

Title	On Integers Defined by a Linear Recurrence Relation of Order Two
Author(s)	UCHIYAMA, Saburo
Citation	数理解析研究所講究録 (1984), 537: 35-41
Issue Date	1984-10
URL	http://hdl.handle.net/2433/98697
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

2 次の一線型回帰数列について

— On Integers Defined by a Linear Recurrence Relation of Order Two —

筑波大 数学 内山三郎 (Saburô UCHIYAMA)

In the course of studying various Diophantine problems the writer had several occasions to encounter the sequence of so-called Pell numbers, that is, a sequence of integers P_n ($n = 0, 1, 2, \dots$) defined by

$$P_0 = 0, \quad P_1 = 1, \quad \text{and} \quad P_{n+1} = 2P_n + P_{n-1} \quad \text{for} \quad n \geq 1.$$

It will be convenient to consider, together with the Pell numbers P_n , the associated numbers Q_n ($n = 0, 1, 2, \dots$) defined by

$$Q_0 = 1, \quad Q_1 = 1, \quad \text{and} \quad Q_{n+1} = 2Q_n + Q_{n-1} \quad \text{for} \quad n \geq 1.$$

Explicit formulae for the P_n and Q_n are

$$P_n = \frac{1}{2\sqrt{2}} \left((1 + \sqrt{2})^n - (1 - \sqrt{2})^n \right),$$

$$Q_n = \frac{1}{2} \left((1 + \sqrt{2})^n + (1 - \sqrt{2})^n \right),$$

and, to collect some simple identities involving P_n and Q_n we note

$$(P_m, P_n) = P_{(m, n)}, \quad P_{m+n} = P_m Q_n + P_n Q_m,$$

$$Q_{m+n} = Q_m Q_n + 2P_m P_n, \quad P_n^2 - P_{n-1} P_{n+1} = (-1)^{n-1},$$

$$P_{2n-1} = P_n^2 + P_{n-1}^2, \quad P_{2n} = 2P_n(P_n + P_{n-1}),$$

$$Q_n^2 - 2P_n^2 = (-1)^n.$$

Here we discuss some arithmetical properties of the (sequences of) Pell numbers P_n .

1) The sequence $(P_n)_{n=1, 2, \dots}$ is uniformly distributed modulo an integer $m > 1$ (in the sense of I. Niven) for $m = 2$ and for no other values of m .

The discriminant of the characteristic polynomial of the defining relation for the P_n is $8 = 2^3$. The sequence (P_n) is uniformly distributed modulo 2 since

$$P_n \equiv n \pmod{2},$$

and is not uniformly distributed modulo 2^h for any $h > 1$, since

$$P_n \equiv 0, 1, 2, \text{ or } 1 \pmod{4}$$

according as

$$n \equiv 0, 1, 2, \text{ or } 3 \pmod{4}.$$

2) The sequence $(\log P_n)_{n=1, 2, \dots}$ is uniformly distributed modulo 1.

This follows from the fact that we have for $n \rightarrow \infty$

$$\log P_{n+1} - \log P_n \rightarrow \log(1 + \sqrt{2}) \notin \mathbb{Q}.$$

3) The sequence $([\log P_n])_{n=1, 2, \dots}$ is uniformly distributed modulo m for every integral $m \geq 2$.

These results can be obtained just as in L. Kuipers, J.-Sh. Shiue, and H. Niederreiter, who proved the corresponding results for the sequence of Fibonacci numbers F_n .

4) By a general result of K. Nagasaka, 'Benford's Law of Anomalous Numbers' is obeyed by the sequence $(P_n)_{n=1,2,\dots}$. Thus, in particular, the frequency of appearance of a $(1 \leq a \leq 9)$ as the left-most digit in the P_n equals $\log_{10}(1 + (1/a))$, the P_n being expressed in the ordinary decimal system.

Here is a small numerical observation.

digit	number of count ($1 \leq n \leq 100$)	number of count ($101 \leq n \leq 200$)	expected number $100 \log_{10}(1 + \frac{1}{a})$
1	30	31	30.1
2	19	17	17.6
3	11	13	12.5
4	9	9	9.7
5	9	8	7.9
6	6	7	6.7
7	6	6	5.8
8	5	4	5.1
9	5	5	4.6

5) It is known that $P_1 = 1$ and $P_7 = 169$ are the only square Pell numbers (apart from $P_0 = 0$). One can hardly prove this fact without appealing to W. Ljunggren's theorem which states that the only solutions in positive integers x, y of the Diophantine equation

$$x^2 - 2y^4 = -1$$

are $x = y = 1$ and $x = 239, y = 13$.

By the way, Ljunggren's proof for his above mentioned result being highly complicated and difficult, there are some authors who express their wish to have a simple and/or elemen-

tary proof of the result. We find that the problem is eventually to prove that $X = 3, Y = 2$ is the only solution in positive integers X, Y of the equation

$$X^4 + 4X^3Y - 6X^2Y^2 - 4XY^3 + Y^4 = 1$$

and that the equation

$$X^4 - 4X^3Y - 6X^2Y^2 + 4XY^3 + Y^4 = 1$$

has no solutions in positive integers X, Y .

