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FREE BOUNDARY PROBLEM FOR UNSTEADY SLAG FLOW IN THE HEARTH

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0. Introduction.

The hearth drainage is one of the most important factors for successful blast furnace operation. The slag is considered to be more difficult to drain than the metal because of its higher viscosity. When the slag surface reaches the level of tap hole, the furnace gas starts to blow out. Then tapping should be stopped. The amount of undrained molten material at the end of tapping is estimated by the shape of the slag surface. In order to determine the influence of tapping conditions due to the shape of the slag surface, the three dimensional problem of the slag flow during tapping was solved by using the finite element method by Ichihara and Fukutake [2]. They concluded that their computational instability was resolved by Kawarada and Natori [11], using the penalty method developed by themselves [5-7, 10].

The objective of this report is to give mathematical justification of penalty formulation, i.e., to prove the convergence of the penalized free boundary to the one of an original problem when we let the penalizing parameter ϵ tend to 0. In section 1, we review the formulation for two dimensional problems of the slag flow, and then we give the penalized formulation by using the method of integral penalty. Section 2 is devoted to the main theorems. In section 3-5, we give the proof of the main theorems.

1. Formulation.

We consider 2-dimensional flow of the slag in a hearth, which is bounded by an impermeable boundary Γ_0 and a free boundary. Γ_0 is consisting of lines $y = 0$, $x = 0$ or $\mathbf{a}$, and on the boundary $x = 0$ a tapping hole is bored near the bottom $y = 0$. On the other hand the free boundary is represented by $y = \phi(t,x)$, and it is denoted by $\Gamma(\phi)$.

In a slag region, which is written by the function ϕ as

$$\Omega(\phi,t) = \{ (x,y) \mid 0 < x < a \text{ and } 0 < y < \phi(t,x) \},$$

velocity vector v of slag satisfies the condition

$$v = - d\nabla u$$

with a potential

$$u = (P - \bar{P}) / (\bar{\rho} g) + y$$

by Darcy's law. Here d is permeability of slag, $\bar{\rho}$ is density of slag, g is the gravitational acceleration, P is a pressure of slag and $\bar{P}$ is a pressure at a standard point.

From the equation of continuity, $\nabla \cdot v = 0$, it follows that

$$\Delta u = 0 \qquad \text{in } \Omega(\phi,t).$$

Boundary conditions for the potential **u** are given such that **u** = y on the free boundary $\Gamma(\phi)$, and $u_{,n} = 0$ on the fixed boundary except the tapping hole, on which it is assumed to be equal to a given function.

On the expression $\phi(t,x)$ of the surface of free boundary, a point $(x, \phi(t,x))$ on the boundary moves to a point $(x - du_{,x}\mathbf{dt},$ $\phi(t,x) - du_{,y}\mathbf{dt})$ after time **dt** past, it is followed by

$$\phi(t,x) - du_{,y}\mathbf{dt} = \phi(t+\mathbf{dt}, x - du_{,x}\mathbf{dt})$$

$$= \phi(t,x) + \phi_{,t}(t,x)\mathbf{dt} - d\phi_{,x}(t,x)u_{,x}\mathbf{dt} + 0(\mathbf{dt})^2.$$

Therefore we have

$$\phi_{,t}(t,x) = d(u_{,x}\phi_{,x} - u_{,y}).$$

While the unit vector $\boldsymbol{n}(t,x)$ of outward normal direction is

$$(1 + \phi_{,x}(t,x)^2)^{-1/2}(-\phi_{,x}(t,x),\ 1).$$

Thus we obtain

$$\phi_{,t}(t,x) = -d(1 + \phi_{,x}{}^2)^{1/2}u_{,n}\mid_{\Gamma(\phi)},$$

$$\phi(0,x) = \bar{\Phi}(x).$$

The above equations are rewritten as equations of p = u − y and ϕ as follows:

$$\Delta p = 0 \qquad \text{in } \Omega(\phi,t),$$

$$p_{,n}\mid_{\Gamma_0} = \pi, \qquad p\mid_{\Gamma(\phi)} = 0,$$

$$\phi_{,t} = -d[1 + (1 + \phi_{,x}{}^2)^{1/2}p_{,n}\mid_{\Gamma(\phi)}],$$

$$\phi(0) = \bar{\Phi}.$$

Here $\pi = -1$ on the bottom of boundary (y = 0), and $\pi = 0$ on the side boundary (x = 0, a) except the tapping hole.

