-1$ and $zn \equiv 0 \pmod m$ showing that 0 and zn belong to the same equivalence class and thus contradicting the last sentence in the preceding paragraph.

In a cyclic SRGD design with prime k and having short orbits, let $\mu(\geq 1)$ be the number of short orbits. Then the number of full orbits is given by

$$\alpha = [\lambda_2 n(k-1) - \mu(k-1)]/\{(k-1)k\} = (\lambda_2 n-\mu)/k. \tag{4.1}$$

Theorem 4.1. When k is prime, for the existence of a cyclic SRGD design having $\mu(\geq 1)$ short orbits, it is necessary and sufficient that

(a) $\mu \equiv \lambda_2 n \pmod k$,

(b) if $(\lambda_2 n-\mu)/k = \alpha$, then there exist integers f_u^i $(0\leq i\leq m-1,\ 1\leq u\leq \alpha)$ such that defining $L = \{0,1,\ldots,n-1\}$, $f_u^i \in L$ for all i,u and in each of the sets

$$\{f_u^i-f_u^0,\ f_u^{i+1}-f_u^1,\ldots,\ f_u^{m-1}-f_u^{m-1-i},\ f_u^0-f_u^{m-i}-1,\ldots,\ f_u^{i-1}-f_u^{m-1}-1 \mid 1\leq u\leq \alpha\},$$
$$1\leq i\leq m-1,$$

where the differences are reduced mod n, every element of $L-\{\beta i\}$ occurs λ_2 times and the element $\beta i \pmod n$ occurs $\lambda_2-\mu$ times $(1\leq i\leq m-1)$, with β being the minimum positive integer satisfying $k\beta+1 \equiv 0 \pmod n$.

Proof. The proof follows along the line of proof of Theorem 2.1. The necessity of (a) is evident from (4.1). To prove the necessity of (b) note that here $k/m = 1$ and hence, as in Theorem 2.1, the α initial blocks in the full orbits must be of the form

(4.2) $\{mf_u^0, mf_u^1+1, \ldots, mf_u^{m-1}+(m-1)\}$, $1 \le u \le \alpha$,

where $0 \le f_u^i \le n-1$ $(0 \le i \le m-1,\ 1 \le u \le \alpha)$. Let

(4.3) $k\beta+1 = zn$,

where clearly $z \le k-1 (= m-1)$. Then zn is contained in the equivalence class S_1. In short orbits, therefore, 0 and zn occur together in μ blocks and, consequently, in full orbits they occur together in $\lambda_2-\mu$ blocks. From (4.2) and (4.3), proceeding exactly along the line of proof of Theorem 2.1, it follows that among

$$\{f_u^1-f_u^0,\ f_u^2-f_u^1, \ldots,\ f_u^{m-1}-f_u^{m-2},\ f_u^0-f_u^{m-1}-1 \mid 1 \le u \le \alpha\},$$

where the differences are reduced mod n, each element of L other than β is repeated λ_2 times while the element β is repeated $\lambda_2-\mu$ times. The further details regarding the necessity of (b) follow in a similar manner. The sufficiency part of the theorem may be proved by retracing the above steps.

Our next result follows along the line of Theorem 2.2.

Theorem 4.2. When k is prime, for the existence of a cyclic SRGD design having $\mu (\ge 1)$ short orbits, it is necessary that $\mu \equiv \lambda_2 n \pmod k$ and either (i) n is odd or (ii) n is even and λ_2 is odd.

Proof. Summing the elements of the sets in Theorem 4.1(b) in two ways, we have, analogously to (2.3), the necessary condition

$$\tfrac{1}{2}n(n-1)\lambda_2 - i\beta\mu \equiv -i\alpha \pmod n,\ 1 \le i \le m-1,$$

whence separate consideration of the cases of odd and even n yields the required result.

The following example illustrates a cyclic SRGD design with short orbits.

Example 4.1. If k=3 and $n \equiv 1 \pmod 6$ then cyclic SRGD designs with short orbits and having parameters

$$v=3n,\ m=3,\ n,\ b=\lambda_2 n^2,\ r=\lambda_2 n,\ k=3,\ \lambda_1=0,\ \lambda_2 \equiv 1,2 \pmod 3$$

may be constructed as follows. By Colbourn and Colbourn (1981), there exists

a cyclic BIB(n,3,1) design for $n \equiv 1 \pmod 6$. Hence there exists a cyclic BIB $(n,3,\lambda_2)$ design for any λ_2. For each initial block $\{b_0,b_1,b_2\}$ of this design take the following two initial blocks:

$$\{(n-1)b_0,(n-1)b_1-n,(n-1)b_2-2n\},\ \{(n-1)b_0,(n-1)b_1-2n,(n-1)b_2-n\},$$

with the entries reduced mod 3n. Then these initial blocks, together with λ_2 copies of an initial block $\{0,n,2n\}$, which has a short orbit, generate a cyclic SRGD design with the desired parameters.

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References

R.C. Bose: On the construction of balanced incomplete block designs, Ann. Eugen. 9 (1939), 353-399.

R.C. Bose, S.S. Shrikhande and K.N. Bhattacharya: On the construction of group divisible incomplete block designs, Ann. Math. Statist. 24 (1953), 167-195.

C.S. Cheng: Optimality of certain asymmetrical experimental designs, Ann. Statist. 6 (1978), 1239-1261.

W.H. Clatworthy: Tables of Two-Associate-Class Partially Balanced Designs, NBS Applied Mathematics Series 63, Washington, D.C., 1973.

M.J. Colbourn and C.J. Colbourn: Cyclic block designs with block size 3, Europ. J. Combin. 2 (1981), 21-26.

H.A. David and F.W. Wolock: Cyclic designs, Ann. Math. Statist. 36 (1965), 1526-1534.

A. Dey and A.K. Nigam: Construction of group divisible designs, J. Ind. Soc. Agric. Statist. 37 (1985), 163-166.

G.H. Freeman: A cyclic method of constructing regular group divisible incomplete block designs, Biometrika 63 (1976), 555-558.

B.C. Huang, T.P. Lin and W.H. Clatworthy: A methodical search method of constructing CSPBIB designs, Proc. Seminar on Combinatorics and Applications in honour of Professor S.S. Shrikhande on his 65th birthday, Indian Statistical Institute, Calcutta, 1982, 167-175.

M. Jimbo and S. Kuriki: On a composition of cyclic 2-designs, Discrete Math. 46 (1983), 249-255.

M. Jimbo and S.A. Vanstone: Recursive constructions for resolvable and doubly resolvable 1-rotational Steiner 2-designs, Utilitas Math. 26 (1984), 45-61.

R. Mukerjee, S. Kageyama and Bhagwandas: Symmetry and classification of certain regular group divisible designs, Graphs and Combinatorics 2 (1986), to appear.

D. Raghavarao: Constructions and Combinatorial Problems in Design of Experiments, John Wiley, New York, 1971.

C.R. Rao: Finite geometries and certain derived results in the theory of numbers, Proc. Nat. Inst. Sci. India 11 (1945), 136-149.

K. Takeuchi: On the optimality of certain type of PBIB designs, Rep. Statist. Appl. Res. JUSE 8 (1961), 28-33.

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