It will be of some interest to note that an application of A. Baker's argument of effectiveness yields the following upper bound for $|X|, |Y|$, where X, Y are any possible integer solutions of these Diophantine equations:

$$\max(|X|, |Y|) < \exp(3^2 \cdot 2^{3522617}) = 10^{10^{6.02548}}$$

It is not hard to prove that $P_0 = 0$ is the only square value of P_{2n} , that is, the equation

$$x^2 - 2y^4 = 1$$

admits only trivial solutions with $y = 0$. In fact, we have

$$P_{n+8} \equiv P_n \pmod{8}; \text{ also } P_{n+20} \equiv P_n \pmod{29}, \text{ since}$$

$$P_{n+20} = Q_n P_{20} + Q_{20} P_n$$

and

$$29 = P_5 |P_{20}, \quad Q_{20} = 22619537 \equiv 1 \pmod{29}.$$

We have, therefore,

$$P_n \equiv 2 \pmod{29} \quad \text{if } n \equiv 2, 8, 22, \text{ or } 28 \pmod{40},$$

$$P_n \equiv 12 \pmod{29} \quad \text{if } n \equiv 4, 6, 24, \text{ or } 26 \pmod{40},$$

$$P_n \equiv 27 \pmod{29} \quad \text{if } n \equiv 12, 18, 32, \text{ or } 38 \pmod{40},$$

$$\begin{aligned}
P_n &\equiv 17 \pmod{29} && \text{if } n \equiv 14, 16, 34, \text{ or } 36 \pmod{40}, \\
P_n &\equiv 2 \pmod{8} && \text{if } n \equiv 10 \pmod{40}, \text{ and} \\
P_n &\equiv 6 \pmod{8} && \text{if } n \equiv 30 \pmod{40}.
\end{aligned}$$

(Note that 2, 12, 27, 17 are quadratic non-residues (mod 29).)

It remains, therefore, only to consider the values of P_n for
 $n \equiv 0, \text{ or } 20 \pmod{40}$.

We have

$$P_{n+10} = Q_n P_{10} + Q_{10} P_n,$$

where

$$Q_{10} = 3363 \equiv 1 \pmod{41}, \quad P_{10} = 2378 \equiv 0 \pmod{41}.$$

Now, let m be the least positive integer such that P_{10m} is either a square or twice a square. If m is odd then

$$P_{10m} = 2 Q_{5m} P_{5m},$$

where

$$P_{5m} \equiv P_5 = 29 \pmod{41},$$

29 being a quadratic non-residue (mod 41). So m must be even, and $P_{5m} = P_{10(m/2)}$ must be a square or twice a square, and we have a contradiction. It follows that $m = 0, P_0 = 0$.

6) It follows from the result of 4) above that there are no Pell numbers P_n which are twice a square, other than $P_2 = 2$ (and $P_0 = 0$).

7) Finally, we should like to give a proof for the fact that $Q_0 = Q_1 = 1$ are the only numbers Q_n which are a square.

Note that $Q_n \equiv 1 \pmod{2}$ for all n . We distinguish two cases according as n is even or odd.

Case of n even: Consider the Diophantine equation

$$x^4 - 2y^2 = 1,$$

which can be rewritten as $(x^2 - 1)(x^2 + 1) = 2y^2$. Since $(x^2 - 1, x^2 + 1) = 2$ and $2 \parallel x^2 + 1$, we must have $x^2 + 1 = 2z^2$ and $x^2 - 1 = w^2$ for some integral z, w . Therefore, the only possibility is $w = 0, x = 1, z = 1, y = 0$ (here, and in what follows also, we have only to consider non-negative values of the unknowns involved), thus giving $Q_0 = 1$.

Case of n odd: Consider the equation

$$x^4 - 2y^2 = -1,$$

which we rewrite as

$$\left(\frac{x^2 - 1}{2}\right)^2 + \left(\frac{x^2 + 1}{2}\right)^2 = y^2.$$

Since $(x^2 - 1)/2$ and $(x^2 + 1)/2$ are coprime and $(x^2 + 1)/2$ is odd, we have for some integers a, b with $(a, b) = 1$, $a + b \equiv 1 \pmod{2}$

$$\frac{x^2 - 1}{2} = 2ab, \quad \frac{x^2 + 1}{2} = a^2 - b^2;$$

this implies

$$x^2 = a^2 - b^2 + 2ab, \quad 1 = a^2 - b^2 - 2ab$$

and so

$$x^2 = (a^2 - b^2)^2 - (2ab)^2.$$

Hence we must have for some integral c, d $2ab = 2cd$, $x = c^2 - d^2$, $a^2 - b^2 = c^2 + d^2$, which gives us $1 = c - d$, $x = c + d$, where $a \equiv c \equiv 1 \pmod{2}$, $b \equiv d \equiv 0 \pmod{2}$. How-

ever, it is known and in fact is not quite difficult to prove that the only integer solutions of the equation

$$x^2 = a^4 - 6a^2b^2 + b^4, \quad (a, b) = 1,$$

are given by $a = 0$ or $b = 0$. Thus we have $b = d = 0$, giving $x = c = 1$ and so $Q_1 = 1$.

This completes the proof of our assertion.

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Cf. Chap. 4, Theorem 3 (pp. 18 - 19).