1.1. An approximation by penalization.

Let the whole hearth be denoted by $B = I \times]0,b[$, $I =]0,a[$, and H be the Heaviside function. Then we penalize the free boundary problem as follows.

We assume a penalized equation in the region $B \setminus \Omega(\phi,t)$ with a penalty parameter $\epsilon > 0$, then the state equations satisfied by functions $p(\epsilon)$ and $\phi(\epsilon)$ of t, x and y are

$$- \Delta p(\epsilon)(t,x,y) + \epsilon^{-1}H(y-\phi(\epsilon)(t,x))p(\epsilon)(t,x,y) = 0,$$

$$p(\epsilon)_{,n} \big|_{\Gamma_0} = \pi,$$

$$p(\epsilon) \big|_{\Gamma_1} = 0.$$

Where Γ_1 is the top of the boundary ∂B, and Γ_0 is extended to the whole rest boundary. For $\phi(\epsilon)$, we have in the distribution sense of $\mathcal{D}'(B)$

$$\epsilon^{-1}H(y - \phi(\epsilon)(t,x))p(\epsilon)(t,x,y)$$

$$\to (- \phi_{,x}(t,x)p_{,x}(t,x,y) + p_{,y}(t,x,y))H'(y - \phi(t,x)),$$

and

$$\epsilon^{-1} \int_y^b H(\eta - \phi(\epsilon)(t,x))p(\epsilon)(t,x,\eta)d\eta$$

$$\to - (1 + \phi_{,x}(t,x)^2)^{1/2}p_{,n}(t,x,\phi(t,x))(1 - H(y - \phi(t,x))),$$

as $\epsilon \to 0$, which are similarly shown as in [3]. Therefore as the governing equations of $\phi(\epsilon)$ we introduce

$$\phi(\epsilon)_{,t}(t,x) = - d(1 - \epsilon^{-1} \int_{\phi(\epsilon)(t,x)}^{b} p(\epsilon)(t,x,y)dy),$$

$$\phi(\epsilon)(0,x) = \bar{\Phi}(x),$$

by the idea of the integrated penalty.

2. Main results.

2.1. Theorem 1.

There exists a unique solution p^0 and ϕ^0 such that

$$\Delta p^0 = 0 \qquad \text{in } \Omega(\phi^0, t),$$

$$p^0_{,n}\,|\,\Gamma_0 = \pi, \qquad p^0\,|\,\Gamma(\phi^0) = 0,$$

$$\phi^0_{,n} = -\,d[1 + (1 + \phi^0_{,x}{}^2)^{1/2} p^0_{,n}\,|\,\Gamma(\phi^0)],$$

$$\phi^0(0) = \bar{\Phi},$$

if $\bar{\Phi}$ is smooth enough and $T > 0$ is small enough.

2.2. Theorem 2.

There exists a unique solution p^ϵ and ϕ^ϵ such that

$$-\,\Delta p^\epsilon(t,x,y) + \epsilon^{-1} H(y - \phi^\epsilon(t,x))\, p^\epsilon(t,x,y) = 0, \quad (x,y) \in B,$$

$$p^\epsilon_{,n}\,|\,\Gamma_0 = \pi, \qquad p^\epsilon\,|\,\Gamma_1 = 0,$$

$$\phi^\epsilon_{,t}(t,x) = -\,d[1 - \epsilon^{-1} \int_{\phi^\epsilon(t,x)}^{b} p^\epsilon(t,x,y)\,dy],$$

$$\phi^\epsilon(0,x) = \bar{\Phi}(x),$$

if $\bar{\Phi}$ is smooth enough, $T > 0$ is small enough (uniformly in ϵ).

2.3. Theorem 3.

If $\bar{\Phi}$ is smooth enough and $T > 0$ is small enough, then

$$\phi^\epsilon \to \phi^0 \qquad \text{as } \epsilon \to 0,$$

in maximum norm.

2.4.

In order to show the theorem 1 and 2 we will use the Nash-Moser's implicit function theorem, which is described as follows:

let E_j, F_j be real Banach space ($j = 0, 1, \cdots, 11$) such that

$$E_0 \supset E_1 \supset \cdots \supset E_{11}, \quad F_1 \supset \cdots \supset F_{11}.$$

And let $\delta > 0$, $V = \{ x \in E_1 \mid |x|_1 < \delta \}$, and

$$\mathcal{F} : V \to F_1 \text{ such that } \mathcal{F}(V \cap E_i) \subset F_i \text{ for any } i.$$

If

(1) [the existence of smoothing operators]

$\exists \theta \geq 1$ and $\exists \mathfrak{S}_\theta \in (E_0 \to E_{11})$; $\forall i, j,\ 0 \leq i \leq j \leq 11 \Rightarrow$

- $x \in E_i \Rightarrow |\mathfrak{S}_\theta x|_j \leq C\theta^{j-i}|x|_i$,
- $x \in E_j \Rightarrow |x - \mathfrak{S}_\theta x|_i \leq C\theta^{i-j}|x|_j$,

(2) [F-differentiability of $\mathcal{F}$]

$\mathcal{F}' \in (V \to L(E_1 \to F_1))$: F-derivative of $\mathcal{F}$,

- $|\mathcal{F}'(x, h)|_1 \leq C|h|_1$ for any $h \in E_1$,
- $|\mathcal{F}(x+h) - \mathcal{F}(x) - \mathcal{F}'(x,h)|_1 \leq C|h|_1^2$,

(3) [the existence of right inverse of $\mathcal{F}$]

$\exists \mathcal{I} \in (V \to (F_1 \to E_0))$;

- $\mathcal{I}(x)F_i \subset E_{i-1}$ $\quad \forall i,\ x \in V \cap E_i$,
- $\mathcal{F}'(x, \mathcal{I}(x,y)) = y$ $\quad \forall x \in V \cap E_2,\ y \in F_2$,
- $|\mathcal{I}(x,y)|_0 \leq C|y|_1$ $\quad \forall y \in F_1$,
- $|\mathcal{I}(x, \mathcal{F}(x))|_{i-1} \leq C(1 + |x|_i)$ $\quad \forall i > 0,\ x \in V \cap E_i$,

(4) $|\mathcal{F}(0)| < \epsilon(C, \delta)$

then

$$\exists x \in V;\ \mathcal{F}(x) = 0.$$

3. Proof of theorem 1.

In order to apply Nash-Moser's implicit function theorem, the problem will be reformulated as an equation of ϕ. Let a function ρ defined in $[0,T] \times I$ be a difference of the free boundary from the initial state, and F be the functional in ρ defined by

$$F(\rho) = \rho_{,t} + d[1 + (1+\phi_{,x}{}^2)^{1/2} P(\rho)_{,n} \mid \Gamma(\phi)],$$

$$\phi = \bar{\Phi} + \rho,$$

where $P(\rho)$ is the solution of

$$-\Delta P(\rho) = 0 \quad \text{in } \Omega(\phi, t),$$

$$P(\rho)_{,n} \mid \Gamma_0 = \pi,$$

$$P(\rho) \mid \Gamma(\phi) = 0.$$

Then the problem is reduced to find a zero point ρ^0 of F.

3.1.

For any $\delta > 0$, let $I_\delta = [\delta, a-\delta]$, then

$$F \in (C^m([0,T] \times I) \to C^{m-1}([0,T] \times I_\delta)).$$

3.2. (The first Frechet derivative of F.)

$$F'(\rho, \sigma) = \sigma_{,t} + d[(1+\phi_{,x}{}^2)^{-1/2} \phi_{,x} \sigma_{,x} P(\rho)_{,n} \mid \Gamma(\phi) + (1+\phi_{,x}{}^2)^{1/2} [P(\rho)_{,n} \mid \Gamma(\phi)]_{,\rho}(\sigma)]$$

and

$$[P(\rho)_{,n} \mid \Gamma(\phi)]_{,\rho}(\sigma) = P'(\rho, \sigma)_{,n} \mid \Gamma(\phi) + P(\rho)_{,ny} \mid \Gamma(\phi)\, \sigma + P(\rho)_{,n} \mid \Gamma(\phi)\, n \cdot n_{,\rho},$$

where $P'(\rho, \sigma)$ is the solution of an equation

$$-\Delta P'(\rho, \sigma) = 0, \quad P'(\rho, \sigma)_{,n} \mid \Gamma_0 = 0,$$

$$P'(\rho,\sigma)\,|\,\Gamma(\phi) = -P(\rho)_{,n}n_y\,|\,\Gamma(\phi)\,\sigma.$$

3.3. (The second Frechet derivative of F.)

$$F''(\rho)(\sigma)^2 = d[(1+\phi_{,x}{}^2)^{-3/2}\sigma_{,x}{}^2P(\rho)_{,n}\,|\,\Gamma(\phi)$$
$$+ 2(1+\phi_{,x}{}^2)^{-1/2}\phi_{,x}\sigma_{,x}[P(\rho)_{,n}\,|\,\Gamma(\phi)]_{,\rho}(\sigma)$$
$$+ (1+\phi_{,x}{}^2)^{1/2}[P(\rho)_{,n}\,|\,\Gamma(\phi)]_{,\rho\rho}(\sigma)^2\ ]$$

where $P''(\rho,\sigma^2)$ appeared in $[P(\rho)_{,n}\,|\,\Gamma(\phi)]_{,\rho\rho}$ is the solution of an equation of the same kind. Therefore $F''(\rho) \in L(C^m \to L(C^m \to C^{m-3}))$ is bounded.

3.4.

For $\rho \in C^{m+4}$, let $J(\rho) \in (C^{m+1-\alpha} \to C^m)$, $\alpha > 0$, be an operator defined by $I(\rho)(\tau) = \sigma$ such that

$$\sigma_{,t} + d\{(1+\phi_{,x}{}^2)^{-1/2}\phi_{,x}P(\rho)_{,n}\,|\,\Gamma(\phi)\,\sigma_{,x}$$
$$+ (1+\phi_{,x}{}^2)^{-1/2}[\,P(\rho)_{,n}\,|\,\Gamma(\phi)\,]_{,\rho}\,\sigma\} = \tau.$$

Thus we can apply the Nash-Moser's implicit function theorem with $E_j = C^{m+4j-4}$, $F_j = C^{m+4j-8-\alpha}$.

Here since $|\,F(0)\,|_{m-4-\alpha} \leq CT^{\alpha}\,|\,F(0)\,|_{m-4}$, we can make $|\,F(0)\,|$ small enough by taking T be small.

3.5. (Uniqueness.)

Let ρ_0 and ρ_1 be zero points of F, and let $\rho = \rho_1 - \rho_0$. Then ρ satisfies

$$\rho_{,t}$$
$$+ d((1+\phi_{1,x}{}^2)^{1/2}+(1+\phi_{0,x}{}^2)^{1/2})^{-1}(\phi_{1,x}+\phi_{0,x})P(\rho_1)_{,n}\,|\,\Gamma(\phi_1)\,\rho_{,x}$$
$$+ d(1+\phi_{0,x}{}^2)^{1/2}B(\rho_0,\rho_1)\,\rho = 0,$$
$$\rho(0) = 0,$$

where $B(\rho_0,\rho_1) = \int_0^1 [\,P(\rho_\theta)_{,n}\,|\,\Gamma(\phi_\theta)\,]_{,\rho}$, $\rho_\theta = \rho_0 + \theta(\rho_1 - \rho_0)$ and $\phi_\theta = \bar{\Phi} + \rho_\theta$. It follows that $\rho = 0$.

4. Proof of theorem 2.

In order to prove the existence and uniqueness of solutions of the penalized problem, let us define F^{ϵ} by

$$F^{\epsilon}(\rho) = \rho_{,t} + d[1 - \epsilon^{-1}\int_0^b H(y-\phi(t,x))P^{\epsilon}(\rho)dy],$$
$$\phi = \bar{\Phi} + \rho,$$

where $P^{\epsilon}(\rho)$ is the solution of an equation

$$-\Delta P^{\epsilon}(\rho) + \epsilon^{-1}H(y-\phi(t,x))P^{\epsilon}(\rho) = 0,$$
$$P^{\epsilon}(\rho)_{,n}|_{\Gamma_0} = \pi, \quad P^{\epsilon}(\rho)|_{\Gamma_1} = 0.$$

Then the penalized problem is reduced to find a zero point ρ^{ϵ} of F^{ϵ}.

4.1.

$$F^{\epsilon\prime}(\rho,\sigma)(t,x) = \sigma_{,t}(t,x)$$
$$- d\epsilon^{-1}[\int_{\phi(t,x)}^{b} P^{\epsilon\prime}(\rho,\sigma)(t,x,y)dy$$
$$- P(\rho)(t,x,\phi(t,x))\,\sigma(t,x)]$$

here $\delta p^{\epsilon} = P^{\epsilon\prime}(\rho,\sigma)$ is the solution of an equation

$$-\Delta\,\delta p^{\epsilon} + \epsilon^{-1}H(y-\phi(t,x))\,\delta p^{\epsilon} = 0,$$
$$\delta p^{\epsilon}{}_{,n}|_{\Gamma_0} = 0, \quad \delta p^{\epsilon}|_{\Gamma_1} = 0,$$
$$\delta p^{\epsilon}{}_{0,n}|_{\Gamma(\phi)} - \delta p^{\epsilon}{}_{1,n}|_{\Gamma(\phi)} = \epsilon^{-1}P^{\epsilon}(\rho)n_y|_{\Gamma(\phi)}\sigma.$$

Thus we have $F^{\epsilon\prime}(\rho,\sigma) \in C^{m-1}$.

4.2.

$$F^{\epsilon\prime\prime}(\rho,\sigma^2)(t,x) = -d\epsilon^{-1}[\int_{\phi(t,x)}^{b} P^{\epsilon\prime\prime}(\rho,\sigma^2)(t,x,y)dy$$
$$- 2P'(\rho,\sigma)(t,x,\phi(t,x))$$
$$- P(\rho)_{,y}(t,x,\phi(t,x))\,\sigma(t,x)]$$

here $\delta^2 p = P^{\epsilon\prime\prime}(\rho,\sigma^2)$ satisfies the equation

$$-\Delta\delta^2 p+\epsilon^{-1}H(y-\phi(t,x))\delta^2 p=0,$$
$$\delta^2 p_{,n}\mid\Gamma_0=0,\quad \delta^2 p\mid\Gamma_1=0,$$
$$\delta^2 p_0\mid\Gamma(\phi)-\delta^2 p_1\mid\Gamma(\phi)=\epsilon^{-1}P^{\epsilon}(\rho)n_y\mid\Gamma(\phi)\sigma^2,$$
$$\delta^2 p_{0,n}\mid\Gamma(\phi)-\delta^2 p_{1,n}\mid\Gamma(\phi)=\epsilon^{-1}[P^{\epsilon}(\rho)n_y]_{,\rho}\,\sigma^2+\cdots.$$

So we have $F^{\epsilon\prime\prime}(\rho,\sigma^2)\in C^{m-1}$.

4.3.

For $\rho\in C^{m+2}$, we can define the operator $I(\rho)\in(C^{m+1-\alpha}\to C^m)$ by $I(\rho)(\tau)=\sigma$ such that

$$F^{\epsilon\prime}(\rho)\sigma=\tau.$$

Thus we can apply the Nash-Moser's implicit function theorem with $E_j=C^{m+2j-2}$, $F_j=C^{m+2j-4-\alpha}$, $\alpha>0$.

5. Convergency of penalized solutions (outline).

5.1.

It is shown in [9] that

$$P(\epsilon,\rho)\longrightarrow P(\rho)(1-H(y-\phi(t,x)))\quad\text{as }\epsilon\to 0.$$

5.2.

For small enough $\epsilon>0$,

$$\|P^{\epsilon}(\rho)\mid\Gamma(\phi)\|_m \le C\epsilon^{1/2}\|P(\rho)_{,n}\mid\Gamma(\phi)\|_{m+1},$$

$$\|(P^{\epsilon}(\rho)+\epsilon^{1/2}P(\rho)_{,n})\mid\Gamma(\phi)\|_m \le C\epsilon\|P(\rho)_{,n}\mid\Gamma(\phi))\|_m,$$

$$\|\int\Gamma(\phi)\perp P^{\epsilon}(\rho)-\epsilon^{1/2}P^{\epsilon}(\rho)\mid\Gamma(\phi)\|_m$$

$$\leq C\epsilon \| P^{\epsilon}(\rho) |\Gamma(\phi)\|_{m+1}.$$

5.3.

$$\|(P^{\epsilon\prime}(\rho,\sigma) - \epsilon^{-1/2}P^{\epsilon}(\rho)n_y\sigma)|\Gamma(\phi)\|_m$$
$$\leq C\epsilon^{1/2}\|(n_y\sigma)|\Gamma(\phi)\|_{m+1,\infty}\|P(\rho)_{,n}|\Gamma(\phi)\|_{m+1},$$
$$\|\int_{\Gamma(\phi)^{\perp}} P^{\epsilon\prime}(\rho,\sigma)dn - \epsilon^{1/2}P^{\epsilon\prime}(\rho,\sigma)|\Gamma(\phi)\|_m$$
$$\leq C\epsilon\|n_y|\Gamma(\phi)\|_{m+1,\infty}\|P(\rho)_{,n}|\Gamma(\phi)\|_{m+1}.$$

5.4.

For $P^{\epsilon}(\rho)$, we have

$$\|\int_{\phi(t,x)}^{b} P^{\epsilon}(\rho)(t,x,y)dy + \epsilon(1+\phi_{,x}{}^2)^{1/2}P(\rho)_{,n}|\Gamma(\phi)\|_1$$
$$\leq C\epsilon^{3/2}\|P^{\epsilon}(\rho)\|_{5/2}$$
$$\leq C(\|\phi\|_{5/2})\epsilon^{3/2}\|\pi\|_{3/2}.$$

Therefore

$$\|\rho_{,t} + d[1 - \epsilon^{-1}\int_{\phi(t,x)}^{b}P^{\epsilon}(\rho)]\| \leq C\epsilon^{1/2}.$$

Let e^{ϵ} denote the error $\rho^{\epsilon} - \rho^0$ in penalization, then

$$e^{\epsilon}{}_{,t} - d\epsilon^{-1}[\int_{\phi^{\epsilon}(t,x)}P^{\epsilon}(\rho^{\epsilon}) - \int_{\phi^0(t,x)}P^{\epsilon}(\rho_0)]$$
$$= O(\epsilon^{1/2}).$$

Now let an operator F be defined by

$$F(\rho) = \int_{\phi(t,x)}^{b}P^{\epsilon}(\rho)(t,x,y)dy,$$

so that

$$e^{\epsilon}{}_{,t} + d[F(\rho^{\epsilon}) - F(\rho^0)] = O(\epsilon^{1/2}).$$

Since $\|F'(\rho,\sigma)\|_m$

$$= \|\int_{\phi(t,x)}P^{\epsilon\prime}(\rho,\sigma)(t,x,y)dy - P^{\epsilon}(\rho)\sigma|\Gamma(\phi)\|_m \leq C\epsilon,$$

we can show that $\nu(t) = \|e^{\epsilon}(t)\|_{C(I)}$ satisfies the inequality

$$\nu(t) \leq C\int_0^t \nu(\tau)d\tau + C\epsilon^{1/2}t, \quad \text{and} \quad \nu(0) = 0.$$

Therefore we have $\nu(t) \leq C\epsilon^{1/2}$ in some finite interval [0,T].

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